

To appear in *Number Theory and the Related Topics* (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Higher Education Press & International Press, Beijing and Boston, 2012.

CONJECTURES AND RESULTS ON $x^2 \pmod{p^2}$ WITH $4p = x^2 + dy^2$

ZHI-WEI SUN

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
<http://math.nju.edu.cn/~zwsun>

In honor of Prof. Keqin Feng on the occasion of his 70th birthday

ABSTRACT. Given a squarefree positive integer d , we want to find *integers* (or rational numbers with denominators not divisible by large primes) a_0, a_1, a_2, \dots such that for sufficiently large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-d}{p}\right) = -1. \end{cases}$$

We give a survey of conjectures and results on this topic and point out the connection between this problem and series for $1/\pi$.

1. INTRODUCTION

Let $d \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ be squarefree. If an odd prime p not dividing d can be written in the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}$ then the Legendre symbol $\left(\frac{-d}{p}\right)$ must be equal to 1. For $d = 1, 2, 3$, it is well known that any odd prime p with $\left(\frac{-d}{p}\right) = 1$ can be written uniquely in the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}^+$ (and $2 \nmid x$ if $d = 1$). (See, e.g., Cox [C].)

Let $p \equiv 1 \pmod{4}$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$. In 1828 Gauss determined $x \pmod{p}$ by showing the congruence

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod{p}.$$

2010 *Mathematics Subject Classification*. Primary 11E25, 11A07; Secondary 05A10, 11B65, 33F05.

Keywords. Representations of primes by binary quadratic forms, binomial coefficients, congruences modulo prime powers, series for $1/\pi$.

Supported by the National Natural Science Foundation (grant 11171140) of China.

In 1986, S. Chowla, B. Dwork and R. J. Evans [CDE] used Gauss and Jacobi sums to prove that

$$\left(\frac{(p-1)/2}{(p-1)/4}\right) \equiv \frac{2^{p-1}+1}{2} \left(2x - \frac{p}{2x}\right) \pmod{p^2},$$

which was first conjectured by F. Beukers. This implies that

$$\left(\frac{(p-1)/2}{(p-1)/4}\right)^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}$$

since

$$\left(2x - \frac{p}{2x}\right)^2 \equiv 4x^2 - 2p \pmod{p^2}.$$

In 2010 J. B. Cosgrave and K. Dilcher [CD] determined $\left(\frac{(p-1)/2}{(p-1)/4}\right) \pmod{p^3}$. A result due to Cauchy and Whiteman (cf. [HW84]) asserts that if $p \equiv 1 \pmod{20}$ then

$$\left(\frac{(p-1)/2}{(p-1)/20}\right) \equiv (-1)^{[5|x]} \left(\frac{(p-1)/2}{3(p-1)/20}\right) \pmod{p}.$$

(Throughout this paper, for an assertion A we let $[A]$ takes 1 or 0 according as A holds or not.) The reader may consult [HW79] for more such congruences of the Cauchy type.

In 1989 K. M. Yeung [Ye] employed the Gross-Koblitz formula ([GK]) for the p -adic Gamma function to prove that if $p \equiv 1 \pmod{6}$ is a prime and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $3 \mid x - 1$ then

$$\left(\frac{(p-1)/2}{(p-1)/6}\right) \equiv \left(2x - \frac{p}{2x}\right) \left(1 - \frac{2^p - 2}{3} + \frac{3^p - 3}{4}\right) \pmod{p^2}.$$

Let $p = mf + 1$ be a prime, where $m, f \in \mathbb{Z}^+$ and $m > 1$. It is interesting to determine $\binom{rf+sf}{rf} \pmod{p^2}$ in terms of parameters arising from representations of p by certain binary quadratic forms where r and s are positive integers with $r + s \leq m$. The reader may consult the survey [HW] and [BEW, Chapter 9] for results and methods on this problem with $m = 3, \dots, 16, 20, 24$. See also [Yo] for related results obtained via the Gross-Koblitz formula.

Let d be a squarefree positive integer. It is known that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number one if and only if $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

Let $d \in \{7, 11, 19, 43, 67, 163\}$. Then $K = \mathbb{Q}(\sqrt{-d})$ has class number one. Let p be an odd prime with $\left(\frac{-d}{p}\right) = 1$. By algebraic number theory, p splits in the ring O_K of algebraic integers in K . As O_K is a principal ideal domain, there are unique positive integers x and y with $x \equiv y \pmod{2}$ such that

$$p = \frac{x + y\sqrt{-d}}{2} \times \frac{x - y\sqrt{-d}}{2}$$

and hence $4p = x^2 + dy^2$.

Let $d \in \mathbb{Z}^+$ be squarefree. Suppose that p is an odd prime with $4p = x^2 + dy^2$ for some $x, y \in \mathbb{Z}$. If both x and y are odd, then $d + 1 \equiv 4p \equiv 4 \pmod{8}$ and hence $d \equiv 3 \pmod{8}$. Thus, if $d \not\equiv 3 \pmod{8}$, then x and y are both even, and $p = (x/2)^2 + d(y/2)^2$.

Let p be an odd prime. In 1977, A. R. Rajwade [Ra] proved that

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + 21x^2 + 112x}{p} \right) = \begin{cases} -2x\left(\frac{x}{7}\right) & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 & \text{if } \left(\frac{p}{7}\right) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Using character sums, Jacobi (cf. [HW]) proved that if $p = 11f + 1$ with $f \in \mathbb{Z}^+$ and $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 2 \pmod{11}$ then

$$x \equiv \frac{\binom{6f}{3f} \binom{3f}{f}}{\binom{4f}{2f}} \pmod{p}.$$

In 1982 J. C. Parnami and Rajwade [PR] showed that

$$\begin{aligned} & \sum_{x=0}^{p-1} \left(\frac{x^3 - 96 \cdot 11x + 112 \cdot 11^2}{p} \right) \\ &= \begin{cases} \left(\frac{2}{p}\right)\left(\frac{x}{11}\right)x & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 & \text{if } \left(\frac{p}{11}\right) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases} \end{aligned}$$

Via elliptic curves with complex multiplication, it is also known that (cf. [RPR], [JM] and [PV]) for $d = 19, 43, 67, 163$ we have

$$\sum_{x=0}^{p-1} \left(\frac{f_d(x)}{p} \right) = \begin{cases} \left(\frac{2}{p}\right)\left(\frac{x}{d}\right)x & \text{if } \left(\frac{p}{d}\right) = 1 \text{ \& } 4p = x^2 + dy^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 & \text{if } \left(\frac{p}{d}\right) = -1, \end{cases}$$

where

$$\begin{aligned} f_{19}(x) &= x^3 - 8 \cdot 19x + 2 \cdot 19^2, \\ f_{43}(x) &= x^3 - 80 \cdot 43x + 42 \cdot 43^2, \\ f_{67}(x) &= x^3 - 440 \cdot 67x + 434 \cdot 67^2, \\ f_{163}(x) &= x^3 - 80 \cdot 23 \cdot 29 \cdot 163x + 14 \cdot 11 \cdot 19 \cdot 127 \cdot 163^2. \end{aligned}$$

See also Williams [W] for other similar results.

Let $p \equiv 1 \pmod{4}$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$. The author [Su3, Conjecture 5.5] conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv \left(\frac{2}{p}\right) \left(2x - \frac{p}{2x}\right) \pmod{p^2},$$

and this was completely proved by the author's twin brother Zhi-Hong Sun [S1] with the help of Legendre polynomials.

Let $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. In combinatorics, the central Delannoy number D_n denotes the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$ and $(1, 1)$ (cf. [S]). It is known that

$$D_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} = [x^n](x^2 + 3x + 2)^n.$$

(Throughout this paper by $[x^n]P(x)$ we mean the coefficient of x^n in the power series of $P(x)$.)

Let $p = 2n + 1$ be an odd prime. For $k = n + 1, \dots, p - 1$, we clearly have

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

For $k = 0, \dots, n$, Z. H. Sun [S1] noted that

$$\begin{aligned} \binom{n+k}{2k} &= \frac{\prod_{0 < j \leq k} (p^2 - (2j-1)^2)}{4^k (2k)!} \\ &\equiv \frac{\prod_{0 < j \leq k} (-(2j-1)^2)}{4^k (2k)!} = \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}. \end{aligned}$$

and this implies van Hamme's observation ([vH])

$$\binom{n}{k} \binom{n+k}{k} = \binom{n+k}{2k} \binom{2k}{k} \equiv \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

Therefore

$$D_{(p-1)/2} = D_n \equiv \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

Since

$$\binom{(p-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p} \quad \text{for } k = 0, 1, \dots, p-1,$$

we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \sum_{k=0}^n (-1)^k \binom{n}{k}^2 = [x^n](1+x)^n(1-x)^n = [x^n](1-x^2)^n \pmod{p}.$$

Thus, if $p \equiv 3 \pmod{4}$ (i.e., $2 \nmid n$) then we have $\sum_{k=0}^{p-1} \binom{2k}{k}^2 / (-16)^k \equiv 0 \pmod{p}$. If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \binom{(p-1)/2}{(p-1)/4} \equiv (-1)^{(p-1)/4} 2x \pmod{p}$$

by applying Gauss' congruence. This determines $\sum_{k=0}^{p-1} \binom{2k}{k}^2 / (-16)^k \pmod{p}$ in a simple way. Similarly, in 2009 the author determined $\sum_{k=0}^{p-1} \binom{2k}{k}^2 / 8^k$ and $\sum_{k=0}^{p-1} \binom{2k}{k}^2 / 32^k$ modulo p by noting that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} &\equiv \sum_{k=0}^n 2^k \binom{n}{k} \binom{n}{n-k} = [x^n] (1+2x)^n (1+x)^n \\ &= [y^n] (y^2 + 3y + 2)^n = D_n \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} &\equiv \sum_{k=0}^n \frac{1}{2^k} \binom{n}{k}^2 = [x^n] \left(1 + \frac{x}{2}\right)^n (1+x)^n \\ &= 2^{-n} [x^n] (x^2 + 3x + 2)^n = \frac{D_n}{2^n} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p}. \end{aligned}$$

Recently the author established the following theorem.

Theorem 1.1. (Z. W. Sun [Su5]) *Let $p \equiv 1 \pmod{4}$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$. Then we can determine $x \pmod{p^2}$ in the following way:*

$$(-1)^{(p-1)/4} x \equiv \sum_{k=0}^{(p-1)/2} \frac{k+1}{8^k} \binom{2k}{k}^2 \equiv \sum_{k=0}^{(p-1)/2} \frac{2k+1}{(-16)^k} \binom{2k}{k}^2 \pmod{p^2}. \quad (1.1)$$

Concerning the representations $p = x^2 + 3y^2$ and $p = x^2 + 7y^2$, we have formulated a conjecture similar to Theorem 1.1.

Conjecture 1.1. (Sun [Su4]) *Let p be an odd prime.*

(i) *If $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod{3}$, then we can determine $x \pmod{p^2}$ in the following way:*

$$\sum_{k=0}^{p-1} \frac{k+1}{48^k} \binom{2k}{k} \binom{4k}{2k} \equiv x \pmod{p^2}. \quad (1.2)$$

(ii) If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then we can determine $y \pmod{p^2}$ via the congruence

$$\sum_{k=0}^{p-1} \binom{k}{3} \frac{k \binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p+1)/4} y \pmod{p^2}. \quad (1.3)$$

(iii) If $\left(\frac{p}{7}\right) = 1$ and $p = x^2 + 7y^2$ with $x, y \in \mathbb{Z}$ and $\left(\frac{x}{7}\right) = 1$, then we can determine $x \pmod{p^2}$ in the following way:

$$\sum_{k=0}^{p-1} \frac{k+8}{63^k} \binom{2k}{k} \binom{4k}{2k} \equiv 8 \left(\frac{p}{3}\right) x \pmod{p^2}. \quad (1.4)$$

For congruences concerning the sum $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$ modulo powers of a prime p (where $m \in \mathbb{Z}$ and $m \not\equiv 0 \pmod{p}$), the reader may consult [PS], [ST1], [ST2], [Su1] and [Su2].

Let $d \in \mathbb{Z}^+$ be squarefree and let p be an odd prime with $\left(\frac{-d}{p}\right) = 1$. If p or $4p$ can be written in the form $x^2 + dy^2$ (in such a case we always assume $x, y \in \mathbb{Z}$ even though we may not mention it to avoid lines overfull), how to determine $x^2 \pmod{p^2}$ in a simple and explicit form? We will focus on this question in this survey and present various conjectures and results on the author's following problems.

Problem 1.1. *Given a squarefree positive integer d , find integers a_0, a_1, \dots such that for sufficiently large primes p we have*

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-d}{p}\right) = -1. \end{cases}$$

If one thinks that the integral condition of a_0, a_1, a_2, \dots in Problem 1.1 is too strict, we may study the following easier problem.

Problem 1.2. *Given a squarefree $d \in \mathbb{Z}^+$, find rational numbers a_0, a_1, a_2, \dots with denominators not divisible by large primes such that for sufficiently large primes p we have*

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-d}{p}\right) = -1. \end{cases}$$

We find that Problems 1.1 and 1.2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.

Now we give suggested solutions to the problems with $d = 7, 15, 427$ as examples.

Conjecture 1.2 (Sun [Su3]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \quad (1.5)$$

Remark 1.1. The congruence modulo p can be easily deduced from (6) in Ahlgren [A, Theorem 5]. Recently the author's twin brother Z. H. Sun [S2] confirmed the conjecture in the case $\left(\frac{p}{7}\right) = -1$ via Legendre polynomials.

The imaginary quadratic fields $\mathbb{Q}(\sqrt{-15})$ and $\mathbb{Q}(\sqrt{-427})$ have class number two. Our next two conjectures provide affirmative answers to Problem 1.1 with $d = 15$ and Problem 1.2 with $d = 427$.

Conjecture 1.3. *For $k = 0, 1, 2, \dots$ let T_k denote the trinomial coefficient defined as the coefficient of x^k in $(x^2 + x + 1)^k$. For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 T_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1, \text{ i.e., } p \equiv 7, 11, 13, 14 \pmod{15}, \end{cases}$$

and

$$\sum_{k=0}^{p-1} (105k + 44)(-1)^k \binom{2k}{k}^2 T_k \equiv p \left(20 + 24 \left(\frac{p}{3}\right) (2 - 3^{p-1}) \right) \pmod{p^3}.$$

Also,

$$\frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (-1)^{n-1-k} (105k + 44) \binom{2k}{k}^2 T_k \in \mathbb{Z}^+ \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 1.4. *Let p be an odd prime with $p \neq 11$. Then*

$$\sum_{n=0}^{p-1} \frac{1}{(-22)^{3n}} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{61}\right) = 1 \text{ \& } 4p = x^2 + 427y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 7x^2 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{61}\right) = -1 \text{ \& } 4p = 7x^2 + 61y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{427}\right) = -1. \end{cases}$$

We also have

$$\sum_{n=0}^{p-1} \frac{11895n + 1286}{(-22)^{3n}} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \equiv 1286p \left(\frac{p}{7}\right) \pmod{p^2}.$$

L. van Hamme [vH] observed that certain hypergeometric series have their p -adic analogues, e.g., motivated by the identity (of Ramanujan's type)

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}$$

he conjectured that

$$\sum_{k=0}^{p-1} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p \left(\frac{-1}{p}\right) \pmod{p^3}$$

for any odd prime p (and this was confirmed by E. Mortenson [M]). The reader may consult [BBC], [Be], [BoBo] and [ChCh] for more Ramanujan-type series for $1/\pi$.

The author has the following viewpoint (or hypothesis) the initial version of which appeared in his message to Number Theory Mailing List (sent on March 30, 2010) available from the website <http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind1003&L=nmbtrthry&T=0&P=1668>.

Philosophy about Series for $1/\pi$. (i) *Given a 'regular' identity of the form*

$$\sum_{k=0}^{\infty} (bk+c) \frac{a_k}{m^k} = \frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we must have

$$\sum_{k=0}^{n-1} (bk+c) a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n . Furthermore, there exist an integer m' (often $m' = m$) and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, \dots\}$ (and with C/\sqrt{d} often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1} \frac{a_k}{m^k} \equiv \begin{cases} \left(\frac{m'}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-d}{p}\right) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$ for any prime $p \equiv 3 \pmod{4}$ with $p \nmid 3m$.

(ii) Let b, c, m, a_0, a_1, \dots be integers with bm nonzero and the series $\sum_{k=0}^{\infty} (bk + c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1} (bk + c) \frac{a_k}{m^k} \equiv p \left(c_0 \left(\frac{-d}{p} \right) + c_1 \left(\frac{d'}{p} \right) \right) \pmod{p^2}$$

for all sufficiently large primes p . If $d' \geq 0$, then

$$\sum_{k=0}^{\infty} (bk + c) \frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with C/\sqrt{d} rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty} (bk + c) \frac{a_k}{m^k} = \frac{\lambda_0 \sqrt{d} + \lambda_1 \sqrt{d_1}}{\pi}.$$

Remark 1.2. We don't have a precise description of the meaning of the word *regular* in the above philosophy, but almost all identities of the stated form are regular.

Now we illustrate the philosophy by the author's next three conjectures.

Conjecture 1.5. For any prime $p > 3$, we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{256^n} \sum_{k=0}^n \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1, \end{cases} \end{aligned}$$

and

$$\sum_{n=0}^{p-1} \frac{6n-1}{256^n} \binom{2n}{n} \sum_{k=0}^n \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k} \equiv -p \pmod{p^2}.$$

Also,

$$\sum_{n=0}^{\infty} \frac{6n-1}{256^n} \binom{2n}{n} \sum_{k=0}^n \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k} = \frac{8\sqrt{3}}{\pi}.$$

For $b, c \in \mathbb{Z}$ the n th generalized central trinomial coefficient $T_n(b, c)$ is defined as $[x^n](x^2 + bx + c)^n$. In [Su8] the author conjectured that

$$\sum_{k=0}^{\infty} \frac{80k+13}{(-168^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(7, 4096) = \frac{14\sqrt{210} + 21\sqrt{42}}{8\pi}.$$

Here are related conjectural congruences.

Conjecture 1.6. *Let $p > 3$ be a prime with $p \neq 7$. Then*

$$\begin{aligned} & \left(\frac{-42}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k} T_k(7, 4096)}{(-168^2)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} \frac{80k + 13}{(-168^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(7, 4096) \equiv p \left(3 \left(\frac{-42}{p} \right) + 10 \left(\frac{-210}{p} \right) \right) \pmod{p^2}.$$

The imaginary quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number eight, so our following conjecture is of particular interest.

Conjecture 1.7. *For any prime $p > 5$, we have*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{2160^n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = 1, p = x^2 + 105y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = -1, 2p = x^2 + 105y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = -1, p = 3x^2 + 35y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = -1, \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = 1, 2p = 3x^2 + 35y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = -1, p = 5x^2 + 21y^2, \\ 10x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = -1, 2p = 5x^2 + 21y^2, \\ 28x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = -1, \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1, p = 7x^2 + 15y^2, \\ 14x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1, \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = 1, 2p = 7x^2 + 15y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-105}{p}\right) = -1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{357n + 103}{2160^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} \\ & \equiv p \left(54 \left(\frac{-1}{p} \right) + 49 \left(\frac{15}{p} \right) \right) \pmod{p^2}. \end{aligned}$$

Also,

$$\sum_{n=0}^{\infty} \frac{357n + 103}{2160^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} = \frac{90}{\pi}.$$

We mention that series for $1/\pi$ are usually difficult to prove; most proofs of them involve modular functions or elliptic integrals.

From Section 2 we will give various conjectures and results on Problems 1.1 and 1.2. Problem 1.1 for $d = 1$ already has a positive answer. We suggest positive answers to Problem 1.1 for

$$d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.$$

We also formulate many conjectures concerning Problem 1.2; in particular, we give explicit conjectural positive answers for those squarefree positive integers d with $\mathbb{Q}(\sqrt{-d})$ having class number at most two except for $d = 187, 403$. Note that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if

$$d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}.$$

For each of

$$d \in \{21, 30, 33, 42, 57, 70, 78, 85, 93, 102, 130, 133, 190\},$$

the quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number four; for these values of d we also provide explicit conjectural positive answers to Problem 1.2 or even Problem 1.1.

2. USING APÉRY POLYNOMIALS AND PRODUCTS OF THREE BINOMIAL COEFFICIENTS

The well-known Apéry numbers (introduced by Apéry in his proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see [Ap] and [P])), are given by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 \quad (n = 0, 1, 2, \dots),$$

They also have close connections to modular forms (cf. K. Ono [O]). As in [Su7] we define Apéry polynomials by

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 x^k \quad (n = 0, 1, 2, \dots).$$

Clearly $A_n(1) = A_n$. The author [Su7] proved that for all $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ we have

$$\sum_{k=0}^{n-1} (2k+1)A_k(x) \equiv 0 \pmod{n}.$$

He also conjectured that $n \mid \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k(x)$ for all $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, which was later confirmed by V. J. W. Guo and J. Zeng [GZ].

The following theorem solves Problem 1.1 for $d = 1$.

Theorem 2.1 (Sun [Su7]). *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k A_k(-2) &\equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (2.1)$$

Remark 2.1. A lemma for the proof of Theorem 1.1 states that for any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

This was first conjectured by the author [Su4] and later confirmed by his twin brother Z.-H. Sun [S2].

Concerning Problem 1.1 for $d = 2, 3$ the author made the following conjecture.

Conjecture 2.1 (Sun [Su7]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = -1, \text{ i.e., } p \equiv 5, 7 \pmod{8}. \end{cases} \quad (2.2)$$

And

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \quad (2.3)$$

Remark 2.2. The author [Su7] proved the mod p version of (2.2) and (2.3), and (2.3) in the case $p \equiv 2 \pmod{3}$.

The following theorem relates sums of Apéry polynomials to sums of products of three binomial coefficients. Obviously, for each $k \in \mathbb{N}$ we have

$$\binom{4k}{k, k, k, k} = \frac{(4k)!}{(k!)^4} = \binom{2k}{k}^2 \binom{4k}{2k}.$$

Theorem 2.2 (Sun [Su7]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}. \quad (2.4)$$

Also, for any p -adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}. \quad (2.5)$$

Recall that Conjecture 1.2 determines $\sum_{k=0}^{p-1} \binom{2k}{k}^3 \pmod{p^2}$ for any odd prime p . By Theorem 2.2, we have

$$\sum_{k=0}^{p-1} (-1)^k A_k(16) \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^3 \pmod{p^2}.$$

Conjecture 2.2. *Let $p \neq 2, 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} A_k(-4) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

Remark 2.3. Let $p \neq 2, 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1, 9 \pmod{20}$ then $p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 3, 7 \pmod{20}$ then $2p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$.

Conjecture 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} A_k(9) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Conjecture 2.4. *Let $p > 3$ be a prime with $p \neq 11$. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(99^2) &\equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{1584^{2k}} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-11}{p}\right) = \left(\frac{2}{p}\right) = 1 \text{ \& } p = x^2 + 22y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } \left(\frac{-11}{p}\right) = \left(\frac{2}{p}\right) = -1 \text{ \& } p = 2x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-11}{p}\right) = -\left(\frac{2}{p}\right). \end{cases} \end{aligned}$$

And

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(99^4) &\equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{396^{4k}} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{29}{p}\right) = \left(\frac{-2}{p}\right) = 1 \text{ \& } p = x^2 + 58y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } \left(\frac{29}{p}\right) = \left(\frac{-2}{p}\right) = -1 \text{ \& } p = 2x^2 + 29y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-58}{p}\right) = -1. \end{cases} \end{aligned}$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (280k+19) \binom{4k}{k,k,k,k} 1584^{2(n-1-k)} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$. Also, for $n = 2, 3, 4, \dots$ we have

$$b_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (26390k+1103) \binom{4k}{k,k,k,k} 396^{4(n-1-k)} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3b_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark 2.4. Ramanujan (cf. [Be, p. 354]) found that

$$\sum_{k=0}^{\infty} \frac{280k+19}{1584^{2k}} \binom{4k}{k,k,k,k} = \frac{2 \times 99^2}{\pi\sqrt{11}}$$

and

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

The reader may consult [BB], [BBC] and [CC] for other known Ramanujan-type series not mentioned in this paper.

Conjecture 2.5. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(-324) &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(-2^{10}3^4)^k} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{13}{p}\right) = \left(\frac{-1}{p}\right) = 1 \text{ \& } p = x^2 + 13y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{13}{p}\right) = \left(\frac{-1}{p}\right) = -1 \text{ \& } 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{13}{p}\right) = -\left(\frac{-1}{p}\right). \end{cases} \end{aligned}$$

We also have

$$\sum_{k=0}^{p-1} \frac{260k + 23}{(-2^{10}3^4)^k} \binom{4k}{k, k, k, k} \equiv 23p \left(\frac{-1}{p}\right) + \frac{5}{3}p^3 E_{p-3} \pmod{p^4},$$

where E_0, E_1, E_2, \dots are Euler numbers. Furthermore, for $n = 2, 3, 4, \dots$ we have

$$c_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (260k + 23) \binom{4k}{k, k, k, k} (-82944)^{n-1-k} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3c_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Conjecture 2.6. Let $p > 3$ be a prime with $p \neq 7$. Then

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(-882^2) &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{10}21^4)^k} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = 1 \text{ \& } p = x^2 + 37y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = -1 \text{ \& } 2p = x^2 + 37y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-37}{p}\right) = -1. \end{cases} \end{aligned}$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$d_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (21460k + 1123) \binom{4k}{k, k, k, k} (-2^{10}21^4)^{n-1-k} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3d_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

We will not list our similar conjectures on $\sum_{k=0}^{p-1} A_k(x) \pmod{p^2}$ with x among

$$-48, 81, 2401, -25920, -\frac{9}{16}, \frac{81}{32}, \frac{81}{256}, -\frac{3969}{256}.$$

Recall that $\mathbb{Q}(\sqrt{-11})$ has class number one. The author's following conjecture provides a positive answer to Problem 1.2 with $d = 11$.

Conjecture 2.7 (Sun [Su3]). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1. \end{cases}$$

Remark 2.5. Recently Z. H. Sun [S3] confirmed the mod p version of the congruence.

Note that [Su4] contains some of the author's conjectural congruences related to Ramanujan-type series. Below we present more such conjectures.

Conjecture 2.8. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-96)^{3k}} \equiv \begin{cases} \left(\frac{-6}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1 \text{ \& } 4p = x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{342k + 25}{(-96)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} \equiv 25p \left(\frac{-6}{p}\right) \pmod{p^3}.$$

Remark 2.6. The conjectural congruences modulo p have been confirmed by Z. H. Sun [S3]. D. V. Chudnovsky and G. V. Chudnovsky [CC] obtained that

$$\sum_{k=0}^{\infty} \frac{342k + 25}{(-96)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} = \frac{32\sqrt{6}}{\pi}.$$

Conjecture 2.9. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-960)^{3k}} \equiv \begin{cases} \left(\frac{p}{15}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{43}\right) = 1 \text{ \& } 4p = x^2 + 43y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{43}\right) = -1. \end{cases}$$

If $p \neq 11$, then

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-5280)^{3k}} \equiv \begin{cases} \left(\frac{-330}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{67}\right) = 1 \text{ \& } 4p = x^2 + 67y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{67}\right) = -1. \end{cases}$$

Remark 2.7. The conjectural congruences modulo p has been confirmed by Z. H. Sun [S3].

Conjecture 2.10. *Let $p > 5$ be a prime with $p \neq 23, 29$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-640320)^{3k}} \equiv \begin{cases} \left(\frac{-10005}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{163}\right) = 1 \text{ \& } 4p = x^2 + 163y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{163}\right) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} \equiv 13591409p \left(\frac{-10005}{p}\right) \pmod{p^3}.$$

Remark 2.8. The conjectural congruence modulo p has been confirmed by Z. H. Sun [S3]. D. V. Chudnovsky and G. V. Chudnovsky [CC] got the formula

$$\sum_{k=0}^{\infty} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} = \frac{3 \times 53360^2}{2\pi\sqrt{10005}},$$

which enabled them to hold the record for the calculation of π during 1989-1994.

Conjecture 2.11. *Let $p > 3$ be a prime with $p \neq 17$. Then*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{68^n} \sum_{k=0}^n \binom{n}{k}^3 64^k \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-12^3)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } 4p = x^2 + 51y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 4p = 3x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{51}\right) = -1. \end{cases}$$

Also,

$$\sum_{n=0}^{p-1} \frac{36n + 19}{68^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 64^k \equiv 19p \left(\frac{p}{17}\right) \pmod{p^2},$$

and

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{51k + 7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 7 \left(\frac{p^a}{3}\right) + \left(\frac{p^{a-1}}{3}\right) \frac{5}{6} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}$$

for every $a \in \mathbb{Z}^+$, where $B_n(x)$ denotes the Bernoulli polynomial of degree n .

Remark 2.9. Ramanujan [R] found that

$$\sum_{k=0}^{\infty} \frac{51k + 7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} = \frac{12\sqrt{3}}{\pi}.$$

Conjecture 2.12. *Let $p > 3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-48)^{3k}} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{41}\right) = 1 \text{ \& } 4p = x^2 + 123y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{41}\right) = -1 \text{ \& } 4p = 3x^2 + 41y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{123}\right) = -1. \end{cases}$$

Also,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{615k + 53}{(-48)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv 53 \left(\frac{p^a}{3}\right) + \left(\frac{p^{a-1}}{3}\right) \frac{5}{12} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}$$

for any positive integer a .

(ii) *Suppose $p > 5$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-300)^{3k}} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = 1 \text{ \& } 4p = x^2 + 267y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = -1 \text{ \& } 4p = 3x^2 + 89y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{267}\right) = -1. \end{cases}$$

Also,

$$\begin{aligned} & \frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{14151k + 827}{(-300)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \\ & \equiv 827 \binom{p^a}{3} + \binom{p^a-1}{3} \frac{13}{150} p^2 B_{p-2} \left(\frac{1}{3} \right) \pmod{p^3} \end{aligned}$$

for any positive integer a .

Remark 2.10. It is known (cf. [CC]) that

$$\sum_{k=0}^{\infty} \frac{615k + 53}{(-48)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} = \frac{96\sqrt{3}}{\pi}$$

and

$$\sum_{k=0}^{\infty} \frac{14151k + 827}{(-300)^{3k}} \binom{2k}{k}^2 \binom{4k}{2k} = \frac{1500\sqrt{3}}{\pi}.$$

3. USING THE POLYNOMIALS $S_n(x) = \sum_{k=0}^n \binom{n}{k}^4 x^k$

In this section we give many conjectures on Problem 1.1 involving the polynomials

$$S_n(x) := \sum_{k=0}^n \binom{n}{k}^4 x^k \quad (n = 0, 1, 2, \dots).$$

Conjecture 3.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-2) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x \ \& \ 2 \mid y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (3k+2) S_k(-2) \equiv \frac{p}{2} \left(1 + 3 \left(\frac{-1}{p} \right) \right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+4) S_k(-2) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.2. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} S_k(-4) \equiv \sum_{k=0}^{p-1} S_k(-64) \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-5}{p}\right) = -1, \text{ i.e., } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (8k+7)S_k(-64) \equiv p \left(\frac{p}{15}\right) \left(3 + 4 \left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k+7)S_k(-64) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.3. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} S_k(4) \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \text{ } (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1, \text{ i.e., } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (24k+17)S_k(4) \equiv p \left(5 + 12 \left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (24k+17)S_k(4) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.4. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} S_k(16) \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ \& } p = x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 7, 13, 23, 37 \pmod{40} \text{ \& } p = 2x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-10}{p}\right) = -1. \end{cases} \end{aligned}$$

When $\left(\frac{-10}{p}\right) = 1$ we have

$$\sum_{k=0}^{p-1} (160k + 129)S_k(16) \equiv 80p \left(\frac{p}{5}\right) \pmod{p^2}.$$

Conjecture 3.5. *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} S_k(1) &\equiv \sum_{k=0}^{p-1} S_k(-9) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

And

$$\begin{aligned} \sum_{k=0}^{p-1} (3k + 2)S_k(1) &\equiv 2p \left(\frac{p}{5}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} (5k + 4)S_k(-9) &\equiv \frac{p}{2} \left(\frac{p}{3}\right) \left(3 + 5 \left(\frac{p}{15}\right)\right) \pmod{p^2}. \end{aligned}$$

Moreover,

$$\frac{1}{2n} \sum_{k=0}^{n-1} (3k + 2)S_k(1) \in \mathbb{Z} \quad \text{and} \quad \frac{1}{2n} \sum_{k=0}^{n-1} (5k + 4)S_k(-9) \in \mathbb{Z}$$

for all $n = 1, 2, 3, \dots$.

Conjecture 3.6. *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} S_k(36) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = 1 \text{ \& } p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 3x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 2x^2 + 15y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } p = 5x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-30}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (8k + 7)S_k(36) \equiv p \left(\frac{p}{15}\right) \left(3 + 4 \left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k + 7)S_k(36) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(196) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 70y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 2x^2 + 35y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } p = 5x^2 + 14y^2, \\ 28x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } p = 7x^2 + 10y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-70}{p}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (120k + 109)S_k(196) \equiv p \left(\frac{p}{7}\right) \left(49 + 60 \left(\frac{-14}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\frac{1}{n} \sum_{k=0}^{n-1} (120k + 109)S_k(196) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.8. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-324) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } p = x^2 + 85y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{17}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } 2p = x^2 + 85y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 5x^2 + 17y^2, \\ 2p - 10x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 2p = 5x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-85}{p}\right) = -1. \end{cases}$$

Provided $p > 3$ we have

$$\sum_{k=0}^{p-1} (34k + 31)S_k(-324) \equiv p \left(\frac{p}{5}\right) \left(17 + 14 \left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (34k + 31)S_k(-324) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.9. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(1296) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{13}\right) = 1 \text{ \& } p = x^2 + 130y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{13}\right) = -1 \text{ \& } p = 2x^2 + 65y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{-2}{p}\right) = \left(\frac{p}{13}\right) = -1 \text{ \& } p = 5x^2 + 26y^2, \\ 2p - 40x^2 \pmod{p^2} & \text{if } \left(\frac{p}{13}\right) = 1, \left(\frac{-2}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 10x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-130}{p}\right) = -1. \end{cases}$$

Provided $p > 3$ we have

$$\sum_{k=0}^{p-1} (130k + 121)S_k(1296) \equiv p \left(\frac{-2}{p}\right) \left(56 + 65 \left(\frac{-26}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (130k + 121)S_k(1296) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.10. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(5776) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{19}\right) = 1 \text{ \& } p = x^2 + 190y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{19}\right) = -1 \text{ \& } p = 2x^2 + 95y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 5x^2 + 38y^2, \\ 2p - 40x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{19}\right) = -1 \text{ \& } p = 10x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-190}{p}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (816k + 769)S_k(5776) \equiv p \left(\frac{p}{95}\right) \left(361 + 408 \left(\frac{p}{19}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (816k + 769)S_k(5776) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.11. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(12) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ \& } p = x^2 + y^2 \text{ (} 3 \nmid x \text{),} \\ \left(\frac{xy}{3}\right)4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \text{ \& } p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (4k+3)S_k(12) \equiv p \left(1 + 2 \left(\frac{3}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (4k+3)S_k(12) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 3.12. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-20) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + y^2 \text{ (} 5 \nmid x \text{),} \\ 4xy \pmod{p^2} & \text{if } p \equiv 13, 17 \pmod{20} \text{ \& } p = x^2 + y^2 \text{ (} 5 \mid x - y \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (6k+5)S_k(-20) \equiv p \left(\frac{-1}{p}\right) \left(2 + 3 \left(\frac{-5}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+5)S_k(-20) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

In 2005 Yifan Yang [Y] found the interesting identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{36^k} \sum_{j=0}^k \binom{k}{j}^4 = \frac{18}{\pi\sqrt{15}}.$$

Motivated by this we give the following easy result.

Theorem 3.1. *Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$. Let p be an odd prime not dividing m . Then*

$$\sum_{k=0}^{p-1} \frac{ak+b}{m^k} \sum_{j=0}^k \binom{k}{j}^4 \equiv - \sum_{k=0}^{p-1} (2ak+2a-b) S_k(m) \pmod{p}.$$

In particular,

$$\sum_{k=0}^{p-1} \frac{\sum_{j=0}^k \binom{k}{j}^4}{m^k} \equiv \sum_{k=0}^{p-1} S_k(m) \pmod{p}.$$

Proof. Let

$$S := \sum_{n=0}^{p-1} (2an+2a-b) S_n(m).$$

Then

$$\begin{aligned} S &= \sum_{k=0}^{p-1} m^k \sum_{n=k}^{p-1} (2a(n+1)-b) \binom{n}{k}^4 \\ &= \sum_{k=0}^{p-1} m^k \sum_{j=0}^{p-1-k} (2a(k+j+1)-b) \binom{k+j}{j}^4 \\ &= \sum_{k=0}^{p-1} m^k \sum_{j=0}^{p-1-k} (2a(k+j+1)-b) \binom{-k-1}{j}^4 \\ &\equiv \sum_{k=0}^{p-1} m^k \sum_{j=0}^{p-1-k} (2a(k+j+1)-b) \binom{p-1-k}{j}^4 \pmod{p} \end{aligned}$$

and hence

$$\begin{aligned} S &\equiv \sum_{k=0}^{p-1} m^{p-1-k} \sum_{j=0}^k (2a(p-k+j)-b) \binom{k}{j}^4 \\ &\equiv - \sum_{k=0}^{p-1} \frac{1}{m^k} \sum_{l=0}^k (2al+b) \binom{k}{l}^4 \pmod{p}. \end{aligned}$$

Note that

$$2 \sum_{l=0}^k l \binom{k}{l}^4 = \sum_{j=0}^k \left(j \binom{k}{j}^4 + (k-j) \binom{k}{k-j}^4 \right) = k \sum_{j=0}^k \binom{k}{j}^4.$$

So we have

$$S \equiv - \sum_{k=0}^{p-1} \frac{ak+b}{m^k} \sum_{j=0}^k \binom{k}{j}^4 \pmod{p}.$$

This completes the proof. \square

Conjecture 3.13. *Let m be among*

$$1, -2, \pm 4, -9, 12, 16, -20, 36, -64, 196, -324, 1296, 5776.$$

For any odd prime p not dividing m we have

$$\sum_{k=0}^{p-1} \frac{\sum_{j=0}^k \binom{k}{j}^4}{m^k} \equiv \sum_{k=0}^{p-1} S_k(m) \pmod{p^2}.$$

Now we give two more conjectures involving $\sum_{k=0}^n \binom{n}{k}^3 x^k$.

Conjecture 3.14. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{4n}{2n}}{16^n} \sum_{k=0}^n \binom{n}{k}^3 \\ & \equiv \begin{cases} \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

When $p \equiv 5, 7 \pmod{8}$, we have

$$\sum_{k=0}^{p-1} (16n + 5) \frac{\binom{4n}{2n}}{16^n} \sum_{k=0}^n \binom{n}{k}^3 \equiv 0 \pmod{p^2}.$$

Conjecture 3.15. (i) *Let $p > 5$ be an prime with $p \neq 11$. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{(-121)^n} \sum_{k=0}^n \binom{n}{k}^3 (-5)^{3k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 15y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases} \end{aligned}$$

(ii) *For any prime $p > 5$ with $p \neq 41$, we have*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{4100^n} \sum_{k=0}^n \binom{n}{k}^3 2^{12k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{41}\right) = 1 \text{ \& } 4p = x^2 + 123y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{41}\right) = -1 \text{ \& } 4p = 3x^2 + 41y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{123}\right) = -1. \end{cases} \end{aligned}$$

(iii) Let $p > 3$ be a prime with $p \neq 53, 89$. Then

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{1000004^n} \sum_{k=0}^n \binom{n}{k}^3 10^{6k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = 1 \text{ \& } 4p = x^2 + 267y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = -1 \text{ \& } 4p = 3x^2 + 89y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{267}\right) = -1. \end{cases}$$

4. USING THE FUNCTION $F_n(x) = \sum_{k=0}^n \binom{n}{k}^3 \binom{2k}{k} x^{-k}$

For $n = 0, 1, 2, \dots$ we define

$$F_n(x) := \sum_{k=0}^n \binom{n}{k}^3 \binom{2k}{k} x^{-k}.$$

Conjecture 4.1. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k F_k(-2) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ \& } p = x^2 + y^2 \text{ (3 \nmid x)}, \\ \left(\frac{xy}{3}\right) 4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \text{ \& } p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (3k+2)(-1)^k F_k(-2) \equiv \frac{p}{2} \left(\frac{-1}{p}\right) \left(3 \left(\frac{p}{3}\right) + 1\right) \pmod{p^2}.$$

Conjecture 4.2. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k F_k(2) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (10k+7)(-1)^k F_k(2) \equiv p \left(5 \left(\frac{-1}{p}\right) + 2\right) \pmod{p^2}.$$

Conjecture 4.3. Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} (-1)^k F_k(4) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

When $p \equiv 1, 5, 7, 11 \pmod{24}$, we have

$$\sum_{k=0}^{p-1} (72k+47)(-1)^k F_k(4) \equiv 0 \pmod{p^2}.$$

Conjecture 4.4. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k F_k(12) \\ \equiv & \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (5k + 3)(-1)^k F_k(12) \equiv \frac{p}{2} \left(\frac{-2}{p}\right) \left(5 \left(\frac{p}{3}\right) + 1\right) \pmod{p^2}.$$

Conjecture 4.5. *Let $p \neq 2, 7$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k F_k(-14) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 21y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } 2p = x^2 + 21y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } p = 3x^2 + 7y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } 2p = 3x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-21}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (18k + 11)(-1)^k F_k(-14) \equiv p \left(\frac{p}{7}\right) \left(2 + 9 \left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Conjecture 4.6. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k F_k(40) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = 1 \text{ \& } p = x^2 + 30y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 2x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } p = 3x^2 + 10y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } p = 5x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-30}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (12k + 7)(-1)^k F_k(40) \equiv p \left(\frac{10}{p}\right) \left(6 + \left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

Conjecture 4.7. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1} (-1)^k F_k(-50) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{11}\right) = 1 \ \& \ p = x^2 + 33y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{11}\right) = -1 \ \& \ 2p = x^2 + 33y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1 \ \& \ p = 3x^2 + 11y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{11}\right) = -1 \ \& \ 2p = 3x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-33}{p}\right) = -1. \end{cases}$$

If $p \neq 5$, then

$$\sum_{k=0}^{p-1} (99k + 58)(-1)^k F_k(-50) \equiv \frac{p}{2} \left(\frac{-1}{p}\right) \left(99 \left(\frac{p}{3}\right) + 17\right) \pmod{p^2}.$$

Conjecture 4.8. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1} (-1)^k F_k(112) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1 \ \& \ p = x^2 + 42y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1, \left(\frac{-2}{p}\right) = \left(\frac{p}{3}\right) = -1 \ \& \ p = 2x^2 + 21y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = -1 \ \& \ p = 3x^2 + 14y^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{-2}{p}\right) = \left(\frac{p}{7}\right) = -1 \ \& \ p = 6x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-42}{p}\right) = -1. \end{cases}$$

If $p \neq 7$, then

$$\sum_{k=0}^{p-1} (180k + 103)(-1)^k F_k(112) \equiv p \left(\frac{p}{7}\right) \left(90 \left(\frac{p}{3}\right) + 13\right) \pmod{p^2}.$$

Conjecture 4.9. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1} (-1)^k F_k(-338) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{19}\right) = 1 \ \& \ p = x^2 + 57y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{19}\right) = -1 \ \& \ 2p = x^2 + 57y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{19}\right) = -1 \ \& \ p = 3x^2 + 19y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1 \ \& \ 2p = 3x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-57}{p}\right) = -1. \end{cases}$$

If $p \neq 13$, then

$$\sum_{k=0}^{p-1} (855k + 482)(-1)^k F_k(-338) \equiv \frac{p}{2} \left(\frac{-1}{p} \right) \left(855 \left(\frac{p}{19} \right) + 109 \right) \pmod{p^2}.$$

Conjecture 4.10. Let $p > 5$ be a prime with $p \neq 13$. Then

$$\sum_{k=0}^{p-1} (-1)^k F_k(1300) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{13}\right) = 1 \ \& \ p = x^2 + 78y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \ \left(\frac{p}{3}\right) = \left(\frac{p}{13}\right) = -1 \ \& \ p = 2x^2 + 39y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{13}\right) = 1, \ \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = -1 \ \& \ p = 3x^2 + 26y^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \ \left(\frac{2}{p}\right) = \left(\frac{p}{13}\right) = -1 \ \& \ p = 6x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-78}{p}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (204k + 113)(-1)^k F_k(1300) \equiv p \left(\frac{p}{39} \right) \left(102 \left(\frac{p}{3} \right) + 11 \right) \pmod{p^2}.$$

Conjecture 4.11. Let $p > 3$ be a prime with $p \neq 31$. We have

$$\sum_{k=0}^{p-1} (-1)^k F_k(-3038) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{31}\right) = 1 \ \& \ p = x^2 + 93y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{31}\right) = 1, \ \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1 \ \& \ 2p = x^2 + 93y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \ \left(\frac{-1}{p}\right) = \left(\frac{p}{31}\right) = -1 \ \& \ p = 3x^2 + 31y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \ \left(\frac{p}{3}\right) = \left(\frac{p}{31}\right) = -1 \ \& \ 2p = 3x^2 + 31y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-93}{p}\right) = -1. \end{cases}$$

If $p \neq 7$, then

$$\sum_{k=0}^{p-1} (1170k + 643)(-1)^k F_k(-3038) \equiv p \left(\frac{p}{31} \right) \left(585 \left(\frac{-1}{p} \right) + 58 \right) \pmod{p^2}.$$

Conjecture 4.12. *Let $p > 3$ be a prime. We have*

$$\sum_{k=0}^{p-1} (-1)^k F_k(4900) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } p = x^2 + 102y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{17}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } p = 2x^2 + 51y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 3x^2 + 34y^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 6x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-102}{p}\right) = -1. \end{cases}$$

If $p \neq 7$, then

$$\sum_{k=0}^{p-1} (561k + 307)(-1)^k F_k(4900) \equiv \frac{p}{2} \left(\frac{-6}{p}\right) \left(561 \left(\frac{p}{51}\right) + 53\right) \pmod{p^2}.$$

5. USING THE FUNCTION $G_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} x^{-k}$

For $n \in \mathbb{N}$ we define

$$G_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} x^{-k}.$$

Those integers $G_n(1)$ ($n = 0, 1, 2, \dots$) are called Domb numbers. In 2004 H. H. Chan, S. H. Chan and Z. Liu [CCL] proved that

$$\sum_{k=0}^{\infty} \frac{5k+1}{64^k} G_k(1) = \frac{8}{\sqrt{3}\pi}.$$

Motivated by his work with Mahler measures and new transformation formulas for ${}_5F_4$ series, M. D. Rogers [Ro] discovered that

$$\sum_{k=0}^{\infty} \frac{3k+1}{32^k} (-1)^k G_k(1) = \frac{2}{\pi},$$

which was independently showed by H. H. Chan and H. Verrill [CV].

Conjecture 5.1. *Let $p > 3$ be a prime. We have*

$$\sum_{k=0}^{p-1} G_k(1) \equiv \sum_{k=0}^{p-1} G_k(-5) \equiv \sum_{k=0}^{p-1} \frac{G_k(1)}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{p^2} \text{ \& } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-15}{p}\right) = -1. \end{cases}$$

If $p \neq 5$, then

$$\sum_{k=0}^{p-1} (5k+4)G_k(1) \equiv 4p \binom{p}{3} \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} (21k+16)G_k(-5) \equiv 16p \binom{p}{5} \pmod{p^2}.$$

Also,

$$\frac{1}{4n} \sum_{k=0}^{n-1} (5k+4)G_k(1) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \dots$$

Conjecture 5.2. *Let $p > 3$ be a prime. We have*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{G_k(1)}{(-2)^k} &\equiv \sum_{k=0}^{p-1} \frac{G_k(1)}{4^k} \equiv \sum_{k=0}^{p-1} \frac{G_k(1)}{16^k} \equiv \sum_{k=0}^{p-1} \frac{G_k(1)}{(-32)^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} G_k(6) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } p = x^2 + 3y^2 \text{ (} x, y \in \mathbb{Z} \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} \sum_{k=0}^{p-1} (3k+2) \frac{G_k(1)}{(-2)^k} &\equiv 2p \left(\frac{-1}{p}\right) \pmod{p^3}, \\ \sum_{k=0}^{p-1} (3k+1) \frac{G_k(1)}{(-32)^k} &\equiv p \left(\frac{-1}{p}\right) + p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} (15k+11)G_k(6) &\equiv p \left(10 \left(\frac{3}{p}\right) + 1\right) \pmod{p^2}. \end{aligned}$$

If $p \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{p-1} (3k+2) \frac{G_k(1)}{4^k} \equiv \sum_{k=0}^{p-1} (3k+1) \frac{G_k(1)}{16^k} \equiv 0 \pmod{p^2}.$$

Conjecture 5.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{G_k(1)}{8^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{G_k(1)}{(-8)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} (2k+1) \frac{G_k(1)}{8^k} \equiv p \pmod{p^4} \quad \text{and} \quad \sum_{k=0}^{p-1} (2k+1) \frac{G_k(1)}{(-8)^k} \equiv p \left(\frac{p}{3}\right) \pmod{p^3}.$$

Conjecture 5.4. *Let p be an odd prime. We have*

$$\begin{aligned} \sum_{k=0}^{p-1} G_k(4) &\equiv \sum_{k=0}^{p-1} (-1)^k F_k(4) \pmod{p^2}, \\ \sum_{k=0}^{p-1} G_k(-12) &\equiv \left(\frac{3}{p}\right) \sum_{k=0}^{p-1} (-1)^k F_k(-14) \pmod{p^2} \quad (p \neq 3, 7), \\ \sum_{k=0}^{p-1} G_k(36) &\equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} (-1)^k F_k(40) \pmod{p^2} \quad (p > 5), \\ \sum_{k=0}^{p-1} G_k(-44) &\equiv \sum_{k=0}^{p-1} (-1)^k F_k(-50) \pmod{p^2} \quad (p \neq 5, 11), \\ \sum_{k=0}^{p-1} G_k(100) &\equiv \sum_{k=0}^{p-1} (-1)^k F_k(112) \pmod{p^2} \quad (p \neq 5, 7), \\ \sum_{k=0}^{p-1} G_k(-300) &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} (-1)^k F_k(-338) \pmod{p^2} \quad (p \neq 3, 5, 13), \\ \sum_{k=0}^{p-1} G_k(1156) &\equiv \sum_{k=0}^{p-1} (-1)^k F_k(1300) \pmod{p^2} \quad (p \neq 5, 13, 17), \\ \sum_{k=0}^{p-1} G_k(-2700) &\equiv \left(\frac{3}{p}\right) \sum_{k=0}^{p-1} (-1)^k F_k(-3038) \pmod{p^2} \quad (p > 7, p \neq 31), \\ \sum_{k=0}^{p-1} G_k(4356) &\equiv \left(\frac{-6}{p}\right) \sum_{k=0}^{p-1} (-1)^k F_k(4900) \pmod{p^2} \quad (p > 11). \end{aligned}$$

$$6. \text{ USING } a_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k$$

For $n \in \mathbb{N}$ define

$$a_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k.$$

Those numbers $a_n(1)$ ($n = 0, 1, 2, \dots$) first appeared in Apéry's proof of the irrationality of $\zeta(2)$.

We find $\sum_{k=0}^{p-1} \binom{2k}{k} a_k(1)/m^k \pmod{p^2}$ related to the representation $4p = x^2 + dy^2$ with

$$(m, d) = (-3, 15), (4, 11), (18, 1), (-28, 35), (36, 19), (72, 10), \\ (147, 15), (-828, 115), (-15228, 235).$$

Conjecture 6.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(1)}{4^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{22k + 9}{4^k} \binom{2k}{k} a_k(1) \equiv 9p \pmod{p^2}.$$

Conjecture 6.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(1)}{36^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1 \text{ \& } 4p = x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{38k + 13}{36^k} \binom{2k}{k} a_k(1) \equiv 13p \pmod{p^2}.$$

Conjecture 6.3. *Let $p \neq 2, 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(1)}{(-28)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = 1 \text{ \& } 4p = x^2 + 35y^2, \\ 2p - 5x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } 4p = 5x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{35}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{10k + 3}{(-28)^k} \binom{2k}{k} a_k(1) \equiv 3p \left(\frac{p}{7}\right) \pmod{p^2}.$$

Conjecture 6.4. *Let $p \neq 2, 3, 23$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(1)}{(-828)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{23}\right) = 1 \text{ \& } 4p = x^2 + 115y^2, \\ 2p - 5x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{23}\right) = -1 \text{ \& } 4p = 5x^2 + 23y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{115}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{190k + 29}{(-828)^k} \binom{2k}{k} a_k(1) \equiv 29p \left(\frac{p}{23}\right) \pmod{p^2}.$$

Conjecture 6.5. *Let $p \neq 5, 47$ be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(1)}{(-15228)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{47}\right) = 1 \text{ \& } 4p = x^2 + 235y^2, \\ 2p - 5x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{47}\right) = -1 \text{ \& } 4p = 5x^2 + 47y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{235}\right) = -1. \end{cases}$$

When $p \neq 3$ we have

$$\sum_{k=0}^{p-1} \frac{682k + 71}{(-15228)^k} \binom{2k}{k} a_k(1) \equiv 71p \left(\frac{p}{47}\right) \pmod{p^2}.$$

Conjecture 6.6. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(-3)}{4^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(9)}{100^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } 4p = x^2 + 27y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} (30k + 13) \frac{\binom{2k}{k} a_k(-3)}{4^k} \equiv p \left(3 + 10 \left(\frac{p}{3}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} (66k + 23) \frac{\binom{2k}{k} a_k(9)}{100^k} \equiv 23p \pmod{p^2}.$$

Conjecture 6.7. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(-27)}{26^{2k}} \equiv \begin{cases} \left(\frac{p}{3}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1. \end{cases}$$

When $p \neq 7$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k(27)}{28^{2k}} \equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Also,

$$\sum_{k=0}^{\infty} (114k + 31) \frac{\binom{2k}{k} a_k(-27)}{26^{2k}} = \frac{338\sqrt{3}}{11\pi} \text{ and } \sum_{k=0}^{\infty} (930k + 143) \frac{\binom{2k}{k} a_k(27)}{28^{2k}} = \frac{980\sqrt{3}}{\pi}.$$

For $n \in \mathbb{N}$ we define

$$a_n^*(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^{n-k} = x^n a_n \left(\frac{1}{x}\right).$$

Conjecture 6.8. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k^*(-8)}{96^k} \\ & \equiv \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (13k + 4) \frac{\binom{2k}{k} a_k^*(-8)}{96^k} \equiv 4p \left(\frac{-2}{p}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty} (13k + 4) \frac{\binom{2k}{k} a_k^*(-8)}{96^k} = \frac{9\sqrt{2}}{2\pi}.$$

Conjecture 6.9. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k^*(-32)}{1152^k} \equiv \begin{cases} \left(\frac{2}{p}\right)(4x^2 - 2p) & \text{if } p \equiv 1 \pmod{3} \text{ \& } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (290k + 61) \frac{\binom{2k}{k} a_k^*(-32)}{1152^k} \equiv p \left(\frac{2}{p}\right) \left(6 + 55 \left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty} (290k + 61) \frac{\binom{2k}{k} a_k^*(-32)}{1152^k} = \frac{99\sqrt{2}}{\pi}.$$

Conjecture 6.10. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k^*(64)}{3840^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} (962k + 137) \frac{\binom{2k}{k} a_k^*(64)}{3840^k} \equiv p \left(\frac{-5}{p}\right) \left(147 - 10 \left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty} (962k + 137) \frac{\binom{2k}{k} a_k^*(64)}{3840^k} = \frac{252\sqrt{5}}{\pi}.$$

7. MISCELLANEOUS THINGS

In this section we give some miscellaneous conjectures related to Problem 1.2.

Conjecture 7.1. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{1}{9^n} \sum_{k=0}^n \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2 \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } 4p = x^2 + 27y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

And

$$\sum_{n=0}^{p-1} \frac{8n+1}{9^n} \sum_{k=0}^n \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2 \equiv p \left(\frac{p}{3}\right) \pmod{p^3}.$$

We also have

$$\sum_{n=0}^{\infty} \frac{8n+1}{9^n} \sum_{k=0}^n \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2 = \frac{3\sqrt{3}}{\pi}.$$

Conjecture 7.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{1}{2^n} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{n=0}^{p-1} \frac{3n-1}{2^n} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} \equiv -p \binom{-6}{p} \pmod{p^2}.$$

We also have

$$\sum_{n=0}^{\infty} \frac{3n-1}{2^n} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} = \frac{3\sqrt{6}}{2\pi}.$$

Conjecture 7.3. *Let p be an odd prime. Then*

$$\begin{aligned} & \binom{-1}{p} \sum_{n=0}^{p-1} \frac{1}{64^n} \sum_{k=0}^n \binom{n}{k} \binom{2k}{n} \binom{2k}{k} \binom{2n-2k}{n-k} 3^{2k-n} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

And

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{21n+1}{64^n} \sum_{k=0}^n \binom{n}{k} \binom{2k}{n} \binom{2k}{k} \binom{2n-2k}{n-k} 3^{2k-n} \\ & \equiv p \binom{-1}{p} \left(4 - 3 \binom{p}{3}\right) \pmod{p^2}. \end{aligned}$$

We also have

$$\sum_{n=0}^{\infty} \frac{21n+1}{64^n} \sum_{k=0}^n \binom{n}{k} \binom{2k}{n} \binom{2k}{k} \binom{2n-2k}{n-k} 3^{2k-n} = \frac{64}{\pi}.$$

Conjecture 7.4. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{1}{(-64)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{13}\right) = 1 \text{ \& } 4p = x^2 + 91y^2, \\ 2p - 7x^2 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{13}\right) = -1 \text{ \& } 4p = 7x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{91}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{n=0}^{p-1} (39n+10) \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \equiv 10p \binom{p}{7} \pmod{p^2}.$$

Conjecture 7.5. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{1}{216^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} 14^{2k-n} \\ \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{67}\right) = 1 \text{ \& } 4p = x^2 + 67y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{67}\right) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} (33165n + 11546) \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} 14^{2k-n} \equiv 11546p \pmod{p^2}.$$

Conjecture 7.6. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1} \frac{1}{(-8)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} 26^{2k-n} \\ \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{23}\right) = 1 \text{ \& } 4p = x^2 + 115y^2, \\ 5x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{23}\right) = -1 \text{ \& } 4p = 5x^2 + 23y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{115}\right) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} \frac{6555n + 3062}{(-8)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} 26^{2k-n} \\ \equiv 2p \left(220 + 1311 \left(\frac{p}{23}\right) \right) \pmod{p^2}.$$

Conjecture 7.7. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1} \frac{1}{(-8)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{3k}{n} \binom{3(n-k)}{n} \\ \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = 1 \text{ \& } 4p = x^2 + 35y^2, \\ 2p - 5x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } 4p = 5x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{35}\right) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} \frac{35n + 18}{(-8)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{3k}{n} \binom{3(n-k)}{n} \equiv 9p \left(7 - 5 \left(\frac{p}{5}\right) \right) \pmod{p^2}.$$

Conjecture 7.8. *Let $p > 3$ be a prime. When $p \neq 11, 17$, we have*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{198^{2n}} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } p = x^2 + 102y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } \left(\frac{p}{17}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } p = 2x^2 + 51y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{2}{p}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 3x^2 + 34y^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 6x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-102}{p}\right) = -1. \end{cases}$$

When $p \neq 23, 59$, we have

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{(-1123596)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{59}\right) = 1 \text{ \& } p = x^2 + 177y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{59}\right) = -1 \text{ \& } 2p = x^2 + 177y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{59}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1 \text{ \& } p = 3x^2 + 59y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{59}\right) = -1 \text{ \& } 2p = 3x^2 + 59y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-177}{p}\right) = -1. \end{cases}$$

Conjecture 7.9. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k}}{64^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } 4p = x^2 + 51y^2, \\ 3x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 4p = 3x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{51}\right) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} (17n + 9) \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k}}{64^k} \equiv \frac{p}{3} \left(34 \left(\frac{p}{17} \right) - 7 \right) \pmod{p^2}.$$

Conjecture 7.10. *Let $p > 5$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{400^n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} 196^{n-k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{11}\right) = 1 \text{ \& } p = x^2 + 22y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{p}{11}\right) = -1 \text{ \& } p = 2x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-22}{p}\right) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} \frac{33n+19}{400^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} 196^{n-k} \equiv 19p \pmod{p^2}.$$

For $n \in \mathbb{N}$ we define

$$P_n^-(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} (-1)^k x^{2k-n}.$$

Conjecture 7.11. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^-(8)}{480^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^-(18)}{5760^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } p = x^2 + 85y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left(\frac{p}{17}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) = -1 \text{ \& } 2p = x^2 + 85y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{5}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } p = 5x^2 + 17y^2, \\ 2p - 10x^2 \pmod{p^2} & \text{if } \left(\frac{p}{5}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 2p = 5x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-85}{p}\right) = -1. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} \frac{1054k+233}{480^k} \binom{2k}{k} P_k^-(8) \equiv p \left(\frac{-1}{p}\right) \left(221 + 12 \left(\frac{p}{15}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{224434k+32849}{5760^k} \binom{2k}{k} P_k^-(18) \equiv p \left(32305 \left(\frac{-1}{p}\right) + 544 \left(\frac{p}{5}\right)\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty} \frac{1054k+233}{480^k} \binom{2k}{k} P_k^-(8) = \frac{520}{\pi}$$

and

$$\sum_{k=0}^{\infty} \frac{224434k + 32849}{5760^k} \binom{2k}{k} P_k^-(18) = \frac{93600}{\pi}.$$

For $n = 0, 1, 2, \dots$, we define

$$P_n^+(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} x^{2k-n}.$$

Recall that $P_n^+(1) = \sum_{k=0}^n \binom{n}{k}^3$ by an identity of V. Strehl (see also D. Zagier [Z]).

Conjecture 7.12. (i) *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^+(9)}{144^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{13}\right) = 1 \ \& \ p = x^2 + 13y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{13}\right) = -1 \ \& \ 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-13}{p}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{65k + 22}{144^k} \binom{2k}{k} P_k^+(9) \equiv 22p \pmod{p^2}.$$

(ii) *Let $p \neq 5$ be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k^2(13, 13)}{1300^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{13}{p}\right) = 1 \ \& \ p = x^2 + 13y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{13}{p}\right) = -1 \ \& \ 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-13}{p}\right) = -1. \end{cases} \end{aligned}$$

And

$$\sum_{k=0}^{p-1} \frac{312k + 91}{1300^k} \binom{2k}{k} T_k^2(13, 13) \equiv 91p \pmod{p^2}.$$

Conjecture 7.13. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^+(-7)}{16^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k^2(3, -3)}{(-108)^k} \equiv \left(\frac{p}{3}\right) \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{(-108)^n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1 \ \& \ p = x^2 + 21y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = -1, \left(\frac{p}{3}\right) = 1 \ \& \ p = 3x^2 + 7y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right) = -1, \left(\frac{p}{7}\right) = 1 \ \& \ 2p = x^2 + 21y^2, \\ 6x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = -1, \ \& \ 2p = 3x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-21}{p}\right) = -1. \end{cases} \end{aligned}$$

We also have

$$\sum_{k=0}^{p-1} \frac{63k+26}{16^k} \binom{2k}{k} P_k^+(-7) \equiv p \left(5 + 21 \left(\frac{p}{7} \right) \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{56k+19}{(-108)^k} \binom{2k}{k} T_k^2(3, -3) \equiv \frac{p}{2} \left(21 \left(\frac{p}{7} \right) + 17 \right) \pmod{p^2}.$$

Conjecture 7.14. *Let $p \neq 43$ be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^+(175)}{(-29584)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = \left(\frac{p}{19}\right) = 1 \text{ \& } p = x^2 + 133y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{19}\right) = -1 \text{ \& } 2p = x^2 + 133y^2, \\ 2p - 28x^2 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1, \left(\frac{-1}{p}\right) = \left(\frac{p}{7}\right) = -1 \text{ \& } p = 7x^2 + 19y^2, \\ 2p - 14x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p}\right) = 1, \left(\frac{p}{7}\right) = \left(\frac{p}{19}\right) = -1 \text{ \& } 2p = 7x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-133}{p}\right) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{8851815k + 1356374}{(-29584)^k} \binom{2k}{k} P_k^+(175) \equiv p \left(1300495 \left(\frac{p}{7} \right) + 55879 \right) \pmod{p^2}.$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{8851815k + 1356374}{(-29584)^k} \binom{2k}{k} P_k^+(175) = \frac{1349770\sqrt{7}}{\pi}.$$

We also find $\sum_{k=0}^{p-1} \binom{2k}{k} P_k^+(x)/m^k \pmod{p^2}$ (with $p > 3$ a prime not dividing $m \in \mathbb{Z} \setminus \{0\}$) related to the representation $4p = x^2 + dy^2$ with

$$\begin{aligned} (m, x, d) = & (2, 16, 6), (4, -1, 7), (4, 80, 10), (5, 64, 10), (6, 16, 2), (6, 240, 30), \\ & (7, -16, 21), (14, 289, 7), (14, 2800, 70), (15, 96, 30), (20, 16, 30), \\ & (21, 576, 42), (24, 400, 2), (25, -16, 33), (25, 384, 2), \\ & (36, 336, 42), (45, 2304, 70), (49, 800, 22), (50, 784, 22), \\ & (56, 16, 42), (144, 720, 70), (169, -16, 57), (441, 7056, 37). \end{aligned}$$

Actually we have many other conjectural congruences and series for $1/\pi$, which cannot be listed here due to the limitation of the length of this survey.

Acknowledgments. The author thanks the two referees for their helpful comments.

REFERENCES

- [A] S. Ahlgren, *Gaussian hypergeometric series and combinatorial congruences*, in: Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics (Gainesville, FL, 1999), pp. 1–12, Dev. Math., Vol. 4, Kluwer, Dordrecht, 2001.
- [Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$* . *Journées arithmétiques de Luminy*, Astérisque **61** (1979), 11–13.
- [BB] N. D. Baruah and B. C. Berndt, *Eisenstein series and Ramanujan-type series for $1/\pi$* , *Ramanujan J.* **23** (2010), 17–44.
- [BoBo] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987.
- [BBC] N. D. Baruah, B. C. Berndt and H. H. Chan, *Ramanujan’s series for $1/\pi$: a survey*, *Amer. Math. Monthly* **116** (2009), 567–587.
- [Be] B. C. Berndt, *Ramanujan’s Notebooks, Part IV*, Springer, New York, 1994.
- [BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.
- [CCL] H. H. Chan, S. H. Chan and Z. Liu, *Domb’s numbers and Ramanujan-Sato type series for $1/\pi$* , *Adv. in Math.* **186** (2004), 396–410.
- [CC] H. H. Chan and S. Cooper, *Rational analogues of Ramanujan’s series for $1/\pi$* , preprint, 2011.
- [CV] H. H. Chan and H. Verrill, *The Apéry numbers, the Almkvist-Zudilin numbers and new series for $1/\pi$* , *Math. Res. Lett.* **16** (2009), 405–420.
- [CDE] S. Chowla, B. Dwork and R. J. Evans, *On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$* , *J. Number Theory* **24** (1986), 188–196.
- [ChCh] D. V. Chudnovsky and G. V. Chudnovsky, *Approximations and complex multiplication according to Ramanujan*, in: *Ramanujan Revisited: Proc. of the Centenary Confer. (Urbana-Champaign, ILL., 1987)*, (eds., G. E. Andrews, B. C. Berndt and R. A. Rankin), Academic Press, Boston, MA, 1988, pp. 375–472.
- [CD] J. B. Cosgrave and K. Dilcher, *Mod p^3 analogues of theorems of Gauss and Jacobi on binomial coefficients*, *Acta Arith.* **142** (2010), 103–118.
- [C] D. A. Cox, *Primes of the Form $x^2 + ny^2$* , John Wiley & Sons, 1989.
- [GK] B. H. Gross and N. Koblitz, *Gauss sums and the p -adic Γ -function*, *Ann. Math.* **109** (1979), 569–581.
- [GZ] V. J. W. Guo and J. Zeng, *Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials*, preprint, <http://arxiv.org/abs/1101.0983>.
- [HW79] R. H. Hudson and K. S. Williams, *Cauchy-type congruences for binomial coefficients*, *Proc. Amer. Math. Soc.* **85** (1982), 169–174.
- [HW84] R. H. Hudson and K. S. Williams, *Binomial coefficients and Jacobi sums*, *Trans. Amer. Math. Soc.* **281** (1984), 431–505.
- [JM] A. Joux et F. Morain, *Sur les sommes de caractères liées aux courbes elliptiques à multiplication complexe*, *J. Number Theory* **55** (1995), 108–128.
- [M] E. Mortenson, *A p -adic supercongruence conjecture of van Hamme*, *Proc. Amer. Math. Soc.* **136** (2008), 4321–4328.
- [O] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -series*, Amer. Math. Soc., Providence, R.I., 2003.
- [PV] R. Padma and S. Venkataraman, *Elliptic curves with complex multiplication and a character sum*, *J. Number Theory* **61** (1996), 274–282.
- [PS] H. Pan and Z. W. Sun, *A combinatorial identity with application to Catalan numbers*, *Discrete Math.* **306**(2006) 1921–1940.
- [P] A. van der Poorten, *A proof that Euler missed... Apéry’s proof of the irrationality of $\zeta(3)$* , *Math. Intelligencer* **1** (1978/79), 195–203.

- [R] S. Ramanujan, *Modular equations and approximations to π* , Quart. J. Math. (Oxford) (2) **45** (1914), 350–372.
- [Ra] A. R. Rajwade, *The Diophantine equation $y^2 = x(x^2 + 21Dx + 112D^2)$ and the conjectures of Birch and Swinnerton-Dyer*, J. Austral. Math. Soc. Ser. A **24** (1977), 286–295.
- [RP] A. R. Rajwade and J. C. Parnami, *A new cubic character sum*, Acta Arith. **40** (1982), 347–356.
- [RPR] D. B. Rishi, J. C. Parnami and A. R. Rajwade, *Evaluation of a cubic character sum using the $\sqrt{-19}$ division points of the curve $y^2 = x^3 - 2^3 \cdot 19x + 2 \cdot 19^2$* , J. Number Theory **19** (1984), 184–194.
- [RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), pp. 223–231, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.
- [Ro] M. D. Rogers, *New ${}_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$* , Ramanujan J. **18** (2009), 327–340.
- [S] N. J. A. Sloane, Sequence A001850 in OEIS (On-Line Encyclopedia of Integer Sequences), <http://www.oeis.org/>.
- [S1] Z. H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915–1929.
- [S2] Z. H. Sun, *Congruences concerning Legendre polynomials II*, preprint arXiv:1012.3898. On-line version available from <http://arxiv.org/abs/1012.3898>.
- [S3] Z. H. Sun, *Congruences concerning Legendre polynomials III*, preprint, arXiv:1012.4234. On-line version available from <http://arxiv.org/abs/1012.4234>.
- [Su1] Z. W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, Sci. China Math. **53** (2010), 2473–2488. <http://arxiv.org/abs/0909.5648>.
- [Su2] Z. W. Sun, *p -adic valuations of some sums of multinomial coefficients*, Acta Arith. **148** (2011), 63–76.
- [Su3] Z. W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219–2238.
- [Su4] Z. W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.
- [Su5] Z. W. Sun, *On sums involving products of three binomial coefficients*, arXiv:1012.3141. <http://arxiv.org/abs/1012.3141>.
- [Su6] Z. W. Sun, *Congruences involving generalized central trinomial coefficients*, preprint, arXiv:1008.3887. <http://arxiv.org/abs/1008.3887>.
- [Su7] Z. W. Sun, *On sums of Apéry polynomials and related congruences*, arXiv:1101.1946. <http://arxiv.org/abs/1101.1946>.
- [Su8] Z. W. Sun, *On sums related to central binomial and trinomial coefficients*, preprint, arXiv:1101.0600. <http://arxiv.org/abs/1101.0600>.
- [ST1] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.
- [ST2] Z. W. Sun and R. Tauraso, *On some new congruences for binomial coefficients*, Int. J. Number Theory **7** (2011), 645–662.
- [vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p -adic Functional Analysis (Nijmegen, 1996), pp. 223–236, Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997.
- [W] K. S. Williams, *Evaluation of character sums connected with elliptic curves*, Proc. Amer. Math. Soc. **73** (1979), 291–299.
- [Y] Y. Yang, *Personal communication*, March, 2011.
- [Ye] K. M. Yeung, *On congruences for binomial coefficients*, J. Number Theory **33** (1989), 1–17.
- [Yo] P. T. Young, *Gauss sums and multinomial coefficients*, J. Number Theory **106** (2004), 13–25.

- [Z] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, in: Groups and Symmetries: from Neolithic Scots to John McKay, CRM Proc. Lecture Notes 47, Amer. Math. Soc., Providence, RI, 2009, pp. 349–366.