Compositae and their properties

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January 14, 2013

Abstract

A new class of functions based on compositions of an integer n and termed compositae is introduced. Main theorems are presented; compositae are written for polynomials, trigonometric and hyperbolic functions, radicals, exponential and log functions. A solution is proposed for the problems of derivation of compositions of ordinary generating functions, Riordan arrays, inverse and reciprocal generating functions, functional equations $A(x) = G(xA(x)^m)$, and identities.

1 Introduction

The computations based on combinatorial objects are an important line of research in enumerative combinatorics and allied fields of mathematics. For example, ordered partitions of a finite set was used to derive the formula for a composition of exponential generating functions [1]. Computations that use compositions of an integer n are found in various problems: derivation of a convolution of convolutions [2] and composition of ordinary generating functions [3], determination of the n-th order derivatives of a composite function [4], generation of ordered root trees [5], etc. However, there is no unified approach to solving composition-based problems. In the work, a unified approach to the above problems is considered the basis for which is a special function termed a composita. The notion of a composita is close to that of a Riordan array [6, 7] and is its degenerate case, for a composita characterizes only one function. Let us introduce the definition of a composita.

Definition 1. A composita of the ordinary generating function $F(x) = \sum_{n>0} f(n)x^n$ is the function

$$F^{\Delta}(n,k) = \sum_{\pi_k \in C_n} f(\lambda_1)f(\lambda_2)\dots f(\lambda_k), \tag{1}$$

where C_n is a set of all compositions of an integer n, π_k is the composition $\sum_{i=1}^{k} \lambda_i = n$ with k parts exactly.

It follows from the definition of a composita that it is defined for a generating function $F(x)$ for which $f(0)=0$. Let us consider a generating function $F(x)=\frac{x}{1-x}=\sum_{n>0} x^n$. On the strength of formula (1), the composita of this function is

$$F^{\Delta}(n,k)=\binom{n-1}{k-1}.$$

For all $n>0$ we have $f(n)=1$; therefore, formula (1) counts the number of compositions of n with exactly k parts.

2 Main theorem

Let us derive a recurrent formula for the composita of a generating function .

Theorem 2.1. For the composita $F^{\Delta}(n,k)$ of the generating function $F(x)=\sum_{n>0} f(n)x^n$, the following recurrent relation holds true:

$$F^{\Delta}(n,k)=\begin{cases} f(n), & k=1,\\ \sum\limits_{i=1}^{n-k+1} f(i)F^{\Delta}(n-i,k-1) & k<n. \end{cases} \tag{2}$$

Proof. The composition π_k at $k=1$ is unique and is equal to n; from whence it follows that $F^{\Delta}(n,1)=f(n)$. Now for $k>1$ we group in formula (1) all products $f(\lambda_1)f(\lambda_2)\ldots f(\lambda_k)$ of the composition π_k with equal λ_1. Let us take $f(\lambda_1)$ out of the brackets; we see that the sum of the products in the brackets is equal to $F^{\Delta}(n-\lambda_1,k-1)$. Then for all values of λ_1 we obtain

$$F^{\Delta}(n,k)=f(1)F^{\Delta}(n-1,k-1)+f(2)F^{\Delta}(n-2,k-1)+\ldots+f(i)F^{\Delta}(n-i,k-1)+\ldots+$$
$$+f(n-(n-k+1))F^{\Delta}(k-1,k-1)$$

Thus, the theorem is proved. $\square$

It can readily be seen that

$$F^{\Delta}(n,n)=f(1)F^{\Delta}(n-1,n-1)=f(1)^n.$$

Formula (2) allows the conclusion that the composita is a characteristic of the generating function $F(x)$. In tabular form, the composita is presented as a triangle as follows:

$$\begin{array}{ccccccccccc}
 & & & & & F^{\Delta}_{1,1} & & & & & \\
 & & & & F^{\Delta}_{2,1} & & F^{\Delta}_{2,2} & & & & \\
 & & & F^{\Delta}_{3,1} & & F^{\Delta}_{3,2} & & F^{\Delta}_{3,3} & & & \\
 & & F^{\Delta}_{4,1} & & F^{\Delta}_{4,2} & & F^{\Delta}_{4,3} & & F^{\Delta}_{4,4} & & \\
 & \iddots & & \vdots & & \vdots & & \vdots & & \ddots & \\
F^{\Delta}_{n,1} & & F^{\Delta}_{n,2} & & \ldots & & \ldots & & F^{\Delta}_{n,n-1} & & F^{\Delta}_{n,n}
\end{array}$$

or, given $F^{\Delta}_{1,n} = f(n)$, $F^{\Delta}_{n,n} = [f(1)]^n$, as

$$\begin{array}{ccccccccccc}
 & & & & & f(1) & & & & & \\
 & & & & f(2) & & f^2(1) & & & & \\
 & & & f(3) & & F^{\Delta}_{3,2} & & f^3(1) & & & \\
 & & f(4) & & F^{\Delta}_{4,2} & & F^{\Delta}_{4,3} & & f^4(1) & & \\
 & \cdot^{\cdot^{\cdot}} & & \vdots & & \vdots & & \vdots & & \ddots & \\
f(n) & & F^{\Delta}_{n,2} & & \dots & & \dots & & F^{\Delta}_{n,n-1} & & f^n(1)
\end{array}$$

Presented below are the first terms of the composita of the generating function $F(x) = \frac{x}{1-x}$ (it is a Pascal triangle):

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

For the given generating function $F(x) = \sum_{n\geq 1} f(n)x^n$, the composita $F^{\Delta}(n,k)$ always exists and is unique.

3 Generating function of a composita

Let us demonstrate that a composita has a generating function. For this purpose, we prove the following theorem.

Theorem 3.1. Let there be an ordinary generating function $A(x) = \sum_{n>0} a(n)x^n$, then the generating function of the composita is equal to

$$A(x)^k = \sum_{n\geqslant k} A^{\Delta}(n,k)x^n$$

Proof.

$$[A(x)]^k = \sum_{n\geqslant k} \sum_{\pi_k \in C_n} f(\lambda_1)f(\lambda_2)\dots f(\lambda_k)x^n = \sum_{n\geqslant k} A^{\Delta}(n,k)x^n.$$

□

This theorem gives grounds to use an apparatus of generating functions for computation of compositae. Let us consider several theorems for computation of compositae.

Theorem 3.2. Let there be a generating function $F(x) = \sum_{n>0} f(n)x^n$, its composita $F^{\Delta}(n,k)$, and constant α. Then the generating function $A(x) = \alpha F(x)$ has the composita

$$A^{\Delta}(n,k) = \alpha^k F^{\Delta}(n,k).$$

Proof.

$$[A(x)]^k = [\alpha F(x)]^k = \alpha^k [F(x)]^k.$$

□

Theorem 3.3. Let there be a generating function $F(x) = \sum_{n>0} f(n)x^n$, its composita $F^{\Delta}(n,k)$, and constant α. The generating function $A(x) = F(\alpha x)$ has the composita

$$A^{\Delta}(n,k) = \alpha^n F^{\Delta}(n,k).$$

Proof. By definition, we have

$$A^{\Delta}(n,k) = \sum_{\pi_k \in C_n} \alpha^{\lambda_1} f(\lambda_1) \alpha^{\lambda_2} f(\lambda_2) \ldots \alpha^{\lambda_k} f(\lambda_k) =$$

$$= \alpha^n \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k) = \alpha^n F^{\Delta}(n,k).$$

□

Theorem 3.4. Let there be a generating function $F(x) = \sum_{n>0} f(n)x^n$, its composita $F^{\Delta}(n,k)$, and generating functions $B(x) = \sum_{n\geqslant 0} b(n)x^n$ and $[B(x)^k] = \sum_{n\geqslant 0} B(n,k)x^n$. Then the generating function $A(x) = F(x)B(x)$ has the composita

$$A^{\Delta}(n,k) = \sum_{i=k}^{n} F^{\Delta}(i,k) B(n-i,k).$$

Proof. Because $a(0) = f(0)b(0) = 0$, the function $A(x)$ has the composita $A^{\Delta}(n,k)$. On the other hand,

$$[A(x)]^k = [F(x)]^k [B(x)]^k.$$

Hence, from the rule of product of generating functions we have

$$A^{\Delta}(n,k) = \sum_{i=k}^{n} F^{\Delta}(i,k) B(n-i,k).$$

□

If for $B(x)$ we have $b(0) = 0$ the formula takes the form:

$$A^{\Delta}(n,k) = \sum_{i=k}^{n-k} F^{\Delta}(i,k) B^{\Delta}(n-i,k).$$

Theorem 3.5. Let there be generating functions $F(x) = \sum_{n>0} f(n)x^n$, $G(x) = \sum_{n>0} g(n)x^n$ and their compositae $F^{\Delta}(n,k)$, $G^{\Delta}(n,k)$. Then the generating function $A(x) = F(x) + G(x)$ has the composita

$$A^{\Delta}(n,k) = F^{\Delta}(n,k) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^{\Delta}(i,j) G^{\Delta}(n-i,k-j) + G^{\Delta}(n,k).$$

Table 1: Known compositae

No	function $F(x)$	composita $F^{\Delta}(n,k)$
1	x^m	$\delta_{n,mk}$, $m>0$
2	$\frac{x}{1-x}$	$\binom{n-1}{k-1}$
3	xe^x	$\frac{k^{n-k}}{(n-k)!}$
4	$\ln(x+1)$	$\frac{k!}{n!}\left[{n \atop k}\right]$
5	e^x-1	$\frac{k!}{n!}\left\{{n \atop k}\right\}$

Proof. From the binomial theorem, we have

$$[A(x)]^k=\sum_{j=0}^{k}\binom{k}{j}[F(x)]^j[G(x)]^{k-j}.$$

$$[F(x)]^j=\sum_{n\geqslant j}F^{\Delta}(n,j),$$

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$$[G(x)]^{k-j}=\sum_{n\geqslant k-j}G^{\Delta}(n,k-j).$$

From the rule of multiplication of series, we obtain

$$A^{\Delta}(n,k)=F^{\Delta}(n,k)+\sum_{j=1}^{k-1}\binom{k}{j}\sum_{i=j}^{n-k+j}F^{\Delta}(i,j)G^{\Delta}(n-i,k-j)+G^{\Delta}(n,k).$$

□

4 Compositae of generating functions

For derivation of a composita of the generating function $A(x)$, we are to find coefficients of the generating function $A(x)^k$. Many similar functions are now available. As an example, Table 1 presents compositae for the generating functions given in [1, 2].

Let us consider the derivation of compositae for polynomials. Let us find a composita of the generating function $A(x)=ax+bx^2$. The composita of the function $F(x)=ax$, according to theorem (3.2), is equal to $a^k\delta_{n,k}$ and is the composita of the function $F(x)=bx^2-b^k\delta_{n,2k}$. After transformations with the use of theorem (3.5) for a composita of the sum of generating function, we obtain

$$A^{\Delta}(n,k)=\binom{k}{n-k}a^{2k-n}b^{n-k}.$$

After derivation of the composita of the function $A(x)=ax+bx^2$, we can obtain a composita of the function $B(x)=ax+bx^2+cx^3$; for this purpose, we can write it as the sum of the

Table 2: Compositae of polynomials

No	function $F(x)$	composita $F^{\Delta}(n,k)$
1	$ax+bx^2$	$\binom{k}{n-k}a^{2k-n}b^{n-k}$
2	$ax+bx^2+cx^3$	$\sum\limits_{j=0}^{k}\binom{k}{j}\binom{j}{n-k-j}a^{k-j}b^{2j+k-n}b^{n-k-j}$
3	$ax+cx^3$	$\binom{k}{\frac{3k-n}{2}}a^{\frac{3k-n}{2}}c^{\frac{n-k}{2}}$
4	$ax+bx^2+dx^4$	$\sum\limits_{j=\lfloor\frac{4k-n}{3}\rfloor}^{k} a^{4-nk-2j}b^{n-4k+3j}d^{k-j}\binom{j}{n-4k+3j}\binom{k}{j}$
5	$ax+bx^2+cx^3+dx^4$	$\sum\limits_{j=0}^{k}\binom{k}{j}\sum\limits_{i=j}^{n-k+j} a^{2j-i}b^{i-j}c^{4(k-j)+i-n}d^{n-3(k-j)-i}\binom{j}{i-j}\binom{k-j}{n-3(k-j)-i}$

functions $B_1(x)=ax$ and $B_2(x)=x(bx+cx^2)$. The composita for the function $B_2(x)$ is thus obtained by simple shift $B_2^{\Delta}(n,k)=A^{\Delta}(n-k,k)$. Next, we can use theorem (3.5)for a composita of the sum of generating functions to obtain a composita of the desired function. With this procedure, compositae for different polynomials were obtained; they are presented in Table 2.

Let us turn to computations of compositae of trigonometric functions. For this purpose, we resort to the Euler identity $e^{ix}=\cos(x)+i\sin(x)$. Let us consider computations of a composita of the sine. Using the expression

$$\sin(x)=\frac{e^{ix}-e^{-ix}}{2i},$$

we obtain $\sin(x)^k$

$$\sin(x)^k=\frac{1}{2^k i^k}\sum_{m=0}^{k}\binom{k}{m}e^{imx}e^{-i(k-m)x}(-1)^{k-m}=\frac{1}{2^k i^k}\sum_{m=0}^{k}\binom{k}{m}e^{i(2m-k)x}(-1)^{k-m}.$$

Hence the composita is equal to

$$\frac{1}{2^k}i^{n-k}\sum_{m=0}^{k}\binom{k}{m}\frac{(2m-k)^n}{n!}(-1)^{k-m}.$$

Taking into account that $n-k$ is an even number and the function is symmetric about k, we obtain the composita of the generating function $\sin(x)$

$$A^{\Delta}(n,k)=\begin{cases}\frac{1}{2^{k-1}n!}\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{m}(2m-k)^n(-1)^{\frac{n+k}{2}-m}, & (n-k)-\text{even}\\ 0, & (n-k)-\text{odd}\end{cases}$$

With this approach, compositae of the generating functions $x\cos(x)$, $\tan(x)$, $\arctan(x)$, $\sinh(x)$, $x\cosh(x)$ were obtained; they are presented in Table 3.

Table 3: Compositae of trigonometric and hyperbolic functions

No	function $F(x)$	composita $F^{\Delta}(n,k)$
1	$\sin(x)$	$(1+(-1)^{n-k})\frac{1}{2^k n!}\sum\limits_{m=0}^{\frac{k}{2}}\binom{k}{m}(2m-k)^n(-1)^{\frac{n+k}{2}-m},$
2	$x\cos(x)$	$\begin{cases}\frac{1+(-1)^{n-k}}{2^k(n-k)!}(-1)^{\frac{n-k}{2}}\sum\limits_{j=0}^{\frac{k-1}{2}}\binom{k}{j}(2j-k)^{n-k}, & n>k\\ 1, & n=k\end{cases}$
3	$\tan(x)$	$\frac{1+(-1)^{n-k}}{n!}\sum\limits_{j=k}^{n}2^{n-j-1}\left\{{n \atop j}\right\}j!(-1)^{\frac{n+k}{2}+j}\binom{j-1}{k-1}$
4	$\arctan(x)$	$\frac{\left((-1)^{\frac{3n+k}{2}}+(-1)^{\frac{n-k}{2}}\right)k!}{2^{k+1}}\sum\limits_{j=k}^{n}\frac{2^j}{j!}\binom{n-1}{j-1}\left[{j \atop k}\right].$
5	$\sinh(x)$	$\frac{1}{2^k}\sum\limits_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(k-2i)^n}{n!}$
6	$x\cosh(x)$	$\frac{1}{2^k}\sum\limits_{i=0}^{k}\binom{k}{i}\frac{(k-2i)^{n-k}}{(n-k)!}$

5 Composition of ordinary generating functions

Let us consider the application of compositae for computation of compositions of ordinary generating functions. For this purpose, we prove the following theorem.

Theorem 5.1. Let there be functions $f(n)$ and $r(n)$ and their generating functions $F(x) = \sum_{n\geqslant 1} f(n)x^n$, $R(x) = \sum_{n\geqslant 0} r(n)x^n$. Then for the composition of the generating functions $A(x) = R(F(x))$, the following expression holds true:

$$a(0) = r(0),$$

$$a(n) = \sum_{k=1}^{n} F^{\Delta}(n,k)r(k). \qquad (3)$$

Proof. So for computations of $A(x) = R(F(x))$, we are to obtain

$$A(x) = R(F(x)) = \sum_{n\geqslant 0} r(n)F(x)^n x^n.$$

Substitution of $\sum_{n\geqslant k} F^{\Delta}(n,k)x^n$ instead of $F(x)^k$ and summation of the coefficients with equal exponents x^n gives us the desired formula:

$$a(0) = r(0),$$

$$a(n) = \sum_{k=1}^{n} F^{\Delta}(n,k)r(k).$$

□

In what follows, for the composition $A(x) = R(F(x))$ we put $a(0) = r(0)$.

Example 5.2. Let there be a generating function $A(x) = \frac{1}{1-ax-bx^2-cx^3}$, where $a, b, c \neq 0$. Then, given the composita of the polynomial $F(x) = ax + bx^2 + cx^3$ (see Table 2) and the formula of composition (3), we obtain

$$a(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} b^{n-k-j}.$$

Example 5.3. Let there be a generating function $A(x) = e^{\sinh(x)}$. Then, given the composita of the polynomial $F(x) = \sinh(x)$ (see Table 3) and the formula of composition (3), we obtain

$$a(n) = \sum_{k=1}^{n} \frac{1}{2^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(k-2i)^n}{n!} \frac{1}{k!}.$$

Definition 2. Let there be a composition of generating functions $A(x) = R(F(x))$. Then the product of two compositae is the composita of the composition $A(x)$ and is denoted as $A^{\Delta}(n, k) = F^{\Delta}(n, k) \circ R^{\Delta}(n, k)$.

Theorem 5.4. Let there be two generating functions $F(x) = \sum_{n>0} f(n)x^n$ and $R(x) = \sum_{n>0} r(n)x^n$ and their compositae $F^{\Delta}(n, k)$ and $R^{\Delta}(n, k)$. Then the expression valid for the product of the compositae $A^{\Delta} = F^{\Delta} \circ R^{\Delta}$ is

$$A^{\Delta}(n, m) = \sum_{k=m}^{n} F^{\Delta}(n, k) R^{\Delta}(k, m). \tag{4}$$

Proof.

$$[A(x)]^m = [G(F(x)]^m = G^m(F(x))$$

Hence, according to the composition rule and considering that the nonzero terms $G^{\Delta}(n, m)$ begins with $n \geqslant m$, we have

$$A^{\Delta}(n, m) = \sum_{k=m}^{n} F^{\Delta}(n, k) G^{\Delta}(k, m).$$

□

Example 5.5. Let us find a composita of the generating function of Fibonacci numbers $A(x) = \frac{x}{1-x-x^2}$. Let us write $A(x) = xR(F(x))$, where $R(x) = \frac{1}{1-x}$, $F(x) = x + x^2$. Then

$$A(x)^m = x^m R(F(x))^m$$

$$R(x)^m = \sum_{n \geqslant 0} \binom{n+k-1}{k-1} x^n.$$

Using the formula of composition (3), we obtain the coefficients of the generating function $R(F(x))^m$:

$$\begin{cases} 1, & n=0, \\ \sum_{k=1}^{n} \binom{k}{n-k}\binom{k+m-1}{m-1}, & n>0. \end{cases}$$

Hence, the composita of the generating function for Fibonacci numbers is equal to

$$\sum_{k=\lceil \frac{n-m}{2} \rceil}^{n-m} \binom{k}{n-m-k}\binom{m+k-1}{m-1}$$

$$G^{\Delta}(n,k) = \sum_{i=k}^{n} \binom{k}{\frac{(m+1)k-i}{m}} (-1)^{\frac{i-k}{m}} \binom{n-i+k-1}{k-1}.$$

Example 5.6. The generating function for Bernoulli numbers is

$$A(x) = \frac{x}{e^x - 1}.$$

This generating function can be represented as the composition $B(F(x))$, where $B(x) = \frac{\ln x}{x}$, $F(x) = e^x - 1$. Let us find expressions for the coefficients of the generating functions $[B(x)]^k$

$$[B(x)]^k = \sum_{n \geqslant 0} \left[{n \atop k} \right] \frac{k!}{n!} x^{n-k}.$$

Hence

$$B(n,k) = \left[{n+k \atop k} \right] \frac{k!}{(n+k)!}$$

Given the composita of the function $F(x)$ (see the previous section),

$$F^{\Delta}(n,k) = \frac{k!}{n!} \left\{ {n \atop k} \right\}.$$

The superposition of the generating functions $A(x)^k = [B(e^x - 1)]^k$ is

$$A(n,m) = \begin{cases} 1, & n=0, \\ \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{k!}{n!} \left[{k+m \atop m} \right] \frac{m!}{(k+m)!}, & n>0. \end{cases}$$

Then the composita of the function $xA(x)$ is equal to

$$A^{\Delta}(n,m) = \begin{cases} 1, & n=m, \\ \frac{m!}{(n-m)!} \sum_{k=1}^{n-m} \frac{k!}{(k+m)!} \left[{k+m \atop m} \right] \left\{ {n-m \atop k} \right\}, & n>m. \end{cases}$$

6 Reciprocal generating functions

Reciprocal generating functions are functions that satisfy the condition [8]:

$$A(x)B(x) = 1.$$

Let us prove the theorem for the composita of a reciprocal function.

Theorem 6.1. Let there be a generating function $B(x)$, $b(0) \neq 0$ and a composita of the function $xB(x)$ – $B^{\Delta}(n,m)$. Then the composita of the function $xA(x)$ is equal to

$$A^{\Delta}(n,m) = \begin{cases} \frac{1}{b_0^m}, & n = m, \\ \frac{1}{b_0^m} \sum\limits_{k=1}^{n-m} (-1)^k \binom{m+k-1}{m-1} \sum\limits_{j=0}^{k} b_0^{k-j} (-1)^{j-k} \binom{k}{j} B^{\Delta}(n-m+j, j), & n > m \end{cases} \tag{5}$$

Proof.

$$[xA(x)]^k = \left[\frac{x}{b_0 + B(x) - b_0} \right]^k.$$

After derivation of the composita of the function $\frac{1}{b_0}(B(x) - b_0)$ and composition of the function $F(x) = \frac{1}{b_0(1+x)}$, we obtain the desired formula. □

Example 6.2. Let us find a composita of the generating function $F(x) = x^2 \csc(x)$. For this purpose, we write

$$F(x) = \frac{x}{1 + \frac{\sin(x)}{x} - 1};$$

from whence we find the composita of the function $\frac{\sin(x)}{x} - 1$ and obtain the composita

$$F^{\Delta}(n,k) = \begin{cases} 1, & n = m, \\ 0, & (n-m)\text{- odd}, \\ \sum\limits_{k=1}^{n-m} \binom{m+k-1}{m-1} \sum\limits_{j=1}^{k} \frac{\binom{k}{j} \sum\limits_{i=0}^{\lfloor \frac{j}{2} \rfloor} (j-2i)^{n-m+j} \binom{j}{i} (-1)^{\frac{n-m}{2}+i+j}}{2^{j-1}(n-m+j)!} & (n-m)\text{- even}, \end{cases}$$

7 Inverse generating functions

Theorem 7.1. Let there be given an ordinary generating function $F(x) = \sum_{n \geq 1} f(n) x^n$ and its composita $F^{\Delta}(n,k)$. Then for the coefficients of the inverse generating function $F^{-1}(x) = \sum_{n \geq 1} a(n) x^n$, the following recurrent expression holds true:

$$a(n) = \begin{cases} \frac{1}{f(1)}, & n = 1, \\ -\frac{1}{f^n(n)} \sum\limits_{k=1}^{n-1} F^{\Delta}(n,k) a(k), & n > 1. \end{cases} \tag{6}$$

Proof. By definition, we have

$$F^{-1}(F(x)) = x.$$

Now, using the formula of composition of ordinary generating function, we write

$$a(1)f(1) = 1.$$

$$\sum_{k=1}^{n} F^{\Delta}(n,k)a(k) = 0,\ n > 1;$$

from whence we obtain the desired formula

$$a(n) = \begin{cases} \frac{1}{f(1)}, & n = 1, \\ -\frac{1}{f^n(n)} \sum\limits_{k=1}^{n-1} F^{\Delta}(n,k)a(k), & n > 1. \end{cases}$$

□

Example 7.2. Let there be given a generating function $F(x) = xe^x$ and its composita $F^{\Delta}(n,k) = \frac{k^{n-k}}{(n-k)!}$. Then for the coefficients of the inverse generating function, the following expression holds true:

$$a(n) = \begin{cases} 1, & n = 1, \\ -\sum\limits_{k=1}^{n-1} \frac{k^{n-k}}{(n-k)!} a(k), & n > 1. \end{cases}$$

Example 7.3. Let there be given a generating function $G(x) = x + \sin(x)$. Let us derive an expression for the coefficients of the inverse generating function. First, we find a composita of the generating function $G(x)$ using theorem (3.5) for a composita of the sum of generating functions:

$$G^{\Delta}(n,k) = \sum_{j=1}^{k} \binom{k}{j} F_{sin}^{\Delta}(n-k+j, j) + \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta, $F_{sin}^{\Delta}(n,k)$ is the composita of the sine (Table 3). Then for the coefficients of the inverse generating function, the following expression holds true:

$$a(n) = \begin{cases} \frac{1}{2}, & n = 1, \\ -\frac{1}{2^n} \sum\limits_{k=1}^{n-1} \left[\sum\limits_{j=1}^{k} \binom{k}{j} F_{sin}^{\Delta}(n-k+j, j) \right] a(k), & n > 1. \end{cases}$$

8 Solution of equations $A(x) = G(xA(x)^m)$

For solution of the functional equation

$$A(x) = xG(A(x)), \tag{7}$$

where $A(x)$ and $G(x)$ are generating functions and $G(0) \neq 0$, we know the Lagrange inversion formula [1] in which the coefficients of the generating functions $A(x)$ and $G(x)$ are related as follows:

$$n[x^n]A(x)^k = k[x^{n-k}]G(x)^n.$$

In the left-hand side, the composita of the generating function $A(x)$ multiplied by n is written. Hence,

$$[x^n]f(x)^k = A^{\Delta}(n,k).$$

Let us put

$$(xG(x))^k = \sum_{n\geqslant 1} G_x^{\Delta}(n,k)x^n;$$

from whence we have

$$G(x)^k = \sum_{n\geqslant k} G_x^{\Delta}(n,k)x^{n-k}.$$

After transformations, the relation takes the form:

$$A^{\Delta}(n,k) = \frac{k}{n} G_x^{\Delta}(2n-k,n). \tag{8}$$

Functional equation (7) by replacing $xB(x) = A(x)$ can be represented in the form:

$$B(x) = G(xB(x)).$$

Because of the unique dependence between the generating function and the composita, formula (8) provides a solution of the backward equation $A(x) = xG(A(x))$ when $A(x)$ is known and $G(x)$ is unknown. Hence,

$$G^{\Delta}(n,k) = \frac{k}{2k-n} A^{\Delta}(k, 2k-n).$$

It should be noted that for $n = k$,

$$G^{\Delta}(n,n) = A^{\Delta}(n,n).$$

This peculiarity means that the right diagonal in transformations of compositae remains unchanged.

Definition 3. The left composita of a generating function $G(x)$ is the composita:

$$A^{\Delta}(n,k) = \frac{k}{2k-n} G^{\Delta}(k, 2k-n),$$

where $G^{\Delta}(n,k)$ is the composita of the generating function $G(x)$.

Definition 4. The right composita of a generating function $G(x)$ is the composita:

$$A^{\Delta}(n,k) = \frac{k}{n} G^{\Delta}(2n-k,n),$$

where $G^{\Delta}(n,k)$ is the composita of the generating function $G(x)$.

Derived relation (8) for solving functional equations can be generalized if a generating function is already the solution of a certain functional equation. Then, on the strength of formula (8), each right composita has its right composita, and each left composita has its left composita. Generalization of the formulae for the left and right compositae allows writing one expression:

$$A_x^{\Delta}(n,k) = \frac{k}{i_{m-1}} G_x^{\Delta}(i_m, i_{m-1}).$$

$$i_m = (m+1)n - mk.$$

Let us prove the following theorem.

Theorem 8.1. Let there be given a generating function $G(x)$, $G(0) \neq 0$ and let the functional equation

$$A(x) = G(xA^m(x))$$

be specified for a set of integers $m \in N$. Then

$$A_x^{\Delta}(n,k) = \frac{k}{i_{m-1}} G_x^{\Delta}(i_m, i_{m-1}),$$

where $A_x^{\Delta}(n,k)$ is the composita of the generating function $xA(x)$, $G_x^{\Delta}(n,k)$ is the composita of the generating function $xG(x)$,

$$i_m = (m+1)n - mk.$$

Proof. Let $m = 0$, then $A(x) = G(xA^0(x))$, and $i_{m-1} = k$, $i_m = n$. We obtain the identity $A^{\Delta}(n,k) = \frac{k}{k} G^{\Delta}(n,k)$. Let $m = 1$, then $A(x) = G(xA(x))$ and from the Lagrange inversion theorem we have $i_{m-1} = n$, $i_m = 2n-k$. So the composita is $A^{\Delta}(n,k) = \frac{k}{n} G^{\Delta}(2n-k, n)$. By induction, we put that for m we have the solution of the equation:

$$A_m(x) = G(xA_m(x)^m) \tag{9}$$

and

$$A_m^{\Delta}(n,k) = \frac{k}{i_{m-1}} G^{\Delta}(i_m, i_{m-1}).$$

Then we find the solution for $m+1$

$$A_{m+1}(x) = A_m(xA_{m+1}(x)).$$

Instead of $A_m(x)$ we substitute the right hand-side of (9):

$$A_{m+1}(x) = G(xA_{m+1}(x)[A_m(xA_{m+1}(x)]^m);$$

from whence it follows that

$$A_{m+1}(x) = G(xA_{m+1}(x)^{m+1}).$$

Table 4: Table of functional equations

Equation	Function	Composita $xA(x)$	OEIS
$A(x) = 1 + xA^{-1}(x)$	$\frac{1+\sqrt{1+4x}}{2}$	$\frac{k}{2k-n}\binom{2k-n}{n-k}$	
$A(x) = 1 + xA^{0}(x)$	$1+x$	$\binom{k}{n-k}$	
$A(x) = 1 + xA^{1}(x)$	$\frac{1}{1-x}$	$\binom{n-1}{k-1}$	A000012
$A(x) = 1 + xA^{2}(x)$	$\frac{1-\sqrt{1-4x}}{2x}$	$\frac{k}{n}\binom{2n-k-1}{n-1}$	A000108
$A(x) = 1 + xA^{3}(x)$		$\frac{k}{n}\binom{3n-2k}{n-k}$	A001764

$$A^{\Delta}_{m+1}(n,k) = \frac{k}{i_m} G^{\Delta}(i_{m+1}, i_m).$$

Now we consider the case $m < 0$. Then the equation takes the form:

$$A(x) = G\left(\frac{x}{A(x)^w}\right),$$

for $w = -m$. Because $G(0) \neq 0$ and $A(0) = G(0)$, the functions $R(x) = \frac{1}{G(x)}$, $F(x) = \frac{1}{A(x)}$ exist. Hence, replacement of the functions $G(x)$ and $A(x)$ by the reciprocal functions gives

$$F(x) = R(xF(x)^w).$$

Thus, the solution of this equations gives us the composita for the function $xF(x)$; from whence, using theorem (6.1)for a composita of reciprocal functions, we obtain the composita of the desired function $A(x)$ for $m < 0$. Thus, the theorem is proved. $\square$

The compositae of the functions $xG(x)$ and $\frac{x}{G(x)}$ specify the conjugate sequences of a composita. As a corollary of the theorem, the left composita of the function $G(x)$ can be found by finding the right composita of the function $\frac{x}{G(x)}$ and then the reciprocal composita of the derived composita.

Table 4 presents a sequence of functional equations for the generating function $G(x) = 1 + x$.

Example 8.2. Let us find a solution of the functional equation

$$A(x) = 1 + xA(x) + x^2A(x)^2 + 2x^3A(x)^3.$$

Then the function $xG(x)$ has the form:

$$xG(x) = x + x^2 + x^3 + 2x^4.$$

The composita of this function in view of the coefficients $a = 1,\ b = 1,\ c = 1,\ d = 2$ is

$$G^{\Delta}(n,k) = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} 2^{n-3(k-j)-i} \binom{j}{i-j} \binom{k-j}{n-3(k-j)-i}.$$

Hence, the desired solution is $a(n)=\frac{1}{n}G^{\Delta}(2n-1,n)$

$$a(n)=\frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\sum_{i=j}^{n+j-1}\binom{j}{i-j}2^{-n+3j-i-1}\binom{n-j}{-n+3j-i-1}$$

Example 8.3. Let us find a solution of the equation

$$A(x)=x\left(e^{A(x)}+e^{A^2(x)}\right).$$

To do this, we are to find a composita of the function

$$G(x)=xe^{x}+xe^{x^2}.$$

For this purpose, it is necessary to find a composita of the sum of the functions $f(x)=xe^x$ and $h(x)=xe^{x^2}$. The composita of the function $f(x)$ is known (Table 1) and is $F^{\Delta}(n,k)=\frac{k^{n-k}}{(n-k)!}$. The composita of the function $h(x)$ can be found as the product of the compositae $F^{\Delta}(n,k)$ and $\delta(n,2k)$; from whence we have

$$H^{\Delta}(n,k)=\frac{k^{\frac{n-k}{2}}\left((-1)^{n-k}+1\right)}{2\left(\frac{n-k}{2}\right)!}.$$

Then, using the theorem for a composita of the sum of generating functions and formula (8), we obtain the desired composita $A^{\Delta}(n,m)$, and the desired function $a(n)=A^{\Delta}(n,1)$ takes the form:

$$\frac{1}{2^{n+1}n}\left[\sum_{k=1}^{n-1}\binom{n}{k}\sum_{i=k}^{n+k-1}\frac{k^{i-k}\;(n-k)^{n+k-i-1}\left((-1)^{n+k-i-1}+1\right)}{(i-k)!\;\left(\frac{n+k-i-1}{2}\right)!}+\frac{2n^{n-1}}{(n-1)!}+\frac{n^{\frac{n-1}{2}}\left((-1)^{n-1}+1\right)}{\left(\frac{n-1}{2}\right)!}\right]$$

Example 8.4. Let us find compositae of radicals of the form $F(x)=1-\sqrt[m]{1-x}$. For this purpose, we write the functional equation for the left composita

$$A(x)=\frac{1-\sqrt[m]{1-\frac{x}{A(x)}}}{\frac{x}{A(x)}}.$$

Hence,

$$A(x)=\frac{x}{1-(1-x)^m}.$$

After transformations we obtain

$$A(x)=\frac{1}{m}\frac{1}{\left(1-\frac{1}{m}\sum\limits_{j=2}^{m}\binom{m}{j}x^{j-1}(-1)^j\right)}.$$

Next, we find the composita $A^{\Delta}(n,m)$ of the function $xA(x)$ with the use of the formula for composition of generating functions with the right composita being the composita of the desired generating function:

$$F^{\Delta}(n,m)=\frac{m}{n}A^{\Delta}(2n-m,n).$$

Let us consider the example for $m=3$: the desired function is $F(x)=1-\sqrt[3]{1-x}$ and the function of the left composita is

$$xA(x)=\frac{1}{3}\frac{x}{1-x+\frac{1}{3}x^2}$$

The composita of the function $G(x)=x-\frac{1}{3}x^2$ is

$$G^{\Delta}(n,k)=\binom{k}{n-k}(-1)^{n-k}\left(\frac{1}{3}\right)^{n-k}.$$

Then the composita of the function $xA(x)$ is equal to

$$A^{\Delta}(n,m)=\begin{cases}(\frac{1}{3})^m, & n=m,\\ (\frac{1}{3})^m\sum\limits_{k=1}^{n-m}\binom{k}{n-m-k}3^{m+k-n}(-1)^{n-m-k}\binom{m+k-1}{m-1}, & n>m.\end{cases}$$

Hence the composita of the function $1-\sqrt[3]{1-x}$ is

$$F^{\Delta}(n,m)=\begin{cases}(\frac{1}{3})^n, & n=m,\\ \frac{m}{n}\sum\limits_{k=1}^{n-m}\binom{k}{n-m-k}3^{-2n+m+k}(-1)^{n-m-k}\binom{n+k-1}{n-1}, & n>m.\end{cases}$$

Thus,

$$\sqrt[3]{1-x}=1-\frac{1}{3}x-\sum_{n>1}\frac{1}{n}\sum_{k=1}^{n-1}\binom{k}{n-k-1}3^{k-2n+1}(-1)^{n-1-k}\binom{n+k-1}{n-1}x^n.$$

Example 8.5. Let us find a composita of the function $\arcsin(x)$; to do this, we use the functional equation

$$A(x)=xA(x)\csc(xA(x));$$

from whence we have $A(x)=\arcsin(x)$. Thus, the right composita of the composita of the function $x^2\csc(x)$ is the composita $\arcsin(x)$.

$$A^{\Delta}(n,m)=\begin{cases}1, & n=m,\\ 0, & (n-m)\text{- odd},\\ \frac{m}{n}\sum\limits_{k=1}^{n-m}\binom{n+k-1}{n-1}\sum\limits_{j=1}^{k}\frac{\binom{k}{j}\sum\limits_{i=0}^{\lfloor\frac{j}{2}\rfloor}(j-2\,i)^{n-m+j}\binom{j}{i}(-1)^{\frac{n-m}{2}+i+j}}{2^{j-1}(n-m+j)!} & (n-m)\text{- even}.\end{cases}$$

9 Riordan array

As is known [6, 7], the Riordan array for the generating functions $G(x)$ and $F(x)$ is a triangle $R_{n,k}$ with properties such that for the generating functions $A(x)$ and $B(x)$ related by $A(x) = G(x)B(F(x))$, the following expression holds true:

$$a(n) = \sum_{k=0}^{n} R_{n,k} b(k).$$

The Riordan array for the functions $G(x)$ and $F(x)$ is denoted by the pair $(G(x), F(x))$. Let us demonstrate that the Riordan array can be derived using the composita of a generating function. For this purpose, we prove the following theorem.

Theorem 9.1. Let there be a generating function $F(x) = \sum_{n>0} f(n)x^n$, its composita $F^{\Delta}(n,k)$, and generating function $G(x) = \sum_{n\geqslant 0} g(n)x^n$. Then for the Riordan array of the generating functions $(G(x), F(x))$, the following expression holds true:

$$R_{n,k} = \begin{cases} g(n), & k = 0, \\ \sum_{i=0}^{n-k} g(i)F^{\Delta}(n-i,k), & k > 0, \end{cases} \tag{10}$$

where $k \leqslant n$.

Proof. Let there be a generating function $B(x) = \sum_{n\geqslant 0} b(n)x^n$. Let us find the expression for the coefficients of the generating function $D(x) = G(x)B(F(x))$

$$d(0) = b(0).$$

$$d(n) = \sum_{k=1}^{n} F^{\Delta}(n,k)b(k).$$

No we find the expression for the coefficients of the generating function $A(x)$

$$a(n) = \sum_{i=0}^{n} g(i)d(n-i).$$

Hence

$$a(n) = g(0)\sum_{k=1}^{n} F^{\Delta}(n,k)b(k) + g(1)\sum_{k=1}^{n-1} F^{\Delta}(n-1,k)b(k) + \ldots + g(n-1)r(1) + g(n)b(0).$$

By grouping the expressions with coefficients $b(i)$ we obtain

$$a(n) = g(n)b(0) + \sum_{i=0}^{n-1} g(i)F^{\Delta}(n-i,1)b(1) + \sum_{i=0}^{n-2} g(i)F^{\Delta}(n-i,2)b(2) + \ldots + g(0)F^{\Delta}(n,n)b(n).$$

Table 5: Riordan arrays

$f(x)\diagdown q(x)$	$\frac{1}{1-x}$	e^x	$\ln(\frac{1}{1-x})$	$\frac{1-\sqrt{1-4x}}{2x}$
$\frac{x}{1-x}$	$\binom{n}{k}$	$\sum\limits_{i=0}^{n-k} \frac{1}{i!}\binom{n-i-1}{k-1}$	$\sum\limits_{i=1}^{n-k} \frac{1}{i}\binom{n-i-1}{k-1}$	$\sum\limits_{i=0}^{n-k} \frac{1}{i+1}\binom{2i}{i}\binom{n-i-1}{k-1}$
xe^x	$\sum\limits_{i=0}^{n-k} \frac{k^{n-k-i}}{(n-k-i)!}$	$\frac{(k+1)^{n-k}}{(n-k)!}$	$\sum\limits_{i=1}^{n-k} \frac{k^{n-k-i}}{i\,(n-k-i)!}$	$\sum\limits_{i=1}^{n-k} \frac{1}{i+1}\binom{2i}{i}\frac{k^{n-k-i}}{(n-k-i)!}$
$\ln(1+x)$	$k!\sum\limits_{i=0}^{n-k} \frac{\left[{n-i \atop k}\right]}{(n-i)!}$	$k!\sum\limits_{i=0}^{n-k} \frac{\left[{n-i \atop k}\right]}{i!(n-i)!}$	$k!\sum\limits_{i=1}^{n-k} \frac{\left[{n-i \atop k}\right]}{i(n-i)!}$	$k!\sum\limits_{i=0}^{n-k} \frac{1}{i+1}\binom{2i}{i}\frac{\left[{n-i \atop k}\right]}{(n-i)!}$
$\frac{1-\sqrt{1-4x}}{2}$	$\sum\limits_{i=0}^{n-k} \frac{k}{n-i}\binom{2(n-i)-k-1}{n-i-1}$	$\sum\limits_{i=0}^{n-k} \frac{k}{i!(n-i)}\binom{2(n-i)-k-1}{n-i-1}$	$\sum\limits_{i=1}^{n-k} \frac{k}{i(n-i)}\binom{2(n-i)-k-1}{n-i-1}$	$\frac{k+1}{n+1}\binom{2n-k}{n-k}$

Thus,

$$R_{n,k} = \begin{cases} g(n), & k = 0, \\ \sum\limits_{i=0}^{n-k} g(i)F^{\Delta}(n-i,k), & k > 0, \end{cases}$$

□

Let us consider the Riordan array $(1, F(x))$. According to formula (10), we obtain the following triangle:

$$\begin{array}{lllll} 1, & & & & \\ 0, & F(1,1), & & & \\ 0, & F(2,1), & F(2,2), & & \\ 0, & F(3,1), & F(3,2), & F(3,3), & \\ 0, & F(4,1), & F(4,2), & F(4,3), & F(4,4) \end{array}$$

It is seen that the composita is a degenerate case of the Riordan array $(1, F(x))$ in which the column $R(n,0)$ is absent.

Corollary 9.2. A Riordan array of the form $(F(x), xF(x))$ is the composita of the function $xF(x)$.

Proof. Substitution of $F(x)$ in place of $G(x)$ in expression (10)gives

$$R_{n,k} = \begin{cases} f(n), & k = 0, \\ \sum\limits_{i=0}^{n-k} f(i)F^{\Delta}(n-i,k), & k > 0, \end{cases}$$

Hence, numbering from $(1,1)$ rather than from $(0,0)$ gives us the expression for composita (2) □

Table 5 presents expressions for the Riordan array for the pair of functions $(g(x), f(x))$ obtained from theorem 9.1; in the cells of Table 5 in which sums are given, the term $R_{n,0} = f(n)$ is taken by default.

10 Identities based on compositae

Theorem 10.1. For any three compositae, the following identity is valid:

$$\sum_{k=m}^{n}\sum_{i=k}^{n} F^{\Delta}(n,i)R^{\Delta}(i,k)G^{\Delta}(k,m) = \sum_{k=m}^{n}\sum_{i=k}^{n} R^{\Delta}(n,i)G^{\Delta}(i,k)F^{\Delta}(k,m). \qquad (11)$$

Proof. The composition of generating functions is an associative operation

$$F(x)\circ(R(x)\circ G(x)) = (F(x)\circ R(x))\circ G(x).$$

Hence, the product of compositae is also associative and identity (11) holds true. □

Theorem 10.2. For the composita $F^{\Delta}(n,m)$ of the generating function $F(x)=\sum_{n>0} f_n x^n$, the following identity is valied:

$$nF^{\Delta}(n,m) = m\sum_{k=1}^{n-m+1} kf(k)F^{\Delta}(n-k,m-1),\ n\geq m>1.$$

Proof. Let us consider the derivative of the generating function of the composita

$$[F(x)^m]' = \sum_{n\geqslant 0}(n+1)F^{\Delta}(n,m)x^n$$

On the other hand,

$$[F(x)^m]' = mF(x)^{m-1}F'(x) = \sum_{n\geqslant 0} m\sum_{k=1}^{n-m+1} kf(k)F^{\Delta}(n-k,m-1)x^n;$$

from whence we have

$$nF^{\Delta}(n,m) = m\sum_{k=1}^{n-m+1} kf(k)F^{\Delta}(n-k,m-1),\ n\geq m>1.$$

□

Theorem 10.3. Let there be given compositae $F^{\Delta}(n,k)$ and $F^{-1\Delta}(n,k)$ of direct $F(x)$ and inverse $F^{-1}(x)$ generating functions. Then the following identity holds true:

$$\sum_{k=m}^{n} F^{\Delta}(n,k)F^{-1\Delta}(k,m) = \sum_{k=m}^{n} F^{-1\Delta}(n,k)F^{\Delta}(k,m) = \delta_{n,m}. \qquad (12)$$

Proof. From $F(F^{-1}(x)) = F^{-1}(F(x)) = x$ we have the identity (12) □

Example 10.4. Let there be given a generating function $F(x) = xe^x$ and a Lambert function $W(x)$ and their compositae $F^{\Delta}(n,k) = \frac{k^{n-k}}{(n-k)!}$, $W^{\Delta}(n,k) = \frac{k\, n^{n-k-1}\,(-1)^{n-k}}{(n-k)!}$. Then

$$\sum_{k=m}^{n} \frac{k\, m^{k-m}\, n^{n-k-1}\ (-1)^{n-k}}{(k-m)!\ (n-k)!} = m \sum_{k=m}^{n} \frac{k^{n-m-1}\ (-1)^{k-m}}{(k-m)!\ (n-k)!} = \delta_{n,m}.$$

After transformations we have the following identity:

$$(n+m)^{n-1} = \sum_{k=0}^{n-1} \binom{n}{k} (m+k)^{n-1}\ (-1)^{n-k+1}.$$

Example 10.5. Let there be given generating functions $F(x) = \frac{1-\sqrt{1-4x}}{2}$ and $G(x) = x - x^2$ and their compositae $F^{\Delta}(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$, $G^{\Delta}(n,k) = \binom{k}{n-k}(-1)^{n-k}$. Then

$$\frac{1}{n}\sum_{k=m}^{n} k(-1)^{k-m}\binom{m}{k-m}\binom{2n-k-1}{n-1} = m\sum_{k=m}^{n}\frac{1}{k}\binom{k}{n-k}\binom{2k-m-1}{k-1}(-1)^{n-k} = \delta_{n,m}.$$

Theorem 10.6. For the composita $G^{\Delta}(n,k)$, the following identity holds true:

$$\frac{r}{mn+r} G^{\Delta}((m+1)n+r, mn+r) = \sum_{k=1}^{n} \frac{k}{n} G^{\Delta}((m+1)n-k, mn) G^{\Delta}(r+k, r),$$

where $n, m > 0$, $r \leqslant n$.

Proof. The solution of the equation $A(x) = G(xA(x)^m)$ allows us to express the composita of the generating function $xA(x)$ in terms of the composita of the generating function $xG(x)$. We write this equation in the form:

$$A(x)^r = [G(xA(x)^m)]^r$$

Then $A^k(x) = \sum_{n\geqslant 0} A(n,k)x^n$; from whence it follow that $[A^k(x)]^m = \sum_{n\geqslant 0} A(n,km)x^n$. On the other hand,

$$A(n,k) = A^{\Delta}(n+k,k) = \frac{k}{mn+k} G^{\Delta}((m+1)n+k, mn+k).$$

Now we find the expression of the composita for $xA(x)^m$

$$A(n,km) = \frac{km}{mn+km} G^{\Delta}((m+1)n+km, mn+km).$$

Hence the desired composita is $A(n-k, km)$

$$A(n-k,km) = \frac{k}{n} G^{\Delta}((m+1)(n-k)+km, m(n-k)+km) =$$

$$= \frac{k}{n} G^{\Delta}((m+1)n-k, mn)$$

Let us write the expression for the right-hand side of the equation using the formula for composition of ordinary generating functions:

$$A(n,r) = \sum_{k=1}^{n} \frac{k}{n} G^{\Delta}((m+1)n-k, mn) G^{\Delta}(r+k, r).$$

Now

$$\frac{r}{mn+r} G^{\Delta}((m+1)n+r, mn+r) = \sum_{k=1}^{n} \frac{k}{n} G^{\Delta}((m+1)n-k, mn) G^{\Delta}(r+k, r),$$

which is what we set out to prove. □

Example 10.7. Let $G(x) = \frac{1}{1-x}$, $G^{\Delta}(n,k) = \binom{n-1}{k-1}$, then the equation

$$A(x) = \frac{1}{1 - xA(x)^m}$$

generates the identity

$$\frac{r}{mn+r} \binom{(m+1)n+r-1}{mn+r-1} = \sum_{k=1}^{n} \frac{k}{n} \binom{(m+1)n-k-1}{mn-1} \binom{r+k-1}{r-1}.$$

$$\frac{r}{n+r} \binom{2n+r-1}{n+r-1} = \frac{1}{n} \sum_{k=1}^{n} k \binom{2n-k-1}{n-1} \binom{k+r-1}{r-1}$$

Example 10.8. Let $G(x) = \frac{e^x-1}{x}$, $G^{\Delta}(n,k) = \frac{k!}{n!} \left\{ {k \atop n} \right\}$, then the equation

$$xA^{m+1}(x) = \exp\left(xA(x)^m\right) - 1.$$

generates the identity

$$\frac{r}{mn+r} \left\{ {mn+n+r \atop mn+r} \right\} \frac{(mn+r)!}{(mn+n+r)!} =$$

$$= \frac{1}{n} \sum_{k=1}^{n} k \left\{ {mn+n-k \atop mn} \right\} \frac{(mn)!}{(mn+n-k)!} \left\{ {k+r \atop r} \right\} \frac{(k)!}{(k+r)!});$$

Example 10.9. Let $G(x) = 1 + x; G^{\Delta}(n,k) = \binom{k}{n-k}$, then the equation

$$A(x) = 1 - xA(x)^m$$

generates the identity

$$\frac{r}{mn+r} \binom{mn+r}{n} = \sum_{k=1}^{n} \frac{k}{n} \binom{mn}{n-k} \binom{r}{k}.$$

11 Conclusion

The composita of the ordinary generating function $F(x) = \sum_{n>0} f(n)x^n$ was derived with resort to compositions of an integer n; the composita is a degenerate case of the Riordan array $(1, F(x))$ and uniquely characterizes this function. Theorems (1-11) allow computations of compositae of generating functions. The proposed apparatus of compositae is applicable to derive compositions of ordinary generating functions, expressions for reciprocal generating functions, recurrent expressions for inverse generating functions, solutions of functional equations, expressions for Riordan arrays, and various identities.

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