The Kalmanson Complex

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Abstract

Let X be a finite set of cardinality n. The Kalmanson complex $\mathcal{K}_n$ is the simplicial complex whose vertices are non-trivial X-splits, and whose facets are maximal circular split systems over X. In this paper we examine $\mathcal{K}_n$ from three perspectives. In addition to the T-theoretic description, we show that $\mathcal{K}_n$ has a geometric realization as the Kalmanson conditions on a finite metric. A third description arises in terms of binary matrices which possess the circular ones property. We prove the equivalence of these three definitions. Using these equivalences we then partially describe the f-vector of $\mathcal{K}_n$.

1 Introduction

A phylogenetic tree is a connected, acyclic graph which presents the common evolutionary history of a group of species (taxa). A *phylogenetic network* generalizes this structure by allowing for the presence of cycles. These networks have become a popular means of conveying recombination, horizontal transfer, and other reticulate events which cannot be represented by a bifurcating tree [9, 2].

In this paper we study the combinatorial structure of phylogenetic networks. As described in [5], these networks are mathematically founded in T-theory and related results on metrics over a finite set. [11] gave a combinatorial interpretation of the output of the Neighbor-Net algorithm, a popular method for estimating phylogenetic networks. Our work is inspired by, but distinct from, these papers. Here we study the space of metrics which underlies phylogenetic networks from a purely combinatorial viewpoint, as a simplicial complex. To our knowledge this is the first attempt to define and study these networks in such a way.

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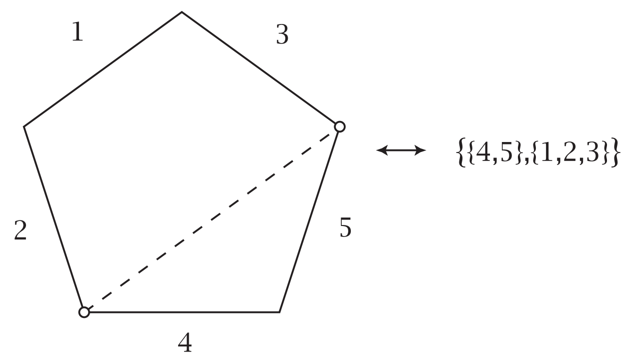

Figure 1: A circular split.

1.1 Background

We begin with some basic concepts from T-theory. For a full introduction, the reader is referred to [3, 8].

Throughout the paper, X is a finite set of cardinality $n \geq 4$. An X*-split* is a bipartition of X; that is, $S = \{A, B\}$ is an X-split if $A \cap B = \emptyset$ and $A \cup B = X$. (When the meaning is obvious, we will simply call S a split.) A and B are called the *blocks* of S, and the *size* of S is defined as $\mathrm{size}(S) := \min\{|A|, |B|\}$. S is *non-trivial* if $\mathrm{size}(S) > 1$ and *minimal* if $\mathrm{size}(S) = 2$.

Let $\mathcal{S}(X)$ be the set of non-trivial X-splits. A *split system* $\mathcal{S} \subset \mathcal{S}(X)$ is a set of splits. [3] introduced the concept of a *circular split system*.[1]

> **Definition 1.** A split system $\mathcal{S}$ is *circular* if there is a permutation $\sigma \in S_n$ such that for each split $S = \{A, B\} \in \mathcal{S}$ there exists $i, j \in [n]$ such that
>
> $$S = \left\{ \left\{ x_{\sigma(\bar{i})}, x_{\sigma(\overline{i+1})}, \ldots, x_{\sigma(\overline{j-1})}, x_{\sigma(\bar{j})} \right\}, \left\{ x_{\sigma(\bar{j})}, x_{\sigma(\overline{j+1})}, \ldots, x_{\sigma(\overline{i-1})}, x_{\sigma(\bar{i})} \right\} \right\}$$
>
> where $\bar{i}$ denotes $i \pmod{n}$.

Circular split systems have a simple geometric interpretation: they are obtained by labeling the edges of a regular n-gon, and connecting its edges with diagonals to form splits (Figure 1). From this we see that a circular split system contains at most $\binom{n}{2}$ distinct splits. A partial converse also holds: any weakly compatible (cf. Section 4) split system containing $\binom{n}{2}$ splits is circular [3].

From Definition 1 we see the set of circular split systems is closed under the operations of taking subsets (any subset of a circular split system is circular) and forming intersections. Hence, it is a simplicial complex. This complex is our main object of study.

[1] Circular split systems are sometimes referred to in the literature as *cyclic split systems*.

Definition 2. The *Kalmanson complex* is the simplicial complex whose vertices are X-splits, and whose facets are maximal circular split systems.

Clearly this complex is unique up to the cardinality of X. Henceforth we write $\mathcal{K}_n$ to denote the Kalmanson complex over a base set of cardinality n.

The complex is named after [10], whose investigations into polynomial time-solvable instances of the traveling salesman problem lead to a geometric realization of $\mathcal{K}_n$ in terms of permuted systems of linear inequalities. Another description of the complex is found in computer science, where $\mathcal{K}_n$ arises as equivalence classes of certain binary matrices which possess the consecutive ones property [4].

1.2 Summary of Results

We investigate how these three descriptions — geometric, combinatorial, and computer scientific — relate to one another. Our main result is to prove that the various formulations of $\mathcal{K}_n$ are combinatorially isomorphic. We then use this result to study the f-vector of $\mathcal{K}_n$ (Theorems 2 and 10). We relate the problem of enumerating the faces of $\mathcal{K}_n$ to a counting problem on certain classes of binary matrices, and exploit a structure theorem of [17] to obtain a new result on the number of triangles in $\mathcal{K}_n$ (Theorem 17). Even in the simplest non-trivial case, this counting problem is seen to possess considerable complexity, and determining a more general method of counting the faces of $\mathcal{K}_n$ remains an interesting open problem.

The remainder of the paper is organized as follows. In Section 2, we review the Kalmanson conditions. These are a set of inequality restrictions on a finite metric which, when satisfied, allow the traveling salesman problem to be solved in polynomial time. We show that these inequalities give a geometric realization of $\mathcal{K}_n$. In Section 3, we study the consecutive ones property for binary matrices. This property is shown to be equivalent to the circularity property for split systems discussed above, giving us a third description of $\mathcal{K}_n$ in terms of equivalence classes of binary matrices. In Section 4, we use these three ways of viewing $\mathcal{K}_n$ to enumerate some of its faces, thus giving a partial characterization of its f-vector. Finally, in Section 5 we offer some concluding remarks, open problems and potential directions for further research.

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2 Geometry of $\mathcal{K}_n$

Throughout this section, we let $D = (d_{ij})_{i,j\in[n]}$ be a symmetric, non-negative matrix with zeros along the diagonal. We refer to matrices possessing this property as *distance matrices.*

Definition 3. Let $D = (d_{ij})$ be a distance matrix and let S_n denote the symmetric group on n letters. The *traveling salesman problem* (TSP) *over* D is

$$\min_{\sigma\in S_n}\left(\sum_{i=1}^{n-1} d_{\sigma(i)\sigma(i+1)} + d_{\sigma(n)}d_{\sigma(1)}\right)$$

For general D, it is well-known that the TSP is NP-hard. However, some special cases have lower complexity. In particular, [10] showed that if D satisfies a certain set of linear inequalities, then the TSP over D possesses a trivial solution.

Theorem 1 ([10])**.** *Let D be a distance matrix. If*

$$\max(d_{ij} + d_{kl}, d_{il} + d_{jk}) \leq d_{ik} + d_{jl} \text{ for all } 1 \leq i < j < k < l \leq n \tag{1}$$

then the identity permutation solves the TSP over D.

The inequalities (1) are referred to as the *Kalmanson conditions*, and a matrix which satisfies them is a *Kalmanson matrix* (or simply Kalmanson.)

It may be that D does not satisfy (1), but that some permutation of the rows and columns of D does. In this case we say that D is a *permuted Kalmanson matrix.* Since permuting D amounts to simply relabeling the underlying distance or cost data, this operation preserves the structure of the problem. [6] give an $O(n^2)$ recognition algorithm for permuted Kalmanson matrices, so we say the TSP is polynomial time-solvable for this class.

Geometrically, (1) comprises a finite intersection of closed half-spaces: a polyhedron. Given a polyhedron $P \subset \mathbb{R}^k$ and a hyperplane $H \subset \mathbb{R}^k$, we say H *supports* P if $H \cap P \neq \emptyset$ and P is completely contained in one of the closed half-spaces defined by H. $F \subset P$ is a *face* of P if $F = P \cap H$ for some supporting hyperplane H of P. The *face lattice* of P is the poset of faces of P ordered by set inclusion.

Recall that a set of polyhedra which intersect along faces is called a *polyhedral fan.* Permuting the indices in (1) generates a polyhedral fan which we denote $\mathcal{P}_n$.

Example 1. For $n = 4$, $\mathcal{P}_n$ is the union of three polyhedra obtained by permuting the indices in (1)

$$\mathcal{P}_n = (d_{ij})_{i,j\in[n]} \text{ such that : } \begin{cases} d_{12} + d_{34} & \leq d_{14} + d_{23} \\ d_{13} + d_{24} & \leq d_{14} + d_{23} \end{cases} \text{or}$$

$$\begin{cases} d_{13} + d_{24} & \leq d_{12} + d_{34} \\ d_{14} + d_{23} & \leq d_{12} + d_{34} \end{cases} \text{or}$$

$$\begin{cases} d_{14} + d_{23} & \leq d_{13} + d_{24} \\ d_{12} + d_{34} & \leq d_{13} + d_{24} \end{cases}$$

Collectively, these define the region of $\mathbb{R}^{\binom{4}{2}}$ containing all 4×4 permuted Kalmanson matrices.

The main claim of this section is that $\mathcal{P}_n$ is a geometric realization of $\mathcal{K}_n$ in the sense that they are combinatorially equivalent.

Theorem 2. *The face lattices of $\mathcal{K}_n$ and $\mathcal{P}_n$ are isomorphic as posets.*

The remainder of this section is devoted to proving the theorem by finding an inclusion-preserving bijection between the faces of these two sets.

In [7] it is shown that the polyhedron defined by (1) decomposes into an n-dimensional lineality space and a pointed cone of dimension $\binom{n}{2} - n$. We are interested in the structure of the latter since it encapsulates the combinatorial data embodied by the polyhedron. The authors give an explicit description of the extreme rays of this cone.

Example 2. For $n = 5$, the rays of the standard Kalmanson polyhedron are

$$V^{(2)} = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \qquad V^{(3)} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$V^{(1,3)} = \left(\begin{array}{c|cc|cc} 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \qquad V^{(1,4)} = \left(\begin{array}{c|ccc|c} 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 0 \end{array}\right) \qquad V^{(2,4)} = \left(\begin{array}{cc|cc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

We see that $V^{(i)}$ and $V^{(i,j)}$ have a structure which is the result of arranging square blocks of zeros along the diagonal. The idea behind the bijection we will establish is that these matrices, along with their permutations, actually encode the non-trivial X-splits. To see this, we associate to each matrix M an index set $I(M) \subset [n]$ which records the diagonal positions of the odd-numbered blocks. Hence, in the above example we have:

$$\begin{array}{lll} I(V^{(2)}) = \{1,2\} & I(V^{(3)}) = \{1,2,3\} & \\ I(V^{(1,3)}) = \{1,4,5\} & I(V^{(1,4)}) = \{1,5\} & I(V^{(2,4)}) = \{1,2,5\} \end{array}$$

These sets each represent blocks of an X-split. For example, $I(V^{(2,4)}) = \{1,2,5\}$ corresponds to the split $\{\{1,2,5\},\{3,4\}\}$.

Remark. Each of the matrices in the Example 2 corresponds to one of the $5 \times (5-3)/2 = 5$ diagonals of a pentagon with sides labeled 1 through 5 in ascending order.

We now formalize this idea. The defining equations for the extreme ray matrices are:

Theorem 3 ([7]). *The cone of Kalmanson matrices is ruled by the symmetric matrices* $V^{(i)} = \left(v_{pq}^{(i)}\right)$, $2 \le i \le n-2$ *and* $V^{(i,j)} = \left(v_{pq}^{(i,j)}\right)$, $1 \le i \le n-3$, $i+2 \le j \le n-1$, *where*

$$v_{pq}^{(i)} := \begin{cases} 1, & 1 \le p < i < q \le n \\ 0, & \textit{otherwise} \end{cases}$$

$$v_{pq}^{(i,j)} := \begin{cases} 1, & 1 \le p \le i < q \le j \textit{ and } i < p \le j < q \le n \\ 0, & \textit{otherwise} \end{cases}$$

Let two index sets $I, J \subset [n]$ be equivalent if $I = J$ or $I = [n] - J$, and let $\mathcal{I}_n$ be the set of such equivalence classes such that $2 \leq |I| \leq n-2$ for all $[I] \in \mathcal{I}_n$. (We clearly have that $\mathcal{I}_n \cong \mathcal{S}(X)$, the set of all non-trivial X-splits.) Next define the matrices $E^{(i)} = \left(e_{pq}^{(i)}\right)$ by the rule

$$e_{pq}^{(i)} = \begin{cases} 1, & p = i \text{ xor } q = i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Lemma 4. *When considered as matrices over the ring $\mathbb{Z}/2\mathbb{Z}$, the following identities hold:*

$$V^{(i)} \equiv \sum_{k=i+1}^{n} E^{(k)}$$

$$V^{(i,j)} \equiv \sum_{k=i+1}^{j} E^{(k)}$$

Proof. Let $f_{pq} = e_{pq}^{(i+1)} + e_{pq}^{(i+2)} + \cdots + e_{pq}^{(n)}$. We have

$$f_{pq} = \begin{cases} 0, & p = q \text{ or } (p \leq i \text{ and } q \leq i) \\ 1, & (p > i \text{ and } q \leq i) \text{ or } (p \leq i \text{ and } q > i) \\ 2, & \text{otherwise} \end{cases} \tag{3}$$

Hence, modulo $2\mathbb{Z}$ we obtain

$$V^{(i)} \equiv (f_{pq}) = \sum_{k=i+1}^{n} E^{(k)}$$

The second identity is established by noting that $V^{(i,j)} \equiv V^{(i)} + V^{(j)} \pmod 2$. $\square$

Let the symmetric group S_n act on $\mathcal{I}_n$ by $\sigma \cdot [I] = [\sigma \cdot I]$, and on the set of $n \times n$ matrices by simultaneous permutation of rows and columns: $\sigma \cdot M = \left(m_{\sigma(i),\sigma(j)}\right)$ for all $\sigma \in S_n$ and $M = (m_{ij})$. Define

$$\mathcal{V}_n := \left\{V^{(i)} : 2 \leq i \leq n-2\right\} \cup \left\{V^{(i,j)} : 1 \leq i \leq n-3, i+2 \leq j \leq n-1\right\}$$

and let $\mathcal{R}_n = \{\sigma \cdot V : \sigma \in S_n, V \in \mathcal{V}_n\}$ be the set of all permutations of the $\mathcal{V}_n$.

From (2) we have that $\sigma \cdot E^{(i)} = E^{(\sigma(i))}$. We therefore obtain via Lemma 4 that

$$\sigma \cdot V^{(i)} \equiv \sum_{k=i+1}^{k} E^{(\sigma(i))} \pmod 2 \tag{4}$$

(and similarly for $V^{(i,j)}$) since the group action commutes with matrix addition. Hence we can write any $M \in \mathcal{R}_n$ as $M = \sum_{i \in I} E^{(i)}$ for some index set $I \subset [n]$. In fact, this representation is unique in $\mathcal{I}_n$.

Lemma 5. *Suppose $M = \sum_{i \in I} E^{(i)}$. Then $M = \sum_{j \in J} E^{(j)}$ if and only if $[I] = [J] \in \mathcal{I}_n$.*

Proof. $\Rightarrow$) Setting $i = 0$ in (3) gives $\sum_{i=1}^{n} E^{(i)} \equiv 0 \pmod 2$. Then

$$\sum_{i=1}^{n} E^{(i)} \equiv 0 \equiv \sum_{i \in I} E^{(i)} - \sum_{j \in J} E^{(j)} \equiv \sum_{i \in I} E^{(i)} + \sum_{j \in J} E^{(j)} \pmod 2 \tag{5}$$

Canceling term-wise from the left- and right-hand sides gives $\sum_{i \in K} E^{(i)} \equiv 0$ for $K := [n] - I - J$. Now, $|K| \neq 1, n-1$ since both cases reduce (5) to $E^{(i)} \equiv 0$ for some i, a contradiction. If $2 \leq |K| \leq n-2$ then $\sum_{i \in K} E^{(i)} \equiv \sigma \cdot V^{(i)} \not\equiv 0$ for some $2 \leq i \leq n-2$. Hence $|K| = 0$ or $|K| = n$.

$\Leftarrow$) We have

$$\sum_{i=1}^{n} E^{(i)} \equiv 0 \equiv \sum_{i \in I} E^{(i)} + \sum_{i \in [n]-I} E^{(i)} \equiv M - \sum_{i \in [n]-I} E^{(i)} \pmod 2$$

□

Finally, let $S : \mathcal{R}_n \to \mathcal{I}_n$ be the map taking each matrix $M \in \mathcal{R}_n$ to the unique $[I] \in \mathcal{I}_n$ such that $M = \sum_{i \in I} E^{(i)}$. Recall that a map between two sets acted on by a common group is *equivariant* if the map commutes with the group action.

Lemma 6. *$S : \mathcal{R}_n \to \mathcal{I}_n$ is equivariant and bijective.*

Proof. Let $\sigma \in S_n$ and $M = \sum_{i \in I} E^{(i)} \in \mathcal{R}_n$ be arbitrary. We have

$$S(\sigma \cdot M) = S\left(\sum_{i \in I} E^{(\sigma(i))}\right) = S\left(\sum_{i \in \sigma \cdot I} E^{(i)}\right) = \sigma \cdot [I] = \sigma \cdot S(M)$$

so S is equivariant.

Injectivity follows from Lemma 5. Now for any $[I] \in \mathcal{I}_n$ let $\sigma \in S_n$ be the permutation such that $\sigma^{-1} \cdot I = \{n - |I| + 1, n - |I| + 2, \ldots, n\}$. Then by the equivariance of S,

$$\begin{aligned} S(\sigma \cdot V^{(n-|I|)}) &= \sigma \cdot S\left(V^{(n-|I|)}\right) \\ &= \sigma \cdot [\{n - |I| + 1, n - |I| + 2, \ldots, n\}] \\ &= [I] \end{aligned}$$

so that S is onto. $\square$

As $\mathcal{I}_n \cong \mathcal{S}(X)$, we get a bijection $T : \mathcal{R}_n \leftrightarrow \mathcal{S}(X)$ between permuted rays of the cone of Kalmanson matrices, and non-trivial X-splits.

Lemma 7. *$\{T(M_1), T(M_2), \ldots, T(M_k)\}$ is a face of $\mathcal{K}_n$ if and only if $\{M_1, M_2, \ldots, M_k\}$ is a face of $\mathcal{P}_n$.*

Proof. Since

$$\begin{aligned} T(V^{(i)}) &= \Big\{ \{1, 2, \ldots, i\}, \{i+1, \ldots, n\} \Big\} \\ T(V^{(i,j)}) &= \Big\{ \{1, 2, \ldots, i, j+1, \ldots, n\}, \{i+1, \ldots, j\} \Big\} \end{aligned}$$

the set $S := \{T(V) : V \in \mathcal{V}_n\}$ is a maximal circular split system with the ordering $(1, 2, \ldots, n)$. Therefore the claim is true when the $M_i \in \mathcal{V}_n$.

Now if $M_1, \ldots, M_k$ are an arbitrary face of $\mathcal{P}_n$, then there is a $\sigma \in S_n$ such that for each $1 \le i \le k$ there exists $V_i \in \mathcal{V}_n$ with $M_i = \sigma \cdot V_i$. Then

$$\begin{aligned} \{T(M_1), \ldots, T(M_k)\} &= \{T(\sigma \cdot V_1), \ldots, T(\sigma \cdot V_k)\} \\ &= \{\sigma \cdot T(V_1), \ldots, \sigma \cdot T(V_k)\} \\ &\subseteq \sigma \cdot S \end{aligned}$$

is a permuted face of $\mathcal{K}_n$. $\square$

Hence, T induces a canonical bijection $\{M_1, M_2, \ldots, M_k\} \mapsto \{T(M_1), T(M_2), \ldots, T(M_k)\}$ between the faces of $\mathcal{K}_n$ and those of $\mathcal{P}_n$. As this map is clearly order-preserving, we have proven Theorem 2.

3 $\mathcal{K}_n$ and the consecutive ones property

Thus far we have defined $\mathcal{K}_n$ abstractly, as a simplicial complex arising in T-theory, and also geometrically in terms of permutations of the Kalmanson conditions. In this section we present a third description of the Kalmanson complex as a set of (equivalence classes of) binary matrices possessing a certain structure. Again, we will show that this formulation is entirely equivalent to the preceding two. Throughout this section, M is taken to be an $m \times n$ binary matrix (entries are zero or one.)

Definition 4. M is said to possess the *consecutive ones property for rows* (C1R) if its columns may be permuted such that the ones in each row occur in blocks. M possesses the *circular ones property for rows* (Circ1R) if its columns may be permuted such that either the ones or the zeros (or both) in each row occur in a block.

Intuitively, a Circ1R matrix has the property that for each of its rows, the ones occur in a block when it is "wrapped around" a cylinder.

Example 3. Consider the matrices

$$\underset{(1)}{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}} \qquad \underset{(2)}{\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}} \qquad \underset{(3)}{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}$$

(1) and (2) are C1R, and (3) is Circ1R. To verify that (2) is C1R, we apply the permutation $(1\,3\,4\,5) \in S_5$ to its columns:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{(1\,3\,4\,5)} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

If M is C1R or Circ1R, then the matrix obtained by replacing any number of rows of M by their binary complement will be Circ1R. This provides justification for the following theorem.

Theorem 8 ([16])**.** *Let M be a binary matrix, and let M' be the matrix obtained by complementing each row in M which has a one in the first column. Then M is C1R if and only if M' is Circ1R.*

A circular split system and a Circ1R binary matrix are, in a sense, identical. To see this, let m be fixed and consider the set of all split systems over X which contain m splits: $\mathcal{S}_m(X) = \{\mathcal{S} \subset \mathcal{S}(X) : |\mathcal{S}| = m\}$. Also, let $\mathcal{M}^0_{m\times n}(\{0,1\})$ be the set of $m \times n$ binary matrices who first column contains all zeros, and let the symmetric group S_m act on it by permutation of rows.

Finally, let $\mathcal{Q}_m = \mathcal{M}^0_{m\times n}(\{0,1\})/\sim$ be the set of equivalence classes under the relation "$M_1 \sim M_2 \iff M_1 = \sigma \cdot M_2$ for some $\sigma \in S_m$". (Note that, as row permutations do nothing to affect the C1R/Circ1R properties, it makes sense to say that a class $[M] \in \mathcal{Q}$ possesses one or both.)

Now define a map $F : \mathcal{S}_m(X) \to \mathcal{Q}_m$ which sends a system of m splits to the class of the binary matrix obtained by converting the splits to a binary vector and stacking them. Formally,

$$\begin{aligned} F : \mathcal{S}_m(X) &\to \mathcal{Q}_m \\ \{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\} &\mapsto \left[(w_{ij})_{i\in[m],j\in[n]}\right] \end{aligned}$$

where

$$w_{ij} = \begin{cases} 1, & j \in A_i \text{ and } 1 \notin A_i \\ 1, & j \notin A_i \text{ and } 1 \in A_i \\ 0, & \text{otherwise} \end{cases}$$

Example 4. Let $n = 5$ and $\mathcal{S} \in \mathcal{S}_3(X)$ be the split system

$$\mathcal{S} = \Big\{\Big\{\{1,2\},\{3,4,5\}\Big\}, \Big\{\{1,3,5\},\{2,4\}\Big\}, \Big\{\{1,4\},\{2,3,5\}\Big\}\Big\}$$

Then

$$F(S) = \left[\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}\right]$$

Lemma 9. *$F : \mathcal{S}_m(X) \to \mathcal{Q}_m$ is a bijection.*

Proof. Let $[M] \in \mathcal{Q}_m$ be given. Simply convert each row of $M \in \mathcal{M}^0_{m\times n}(\{0,1\})$ to an X-split in the obvious way. The resulting split system $\mathcal{S}$ gives $F(\mathcal{S}) = [M]$, so F is onto.

Now suppose $F(\mathcal{S}_1) = F(\mathcal{S}_2)$ for two split systems $\mathcal{S}_1, \mathcal{S}_2$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ represent the same splits up to ordering. But then $\mathcal{S}_1 = \mathcal{S}_2$ so F is one-to-one. $\square$

Having established the bijection, it is easy to see that Circ1R and circularity are analogous properties for $\mathcal{Q}$ and $\mathcal{S}_m(X)$, respectively.

Theorem 10. *Let $\mathcal{S} \in \mathcal{S}_m(X)$ be an arbitrary split system. Then $\mathcal{S}$ is circular iff $F(\mathcal{S})$ is Circ1R.*

Proof. A split system $\mathcal{S}$ is circular iff there is a $\sigma \in S_n$ such that each split $S \in \mathcal{S}$ is of the form

$$S = \left\{ \{\sigma(\bar{i}), \sigma(\overline{i+1}), \dots \sigma(\bar{j})\}, \{\sigma(\overline{j+1}), \dots, \sigma(\overline{i-1})\} \right\}$$

where $\bar{i}$ denotes $i \pmod n$. This occurs iff applying σ to the columns of $F(S)$ yields a class of matrices $[M] \in \mathcal{Q}_m$ whose ones appear consecutively. Since the first column of M is the zero vector, M is C1P iff it is Circ1R by Theorem 8. $\square$

Corollary 11. $\mathcal{S} \in \mathcal{S}_m(X)$ *is circular iff $F(\mathcal{S})$ is C1R.*

The preceding theorem enables us to furnish another description of $\mathcal{K}_n$: it is the poset of all Circ1R binary matrices (up to row permutation) which possess an initial column of zeros, and at least two ones and two zeros in each row, ordered by inclusion of the set of row vectors corresponding to each matrix.

4 f-vector

In this section we will harness the three descriptions of $\mathcal{K}_n$ to study its combinatorial structure in greater detail.

Definition 5. Let Δ be a simplicial complex of dimension $d-1$, and let f_i denote the number of i-dimensional faces of Δ. The *$f-$vector of* Δ is the vector $f = (f_0, f_1, \dots, f_{d-1})$.

Thus, f_0 counts the vertices of Δ and f_{d-1} counts the facets. By geometric analogy, f_1, f_2 and f_{d-2} are called the *edges*, *triangles*, and *ridges* of Δ, respectively.

n	f-vector
4	$[3, 3]$
5	$[10, 45, 90, 60, 12]$
6	$[25, 300, 1755, 4725, 6390, 4860, 2160, 540, 60]$
7	$[56, 1540, 19950, 121485, \dots, 360]$
8	$[119, 7021, 178878, \dots, 2520]$
9	$[246, 30135, 1409590, \dots, 20160]$
k	$\left[2^{k-1} - k - 1, \binom{f_1}{2}, \dots, \frac{k!(k-3)}{4}, \frac{(k-1)!}{2}\right]$

Table 1: Computational results for the Kalmanson complex.

For small n, the f-vector may be computed directly. Results for $n = 4, \dots, 9$ are presented in Table 1. To generate these sequences, we wrote software to enumerate circular split systems using each of the three approaches described above. In all cases the results agreed, providing empirical evidence for the stated equivalences.

We now theoretically explain portions of the f-vector. First we restate some additional definitions and results from [3, 8] which will prove useful in enumerating the faces of $\mathcal{K}_n$. For the remainder of the section, $S_i \in \mathcal{S}(X)$ represents a split and the identity $S_i = \{A_i, B_i\}$ is implicit.

> **Definition 6.** A split system $\mathcal{S}$ is called *weakly compatible* if for all triples $S_1, S_2, S_3 \in \mathcal{S}$ there do not exist points $a, a_1, a_2, a_3 \in X$ such that $a \in A_1 \cap A_2 \cap A_3$ and $a_i \in A_j \iff i = j$.

Weak compatibility enforces a sort of convexity condition on $\mathcal{S}$ by requiring that, for any triple of points in the split system, there is no point which mutually separates them (Figure 2).

For any two splits S_1, S_2 we define a binary operation $\sqcup$ by $\{A_1, B_1\} \sqcup \{A_2, B_2\} = \{A_1 \cap A_2, B_1 \cup B_2\}$.

> **Lemma 12.** *The splits S_1, S_2 and $S_1 \sqcup S_2$ are weakly compatible.*
>
> *Proof.* Let $S_3 = S_1 \sqcup S_2$. If there exist a, a_1, a_2, a_3 as in Definition 6, then $a_3 \in A_3 - (A_1 \cup A_2) \neq \emptyset$. But $A_3 = A_1 \cup A_2$, a contradiction. □

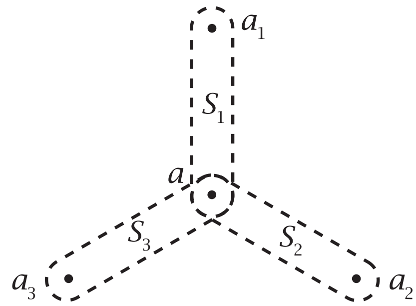

Figure 2: A system of splits which is not weakly compatible.

Theorem 13 ([8])**.** *Let $\mathcal{S}$ be a split system and let $\mathcal{S}'$ be the split system*

$$\mathcal{S}' := \mathcal{S} \cup \Big\{ S_1 \sqcup S_2 : S_1, S_2 \in \mathcal{S} \text{ and } A_i \cap B_j \neq 0,\ \ i, j \in \{1, 2\} \Big\}$$

Then $\mathcal{S}$ is contained in a circular split system if and only if $\mathcal{S}'$ is weakly compatible.

Corollary 14. *A circular split system is weakly compatible.*

4.1 Low (Co-)Dimensional Faces

Enumerating the vertices, edges, ridges and facets of $\mathcal{K}_n$ is now straightforward.

Theorem 15. *Let $f = (f_0, \ldots, f_{d-1})$ denote the f-vector of $\mathcal{K}_n$. Then*

$$f_0 = 2^{n-1} - n - 1 \tag{6}$$

$$f_1 = \binom{f_0}{2} \tag{7}$$

$$f_{d-2} = \left[\binom{n}{2} - n \right] \times f_{d-1} \tag{8}$$

$$f_{d-1} = \frac{(n-1)!}{2} \tag{9}$$

Proof. f_0 counts the number of non-trivial X-splits. There are

$$\sum_{k=2}^{n-2} \binom{n}{k} = 2^n - 2n - 2$$

binary words on n letters which contain at least two zeros and two ones. Since each word and its complement correspond to the same split, we divide by two to obtain f_0.

Equation (7) asserts that every pair of splits $\mathcal{S}_1, \mathcal{S}_2$ is contained in a circular split system. By Lemma 12, the splits S_1, S_2 and $S_1 \sqcup S_2$ are weakly compatible. Then by Theorem 13 $\{S_1, S_2\}$ is a cyclic split system.

Each facet of $\mathcal{K}_n$ corresponds to a circular ordering; that is, an edge labeling of the regular n-gon. Such labelings are unique up to dihedral symmetry. There are $(n-1)!$ labelings up to rotation, and half that number when accounting for reflection. This yields (9).

To prove (8), let $F \subset \mathcal{K}_n$ be a facet spanned by vertices $v_1, \dots, v_d \in X$; without loss of generality assume the circular ordering corresponding to F is $(1, 2, \dots, n)$. Let u be another vertex distinct from the v_i, with corresponding split

$$S_u = \Big\{ \{1, u_2, \dots, u_j\}, \{u_{j+1}, \dots, u_n\} \Big\}$$

Finally, let $i = \min\{i : u_i \neq i\}$, which exists by the assumption that S_u is not circular with respect to the given ordering. Now, the splits S_u and

$$\begin{aligned} S_1 &= \Big\{ \{u_i - 1, u_i\}, X - \{u_i - 1, u_i\} \Big\} \in F \\ S_2 &= \Big\{ \{u_i, u_i + 1\}, X - \{u_i, u_i + 1\} \Big\} \in F \end{aligned}$$

are weakly incompatible: denoting the first blocks of each by A_u, A_1, A_2 we have $A_u \cap A_1 \cap A_2 = \{u_i\}$ while

$$\begin{aligned} 1 &\in A_u - (A_1 \cup A_2) \\ u_i - 1 &\in A_1 - (A_u \cup A_2) \\ u_i + 1 &\in A_2 - (A_u \cup A_1) \end{aligned}$$

Therefore, any collection of $d-1$ vertices of F spans a unique face of codimension two. As described in Section 2, each facet contains $\binom{n}{2} - n$ vertices. $\square$

4.2 Triangles

The computations in Theorem 15 were aided by the fact that $\mathcal{K}_n$ is connected in dimension two and totally disconnected in codimension two. Enumerating the faces in the remaining cases is more challenging. To illustrate the issues involved, we demonstrate how to compute f_2, the number of triangles in $\mathcal{K}_n$.

Example 5. The split system

$$\Big\{ \Big\{ \{1,2\}, \{3,4,5\} \Big\}, \Big\{ \{1,3\}, \{2,4,5\} \Big\}, \Big\{ \{1,4\}, \{2,3,5\} \Big\} \Big\}$$

is not weakly compatible, so it is not a triangle of $\mathcal{K}_n$. By contrast, the split system

$$\Big\{ \Big\{ \{1,2\}, \{3,4,5\} \Big\}, \Big\{ \{2,3\}, \{1,4,5\} \Big\}, \Big\{ \{4,5\}, \{1,2,3\} \Big\} \Big\}$$

is circular with respect to two orderings: $(1,2,3,4,5)$ and $(1,2,3,5,4)$. It is therefore a triangle of $\mathcal{K}_n$ which is contained in two facets.

Our main tool for computing f_2 will be Corollary 11, in conjunction with a structure theorem of [17] which completely characterizes the C1R matrices.

Definition 7. Let M be a matrix. The *configuration of* M is the set of matrices obtained by permuting the rows and/or columns of M (not necessarily by the same permutation.)

Example 6. The configuration of the 2×2 identity matrix is the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Theorem 16 ([17]). *A binary matrix M is C1R if and only if it does not contain as a submatrix* *any configuration of* $M_{I_n}, M_{II_n}, M_{III_n}, M_{IV}, M_V$, $1 \le n < \infty$, *where*

$$M_{I_n} = \begin{array}{c} \\ r_1 \\ r_2 \\ \vdots \\ r_n \\ r_{n+1} \\ r_{n+2} \end{array} \begin{array}{c} \begin{array}{ccccccc} c_1 & c_2 & c_3 & \cdots & c_n & c_{n+1} & c_{n+2} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad M_{IV} = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$M_{II_n} = \begin{array}{c} \\ r_1 \\ r_2 \\ \vdots \\ r_n \\ r_{n+1} \\ r_{n+2} \\ r_{n+3} \end{array} \begin{array}{c} \begin{array}{cccccccc} c_1 & c_2 & c_3 & \cdots & c_n & c_{n+1} & c_{n+2} & c_{n+3} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad M_V = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$M_{III_n} = \begin{array}{c} \\ r_1 \\ r_2 \\ \vdots \\ r_n \\ r_{n+1} \\ r_{n+2} \end{array} \begin{array}{c} \begin{array}{cccccccc} c_1 & c_2 & c_3 & \cdots & c_n & c_{n+1} & c_{n+2} & c_{n+3} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

In the specific case of f_2, where we are counting $3 \times n$ matrices, only two forbidden submatrices pertain:

$$M_{I_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M_{III_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

For a matrix M, we write $\mathrm{col}\,(M)$ to denote the set of column vectors of M. Let $I = \mathrm{col}\,(M_{I_1})$ and $III = \mathrm{col}\,(M_{III_1})$. Note that I and III are "closed" under the operation of row permutation. Hence, by Corollary 11 and Theorem 16,

$$[M] \in \mathcal{Q}_3 \iff |\mathrm{col}(M) \cap I| < 3 \text{ and } |\mathrm{col}(M) \cap III| < 4$$

Accordingly, let

$$F_{i,j} = \{[M] \in \mathcal{Q}_3 : |\mathrm{col}(M) \cap I| = i \text{ and } |\mathrm{col}(M) \cap III| = j\} \tag{10}$$

Then

$$f_2 = |\mathcal{Q}_3| = \sum_{\substack{0 \le i \le 2 \\ 0 \le j \le 3}} |F_{i,j}| \tag{11}$$

Enumerating $F_{i,j}$ involves carefully counting the number of classes of $\mathcal{Q}_3$ while keeping track of how many columns from the sets

$$I = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\} \qquad III = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

appear in each equivalence class.

4.2.1 Sample Calculation: $|F_{0,3}|$

The counting argument is straightforward but tedious. We illustrate the calculation of $|F_{0,3}|$; the remaining cases are similar and are proved in [15].

Let $\mathbb{P}_n$ denote the set of ordered partitions of the integer n. That is, for a k-tuple $(x_1, \ldots, x_k)$ we have

$$(x_1, \ldots, x_k) \in \mathbb{P}_n \iff \sum_{i=1}^{k} x_i = n \text{ and } x_i \ge 1 \text{ for all } i$$

To simplify the notation, we take summation over $\mathbb{P}_{n-1}$ for granted wherever there is no chance of confusion: instead of e.g.

$$\sum_{\substack{(a,b,c,d) \in \mathcal{P}_{n-1} \\ a > 1}} \binom{n-1}{a, b, c, d}$$

we will simply write

$$\sum_{a>1} \binom{n-1}{a, b, c, d}$$

Now let $[M] \in F_{0,3}$. We consider two cases:

1. First, if $(1,1,1)^T \notin \operatorname{col}(M) \cap III$ then M is of the form

$$\begin{array}{cccccccc}
 & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{a}} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{b}} & \multicolumn{3}{c}{\overbrace{\hphantom{0\quad 0\quad 0}}^{c}}
\end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where a, b, c count the instances of the columns in $III - \{(1,1,1)\}$. To prevent the occurrence of a trivial split (row containing < 2 ones) we require $\min(a,b,c) > 1$. Hence there are

$$(1/6) \sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c} \tag{12}$$

such classes, where the factor of $1/6$ reflects the fact that each arrangement is equivalent to six others obtained by permuting the labelings a, b, c. We see that this counts the number ways of arranging $n-1$ ones into three unlabeled rows where each must contain at least two ones.[2] We also have the possibility that M contains additional zero columns:

$$\begin{array}{cccccccc}
 & \overline{d} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{a}} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{b}} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{c}}
\end{array}$$
$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

By an entirely analogous argument we count

$$(1/6) \sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c,d} \tag{13}$$

such classes.

2. In the second case we have $(1,1,1)^T \in \operatorname{col}(M) \cap III$. These are matrices of the form

$$\begin{array}{cccccccc}
 & \multicolumn{3}{c}{\overbrace{\hphantom{0\quad 0\quad 0}}^{c}} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{a}} & \multicolumn{2}{c}{\overbrace{\hphantom{0\quad 0}}^{b}}
\end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

We must have that the number of $(1,1,1)^T$ columns is greater than one, or else we have a trivial split. Also, if $a = b$ then matrix is equivalent to the arrangement obtained by swapping

[2]This number is also known as an associated Stirling number of the second kind, cf. A000478 [12].

the columns labeled a and b and permuting their respective rows. We therefore count

$$\sum_{\substack{a>b\\c>1}} \binom{n-1}{a,b,c} + (1/2)\sum_{\substack{a=b\\c>1}} \binom{n-1}{a,b,c} \tag{14}$$

such classes. In the exact same manner, we also obtain

$$\sum_{\substack{a>b\\c>1}} \binom{n-1}{a,b,c,d} + (1/2)\sum_{\substack{a=b\\c>1}} \binom{n-1}{a,b,c,d} \tag{15}$$

classes by allowing for the presence of addition zero columns.

In equations (12)–(13) and (14)–(15) we see that the multinomial summations frequently occur in pairs, where the summations are over different subsets of $\mathbb{P}_{n-1}$. In this case we economize on notation by parenthesizing the final summation variable, e.g.:

$$(1/6)\sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c,(d)} \equiv (1/6)\sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c} + (1/6)\sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c,d}$$

Using this shorthand and equations (12)–(15), we have

$$|F_{0,3}| = (1/6)\sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c,(d)} + \sum_{\substack{a>b\\c>1}} \binom{n-1}{a,b,c,(d)} + (1/2)\sum_{\substack{a=b\\c>1}} \binom{n-1}{a,b,c,(d)}$$

4.2.2 General Formula

Repeating these counting arguments for the remaining $|F_{i,j}|$ yields the following formula.

Theorem 17. *The number of triangles in $\mathcal{K}_n$ is*

$$
\begin{aligned}
f_2 = &\sum_{\substack{a>b\\c>1}} \binom{n-1}{a,b,c,(d)} + (1/2)\sum_{\substack{a=b\\c>1}} \binom{n-1}{a,b,c,(d)} + (1/6)\sum_{\min(a,b,c)>1} \binom{n-1}{a,b,c,(d)} + \\
&\sum_{a>1} \binom{n-1}{a,b,c,d} + \sum_{\min(a,c)>1} \binom{n-1}{a,b,c,(d)} + 5\sum_{\substack{a>b\\c>1}} \binom{n-1}{a,b,c,d,(e)} + (5/2)\sum_{\substack{a=b\\c>1}} \binom{n-1}{a,b,c,d,(e)} + \\
&\sum_{a>b>1} \binom{n-1}{a,b,c} + (1/2)\sum_{a=b>1} \binom{n-1}{a,b,c} + 3\sum_{a>b} \binom{n-1}{a,b,c,d,(e)} + (3/2)\sum_{a=b} \binom{n-1}{a,b,c,d,(e)} + \\
&2\sum_{a>b>1} \binom{n-1}{a,b,c} + \sum_{a>b=1} \binom{n-1}{a,b,c,(d)} + \sum_{a=b>1} \binom{n-1}{a,b,c} + 9\sum_{a>b>1} \binom{n-1}{a,b,c,d} + \\
&2\sum_{a=b} \binom{n-1}{a,b,c,d,e,(f)} + (9/2)\sum_{a=b>1} \binom{n-1}{a,b,c,d} + 4\sum_{a>b=1} \binom{n-1}{a,b,c,d} + \sum_{c=1} \binom{n-1}{a,b,c,d,(e)} + \\
&\sum_{\substack{a=b=1\\c>d}} \binom{n-1}{a,b,c,d} + (1/2)\sum_{\substack{a=b=1\\c=d}} \binom{n-1}{a,b,c,d} + 4\sum_{a>b} \binom{n-1}{a,b,c,d,e,(f)}
\end{aligned}
$$

Proof. See [15]. □

The first ten entries of this sequence, $n = 5, \ldots, 14$ are

$$90, 1755, 19950, 178878, 1409590, 10270585, \\ 71110930, 475443364, 3100707610, 19856761015, \ldots$$

We have verified this formula computationally up to $n = 10$ (the largest n for which the calculations terminated) using the mathematics software SAGE [14]. Source code for this and related f-vector calculations may be downloaded from: `https://github.com/terhorst/kalmanson`.

5 Conclusion

At present we do not have a way to generalize the theorem to faces of arbitrary dimension. The method used in Theorem 17 is fairly straightforward, though tedious. The next case of tetrahedra ($k = 4$) becomes considerably more difficult, as there are now 7 avoided Tucker matrices to consider: $M_{I_1}, M_{I_2}, M_{II_1}, M_{III_1}, M_{III_2}, M_{IV}, M_V$. The connection to the Tucker theorem suggests a possible application of results on avoided configurations (see [1] for a survey), but most results in that literature are of an extremal, as opposed to enumerative, variety. In [13] some matrices avoiding small configurations are counted, but we are not aware of a general method of enumerating matrices which avoid configurations of arbitrary dimensions. We view this as in interesting problem in enumerative combinatorics which merits further study.

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