CERN-PH-TH/2011-061
DCPT-11/11
IPhT–T11/037

The super-correlator/super-amplitude duality: Part II

Burkhard Eden[a], **Paul Heslop**[a], **Gregory P. Korchemsky**[b], **Emery Sokatchev**[c,d,e]

[a] *Mathematics department, Durham University, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom*

[b] *Institut de Physique Théorique*[1], *CEA Saclay, 91191 Gif-sur-Yvette Cedex, France*

[c] *Physics Department, Theory Unit, CERN, CH -1211, Geneva 23, Switzerland*

[d] *Institut Universitaire de France, 103, bd Saint-Michel F-75005 Paris, France*

[e] *LAPTH*[2], *Université de Savoie, CNRS, B.P. 110, F-74941 Annecy-le-Vieux, France*

Abstract

We continue the study of the duality between super-correlators and scattering super-amplitudes in planar $\mathcal{N}=4$ SYM. We provide a number of further examples supporting the conjectured duality relation between these two seemingly different objects. We consider the five- and six-point one-loop NMHV and the six-point tree-level NNMHV amplitudes, obtaining them from the appropriate correlators of strength tensor multiplets in $\mathcal{N}=4$ SYM. In particular, we find exact agreement between the rather non-trivial parity-odd sector of the integrand of the six-point one-loop NMHV amplitude, as obtained from the correlator or from BCFW recursion relations. Together these results lead to the conjecture that the integrands of any N^{k}MHV amplitude at any loop order in planar $\mathcal{N}=4$ SYM can be described by the correlators of stress-tensor multiplets.

[1] Unité de Recherche Associée au CNRS URA 2306
[2] Laboratoire d'Annecy-le-Vieux de Physique Théorique, UMR 5108

1 Introduction

In the maximally supersymmetric Yang-Mills theory in four dimensions ($\mathcal{N} = 4$ SYM) there is a duality between scattering amplitudes and Wilson loops with light-like edges. This was first noticed at strong coupling [1] via the AdS/CFT correspondence [2], and soon after confirmed also at weak coupling [3–6] directly within the field theory.

Recently, it has been realised that both objects, Wilson loops and MHV amplitudes, can be obtained from the light-cone limit of correlation functions of certain gauge invariant scalar composite operators, which are the bottom component of the $\mathcal{N} = 4$ stress-tensor multiplet $\mathcal{T}$ [7, 8].

In other recent developments, a procedure for computing the *integrand* of all scattering amplitudes in the theory (i.e. for all helicities and at all loop orders) has been derived [9] in terms of momentum twistor variables [10] using BCFW recursion relations [11] generalized to loop level [9, 12]. Supersymmetric generalisations of the polygonal Wilson loops have also been suggested as a dual to non-MHV amplitudes in two publications [13, 14] (see also recent comments in [15]).

For MHV amplitudes the equivalence to correlation functions of scalar operators holds for the integrands, which was verified in [16]. In this and the twin paper [17] we propose to extend this duality to all non-MHV super-amplitudes and to the light-cone limit of the super-correlators of stress-tensor multiplets, respectively. We argue that the integrands of all planar amplitudes are contained as a subsector of the correlation functions. As an illustration of our proposal, in [17] we demonstrated that the n-point tree-level NMHV super-amplitudes can be obtained from the aforementioned correlators, computed at tree level. Here we continue our investigation of the new duality. We set out to show the same for the NMHV five- and six-point amplitudes at one loop, and for the NNMHV six-point amplitudes at tree-level.

The conjectured duality can be formulated as follows. Consider the super-correlation functions of n energy momentum supermultiplets $\langle\mathcal{T}(1)\dots\mathcal{T}(n)\rangle$ in $\mathcal{N}$=4 superspace in the limit in which consecutive points become light-like separated. This correlator depends both on the chiral (θ) and anti-chiral ($\bar\theta$) odd coordinates of $\mathcal{N} = 4$ superspace. To be able to compare it to the super-amplitudes $\mathcal{A}_n$ defined in *chiral* dual superspace (x, θ), we set all $\bar\theta = 0$. Further, before taking the light-cone limit, we divide the correlator by its bottom component $\langle\mathcal{T}(1)\dots\mathcal{T}(n)\rangle^{\rm tree}_{n;0}$, obtained by setting $\theta = \bar\theta = 0$ and computed at tree level. This removes the pole singularities due to propagator factors. Then we claim that the light-cone limit of the ratio of correlators is equivalent to the *square* of the planar super-amplitudes $\mathcal{A}_n$ divided by the tree-level MHV amplitude[2]

$$\lim_{x^2_{i\,i+1}\to 0} \frac{\langle\mathcal{T}(1)\dots\mathcal{T}(n)\rangle}{\langle\mathcal{T}(1)\dots\mathcal{T}(n)\rangle^{\rm tree}_{n;0}}\bigg|_{\bar\theta_i=0} = \Big(\mathcal{A}_n/\mathcal{A}^{\rm tree}_{n;\rm MHV}\Big)^2 . \tag{1.1}$$

At the moment this is slightly schematic and much of this equation needs to be defined more carefully in order for the reader to be able to properly interpret it (e.g. on what

[2]To obtain a well-defined ratio the delta functions imposing the super-momentum conservation are removed from the amplitudes.

(super)space are the two sides defined and how are they related etc.) and we will do this carefully in the next section. Note now however one intriguing feature. The left-hand side is *not equal to* the correlation function directly but the correlation function with a coupling dependent rescaling of the odd coordinates $\theta \to a^{-1/4}\theta$, with $a = g^2 N_c/\pi^2$ being the 't Hooft coupling. A similar rescaling of odd coordinates was performed in [14] to compare the supersymmetric Wilson loop with the superamplitude.

The other thing to note now is that the quantities on both sides of the duality (1.1) diverge at loop level and need regularising. However, on the correlation function side loop corrections can be computed by considering integrals of tree-level correlators with multiple insertions of the $\mathcal{N} = 4$ SYM Lagrangians (itself a member of the stress-tensor multiplet). This enables us to define the left-hand side of the duality at the level of the integrand via a tree-level *rational* correlation function. It can then be compared with the *rational* integrand for the entire super-amplitude, and we find complete agreement for every test performed so far.

Together these results lead to the conjecture that the integrands of any N^{k}MHV amplitude at any loop order in planar $\mathcal{N} = 4$ SYM can be described by the correlators of stress-tensor multiplets.

The paper is organised as follows. In Section 2 we give a summary of the formulation of the new duality (for more detail see [17]). In Section 3 we exploit the off-shell one- and two-loop 4-point correlator results of [18] (lifted to $\mathcal{N} = 4$ as in [19, 20]) to obtain the five-point tree NMHV, the five-point one-loop NMHV and the six-point tree NNMHV amplitudes from our conjecture (1.1). In Section 4 we construct the six-point one-loop NMHV integrand from (1.1), which is much more involved than the five-point case because there is a large parity-odd sector. Using the same techniques as in [16] we verify exact agreement with the result of [9, 21] based on BCFW recursion relations.

2 The duality

The correlation functions in the new duality naturally depend on chiral and anti-chiral Grassmann odd variables, while the amplitudes are usually formulated on chiral superspaces [22, 23]. We argue in [17] and here that the amplitudes are found in the purely left-handed sector of the correlators. In the present paper we focus on explicit calculations; the interested reader can find a more complete exposition of the various superspaces and superfields in [17].

The field content of the $\mathcal{N} = 4$ super Yang-Mills theory comprises six real scalars, four complex Majorana-Weyl fermions and the gauge potential A_μ. The associated field strength, the scalars and the fermions all transform in the adjoint representation of the gauge group, which we assume to be $SU(N_c)$. A particularly useful way of presenting the multiplet on shell is via $\mathcal{N} = 4$ *analytic superspace* [24]. In this formalism the entire multiplet can be sandwiched into a single scalar superfield, charged under a $U(1)$ subgroup of $SU(4)$

$$W_{\mathcal{N}=4}(z)\,, \qquad z = \{x^{\dot\alpha\alpha},\, \rho^{\alpha a},\, \bar\rho_{a'}{}^{\dot\alpha},\, y_{a'}{}^{a}\}\,, \qquad a \in \{1,2\}\,, \quad a' \in \{3,4\}\,. \tag{2.1}$$

Here ρ and $\bar\rho$ are odd variables [3] and y is an additional bosonic coordinate related to the internal $SU(4)$ symmetry group. The ρ variables are harmonic projections of the full Minkwoski superspace variables θ. For instance $\rho_i^{\alpha a} := \theta_i^{\alpha a} + \theta_i^{\alpha a'} y_{i\,a'}{}^{a}$ (more information can be found in appendix E).[4]

$\mathcal{N}$=4 analytic superspace is the most convenient formalism for packaging together correlation functions in $\mathcal{N}$=4 SYM since it manifests the full superconformal symmetry of the problem enabling one to completely solve the superconformal Ward identities and write any correlation function in a fully superconformal way [25]. On the other hand, $\mathcal{N}=4$ SYM does not have an off-shell superspace description and so in order to perform actual perturbative calculations one needs to use $\mathcal{N}$=2 harmonic superspace [26] and then lift the results to $\mathcal{N}$=4 analytic superspace.

The stress-tensor multiplet contains, among others, the following components in its $\rho, \bar\rho$ expansion:

$$\mathcal{T}(x,\rho,\bar\rho,y) \;=\; \mathrm{tr}(W^2_{\mathcal{N}=4}) = \mathcal{O} + \ldots - 4\rho^4\mathcal{L} + \ldots - 4\bar\rho^4\bar{\mathcal{L}} + \ldots + (\rho\sigma^\mu\bar\rho)(\rho\sigma^\nu\bar\rho)T_{\mu\nu} + \ldots\,, \tag{2.2}$$

where

$$\rho^4 \;=\; \frac{1}{12}\,(\rho^2)^{\alpha\beta}\,(\rho^2)_{\alpha\beta}\,, \qquad (\rho^2)^{\alpha\beta} \;=\; \rho^{a\alpha}\rho_a^{\beta} \tag{2.3}$$

and so $\rho^4 \;=\; -(\theta^+)^4/12$ in the notation of [17]. Here the lowest component $\mathcal{O} = \mathrm{tr}(\phi^2) = \mathcal{T}(x,0,0,y)$ is a scalar bilinear operator in the $\mathbf{20'}$ of $SU(4)$. The top spin component $T_{\mu\nu}$ is the stress tensor of the theory, which gives the multiplet its name. Another component of $\mathcal{T}$, of crucial importance in what follows, is the chiral on-shell $\mathcal{N}=4$ SYM Lagrangian $\mathcal{L}$ appearing at ρ^4 (as well as its PCT conjugate $\bar{\mathcal{L}}$ at $\bar\rho^4$). For ease in later formulae, we absorb the nilpotent factor into the definition of $\mathcal{L}$:

$$\rho^4\mathcal{L} \;\to\; \mathcal{L}\,. \tag{2.4}$$

Due to a residual $\mathcal{Z}_4$ R-symmetry of the theory (the centre of $SU(4)$), the expansion of n-point functions of the stress-tensor multiplet $\mathcal{T}$ in terms of the Grassmann variables is organised in powers $\rho^m\bar\rho^n$ with $m-n=4k$ divisible by four [27].

In the present article we will not be interested in the right-handed spinors $\bar\rho$ which we put to zero. The right-handed Poincaré supersymmetry $\bar Q$ and the left-handed conformal supersymmetry S of the model are explicitly broken by this choice[5].

---

[3]The variables $\rho, \bar\rho$ each have just 4 components. This is half the number one would expect for $\mathcal{N}$=4 supersymmetry. This is thus similar to chiral superspace in which (the anti-chiral) half of the odd coordinates are dropped.

[4] In [17] we used the alternative, harmonic superspace notation. There the variables y are part of an $SU(4)$ harmonic matrix $u_A^{+a}, u_A^{-a'}$, and ρ^a_α is equivalent to $\theta^{+a}_\alpha = \theta^A_\alpha u_A^{+a}$.

[5]However, since for the dual amplitudes in formula (1.1), the full dual superconformal symmetry [23] is present at tree level [28], then for the correlation functions also in the Born approximation and in the light-like n-gon limit these symmetries should be “magically” restored.

Then the entire correlator at $\bar\rho = 0$ is expanded in terms of polynomials in ρ, homogeneous of degree $4\,k$:

$$\langle \mathcal{T}(1)\dots\mathcal{T}(n)\rangle|_{\bar\rho_i=0} = \sum_{k=0}^{n-4} G_{n;k}(1,\dots,n;a) \ . \tag{2.5}$$

In what follows we will not display the restriction $\bar\rho_i = 0$ explicitly, but it will always be assumed. We can use the left-handed Poincaré supersymmetry Q and the right-handed conformal supersymmetry $\bar S$ to simultaneously put $\rho_i = 0$ at any four points. This explains the range $0 \le k \le n-4$. For example, at five points we could put $\rho_1 = \ldots = \rho_4 = 0$ leaving only ρ_5, so that the only possible terms in the expansion have $(\rho_5)^0$ and $(\rho_5)^4$ times some functions of the bosonic coordinates $x^{\dot\alpha\alpha}$ and $y^a_{a'}$. The dependence on the full set of ρ_i can eventually be reconstructed by the inverse supersymmetry transformation.

We also have an expansion in the 't Hooft coupling $a = g^2 N_c/\pi^2$ and so we write the full n-point correlator as the double expansion

$$\langle \mathcal{T}(1)\dots\mathcal{T}(n)\rangle = \sum_{k=0}^{n-4}\sum_{l=0}^{\infty} a^{l+k}\, G^{(l)}_{n;k}(1,\dots,n) \ , \tag{2.6}$$

so that we denote by $G^{(l)}_{n;k}$ the n-point correlator at Grassmann level $O(\rho^{4k})$ and at l loops.

The lowest contribution to $G_{n;k}$ – so $G^{(0)}_{n;k}$, which we shall call the Born level – comes at $O(a^k)$ from $(k+1)$-loop graphs w.r.t. ordinary momentum space loop counting. The (l) counter labels the order beyond Born approximation. In the correlation functions $G^{(l)}_{n;k}$ thus carries $a^{(l+k)}$, quite different from the corresponding amplitude as we discuss shortly.

However, it is natural to gather together all the (l) contributions to the correlator (even though they occur at different powers of the coupling). So for example we will define

$$G^{(l)}_n(1,\dots,n) := \sum_{k=0}^{n-4} G^{(l)}_{n;k}(1,\dots,n) \tag{2.7}$$

to be simply the sum of all the (l) contributions to the n-point correlator.

Now we compare the expansion of the correlator with the total colour ordered n-point planar scattering amplitude $\mathcal{A}_n$ (i.e. the sum of the MHV, NMHV, ... parts). This has an expansion very similar to the correlator (2.5):

$$\frac{\mathcal{A}_n}{\mathcal{A}^{\text{tree}}_{n\,\text{MHV}}} = \sum_{k=0}^{n-4} \widehat{\mathcal{A}}_{n;k} \ , \tag{2.8}$$

where the ratio is understood in the sense of removing the momentum and supercharge conservation delta functions. The amplitude is a function of three equivalent sets of variables. These can either be λ^α_i, $\tilde\lambda^{\dot\alpha}_i$ and η^A_i (with $A = 1,2,3,4$) of the chiral on-shell superspace [22], or $x^{\dot\alpha\alpha}_i, \theta^A_{i\,\alpha}$ of the chiral dual superspace [23], or $\lambda^\alpha_i, \mu_{i\,\dot\alpha}, \chi^A_i$ of momentum supertwistor space [29]. The bosonic variables x are "T-dual" to the outgoing on-shell particle momenta [1,30]:

$$(p_i)^{\alpha}_{\dot\alpha} = \lambda^{\alpha}_i \bar\lambda_{i\,\dot\alpha} = (x_i - x_{i+1})^{\alpha}_{\dot\alpha} = (x_{i\,i+1})^{\alpha}_{\dot\alpha} \,. \tag{2.9}$$

For the purpose of comparing with super-correlators, it is most convenient to use the momentum supertwistor odd variable $\chi^A = \lambda^\alpha \theta^A_\alpha$. It is a Lorentz scalar but it carries a four-component internal index $A = (a, a')$. Hence it has the same number of odd components as ρ^α_a.

The loop expansion is more straightforward than for the correlator, we have the double expansion

$$\frac{\mathcal{A}_n}{\mathcal{A}^{\rm tree}_{n\,{\rm MHV}}} = \sum_{l=0}^{\infty} a^l\ \widehat{\mathcal{A}}^{(l)}_n = \sum_{k=0}^{n-4}\sum_{l=0}^{\infty} a^l\ \widehat{\mathcal{A}}^{(l)}_{n;k} \,. \tag{2.10}$$

Unlike the analogous correlator expansion (2.6) all l loop contributions come with a^l.

Our conjecture is roughly that "the square of the amplitude is equal to the correlation function in the light-like limit". More concretely then we write

$$\boxed{\lim_{x^2_{i\,i+1}\to 0} \sum_{l\geq 0} a^l \frac{G^{(l)}_n}{G^{\rm tree}_{n;0}} = \left(\sum_{l=0}^{\infty} a^l\ \widehat{\mathcal{A}}^{(l)}_n\right)^2} \tag{2.11}$$

in the planar limit, which is just a rewriting of equation (1.1) in the introduction without the coupling dependent rescaling of theta.

Note that although the right-hand side is simply the full superamplitude, the left-hand side is *not* the correlator simply due to the fact that the powers of the coupling are not correct (see the discussion below (2.6)) and this is why we write the explicit expansion on both sides. A similar issue arises [14] when comparing the super Wilson loop to amplitudes.

There are a few more ingredients we need in order to properly interpret this equation. Firstly, on the left-hand side the correlator is defined in analytic superspace, with variables x, y and ρ, whereas on the right-hand side the variables are x, χ. In order to make sense of the equation we need to identify these variables. We will find that the Grassmann variables are identified as follows (a fact which follows straightforwardly from the known expressions of both variables in terms of the standard Minkowski superspace variable θ and is derived in appendix E)

$$\chi_i = \langle i|(\rho_i - \rho_{i\,i+1}\, y^{-1}_{i\,i+1}\, y_i)\,, \qquad \chi'_i = \langle i|\rho_{i\,i+1}\, y^{-1}_{i\,i+1}\,, \qquad \langle i| = \lambda^\alpha_i \,. \tag{2.12}$$

The labels in the last equation exclusively indicate the point in superspace to which the variables belong. We surpress the Lorentz and internal indices. They link up naturally if we keep their positions always as given in (2.1) together with $(y^{-1})_a{}^{a'}$. So for example $\rho_{i\,i+1} y^{-1}_{i\,i+1} y_i$ stands for $\rho^{\alpha a}_{i\,i+1}(y^{-1}_{i\,i+1})_a{}^{a'} y_{i\,a'}{}^b$ etc. Further, we have split the $SU(4)$ index A into its $SU(2)\times SU(2)$ subgroup pieces, so $\chi^A = (\chi^a, \chi^{a'})$ which are in turn denoted by (χ, χ').

The second thing we need to know is how to regularise, since as it stands, both sides of the duality relation (2.11) diverge. For generic x_i, all n-point functions of $\mathcal{T}$ are finite and (super)conformal order by order in perturbation theory. The limit $x^2_{i\,i+1} \to 0\,, i \in \{1,\ldots,n\}$ (with the cyclic identification $x_{n+1} = x_1$) puts the n operators at the vertices of an n-gon with light-like edges. In this limit, the correlators develop two kinds of singularities. There are power singularities as for the tree-level correlator

$$G^{(0)}_{n;0}(1,\ldots,n)|_{\bar\rho_i=0} \;=\; \frac{N_c^2-1}{(4\pi^2)^n}\,\frac{y^2_{12}}{x^2_{12}}\,\frac{y^2_{23}}{x^2_{23}}\,\ldots\,\frac{y^2_{n1}}{x^2_{n1}}\,. \tag{2.13}$$

In this formula we have displayed only the most singular term of the connected tree, which turns out to be the highest power singularity also in the loop corrections to G_n. Hence the ratio on the left-hand side of (2.11) is free of power singularities.

But the conformal loop integrals found in the perturbative corrections to G_n develop logarithmic divergences when their external points become null separated. These "pseudo-conformal" integrals require regularisation.

This issue has already been encountered for the MHV duality [8]. At the MHV one- and two-loop level (so $\bar\rho_i = \rho_i = \chi_i = 0$ and up to $O(a^2)$) our conjecture (2.11) yields

$$\begin{aligned}\lim_{x^2_{i\,i+1}\to 0}\frac{G_{n;0}}{G^{(0)}_{n;0}}\,(x_1,\ldots,x_n) \;&=\; 1 + 2\,a\,\widehat{\mathcal{A}}^{(1)}_{n;0}(x_1,\ldots,x_n) \\ &\quad + 2\,a^2\left(\widehat{\mathcal{A}}^{(2)}_{n;0}(x_1,\ldots,x_n) + \frac{1}{2}(\widehat{\mathcal{A}}^{(1)}_{n;0}(x_1,\ldots,x_n))^2\right) + O(a^3)\end{aligned} \tag{2.14}$$

where we have simply input the amplitude expansion (2.10) into the right-hand side of the conjecture (2.11) and expanded the square. We recall that $\widehat{\mathcal{A}}^{(\ell)}_{n;0}$ stands here for $l-$loop correction to the ratio of $n-$particle MHV amplitude to its tree-level expression.

This was demonstrated in [8] for all n-point one-loop MHV amplitudes and the four- and five-point MHV two-loop amplitudes in a non-standard regularisation scheme: the integrand of the loop level correlator was evaluated in four dimensions and to regularise only the measure of the integration over the insertion points was modified to $D = 4 - 2\,\epsilon$ dimensions (with $\epsilon < 0$). This non-standard x-space regularisation precisely mimics the usual p-space infrared prescription for the amplitudes.

But more is true: at the level of the integrands we can stay in exactly four dimensions because we need not worry about singularities. Exact equivalence holds for the integrands themselves, which was verified for the MHV five- and six-point one- and two-loop amplitudes and conjectured for all other cases in [16].

So how can we unambiguously define an integrand for a loop level correlator? A crucial point that enables us to do so (and hence to compare with amplitude integrands) is that loop corrections to such n-point correlators can be computed by means of multiple

Lagrangian insertions[6,7] [8, 16, 18, 31] so that we have:

$$\langle \mathcal{T}(1)\ldots\mathcal{T}(n)\rangle^{(l)} = \frac{1}{l!}\int d\mu_{0_1}\ldots d\mu_{0_l}\langle\mathcal{T}(0_1)\ldots\mathcal{T}(0_l)\mathcal{T}(1)\ldots\mathcal{T}(n)\rangle^{(0)}\,, \tag{2.15}$$

where the bracketed superscript (l) indicates that this is the l loop contribution and where $d\mu := d^4x\,d^4\rho$. The second equality follows from (2.2) and (2.4), the Grassmann integral just picks the ρ^4 component of the superfield $\mathcal{T}$. On the right-hand side the integrand is itself a correlator and furthermore a Born level correlator. Therefore this Born level correlator provides an unambiguous definition of the integrand which we can compare with the integrands coming, for example from the amplitude integrand results of [9, 21].

Further rewriting this in terms of the ρ^{4k} expansion terms in (2.6) we thus have that

$$G^{(l)}_{n;k}(1,\ldots,n) = \frac{1}{l!}\int d\mu_{0_1}\ldots d\mu_{0_l}\,G^{(0)}_{(n+l);(k+l)}(0_1,\ldots,0_l;1,\ldots,n)\,, \tag{2.16}$$

where the semicolon after 0_l distinguishes the loop integration variables from the outer points. On the right-hand side we have the same type of object $G_{n;k}$ as on the left-hand side, but at tree level and at a higher Grassmann level. However, in the light-cone limit the points x_i (with $i = 1,\ldots,n$) form a light-like polygon while the points x_{0_k} (with $k = 1,\ldots,\ell$) remain in arbitrary positions.

So in summary all loop-level integrands of correlation functions can be written in terms of tree-level higher point correlation functions and hence via the duality the integrand of any amplitude at any loop order can be obtained from tree-level stress-tensor multiplet correlators.

For example, at one and two loops, we have the MHV amplitude/correlator duality

$$\lim_{x^2_{i\,i+1}\to 0}\frac{G^{(1)}_{n;0}}{G^{(0)}_{n;0}}(1,\ldots,n) = \int d^4x_0 d^4\rho_0 \lim_{x^2_{i\,i+1}\to 0}\frac{G^{(0)}_{n+1;1}}{G^{(0)}_{n;0}}\big(0;1,\ldots,n\big) = 2\,\widehat{\mathcal{A}}^{(1)}_{n;0}(1,\ldots,n)\,, \tag{2.17}$$

which we interpret as the integrand identity

$$\int d^4\rho_0 \lim_{x^2_{i\,i+1}\to 0}\frac{G^{(0)}_{n+1;1}}{G^{(0)}_{n;0}}\big(0;1,\ldots,n\big) = 2\,\widehat{A}^{(1)}_{n+1;0}(0;1,\ldots,n)\,. \tag{2.18}$$

Here the integrand of the amplitudes (divided by the tree-level MHV amplitude) is denoted by $\widehat{A}$ with the integration points included in the list of arguments before the semicolon,

[6] As mentioned, all the $\mathcal{N}=4$ results in this paper are actually derived from calculations with $\mathcal{N}=2$ superfields [26], either in this work or in the literature that we quote. The insertion procedure that is actually used is differentiation with respect to the coupling constant in the $\mathcal{N}=2$ harmonic superspace formalism, whose essential details are briefly summarised in Appendix B. These $\mathcal{N}=2$ results are then uplifted to $\mathcal{N}=4$ analytic superspace.

[7] In this paper we do the Wick rotation before deriving Feynman rules so that amongst other changes the factor i^l disappears from the corresponding formula in [17].

whereas the corresponding integral is denoted by $\widehat{\mathcal{A}}$; so for example we have

$$\widehat{\mathcal{A}}^{(1)}_{n;0}(x_1,\ldots,x_n) \;=\; \int d^4x_0\, \widehat{A}^{(1)}_{n+1;0}(x_0;x_1,\ldots,x_n)\,. \tag{2.19}$$

Similarly at two loops we have the integrand identity

$$\begin{aligned}
&\frac{1}{2}\int d^4\rho_0\, d^4\rho_{0'} \lim_{x^2_{i\,i+1}\to 0} \frac{G^{(0)}_{n+2;2}}{G^{(0)}_{n;0}}\,(0,0';1,\ldots,n) \\
&= 2\left(\widehat{A}^{(2)}_{n+2;0}(x_0,x_{0'};x_1,\ldots,x_n) + \frac{1}{2}\,\widehat{A}^{(1)}_{n+1;0}(x_0;x_1,\ldots,x_n)\,\widehat{A}^{(1)}_{n+1;0}(x_{0'};x_1,\ldots,x_n)\right)\,.
\end{aligned} \tag{2.20}$$

In (2.18) we have used the Lagrangian component of an additional $\mathcal{T}(0)$ operator at point 0 to obtain the one-loop correction to the n-point $O(\rho^0)$ correlator. The outer points were put onto a light-like n-gon while the insertion point is integrated out. On the other hand, before integration and without any light-like limit this is, of course, just a specific Grassmann component of an $(n+1)$-point function of $\mathcal{T}$'s. Then according to the duality (2.11), we can take this same component of the correlator in an $(n+1)$-gon limit to obtain the $(n+1)$-point NMHV tree-level amplitude [17]. Once again, this correspondence holds at the level of the integrands. In the same way, the $O(\rho^8)$ part of an $(n+2)$-point function of $\mathcal{T}$'s can yield

- the two-loop n-point MHV amplitude, if two points are treated as insertions and integrated out while the others are put onto an n-gon with light-like edges. This is the situation in equation (2.20).
- the one-loop $(n+1)$-point NMHV amplitude, if one point is treated as an insertion and integrated out, while the others are put onto an $(n+1)$-gon.
- the tree-level $(n+2)$-point NNMHV amplitude in an $(n+2)$-gon limit without any integrations.

The possibility of obtaining various amplitudes from the same generating object is reminiscent of the supersymmetric Wilson loop of [14].

In the rest of the paper we provide a number of explicit examples of the duality (2.11), at tree and at loop level.

3 Five-point one-loop NMHV and six-point tree NNMHV

In this section we explore the one- and two-loop corrections to the simplest correlator of the lowest-dimension components $\mathcal{O} = \mathrm{tr}(\phi^2)$ of the stress-tensor multiplets (see (2.2)), the purely bosonic correlator $G_{4;0} = \langle \mathcal{O}(1)\ldots\mathcal{O}(4)\rangle$. We show that the loop corrections to this four-point correlator, interpreted as Lagrangian insertions [18,31] (see (2.16)), can give rise to several superamplitudes. So the integrand of the one-loop four point correlator $G_{4;0}^{(1)} = \int d\mu_5\, G_{5;1}^{(0)}$ and the integrand of the two-loop four-point correlator $G_{4;0}^{(2)} = \frac{1}{2}\int d\mu_5\, d\mu_6\, G_{6;2}^{(0)}$ yield the following amplitudes (see Fig. 1):

$$
\begin{aligned}
G_{5;1}^{(0)} \quad &\rightarrow \quad \begin{cases} \mathrm{MHV}_4^{(1)} & \\ \mathrm{NMHV}_5^{(0)} & \quad \text{Section 3.3} \end{cases} \\
G_{6;2}^{(0)} \quad &\rightarrow \quad \begin{cases} \mathrm{MHV}_4^{(2)} & \\ \mathrm{NMHV}_5^{(1)} & \quad \text{Section 3.4} \\ \mathrm{NNMHV}_6^{(0)} & \quad \text{Section 3.5} \end{cases}
\end{aligned}
\tag{3.1}
$$

Which amplitude is realised depends on how many $\mathcal{T}$ operators are placed on an n-gon with light-like edges, with the others treated as Lagrangian insertions and integrated out.

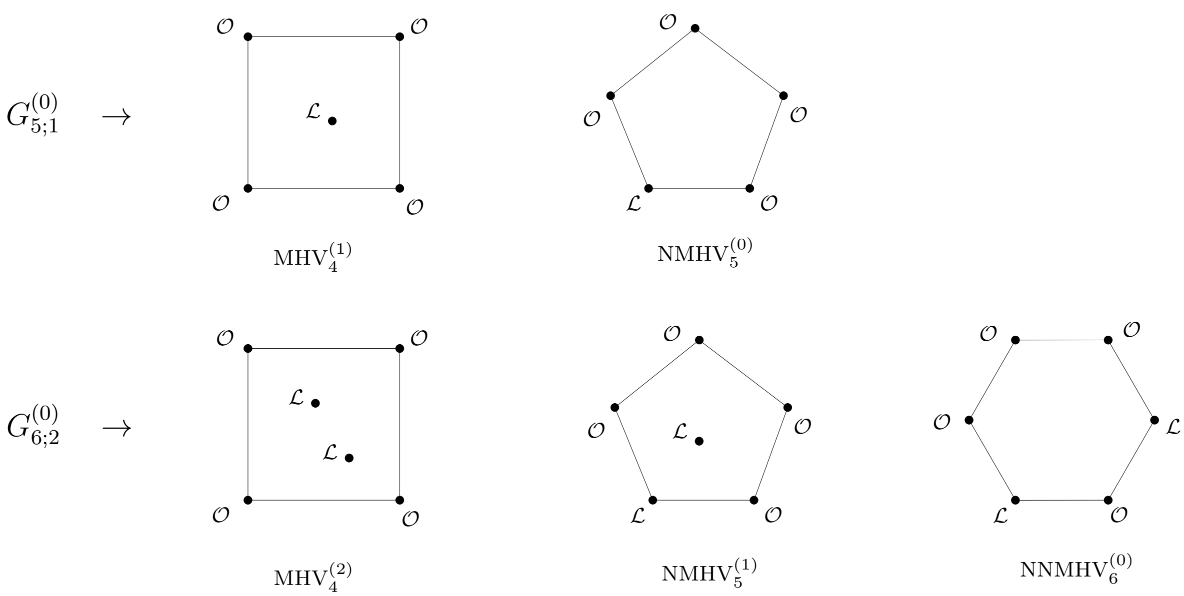

Figure 1: The different light-cone limits taken for the points of the correlators $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{L}\rangle$ and $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{L}\mathcal{L}\rangle$. Operators at neighbouring vertices of a polygon are light-like separated, whereas those inside the polygon are located at arbitrary points.

3.1 Loop corrections to the four-point correlator $G_{4;0}$

We now describe the loop corrections to the four-point correlators and the related integrands (themselves higher-point tree-level correlators) and in later subsections we will relate them in various light-like limits to the respective amplitudes.

Two- and three-point functions of stress-tensor multiplets $\mathcal{T}$'s do not receive quantum corrections [27]. The simplest non-trivial object to study is thus indeed $G_{4;0}$. The form of its loop corrections is heavily restricted by $\mathcal{N}=4$ superconformal symmetry. This "partial non-renormalisation" [20], which we review in appendix A allows a remarkably simple writing of these loop corrections, and of the related higher point Born level correlators. Here we simply display the result of the computations originally done in [18, 19, 32–34].

The one-loop four-point correlator is given as the integral of a certain five-point correlator whereas the two-loop four-point correlator is the integral of a one-loop five-point correlator, or alternatively of a Born level six-point correlator as follows:

$$G^{(1)}_{4;0}(1,2,3,4) = \int d^4x_5 d^4\rho_5\, G^{(0)}_{5;1}(1,2,3,4,5) \tag{3.2}$$

$$\begin{aligned} G^{(2)}_{4;0}(1,2,3,4) &= \int d^4x_5 d^4\rho_5 G^{(1)}_{5;1}(1,2,3,4,5) \\ &= \frac{1}{2}\int d^4x_5 d^4\rho_5 \int d^4x_6 d^4\rho_6\, G^{(0)}_{6;2}(1,2,3,4,5,6)\,. \end{aligned} \tag{3.3}$$

The integrands themselves are given by the simple formulae (see appendix A)

$$G^{(0)}_{5;1}(1,2,3,4,5)|_{(\rho_5)^4} = \frac{2(N_c^2-1)}{(4\pi^2)^5} \times I \times (\rho_5)^4 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{3.4}$$

$$\begin{aligned} G^{(0)}_{6;2}(1,2,3,4,5,6)|_{(\rho_5)^4(\rho_6)^4} &= \frac{4(N_c^2-1)}{(4\,\pi^2)^6} \times I \times (\rho_5)^4(\rho_6)^4 \\ &\times x_{13}^2 x_{24}^2 \frac{\frac{1}{96}\sum_\sigma x^2_{\sigma(1)\sigma(2)} x^2_{\sigma(3)\sigma(4)} x^2_{\sigma(5)\sigma(6)}}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2} \end{aligned} \tag{3.5}$$

where I is the rational prefactor (this universal prefactor is a consequence of superconformal symmetry described further in appendix A)

$$\begin{aligned} I = &\frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}(1-s-t) + \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}(t-s-1) \\ &+ \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2}(s-t-1) + \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} s + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} t \end{aligned} \tag{3.6}$$

where we have introduced the conformal cross-ratios

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{3.7}$$

Note that the x-space factor in the expression for $G^{(0)}_{5;1}$ is simply the one-loop box integrand, whereas the x terms in $G^{(0)}_{6;2}$ arise from the two-loop ladder and one-loop box squared terms in $F^{(2)}$ (A.8). The sum in (3.5) is over all permutations σ of points 1 to 6. There is a 48-fold redundancy in writing it like this since there are only 15 different terms in the sum, so we divide by 48 in order to account for this; the remaining factor 1/2 adjusts the normalisation to meet the result (A.8).

Amplitude integrands can be obtained by taking different light-like limits of these correlation functions as we now investigate.

3.2 The $G_{4;0} \leftrightarrow \text{MHV}_4$ duality

In this subsection we merely reproduce one of the results of [8] as an illustration of the general procedure. The one- and two-loop corrections to G_4 are given in (3.2), (3.3). To compare with four-point MHV amplitudes we need to put the four points of this correlator on the light-like square $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0$, which amongst other things, creates pole singularities in the prefactor I (3.6). According to the duality (2.11) we need to divide the correlator by the connected tree-level correlator in order to remove these poles, i.e. by

$$G_{4;0}^{(0)} = \frac{N_c^2 - 1}{(4\pi^2)^4} \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \text{subleading}\,. \tag{3.8}$$

Remarkably, this is equal to the leading singularity in the prefactor $I(x_1, \ldots, y_4)$ (up to the factor $(N_c^2 - 1)/(4\,\pi^2)^4$) in the light-like limit, so that we obtain from (3.2-3.5)

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{4;0}}{G_{4;0}^{(0)}} = 1 + \lim_{x_{i,i+1}^2 \to 0} \left(a \frac{G_{4;0}^{(1)}}{G_{4;0}^{(0)}} + a^2 \frac{G_{4;0}^{(2)}}{G_{4;0}^{(0)}} \right) + O(a^3) \tag{3.9}$$

$$= \lim_{x_{i,i+1}^2 \to 0} \Bigg[1 + \frac{2\,a}{16\pi^2} \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{3.10}$$

$$+ \frac{2\,a^2}{(16\pi^2)^2} \int d^4x_5\, d^4x_6 \left(x_{13}^2 x_{24}^2 \frac{\frac{1}{96} \sum_\sigma x_{\sigma(1)\sigma(2)}^2 x_{\sigma(3)\sigma(4)}^2 x_{\sigma(5)\sigma(6)}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2} \right) \Bigg].$$

Here in the light-like limit at one loop we immediately recognise the massless one-loop box function, whereas at two loops, in the sum over permutations, many terms are subleading in the light-like limit, and we are left with the massless one-loop box squared together with the massless two-loop ladder diagram. The integrands occurring here exactly match the integrands of the one- and two-loop MHV_4 amplitudes [35, 36] on taking the square.

3.3 The $G_{5;1}^{(0)} \leftrightarrow \text{NMHV}_5^{(0)}$ duality

The simplest non-trivial example of the duality for tree-level amplitudes concerns the five-point NMHV case. Note that this is the anti-MHV amplitude which is related to the MHV amplitude by parity. In the correlator picture parity symmetry (in the sense of the scattered particles, not in the sense of the fields) is far from obvious and thus even this case is quite a non-trivial check of the duality.

According to the duality conjecture, to reproduce the NMHV 5-point tree-level correlator, we need to take the correlator $G_{5;1}^{(0)}$ and put all five points on the light-cone.

This should be compared with the previous subsection. There we were taking the same correlator $G_{5;1}^{(0)}$ but thinking of it as the integrand of the four-point one-loop correlator. The integration point x_5 was thus in an arbitrary position and we reproduced the four-point one-loop MHV amplitude (essentially the massless box function).

Now, x_5 has become the fifth point of a light-like pentagon, together with the external points $x_1, \ldots, x_4$. In other words, in the new light-cone limit $x_{45}^2 = x_{51}^2 = 0$, while $x_{41}^2 \neq 0$. As before, the light-cone poles are compensated by dividing out the tree-level correlator

$$G_{5;0}^{(0)} = \frac{N_c^2 - 1}{(4\pi^2)^5} \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{51}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} + \text{subleading}\,. \tag{3.11}$$

One can easily check from its expression (3.4) that

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{5;1}^{(0)}}{G_{5;0}^{(0)}}\Bigg|_{\rho_5^4} = 2\, \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{25}^2 x_{35}^2} \frac{y_{41}^2}{y_{45}^2 y_{51}^2}\, \rho_5^4\,. \tag{3.12}$$

Let us now compare this to the five-point tree-level amplitude

$$\frac{\mathcal{A}_5^{(0)}}{\mathcal{A}_{5;\mathrm{MHV}}^{\mathrm{tree}}} = 1 + \widehat{\mathcal{A}}_{5;1}^{(0)}(1, \ldots, 5) = 1 + R_{12345}\,. \tag{3.13}$$

The R invariant [23] on the right-hand side of (3.13) corresponds to the NMHV tree-level. A general expression for any R invariant in terms of momentum supertwistors was given in [13]. The case $R_{12345}|_{\rho_5^4}$ is evaluated in Appendix F using the relation (2.12) between the χ and ρ variables. Here we merely state the result (F.8):

$$R_{12345}|_{\rho_5^4} = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{25}^2 x_{35}^2} \frac{y_{41}^2}{y_{45}^2 y_{51}^2}\, \rho_5^4\,.$$

Finally, we compare with (3.12) finding

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{5;1}^{(0)}}{G_{5;0}^{(0)}}\Bigg|_{\rho_5^4} = 2\, \frac{\mathcal{A}_{5;1}^{(0)}}{\mathcal{A}_{5;\mathrm{MHV}}^{\mathrm{tree}}}\Bigg|_{\rho_5^4} \tag{3.14}$$

in perfect agreement with the conjectured duality relation (2.11). The combined $Q, \bar{S}$ supersymmetries are powerful enough to restore the full dependence on the left-handed Grassmann coordinates $\rho_1, \ldots, \rho_4$.

3.4 The $G_{5;1}^{(1)} \leftrightarrow \mathrm{NMHV}_5^{(1)}$ duality

As shown in equation (3.3) the integrand of the one-loop five-point correlator $G_{5;1}^{(1)}$ in the gauge $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$ is given by the 6-point tree-level correlator $G_{6;2}^{(0)}$.

Next, we take the pentagon light-cone limit $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{45}^2 = x_{51}^2 = 0$ of this correlator $G^{(0)}_{6;2}$, given in (3.5) and divide out the free correlator (3.11). The result is

$$
\begin{aligned}
\lim_{x^2_{i,i+1}\to 0} \left.\frac{G^{(1)}_{5;1}}{G^{(0)}_{5;0}}\right|_{\rho_5^4} &= \lim_{x^2_{i,i+1}\to 0} \int d^4x_6 d^4\rho_6 \left.\frac{G^{(0)}_{6;2}}{G^{(0)}_{5;0}}\right|_{\rho_5^4\rho_6^4} \\
&= 4\int d^4x_6\, \frac{x^2_{13}x^2_{24}}{x^2_{14}x^2_{25}x^2_{35}}\; \frac{y^2_{41}}{y^2_{45}y^2_{51}}\; \rho_5^4\; \left[\frac{1}{2}\, g(6;1,2,3,4) + (\text{cyclic})\right] \\
&= 4\, \widehat{\mathcal{A}}^{(0)}_{5;1}\; \widehat{\mathcal{A}}^{(1)}_{5;0}|_{\text{even}}
\end{aligned}
\tag{3.15}
$$

where "+ (cyclic)" means we add 4 terms obtained by cycling the points $1,\ldots,5$ and

$$
g(6;1,2,3,4) = \frac{1}{4\pi^2}\frac{x^2_{13}x^2_{24}}{x^2_{16}x^2_{26}x^2_{36}x^2_{46}} \tag{3.16}
$$

is the one-loop box integrand with integration point x_6. The third line in (3.15) follows because the sum of box integrands $g(6;i,j,k,l)$ in the second line is the same as the integrand of the *even part* of the five-point one-loop MHV amplitude [37] and hence also the same as the even part of the integrand $\widehat{A}^{(1)}_{5;0}$ of the one-loop $\text{NMHV}_5 = \overline{\text{MHV}}_5$ amplitude. Note that although this formula is displayed as an integral identity, we really mean the identity of the corresponding integrands.

Does this agree with our duality conjecture (2.11)? Expanding out the duality relation, we predict

$$
\lim_{x^2_{i,i+1}\to 0} \frac{G^{(1)}_{5;1}}{G^{(0)}_{5;0}}(1,\ldots,5) \;=\; 2\left(\widehat{\mathcal{A}}^{(0)}_{5;1}(1,\ldots,5)\,\widehat{\mathcal{A}}^{(1)}_{5;0}(1,\ldots,5) \;+\; \widehat{\mathcal{A}}^{(1)}_{5;1}(1,\ldots,5)\right)\,. \tag{3.17}
$$

On the other hand, since at five points the NMHV amplitude is in fact an $\overline{\text{MHV}}$ amplitude, it is equal to the tree level NMHV amplitude multiplied by the complex conjugate of the MHV one-loop ratio. Under complex conjugations the even part is invariant, but the parity odd part gets a minus sign. We have therefore that $\widehat{\mathcal{A}}^{(1)}_{5;1} = \widehat{\mathcal{A}}^{(0)}_{5;1}\left(\widehat{\mathcal{A}}^{(1)}_{5;0}|_{\text{even}} - \widehat{\mathcal{A}}^{(1)}_{5;0}|_{\text{odd}}\right)$. Then in the sum of terms in (3.17) the parity odd terms cancel and the prediction is in precise agreement with what we find in (3.15). For completeness, we display the conjecture [9,21] for the integrand $\widehat{A}^{(1)}_{5;1}$ in terms of momentum twistors at the end of Section 4.6. It does indeed satisfy this conjugacy property.

3.5 The $G^{(0)}_{6;2} \leftrightarrow \text{NNMHV}^{(0)}_6$ duality

This is a particular case of the duality between $\overline{\text{MHV}}$ amplitudes and the maximally nilpotent part of correlators. Due to the simple fact that we are taking the square of the amplitude there are (NMHV tree)×(NMHV tree) terms in addition to the NNMHV tree part. In this respect the example is similar to the last one, but it is interesting in its own right since the two contributions are distinct.

The maximally nilpotent part of the superamplitude/supercorrelator duality conjecture (2.11) yields the prediction:

$$\lim_{x_{i,i+1}^2\to 0} \frac{G_{n;n-4}}{G_{n;0}^{\rm tree}} = \sum_{k=0}^{n-4} \widehat{\mathcal{A}}_{n;k}\ \widehat{\mathcal{A}}_{n;n-4-k}\ . \tag{3.18}$$

In particular, at Born level and six points we expect to find

$$\begin{aligned}\lim_{x_{i,i+1}^2\to 0} \frac{G^{(0)}_{6;2}}{G^{(0)}_{6;0}} &= 2\,\widehat{\mathcal{A}}^{(0)}_{6;2} + (\widehat{\mathcal{A}}^{(0)}_{6;1})^2 \\ &= 2\,\widehat{\mathcal{A}}^{\rm tree}_{6;\overline{\rm MHV}} + (\widehat{\mathcal{A}}^{\rm tree}_{6;{\rm NMHV}})^2\,. \end{aligned}\tag{3.19}$$

In order to check this we will first take the hexagon light-like limit of the correlator (3.5) and then evaluate the $\overline{\rm MHV}$ and the additional $({\rm NMHV})^2$ part of the amplitude.

3.5.1 The hexagon limit of the correlator $G^{(0)}_{6;2}$

The fully off-shell correlator $G^{(0)}_{6;2}$ is given in (3.5). We have already seen how taking a four-point light-like limit of this leads to the two-loop four-point MHV integrand, and how taking the pentagon light-like limit leads to the five-point one-loop NMHV integrand. Now we wish to take the hexagon light-like limit in order to obtain the 6-point tree-level NNMHV amplitude.

The hexagon limit creates new light-cone pole singularities, at $x_{45}^2 = x_{56}^2 = x_{61}^2 = 0$. To cancel these we divide by the free correlator

$$G^{(0)}_{6;0} = \frac{N_c^2-1}{(4\pi^2)^6}\frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{56}^2 y_{61}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{61}^2} + \text{subleading}\,. \tag{3.20}$$

Only four terms (of the 15 in (3.5)) remain in the hexagon limit. The result is

$$\begin{aligned}\lim_{x_{i,i+1}^2\to 0} \frac{G^{(0)}_{6;2}}{G^{(0)}_{6;0}}\bigg|_{\rho_5^4\rho_6^4} &= 2\,\frac{y_{14}^2}{y_{45}^2 y_{56}^2 y_{61}^2}\ \rho_5^4\,\rho_6^4 \\ &\times \frac{x_{13}^2 x_{24}^2}{x_{14}^2}\Big(\frac{x_{14}^2}{x_{15}^2 x_{35}^2 x_{26}^2 x_{46}^2} + \frac{x_{13}^2}{x_{15}^2 x_{35}^2 x_{26}^2 x_{36}^2} + \frac{x_{14}^2}{x_{15}^2 x_{25}^2 x_{36}^2 x_{46}^2} + \frac{x_{24}^2}{x_{25}^2 x_{35}^2 x_{26}^2 x_{46}^2}\Big)\,. \end{aligned}\tag{3.21}$$

Next, we must compute both terms in the right-hand side of (3.19) in analytic superspace and compare the result to (3.21).

3.5.2 Evaluating $\widehat{\mathcal{A}}^{\rm tree}_{n;\overline{\rm MHV}}$

Our first task is to find $\widehat{\mathcal{A}}^{\rm tree}_{n;\overline{\rm MHV}}$ and translate it into analytic superspace. The invariant $\widehat{\mathcal{A}}^{\rm tree}_{n;\overline{\rm MHV}}$ is simply the $\overline{\rm MHV}$ superamplitude divided by the MHV superamplitude. In order to find this we will employ the Nair η variables [22] rather than the χ variables used in the rest of the text. These are related to the analytic ρ variables in Appendix E. We take a digression and present a derivation valid for the n-point case although the explicit check against the correlator will finally only be done for $n = 6$.

The $\overline{\mathrm{MHV}}$ superamplitude can be written as

$$\mathcal{A}^{\text{tree}}_{n;\overline{\mathrm{MHV}}} = \delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\ \delta^{(8)}\left(\sum_i \tilde{\lambda}_i \frac{\partial}{\partial \eta_i}\right) \prod_j \frac{\eta_j^4}{[j j+1]}\ . \tag{3.22}$$

This form can be found by considering the standard form of the anti-MHV superamplitude in terms of Fourier transformed Nair coordinates $\tilde{\eta}$ and performing the explicit Fourier transform back to η's. We adopt the usual notation

$$\langle ij \rangle\ =\ \lambda_i^\alpha \lambda_{j\,\alpha}\,, \qquad [ij]\ =\ \tilde{\lambda}_i^{\dot\alpha} \tilde{\lambda}_{j\,\dot\alpha}\,, \qquad x_{i\,i+1}^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha} \tag{3.23}$$

for products of the twistor variables parametrising the light-like distances $x_{i\,i+1}$. In the two-component contractions we do not introduce a weight factor of 1/2, but we choose the normalisation (see Eq. (F.3) below)

$$\eta^4\ =\ \frac{1}{4!}\,\epsilon_{ABCD}\,\eta^A\,\eta^B\,\eta^C\,\eta^D\ =\ \frac{1}{4}\eta^2\,\eta'^2 \tag{3.24}$$

with $\eta^2 = \epsilon_{ab}\eta^a\eta^b$ and similar for η'^2.

We wish to consider (3.22) in the gauge (on analytic superspace) $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$. Using (E.5) this translates into

$$\eta_2' = \eta_2 = 0 \qquad \eta_1 + \eta_1'\,y_1 = 0 \qquad \eta_3 + \eta_3'\,y_4 = 0\ . \tag{3.25}$$

In this gauge we find that the $\overline{\mathrm{MHV}}$ superamplitude becomes simply

$$\mathcal{A}^{\text{tree}}_{n;\overline{\mathrm{MHV}}} = \frac{1}{4}\,\frac{[23]^2[12]^2 y_{14}^2 (\eta_1')^2(\eta_3')^2\eta_4^4\ldots\eta_n^4}{[12][23]\ldots[n1]}\ = \frac{\delta^{(8)}(\sum_i \lambda_i\,\eta_i)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \times \widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}}\ , \tag{3.26}$$

and we have used the definition of $\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}}$ in this gauge as the aforementioned ratio. In the above we have split the four-component η^A into two two-component η and η', as previously. We can use the delta function to eliminate two more η^A from $\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}}$. We choose to eliminate η_{n-1} and η_n, after which writing $\delta^{(8)}(\sum_i \lambda_i\,\eta_i) = \eta_{n-1}^4\,\eta_n^4\,\langle n-1\,n\rangle^4 + \ldots$ yields a unique expression for $\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}}$:

$$\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}} = \frac{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}{[12][23]\ldots[n1]} \times \frac{[12]^2[23]^2}{\langle n-1\,n\rangle^4} \times y_{14}^2 \times (\eta_1')^2(\eta_3')^2\eta_4^4\ldots\eta_{n-2}^4 \tag{3.27}$$

Now that we have $\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\mathrm{MHV}}}$ in terms of η, η', we just need to re-express it in terms of the analytic ρ variables putting in the expression (E.5) for $\eta(\rho), \eta'(\rho)$. We can start with

η'_3 and work upwards as follows:

$$\eta'_3 = \frac{\langle 4|\rho_5\, y_{45}^{-1}}{\langle 34\rangle} \tag{3.28}$$

$$\eta_4 + \eta'_4 y_5 = -\frac{\langle 3|\rho_5}{\langle 34\rangle} \tag{3.29}$$

$$\eta'_4 = \frac{\langle 5|\rho_6\, y_{56}^{-1}}{\langle 54\rangle} + O(\rho_5) \tag{3.30}$$

$$\eta_5 + \eta'_5 y_6 = -\frac{\langle 4|\rho_6}{\langle 45\rangle} + O(\rho_5) \tag{3.31}$$

$$\eta'_5 = \frac{\langle 6|\rho_7\, y_{67}^{-1}}{\langle 65\rangle} + O(\rho_5, \rho_6) \tag{3.32}$$

$$\dots \tag{3.33}$$

where the $O(\rho_5, \rho_6)$ terms indicate terms proportional to ρ_5 or ρ_6. Such terms in $\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\text{MHV}}}$ can be ignored: For example, all possible occurrences of ρ_5 are saturated by $(\eta'_3)^2(\eta_4 + \eta'_4 y_4)^2$. In the end then we can safely substitute the following:

$$\eta'_{j-1} = \frac{\langle j|\rho_{j+1}\, y_{j\,j+1}^{-1}}{\langle j-1\, j\rangle} \qquad j = 4\dots n-2 \tag{3.34}$$

$$\eta_j + \eta'_j y_{j+1} = -\frac{\langle j-1|\rho_{j+1}}{\langle j-1\, j\rangle} \qquad j = 4\dots n-2 \tag{3.35}$$

$$\eta'_{n-2} = \frac{\langle n-1|\rho_n\, y_{n-1\,n}^{-1}}{\langle n-2\, n-1\rangle} \tag{3.36}$$

$$\eta'_1 = \frac{\langle n|\rho_n\, y_{n1}^{-1}}{\langle n1\rangle} \tag{3.37}$$

which, using that

$$\eta_j^4 = \frac{1}{4}\,{\eta'_j}^2\eta_j^2 = \frac{1}{4}\,{\eta'_j}^2(\eta_j + \eta'_j y_{j+1})^2 \tag{3.38}$$

and[8]

$$\frac{1}{4}\,(\eta'_{j-1})^2(\eta_j + \eta'_j y_{j+1})^2 = \frac{\rho_{j+1}^4}{\langle j\, j-1\rangle^2 y_{j\,j+1}^2} \qquad j = 4\dots n-2 \tag{3.39}$$

gives us

$$(\eta'_1)^2(\eta'_3)^2\eta_4^4\dots\eta_{n-2}^4 = \frac{\rho_5^4\dots\rho_n^4\langle n-1n\rangle^4}{\langle 34\rangle^2\langle 45\rangle^2\dots\langle n1\rangle^2\, y_{45}^2\, y_{56}^2\dots y_{1n}^2}\ . \tag{3.40}$$

Inserting this into (3.27) finally yields the n-point $\overline{\text{MHV}}$ invariant in analytic superspace

$$\widehat{\mathcal{A}}^{\text{tree}}_{n;\overline{\text{MHV}}} = \frac{x_{13}^4 x_{24}^4}{x_{13}^2 x_{24}^2\dots x_{n2}^2} \times \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{y_{12}^2 y_{23}^2\dots y_{1n}^2} \times \rho_5^4\dots\rho_n^4\ , \tag{3.41}$$

where we have simply relied on $\langle i\, i+1\rangle[i\, i+1] = x_{i\,i+2}^2$.

[8]Once again this equation is valid only in the chain of substitutions.

3.5.3 Evaluating $(\widehat{\mathcal{A}}^{\textbf{tree}}_{6;\text{NMHV}})^2$

We now wish to compute the other contributions according to the duality conjecture at this level (3.18). For simplicity we concentrate on the six point case, and we wish to find the contribution of $(\widehat{\mathcal{A}}^{\text{tree}}_{6;\text{NMHV}})^2$ to the correlation function according to the duality conjecture (3.19). At six points we have [23]

$$\widehat{\mathcal{A}}^{\text{tree}}_{6;\text{NMHV}} = R_{61234} + R_{61245} + R_{62345} = R_5 + R_3 + R_1 \ , \tag{3.42}$$

where in the third expression we are defining (for six points only) R_i to be the invariant R_{jklmn} which does not contain the index i. Therefore

$$(\widehat{\mathcal{A}}^{\text{tree}}_{6;\text{NMHV}})^2 = (R_5 + R_3 + R_1)^2 = 2R_1R_3 + 2R_1R_5 + 2R_3R_5 \ . \tag{3.43}$$

In this section – like in the rest of the article – we employ momentum supertwistor variables. One of these is the χ_i parameter that we have frequently mentioned. The second variable is a projective four-vector $Z_i = (\lambda_i, \mu_i)$ with

$$\mu_{i\,\dot\alpha} \ = \ \lambda_i^{\alpha}(x_i)_{\alpha\dot\alpha} \ , \tag{3.44}$$

or conversely

$$(x_i)_{\alpha\dot\alpha} = \frac{\lambda_{i\,\alpha}\,\mu_{i-1\,\dot\alpha} - \lambda_{i-1\,\alpha}\,\mu_{i\,\dot\alpha}}{\langle i-1, i\rangle} \tag{3.45}$$

An n-point amplitude can be parametrised by a set $\{Z_1, \ldots, Z_n\}$ with the association [10]

$$x_i \leftrightarrow (Z_i, Z_{i+1})\,. \tag{3.46}$$

The on-shell constraints $x^2_{i,i+1} = 0$ are solved by construction, which can be seen for example from the defining relation for the four-bracket of twistors:

$$\langle i\, j\, k\, l\rangle \ = \ \text{Det}\,(Z_i\, Z_j\, Z_k\, Z_l) \ = \ \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \tag{3.47}$$

If the twistors pertain to two points x_i, x_j this becomes

$$\langle i-1\, i\; j-1\, j\rangle \ = \ \langle i-1\, i\rangle\, \langle j-1\, j\rangle\, x^2_{ij} \tag{3.48}$$

so that $x^2_{i,i+1} = 0$ due to the doubling of Z_i in the determinant on the left hand side.

Each R_i has the form $R_i = c_i(\Sigma_i)^4$ where Σ is defined in (F.2) in Appendix F and c is the bosonic factor of $(\Sigma)^4$ from (F.1). Hence, we find that the nilpotent pieces we need to compute arise from terms like $(\Sigma_1)^4(\Sigma_3)^4$, $(\Sigma_1)^4(\Sigma_5)^4$ and $(\Sigma_3)^4(\Sigma_5)^4$. As before, we want to use the gauge

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0 \ . \tag{3.49}$$

The identification between the χ and the ρ parameters is

$$\chi_i + \chi_i'\, y_i = \langle i|\rho_i\,, \qquad \chi_i + \chi_i'\, y_{i+1} = \langle i|\rho_{i+1}\ , \tag{3.50}$$

or inversely (cf. (2.12))

$$\chi_i\ = \langle i|(\rho_i - \rho_{i\,i+1}\, y_{i\,i+1}^{-1}\, y_i)\,, \qquad \chi_i'\ = \langle i|\rho_{i\,i+1}\, y_{i\,i+1}^{-1}\ .$$

In terms of supertwistor variables the gauge (3.49) reads

$$\chi_1^A = \chi_2^A = \chi_3^A = 0 \qquad \chi_4 + \chi_4'\, y_4 = 0 \qquad \chi_6 + \chi_6'\, y_1 = 0\ . \tag{3.51}$$

This gives us (all Σ_i^A in the next formula carry a four-component index)

$$\begin{aligned}
\Sigma_1 &= \chi_4\, \langle 5623\rangle + \chi_5\, \langle 6234\rangle + \chi_6\, \langle 2345\rangle \\
\Sigma_3 &= \chi_4\, \langle 5612\rangle + \chi_5\, \langle 6124\rangle + \chi_6\, \langle 1245\rangle \\
\Sigma_5 &= \chi_4\, \langle 6123\rangle + \chi_6\, \langle 1234\rangle \\
\tilde\Sigma_3 := \Sigma_3 - \frac{\langle 6124\rangle}{\langle 6234\rangle}\,\Sigma_1 &= \Big(\chi_4\, \langle 6123\rangle\langle 6245\rangle + \chi_6\, \langle 2456\rangle\langle 4123\rangle\Big)\langle 6234\rangle^{-1}
\end{aligned} \tag{3.52}$$

where in the last equation we define a linear combination, $\tilde\Sigma_3$ of Σ_3 and Σ_1 which is independent of χ_5^A and we have simplified the expression using the twistor identity

$$\langle abij\rangle\langle abkl\rangle + \langle abik\rangle\langle ablj\rangle + \langle abil\rangle\langle abjk\rangle = 0\ . \tag{3.53}$$

In our gauge, due to the relations (3.51) χ_4^A and χ_6^A have only two independent components each, say, χ_4' and χ_6', therefore $(\Sigma_5)^4$ saturates all the χ_4^A and χ_6^A variables giving (c.f. the derivation of (F.7) in Appendix F)

$$(\Sigma_5)^4 = \frac{1}{4}\,\langle 6123\rangle^2\langle 1234\rangle^2(\chi_4')^2(\chi_6')^2\, y_{14}^2\ . \tag{3.54}$$

So one need only consider χ_5 terms when multiplied by Σ_5. One then quickly finds

$$\begin{aligned}
(\Sigma_1)^4(\Sigma_5)^4 &= \frac{1}{4}\,\langle 6123\rangle^2\langle 1234\rangle^2\langle 6234\rangle^4(\chi_4')^2(\chi_6')^2\chi_5^4\, y_{14}^2\,, \\
(\Sigma_3)^4(\Sigma_5)^4 &= \frac{1}{4}\,\langle 6123\rangle^2\langle 1234\rangle^2\langle 6124\rangle^4(\chi_4')^2(\chi_6')^2\chi_5^4\, y_{14}^2\,, \\
(\Sigma_1)^4(\Sigma_3)^4 =\ (\Sigma_1)^4(\tilde\Sigma_3)^4 &= \frac{1}{4}\,\langle 6123\rangle^2\langle 1234\rangle^2\langle 2456\rangle^4(\chi_4')^2(\chi_6')^2\chi_5^4\, y_{14}^2\,.
\end{aligned} \tag{3.55}$$

Then we input the bosonic factors c_i from (F.1) and also rewrite χ in terms of ρ in our gauge, using (3.50) to obtain

$$\frac{1}{4}(\chi_4')^2(\chi_6')^2\chi_5^4\, y_{14}^2 =\ \langle 45\rangle^2\langle 56\rangle^2 \times \frac{y_{14}^2}{y_{16}^2 y_{45}^2 y_{56}^2} \times \rho_5^4\, \rho_6^4\ . \tag{3.56}$$

Further, we use (3.48) to rewrite some of the twistor factors in terms of x space variables and we find

$$
\begin{aligned}
R_1R_3 =&\ \frac{x_{13}^2x_{24}^2}{x_{15}^2x_{35}^2x_{26}^2x_{46}^2}\times\frac{\langle 4562\rangle^2\langle 6123\rangle\langle 1234\rangle}{\langle 6124\rangle\langle 2346\rangle\langle 5623\rangle\langle 4512\rangle}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\\
R_1R_5 =&\ \frac{x_{13}^2x_{24}^2}{x_{15}^2x_{35}^2x_{26}^2x_{46}^2}\times\frac{\langle 2346\rangle^2\langle 4561\rangle\langle 5612\rangle}{\langle 4562\rangle\langle 6124\rangle\langle 3461\rangle\langle 2356\rangle}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\\
R_3R_5 =&\ \frac{x_{13}^2x_{24}^2}{x_{15}^2x_{35}^2x_{26}^2x_{46}^2}\times\frac{\langle 6124\rangle^2\langle 2345\rangle\langle 3456\rangle}{\langle 2346\rangle\langle 4562\rangle\langle 1245\rangle\langle 6134\rangle}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\ .
\end{aligned}
\tag{3.57}
$$

The sum of these three terms can be simplified by rewriting the momentum twistor conformal invariants in terms of six complex variables z_i by using the replacement [38]

$$\langle ijkl\rangle = z_{ij}z_{ik}z_{il}z_{jk}z_{jl}z_{kl} \tag{3.58}$$

where $z_{mn} = z_m - z_n$. [9] The advantage of these variables is that identities such as (3.53) become manifest, in this case $z_{wx}z_{tv} + z_{xv}z_{tw} + z_{vw}z_{tx} = 0$. In this way we get

$$
\begin{aligned}
R_1R_3 =&\ \frac{x_{13}^2x_{24}^2}{x_{35}^2x_{46}^2x_{15}^2x_{26}^2}\times\frac{z_{13}^2z_{54}z_{56}}{z_{35}z_{51}z_{14}z_{36}}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\\
R_1R_5 =&\ \frac{x_{13}^2x_{24}^2}{x_{35}^2x_{46}^2x_{15}^2x_{26}^2}\times\frac{z_{51}^2z_{32}z_{34}}{z_{13}z_{35}z_{52}z_{14}}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\\
R_3R_5 =&\ \frac{x_{13}^2x_{24}^2}{x_{35}^2x_{46}^2x_{15}^2x_{26}^2}\times\frac{z_{35}^2z_{16}z_{12}}{z_{51}z_{13}z_{36}z_{52}}\times\frac{y_{14}^2}{y_{16}^2y_{45}^2y_{56}^2}\times\rho_5^4\rho_6^4\ .
\end{aligned}
\tag{3.59}
$$

Finally we are interested in the sum of these terms. It turns out that although each term individually is complicated (at least when expressed in x space) the sum of terms has a very simple form. We have that

$$
\begin{aligned}
\frac{z_{51}^2z_{32}z_{34}}{z_{13}z_{35}z_{52}z_{14}}+\frac{z_{35}^2z_{16}z_{12}}{z_{51}z_{13}z_{36}z_{52}}+\frac{z_{13}^2z_{54}z_{56}}{z_{35}z_{51}z_{14}z_{36}} &\equiv \frac{z_{12}z_{45}}{z_{14}z_{25}}+\frac{z_{16}z_{34}}{z_{14}z_{36}}+\frac{z_{23}z_{56}}{z_{25}z_{36}}\\
&= \frac{x_{13}^2x_{46}^2}{x_{36}^2x_{14}^2}+\frac{x_{35}^2x_{26}^2}{x_{36}^2x_{25}^2}+\frac{x_{15}^2x_{24}^2}{x_{14}^2x_{25}^2}
\end{aligned}
\tag{3.60}
$$

where the first line is an algebraic identity, and in the second line we have replaced the z (via the momentum twistors) back with the x's. Remarkably all parity-odd pieces (which appear in the R invariants themselves) completely cancel in this expression.

Putting this result (3.59), (3.60) first into (3.43) and then together with the expression (3.41) for $\widehat{\mathcal{A}}^{\rm tree}_{6;\overline{\rm MHV}}$, we obtain the right-hand side of the duality relation (3.19):

$$
\begin{aligned}
2\widehat{\mathcal{A}}^{\rm tree}_{6;\overline{\rm MHV}} + (\widehat{\mathcal{A}}^{\rm tree}_{6;\rm NMHV})^2 =&\ 2\,\frac{y_{41}^2}{y_{45}^2y_{56}^2y_{61}^2}\ \rho_5^4\,\rho_6^4\\
&\times\frac{x_{13}^2x_{24}^2}{x_{14}^2}\Big(\frac{x_{14}^2}{x_{15}^2x_{35}^2x_{26}^2x_{46}^2}+\Big[\frac{x_{13}^2}{x_{15}^2x_{35}^2x_{26}^2x_{36}^2}+\frac{x_{14}^2}{x_{15}^2x_{25}^2x_{36}^2x_{46}^2}+\frac{x_{24}^2}{x_{25}^2x_{35}^2x_{26}^2x_{46}^2}\Big]\Big)\ .
\end{aligned}
\tag{3.61}
$$

Remarkably this is in perfect agreement with the correlator prediction (3.21).

[9] In fact it is practically much more straightforward to make the replacement $\langle ijkl\rangle \to \epsilon_{ijklmn}z_{mn}$ which appears to give the same result for conformally invariant objects.

4 The six-point one-loop NMHV amplitude

In the previous section we have illustrated how the off-shell calculation [18] of the tree-level correlator $G^{(0)}_{6;2} = \langle \mathcal{O}(1)\,\mathcal{O}(2)\,\mathcal{O}(3)\,\mathcal{O}(4)\,\mathcal{L}(5)\,\mathcal{L}(6)\rangle^{(0)}$ at $\rho_i = 0$, $i \in \{1, \ldots, 4\}$ can yield three different amplitude integrands:

- *MHV four-point two-loop amplitude*
  in the square light-cone limit $x^2_{i\,i+1} \to 0$, $1 \le i \le 4$, under the double integral $\int d^4x_5 d^4\rho_5\, d^4x_6 d^4\rho_6$.
- *NMHV five-point one-loop amplitude*
  in the pentagon light-cone limit $x^2_{i\,i+1} \to 0$, $1 \le i \le 5$ and under $\int d^4x_6 d^4\rho_6$.
- *NNMHV six-point tree amplitude*
  in the hexagon light-cone limit $x^2_{i\,i+1} \to 0$, $1 \le i \le 6$ without any integration.

In [8, 16], the methods of [18] were applied to find $G^{(0)}_{7;2} = \langle \mathcal{O}(1) \ldots \mathcal{O}(5)\,\mathcal{L}(6)\,\mathcal{L}(7)\rangle^{(0)}|_{\rho_6^4\,\rho_7^4}$ and $G^{(0)}_{8;2}|_{\rho_7^4\,\rho_8^4} = \langle \mathcal{O}(1) \ldots \mathcal{O}(6)\,\mathcal{L}(7)\,\mathcal{L}(8)\rangle|_{\rho_7^4\,\rho_8^4}$ in order to demonstrate the duality between these correlation functions put on the light-cone and the MHV two-loop five- and six-point amplitudes. The *integrands* for the MHV amplitudes derived from the correlators turned out to be equal to those predicted from BCFW recursion rules in [9].

The cases studied in the last section already provide very non-trivial evidence for the duality beyond MHV. However, to hopefully remove any further doubt, we here give an example which shows that the duality applies simultaneously beyond both MHV/$\overline{\text{MHV}}$ sectors and beyond the tree-level sector and in particular can correctly relate the full integrands even including a highly non-trivial parity odd piece. To this end we want to obtain the *NMHV six-point one-loop amplitude* from a new hexagon light-cone limit of the correlator $G^{(0)}_{7;2}$. The pentagon light-like limit of this correlator was already found in [8] to be dual to the *MHV five-point two-loop amplitude integrand*.

Unlike the cases considered in previous sections, since there is an additional outer point, superconformal symmetry alone is not sufficiently powerful to reconstruct the full $\mathcal{N} = 4$ correlator/amplitude from a single $\rho_i^4 \rho_j^4$ component. We therefore show how to do this reconstruction starting from certain $\mathcal{N} = 2$ projections with five hypermultiplet bilinears and one Lagrangian component. It is then enough to check for one Grassmann component that the integrand as computed from the correlator is equal to the momentum twistor expression in [21]. In the following subsection we start by building up some technology needed to master the large parity-odd sector of the calculation in a manifestly conformal way.

Obviously, according to our duality we could also construct the *NNMHV seven-point tree level* amplitude from this correlator $G^{(0)}_{7;2}$, although we refrain from doing so because of the volume of that calculation. It is probably more striking to see the duality at work at NMHV one-loop level in a non-trivial case anyhow.

4.1 An x-space toolkit for 6-point one-loop amplitudes

We wish eventually to construct the correlator $G^{(0)}_{7;2}$ and compare with the six point NMHV 1-loop integrand. We therefore have a total of 7 points, the integration point (which we label 0) and the six outer points which will be light-like separated.) We first develop some techniques for writing down conformal invariants of this form in the hexagon light-like limit.

The pseudo-conformal one-loop integrals that we will encounter at six-points in the rest of this section are the pentagons p_i and boxes g_{ij} defined as

$$\begin{aligned} p_1 &= \frac{1}{4\pi^2}\int \frac{d^4x_0\, x_{10}^2}{x_{20}^2x_{30}^2x_{40}^2x_{50}^2x_{60}^2} \qquad\qquad g(1,2,3,4) = \frac{1}{4\pi^2}\int \frac{d^4x_0}{x_{10}^2x_{20}^2x_{30}^2x_{40}^2}, \\ g_{12}^{\rm 1m} &= g(3,4,5,6)\,, \qquad g_{13}^{\rm 2mh} = g(2,4,5,6)\,, \qquad g_{14}^{\rm 2me} = g(2,3,5,6)\,. \end{aligned} \tag{4.1}$$

The labels indicate the factors which are missing from the maximal denominator $x_{10}^2 \ldots x_{60}^2$. Cyclic shifts yield six such integrals in the first three cases, while there are only three two-mass easy boxes. Thus we have a total of 21 integrals.

The hexagon light-cone limit $x_{i\,i+1}^2 \to 0$, $i \in \{1,\ldots,6\}$ permits three finite cross ratios

$$u_1 = \frac{x_{13}^2x_{46}^2}{x_{14}^2x_{36}^2}\,, \qquad u_2 = \frac{x_{15}^2x_{24}^2}{x_{14}^2x_{25}^2}\,, \qquad u_3 = \frac{x_{26}^2x_{35}^2}{x_{25}^2x_{36}^2}\,. \tag{4.2}$$

This definition puts those x_{ij}^2 into the denominator in which the points are at opposite corners of the hexagon. Cyclic shifts along the hexagon therefore permute these u's but do not invert them.

On several occasions we find the following same fixed combinations of the 21 scalar integrals with polynomials of u_1, u_2, u_3 making an appearance. It is therefore convenient to introduce these combinations as a second basis for the 1 loop integrals:

$$\begin{aligned} \tilde{p}_1 &= (1-u_3)\,\frac{x_{24}^2x_{35}^2x_{46}^2}{x_{14}^2}\; p_1\,, \\ \tilde{g}_{12} &= (1-u_1+u_2-u_3)\, x_{35}^2x_{46}^2\; g_{12}^{\rm 1m}\,, \\ \tilde{g}_{13} &= (1-u_1-u_2-u_3+2u_2u_3)\, x_{25}^2x_{46}^2\; g_{13}^{\rm 2mh}\,, \\ \tilde{g}_{14} &= (1-u_3)(1-u_1-u_2-u_3)\, x_{25}^2x_{36}^2\; g_{14}^{\rm 2me}\,, \end{aligned} \tag{4.3}$$

with the 17 others defined in the obvious way by cyclic shifts.

There is an interesting *integrand identity* involving these combinations with coefficients ± 1 only

$$0 = \sum_{i=1}^{6} \left(-\,\tilde{p}_i \,+\, \tilde{g}_{i\,i+1} \,-\, \tilde{g}_{i\,i+2}\right) + \sum_{i=1}^{3} \tilde{g}_{i\,i+3}\,. \tag{4.4}$$

Putting all terms under a common integral over x_0 and factorising produces a numerator polynomial with 87 terms, all composed of seven x_{ij}^2 factors. This is of conformal form;

every term has weight two at all points. Upon substituting rational numbers we may verify that the polynomial identically vanishes in the hexagon light-cone limit, regardless of the choice of x_0. This could in principle be shown by expanding the Lorentz invariant $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ in its components, although in practice this is hardly possible by the sheer size of the problem. It can also be shown in momentum twistors and is presumably related to the Gram determinant.

If a regulator is introduced, the sum of integrals in (4.4) should not receive non-vanishing singular or finite contributions — after all we are integrating over zero. On the other hand, in dimensional regularisation there *can* be non-zero contributions at $O(\epsilon)$ and beyond, simply because the numerator polynomial ceases to vanish outside $D = 4$.

The sorts of objects which arise in perturbative computations of correlation functions are traces over $x_{\alpha\dot{\alpha}}$, and much of this section will be devoted to understanding how to massage such objects into the most useful forms. The basic conformal covariants of trace type are $\mathrm{Tr}(x_{ij}\tilde{x}_{jk}x_{kl}\ldots\tilde{x}_{mi})$. Conformal covariance is guaranteed by the characteristic repetition of points between neighbouring differences. Due to the index contractions we must have an even number of entries in the trace. In such a trace we can always use the manifestly conformal identity

$$x_{13}\tilde{x}_{32}x_{24} \;=\; -x_{12}\tilde{x}_{23}x_{34} - x_{23}^2\, x_{14} \tag{4.5}$$

(here 1,2,3,4 represent any four points) and its conjugate to put any chain of differences into ascending point ordering. Four-traces of conformal type do not have a parity-odd sector so that they immediately reduce to products of squares. For seven points the longest trace without point repetitions has six entries. There are then seven such cases according to which point is omitted.

Hence all traces appearing in the calculation reduce to $\mathrm{Tr}(x_{12}\tilde{x}_{23}x_{34}\tilde{x}_{45}x_{50}\tilde{x}_{01})$ and similar objects. Point 0 will later be an integration variable, so that we would like to take it out of the trace by tensor decomposition in a way that manifestly preserves conformal invariance. To this end we write the ansatz

$$x_{50}\tilde{x}_{01} \;=\; \sum_{i=2,3,4,6} a_i\, y_i\,, \qquad y_i \;=\; x_{5i}\tilde{x}_{i1} \tag{4.6}$$

because the left hand side is a four-component object and the basis elements on the right hand side transform in the same way to the left and to the right as the left hand side does. Multiplying up by the conjugates of the y_i we obtain four equations which are indeed invertible. They express the a_i in terms of x_{ij}^2 because the conformal traces of length four do not contain a parity-odd part. *Mathematica* can easily solve the system. The a_i are found to be polynomials of 27 terms, each composed of six x_{ij}^2 factors, divided by the common denominator $x_{14}^4 x_{25}^4 x_{36}^4\,\Delta$, where $\Delta = (1 - u_1 - u_2 - u_3)^2 - 4\,u_1u_2u_3$.

The numerator polynomials have the appropriate conformal weights. In particular, there is exactly one x_{i0}^2 in every numerator term. The decomposition can now be substituted into the original six-trace thereby expressing it in terms of x_{ij}^2 and the trace 1234561. Since the common denominator is cyclically invariant it is not hard to derive the decomposition of the other traces involving point 0 from this case.

Finally, the parity-odd part of the trace 1234561 is related to the symbol $\sqrt{\Delta}$ (which appears for example in the formulae (F.14) for the ρ_6^4 component of the R invariants) simply as[10]

$$\frac{2}{x_{14}^2 x_{25}^2 x_{36}^2}\,\mathrm{Tr}(x_{12}\tilde{x}_{23}x_{34}\tilde{x}_{45}x_{56}\tilde{x}_{61}) - (1-u_1-u_2-u_3) \;=\; \sqrt{\Delta}\,. \tag{4.7}$$

Since the trace 1234561 is essentially unique this is common to all the R invariants. The sign of the square root in the right-hand side of (4.7) is in fact always positive in the last formula unless Δ is real and negative. Nevertheless, we should keep in mind that the sign of the parity-odd part of the trace is reversed under the exchange of x for $\tilde{x}$ implying that $\sqrt{\Delta}$ is anti-cyclic under shifts.

Finally the six trace can be written in terms of Lorentz objects as

$$\begin{aligned}\sqrt{\Delta} \;&=\; -\frac{2}{x_{14}^2 x_{25}^2 x_{36}^2}\Big(x_{26}^2\,\epsilon(x_{16},x_{36},x_{46},x_{56}) - x_{36}^2\,\epsilon(x_{16},x_{26},x_{46},x_{56}) + \\ &\qquad x_{46}^2\,\epsilon(x_{16},x_{26},x_{36},x_{56})\Big)\,.\end{aligned} \tag{4.8}$$

4.2 MHV$_6^{(1)}$ revisited

In [8] an x space form of the MHV n-point 1-loop amplitudes was derived by evaluating correlators in terms of $\mathcal{N}=2$ superfields. These expressions naturally contained a sum over non-conformal four-traces like $\mathrm{Tr}(x_{10}\tilde{x}_{30}x_{40}\tilde{x}_{50})$. So as an application of the techniques outlined in the previous section, let us first try to recast the integrand of the MHV 6-point 1-loop amplitude (here taken from the correlator calculation in [8] in the hexagonal light-like limit - the one-loop correlator and its integrand are $G_{6;0}^{(1)} = \int G_{7;1}^{(0)}$) into a convenient form using the aforementioned ideas about tensor decomposition. We first try to write the MHV integrand in terms of one-loop integrals of the form

$$\begin{aligned}\widehat{A}_{6;0}^{(1)}(x_0;x_1,\ldots,x_6) \;&=\; a_1\,\frac{x_{10}^2 x_{20}^2 x_{35}^2 x_{46}^2}{x_{10}^2\ldots x_{60}^2} + \ldots \\ &\quad+\; b_1\frac{x_{10}^2\,\mathrm{Tr}(x_{23}\tilde{x}_{34}x_{45}\tilde{x}_{56}x_{60}\tilde{x}_{02})}{x_{10}^2\ldots x_{60}^2} + \ldots\end{aligned} \tag{4.9}$$

where the dots indicate the 17 other possible terms with a numerator composed of four x_{ij}^2 (corresponding to 6 one-mass boxes, 6 two-mass hard boxes and 6 two-mass easy boxes - since the latter can come multiplied by two different external factors each) and the 5 other trace terms obtained by cyclic permutations of the outer points 123456. The existence of a solution follows from the analysis in [16]. The number coefficients a_i, b_i can be found numerically. The solution is unique up to one free parameter, which we fix by imposing

[10] The term $1-u_1-u_2-u_3$ removes the parity-even part of the trace.

manifest cyclicity. The result is then

$$\widehat{A}^{(1)}_{6;0}(x_0;x_1,\ldots,x_6) \;=\; 2\,x_{35}^2x_{46}^2\; g_{12}^{\rm 1m} - x_{25}^2x_{46}^2\; g_{13}^{\rm 2mh} + (1-u_3)\,x_{25}^2x_{36}^2\; g_{14}^{\rm 2me} \\ -\;\int \frac{d^4x_0\;{\rm Tr}(x_{23}\tilde{x}_{34}x_{45}\tilde{x}_{56}x_{60}\tilde{x}_{02})}{x_{20}^2\ldots x_{60}^2} + ({\rm cyclic})\,. \tag{4.10}$$

(The terms with the two-mass easy box double in the cyclic sum.) Now we can use the tensor decomposition results as outlined around equation (4.6) to rewrite this in terms of the 1234561 trace (which in turn we write in terms of $\sqrt{\Delta}$ using (4.7)) and scalar integrals:

$$\widehat{A}^{(1)}_{6;0}(x_0;x_1,\ldots,x_6) \;=\; x_{35}^2x_{46}^2\; g_{12}^{\rm 1m} + \frac{1}{2}(1-u_3)\,x_{25}^2x_{36}^2\; g_{14}^{\rm 2me} + ({\rm cyclic}) \\ +\;\frac{1}{\sqrt{\Delta}}\;\sum_{i=1}^{6}(-1)^i\;(\tilde{p}_i+\tilde{g}_{i\,i+2}) \tag{4.11}$$

(Note that cycling doubles the two-mass easy terms). The alternating sign $(-1)^i$ in the sum of integrals in the last line compensates the anti-cyclicity of $\sqrt{\Delta}$ (see above) to yield a cylically invariant result.

We would like to point the reader's attention to the fact that the parity-odd part curiously contains the rescaled integrals of the $\tilde{p},\tilde{g}$ basis, all with coefficients $0,\pm1$.

4.3 NMHV$_6^{(0)}$

The next amplitude at six-points we wish to obtain from correlation functions – as a warm up to the case of interest – is the 6-point NMHV tree-level amplitude which can be found from the correlator, $G^{(0)}_{6;1}$ in the hexagonal limit. This is the simplest non-trivial example of the tree-level NMHV correlators considered in the companion paper []. Here we will only consider the component with ρ_6 turned on ie $G^{(0)}_{6;1}|_{\rho_6^4}$ and later will reconstruct the entire result from this.

As usual we compute the $\mathcal{N}$=4 correlator perturbatively using relevant $\mathcal{N}=2$ correlators and uplifting to $\mathcal{N}$=4. Here the $\mathcal{N}=2$ correlator we need is

$$\lim_{x_{i,i+1}^2\to 0}\langle O_1\,\tilde{O}_2\,O_3\,\tilde{O}_4\,\widehat{O}_5\,\mathcal{L}_6^{\mathcal{N}=2}\rangle^{(0)} \tag{4.12}$$

with

$$O\;=\;{\rm Tr}(q^2)\,,\quad \tilde{O}\;=\;{\rm Tr}(\tilde{q}^2)\,,\quad \widehat{O}\;=\;2\,{\rm Tr}(q\,\tilde{q})\,,\quad \mathcal{L}_{\mathcal{N}=2}\;=\;-\frac{1}{4\,g^2}\,{\rm Tr}(\widehat{W}^2_{\mathcal{N}=2}) \tag{4.13}$$

where q is the hypermultiplet and $\widehat{W}$ the field strength multiplet. See Appendix B for some facts about $\mathcal{N}=2$ superfields. Note that the most convenient form of the $\mathcal{N}=2$ action (as given in the appendix) uses a field rescaling of the Yang-Mills prepotential V as opposed to standard conventions. At the linearised level this amounts to $\widehat{W}_{\mathcal{N}=2,{\rm lin}}=g\,W_{\mathcal{N}=2,{\rm lin}}$.

The correlator calculation proceeds in almost the same way as in the case of the MHV n-point one-loop amplitudes described in [8]. For the sake of brevity we only point out some differences between the discussion given there and the new case considered here.

Both the MHV five-point one-loop amplitude and the NMHV six-point tree level can be found from this $\mathcal{N} = 2$ correlator. In order to obtain the MHV amplitude we put the positions of the hypermultiplet bilinears at the vertices of a pentagon with light-like edges. The position of the Lagrangian operator is then the integration variable of the one-loop MHV integrand.

In order to obtain the six-point NMHV tree-level amplitude we rather place all six operators at the vertices of a hexagon. Due to the different light-cone limit the range of relevant diagrams becomes slightly smaller and the cancellation of harmonics with negative charge follows a different pattern; for instance, the limit selects exactly one "TT-block" (c.f. [8]). In close parallel to the MHV cases, the light-cone limit is blind to the actual hypermultiplet projections (i.e. the positioning of $O, \tilde{O}, \widehat{O}$) at points 1...5, up to a constant of proportionality. The parity-odd terms sum into $\sqrt{\Delta}$ via formula (4.8) and the result, after lifting to $\mathcal{N}$=4 using the techniques in appendix D is simply

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{6;1}^{(0)}}{G_{6;0}^{\text{tree}}}\Bigg|_{\rho_6^4} = 2\,\Big(R_1 + R_3 + R_5\Big)\Big|_{\rho_6^4} \quad . \tag{4.14}$$

As always at 6-points, we use the symbol $R_1 = R_{23456}$ (and cyclic) to simplify the notation. It is crucial to note that since we are only considering the ρ_6^4 component here, we can not immediately reconstruct the full correlator using superconformal symmetry (which allows us to freely set 4 ρ's to zero – a fact we have used extensively throughout section 3 – but no more than 4). Nevertheless we will find that the formula (4.14) does in fact lift up to the full $\mathcal{N} = 4$ correlator as will be explained in Section 4.5 below. The result is then in full agreement with our duality conjecture because $R_1 + R_3 + R_5 = \widehat{A}_{6;\text{NMHV}}^{\text{tree}}(x_1, \ldots, x_6)$.

4.4 $\mathbf{NMHV}_6^{(1)}$

Let us finally turn to the main aim of this section, the one-loop correction to the 6-point correlator $G_{6;1}^{(1)}$ at order ρ^4 in the light-like limit, which should give the 6-point NMHV 1 loop amplitude according to our duality, and which we can obtain as an integral of the 7-point tree-level correlator $G_{7;2}^{(0)}$ over the position of the additional Lagrangian at point 0. Again we will first concentrate on a particular component by switching off $\rho_1, \ldots, \rho_5$ and later explain how to obtain the full result from this. So we wish to compute

$$\lim_{x_{i,i+1}^2 \to 0} G_{6;1}^{(1)}(1, \ldots, 6)|_{\rho_6^4} = \int d\mu_0 \lim_{x_{i,i+1}^2 \to 0} G_{7;2}^{(0)}(0; 1, \ldots, 6)|_{\rho_0^4 \rho_6^4} \, . \tag{4.15}$$

Again we can't do the perturbative calculation in $\mathcal{N}$=4 directly, so instead we compute the relevant $\mathcal{N}$=2 correlator and reconstruct $G_{7;2}^{(0)}$. Here the $\mathcal{N} = 2$ correlator is

$$\langle \mathcal{L}_0^{\mathcal{N}=2}\, O_1\, \tilde{O}_2\, O_3\, \tilde{O}_4\, \widehat{O}_5\, \mathcal{L}_6^{\mathcal{N}=2} \rangle^{(0)} \ . \tag{4.16}$$

As before, the five hypermultiplet bilinears and the Lagrangian at point 6 are at the vertices of the hexagon 1234561 with light-like edges. As in previous cases, this calculation is simply a different light-cone limit of a previously studied correlator, $G^{(0)}_{7;2}(x_0, x_{0'}; x_1, \dots, x_5)$ which was used in [8] to study the five-point two-loop MHV amplitude integrand.

This time however by considering this new limit one selects a somewhat different set of graphs. Harmonic analyticity still implies the absence of the "TT blocks", and we can avoid the systematic use of the cyclic identity for harmonics by the same identification tricks as in [8]. Since x_{60} is not on the light-cone, the superconformal reconstruction technique first developed in [18] does not interfere with the light-like limit. The technique directly yields conformally covariant traces of the type $x_{16}\tilde{x}_{62}x_{20}\tilde{x}_{03}\dots$ where the points 6 and 0 alternate so that the complete traces must have length 4,8,12 etc. Putting the points into ascending order (with point 0 beyond point 6) by the manipulation (4.5), we end up with x^2_{ij} and the six-traces discussed in Section 4.1.

Taking the light-like limit for the x^2_{ij} terms is straightforward, but the trace terms are more problematic. In general off the light cone, there are six independent harmonic "channels". But similarly to what we have seen in previous cases on the light cone most of these channels are subleading, and only the pentagon one $y^2_{12}\dots y^2_{61}$ (or in this case the $\mathcal{N}$=2 analogue remains.) This vanishing of all but the pentagon one becomes manifest upon tensor reducing the traces involving x_0. In this channel we can split the remaining six-trace into its even and odd parts, whereby the even part reduces to the integrals p, g times some coefficients similar to those defining the $\tilde{p}, \tilde{g}$ basis, while the parity-odd part still has the terms $x^2_{10}\dots x^2_{50}(x^2_{60})^3$ in the common denominator and thus seems to diverge. It is possible, though, to take the product of the vanishing polynomial (4.4) with some smaller expression out of the numerator in such a way that x^4_{60} factors out. The odd-part then reduces to the usual 21 scalar integrals, too.

At this point we get the one-loop correlator in a much nicer form and which looks like it should simplify further, but we have not yet obtained a concise form for it.

The duality conjecture (2.11) predicts a relation between the correlator and the square of the amplitude. Expanding out the square at this order we thus expect a term of the form (NMHV tree)(MHV one-loop) in the duality relation. Notice that here both factors have a parity-even and a parity-odd part. We expect the correlator to be related to the NMHV six-point one-loop amplitude integrand. We know that any amplitude can be written (at the level of the *integral* in the four-dimensional limit) as a combination of boxes times coefficients [39]. In this context the amplitude is given as [23]

$$\widehat{\mathcal{A}}^{(1)}_{6;1}(x_1, \dots, x_6) \;=\; (R_1 + R_4)\, x^2_{35}x^2_{46}\; g^{1\mathrm{m}}_{12} + (R_3 + R_6)\, x^2_{25}x^2_{46}\; g^{2\mathrm{mh}}_{13} + (\text{cyclic})\,. \qquad (4.17)$$

Here it is important that in this equation both sides are the *integrals* not the integrands as we shall see. So we now have expressions for both the correlator and the amplitude in terms of box and pentagon integrals p, g. Mellin-Barnes representations for the p, g integrals are straightforward to derive, so with the help of the *MB.m* package [40] we have been able to check to satisfactory precision (exact for the singularities, about 0.0(2) for the finite part) that as integrals (where we regularise via standard dimensional regularisation for the amplitude and the analogous regularisation for the correlator and where we have lifted to $\mathcal{N}=4$) we have

$$\begin{aligned} &\frac{G^{(1)}_{6;1}}{G^{(0)}_{6;0}}(x_1,\dots,x_6)|_{\rho_6^4} \\ &= 2\left[(R_1+R_3)|_{\rho_6^4}\,\widehat{\mathcal{A}}^{(1)}_{6;0}(x_1,\dots,x_6)+\widehat{\mathcal{A}}^{(1)}_{6;1}(x_1,\dots,x_6)|_{\rho_6^4}\right]\,. \end{aligned} \tag{4.18}$$

just as predicted by the duality (the first term arises from expanding out the square of the amplitude to this order).

Now comes the important question of whether this duality can be promoted to an *integrand* identity. The integrand for the correlator as we define it is simply the correlator $G^{(0)}_{7;2}$, via the insertion formula (2.16) $G^{(1)}_{6;1}=\int d\mu G^{(0)}_{7;2}$. We do not expect the naive NMHV integrand, $\widehat{R}^{(1)}_{6;1}$, obtained from (4.17) by simply removing the integrations from the boxes to lead to an integrand identity. Instead we have at the integrand level:

$$\begin{aligned} &\int d^4\rho_0\frac{G^{(0)}_{7;2}(0;1,\dots,6)|_{\rho_0^4\rho_6^4}}{G^{(0)}_{6;0}(1,\dots,6)} = \\ &2\left[(R_1+R_3)|_{\rho_6^4}\,\widehat{A}^{(1)}_{6;0}(x_0;x_1,\dots,x_6)\;+\;\widehat{R}^{(1)}_{6;1}(x_0;x_1,\dots,x_6)|_{\rho_6^4}\;+\;r|_{\rho_6^4}\right] \end{aligned} \tag{4.19}$$

where r must vanish upon integration and $\widehat{A}^{(1)}_{6;1}=\widehat{R}^{(1)}_{6;1}+r$ becomes the prediction of our duality for the true $\mathrm{NMHV}^{(1)}_6$ integrand. Note that r has both a parity even and a parity odd part. Suppose we write $\sqrt{\Delta}=\Delta/\sqrt{\Delta}$. The parity-odd part of the last formula then becomes a linear equation which we can easily solve for the parity-odd part of the remainder r. The solution takes a simple form if the integrand identity (4.4) between the $\tilde{p},\tilde{g}$ integrals is used to eliminate $\tilde{p}_6$ from the one-loop MHV amplitude (4.11):

$$\begin{aligned} r_{\rm odd}|_{\rho_6^4} \;=\;& \frac{1}{\sqrt{\Delta}}\,R_{2\,\rm even}|_{\rho_6^4}\Big(2\;(\tilde{p}_2+\tilde{p}_5)-(\tilde{g}_{12}+\tilde{g}_{23}+\tilde{g}_{45}+\tilde{g}_{56}) \\ &\qquad\qquad +(\tilde{g}_{24}+\tilde{g}_{35}+\tilde{g}_{51}+\tilde{g}_{62})-2\;\tilde{g}_{25}\Big) \\ +\;& \frac{1}{\sqrt{\Delta}}\,R_{3\,\rm even}|_{\rho_6^4}\Big(2\;(\tilde{p}_1+\tilde{p}_4)-(\tilde{g}_{12}+\tilde{g}_{34}+\tilde{g}_{45}+\tilde{g}_{61}) \\ &\qquad\qquad +(\tilde{g}_{13}+\tilde{g}_{24}+\tilde{g}_{46}+\tilde{g}_{51})-2\;\tilde{g}_{14}\Big)\,. \end{aligned} \tag{4.20}$$

Here $R_{2,3\ \rm even}|_{\rho_6^4}$ refers to the expressions (F.14) with $\sqrt{\Delta}$ put to zero. Miraculously, if we upgrade the even part of $R_{2,3}|_{\rho_6^4}$ to the full expressions including the parity-odd square root terms then (4.19) turns into an *integrand identity* also in the parity-even sector. We find

$$
\begin{aligned}
r|_{\rho_6^4} &= \frac{1}{\sqrt{\Delta}}\, R_2|_{\rho_6^4}\Big(2\ (\tilde p_2+\tilde p_5)-(\tilde g_{12}+\tilde g_{23}+\tilde g_{45}+\tilde g_{56}) \\
&\qquad\qquad +(\tilde g_{24}+\tilde g_{35}+\tilde g_{51}+\tilde g_{62})-2\ \tilde g_{25}\Big) \\
&+ \frac{1}{\sqrt{\Delta}}\, R_3|_{\rho_6^4}\Big(2\ (\tilde p_1+\tilde p_4)-(\tilde g_{12}+\tilde g_{34}+\tilde g_{45}+\tilde g_{61}) \\
&\qquad\qquad +(\tilde g_{13}+\tilde g_{24}+\tilde g_{46}+\tilde g_{51})-2\ \tilde g_{14}\Big)\ .
\end{aligned}
\tag{4.21}
$$

Mathematica's inbuilt factorisation algorithm can show this algebraically; alternatively one can substitute numbers. The *MB.m* package shows to good precision that the two sums of integrals in (4.21) separately integrate to zero. We remark that both sums can be made to contain only coefficients $0, \pm 1$ by adding in the integrand identity (4.4).

In summary then we have a clear prediction for the $\mathrm{NMHV}_6^{(1)}$ integrand albeit only in the ρ_6 sector. We now turn to the question of obtaining the full integrand in all sectors, ie obtaining the full ρ dependence of r.

4.5 Reconstructing the full supercorrelator/ NMHV amplitude

We now wish to try to reconstruct the full one-loop correlator $G_{6;1}^{(1)}$ with its full ρ dependence in the hexagon limit from the results of the previous section where the ρ_6 projection was derived from $\mathcal{N}=2$ correlators. In fact we eventually discuss the integrand, which is simply the tree correlator $G_{7;2}^{(0)}$.

Our conjecture is that the hexagon light-cone limit will reproduce the NMHV six-point one-loop amplitude and related terms and indeed we have already seen this at the level of the integral and for the ρ_6 projection.

In order to perform this reconstruction, we assert that the correlation function in the hexagon limit must be a linear combination of NMHV 6-point R invariants, $R_1 \dots R_6$ times pseudoconformal integrals. This is simply because the correlator is superconformally invariant, and in the hexagon limit R_i are all the superconformal nilpotent invariants at this order. (The analysis is identical to the corresponding amplitude analysis performed in [23] translated to analytic superspace.) It could be that there are more nilpotent invariants off the light-like limit, but we will not look into this issue here.

Since we know that $R_1 + R_3 + R_5 = R_2 + R_4 + R_6$, only $R_1 \dots R_5$ are independent and we therefore have in principle the following expansion for $G_{6;1}$ (indeed the same expansion is valid at any loop order so for the moment we do not specify the loop order)

$$
\begin{aligned}
G_{6;1} &= R_1\left(y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{56}^2 y_{61}^2\, f_{11}(x) + y_{12}^4 y_{34}^4 y_{56}^4\, f_{12}(x) + \dots\right) \\
&+ R_2\left(y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{56}^2 y_{61}^2\, f_{21}(x) + y_{12}^4 y_{34}^4 y_{56}^4\, f_{22}(x) + \dots\right) \\
&+ \dots \\
&+ R_5\left(y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{56}^2 y_{61}^2\, f_{51}(x) + y_{12}^4 y_{34}^4 y_{56}^4\, f_{52}(x) + \dots\right)
\end{aligned}
\tag{4.22}
$$

The presence of these y terms is a straightforward consequence of the $SU(4)$ R symmetry. One needs to write down any monomial in y_{ij}^2 for which all indices 1,2, ..., 6 appear precisely twice. There are a total of 130 distinct non-vanishing harmonic structures with charge 2 at every point, of which we have displayed the hexagon one (which will finally be the only one to survive) and — for illustration purposes — one other term.

However considerations of this formula projected on the ρ_6^4 component vastly reduce the number of independent functions. The ρ_6^4 components of the R invariants, which are displayed in equation (F.14) have a universal form

$$R_i|_{\rho_6^4} \;=\; R_{i6}(u_1,u_2,u_3,\sqrt{\Delta})\,\frac{\rho_6^4\,x_{24}^2}{x_{26}^2\,x_{46}^2}\,\frac{y_{15}^2}{y_{16}^2\,y_{56}^2}\,, \qquad R_{56}\;=\;R_{66}\;=\;0\,, \tag{4.23}$$

where this equation defines the functions R_{ij} for all i,j by cyclic rotations. The ρ_i^4 components of the R invariants introduce singularities in y^2 into (4.22). Since the whole correlator is supposed to contain only finite dimensional representations of the internal symmetry group we require the absence of such poles; in other words we demand "harmonic analyticity". This is a rather strong constraint which puts most of the unknown functions in (4.22) to zero.

Next, we also know that in the light-like limit the y dependence of $G_{6;1}|_{\rho_6^4}$ is just given by the pentagon harmonic structure $y_{12}^2y_{23}^2y_{34}^2y_{45}^2y_{51}^2$ (this is admittedly currently an observation on the relevant $\mathcal{N}=2$ correlators rather than a proven feature – it is certainly true in the tree and one-loop cases we are interested in as seen in the previous two sections and it seems likely to be true in general at any loop order).

But then taking the ρ_6^4 projection of (4.22) together with these facts we see that the only allowed harmonic structure in (4.22) is the the hexagon y-structure. Namely we have

$$\lim_{x_{ii+1}^2\to 0} G_{6;1} \;=\; y_{12}^2y_{23}^2y_{34}^2y_{45}^2y_{56}^2y_{61}^2\,\sum_{i=1}^{5} R_i\,f_i(x)\,. \tag{4.24}$$

The $f_i(x)$ are obviously subject to cyclic invariance.

Finally we can determine the 5 functions $f_i(x)$ by using the perturbative computations of the previous two sections, where we computed the ρ_6 component of this correlator at tree-level and one-loop, $G_{6;1}^{(l)}|_{\rho_6^4}$, for $l=0,1$ together with its cyclic shifts.

In particular from here we can reconstruct the full function r (recall r is the difference between our prediction for the full amplitude integrand and the naive integrand $\widehat{R}_{6;1}^{(1)}$ involving box integrands only.)

We certainly can not simply take the result (4.20) for r and remove the $|_{\rho_6^4}$ since for one thing the result is not cyclically invariant. Similarly to the result for $G_{6;1}$ (4.24) we have that $r=\sum_i R_i g_i(x)$ for $g_i(x)$ to be determined. But the results of the last section, give us (see (4.21)

$$\sum_{i=1}^{5} R_{i6}\,g_i(x) \;=\; r|_{\rho_6^4} \;=\; \text{known} \tag{4.25}$$

and the five cyclic shifts of this equation, which constitutes a system of six equations for five unknowns $g_i(x)$. We have sought a solution for five of the equations, separately for the even and the odd part of the ρ_i^4 components of the R invariants, using their explicit dependence on u_1, u_2, u_3. The sixth equation is satisfied in both cases thanks to the identity (4.4) between the 21 scalar integrals. The solutions as computed from the even and the odd part look different at the first glimpse, but upon eliminating the $\tilde{p}_5$ integral by equation (4.4) one finds the same solution in both cases. Curiously, the final form of r is

$$\begin{aligned} r \;=\; & \frac{1}{\sqrt{\Delta}}\,(R_1 - R_4)\Big(2\;(\tilde{p}_1 + \tilde{p}_4) - (\tilde{g}_{12} + \tilde{g}_{34} + \tilde{g}_{45} + \tilde{g}_{61}) \\ & \qquad\qquad + (\tilde{g}_{13} + \tilde{g}_{24} + \tilde{g}_{46} + \tilde{g}_{51}) - 2\;\tilde{g}_{14}\Big) \\ + \; & \frac{1}{\sqrt{\Delta}}\,(R_2 - R_5)\Big(2\;(\tilde{p}_3 + \tilde{p}_6) - (\tilde{g}_{23} + \tilde{g}_{34} + \tilde{g}_{56} + \tilde{g}_{61}) \\ & \qquad\qquad + (\tilde{g}_{13} + \tilde{g}_{35} + \tilde{g}_{46} + \tilde{g}_{62}) - 2\;\tilde{g}_{36}\Big)\,. \end{aligned} \tag{4.26}$$

As in formula (4.20) the two sums of scalar integrals can be made to contain only coefficients $0, \pm 1$ by the integrand identity (4.4). The complete expression (4.26) is cyclically invariant: after a shift (respecting the anti-cyclicity of $\sqrt{\Delta}$) one may replace $(R_3 - R_6) = -(R_1 - R_4) + (R_2 - R_5)$ and finally use (4.4) to restore the original form.

While the $\tilde{p}, \tilde{g}$ integrals in the integrand identity (4.4), the linear combination in the parity-odd part of the MHV amplitude (4.11), and the two sums of integrals in (4.26) add up to zero at $O(1/\epsilon^2)$, $O(1/\epsilon)$, $O(1)$ in all four cases, the *MB.m* package clearly indicates otherwise[11] at $O(\epsilon)$. Since we have employed (4.4) in many places in our analysis, our result (4.26) is not necessarily valid with respect to $O(\epsilon)$ corrections in dimensional regularisation or related schemes.

Summing up then we have that the integrand of $G^{(1)}_{6;1}$, namely $G^{(0)}_{7;2}$ is given by:

$$\begin{aligned} & \lim_{x^2_{i\,i+1}\to 0} \int d^4\rho_0 \frac{G^{(0)}_{7;2}(0;1,\ldots,6)}{G^{(0)}_{6;0}(1,\ldots,6)}\Bigg|_{\rho_0^4} \\ & \qquad = 2\left(\widehat{A}^{(0)}_{6;1}(1,\ldots,6)\,\widehat{A}^{(1)}_{6;0}(0;1,\ldots,6) \,+\, \widehat{R}^{(1)}_{6;1}(0;1,\ldots,6) \,+\, r\right) \end{aligned} \tag{4.27}$$

as an integrand identity in four dimensions.

[11]The cancellation of the singularities works to very high precision, while the finite parts typically yield 0.0(2). At $O(\epsilon)$ we consistently found non-vanishing contributions to all four sums; for some sample points even rather large numbers. The first three digits were always significant according to the error estimates supplied by the package.

4.6 Match with the amplitude integrand proposal

We now wish to compare our predicted amplitude integrand to the corresponding expression in [21] which is written in terms of momentum twistors

$$Z_i = (\lambda_i, \mu_i)\,, \qquad \mu_{i\dot\alpha} = \lambda_i^\alpha (x_i)_{\alpha\dot\alpha}\,, \qquad \tilde{x}_i^{\dot\alpha\alpha} = \frac{\lambda_i^\alpha \mu_{i-1}^{\dot\alpha} - \lambda_{i-1}^\alpha \mu_i^{\dot\alpha}}{\langle i-1\, i\rangle}\,. \tag{4.28}$$

A local form of the output of generalised BCFW rules [11] for the integrand of all one-loop NMHV amplitudes was given in [9]. Specialising to the six-point case this gives

$$\begin{aligned} \widehat{A}^{(1)}_{6;1}(x_0; x_1, \dots, x_6) \;&=\; \sum_{i=1}^{6} \left(J_{i+1,i+2,i+4} + J_{i+3,i+4,i+1} + J_{i+5,i+1,i+3}\right) R_i \\ &+\; \sum_{i=1}^{6} J_{i,i+2}\, R_{i+1} \;+\; J_{i,i+1}(R_1 + R_3 + R_5)\,, \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} J_{i,j} \;&=\; \frac{\langle AB\,(i-1\;i\;i+1) \cap (j-1\;j\;j+1)\rangle \langle Xij\rangle}{\langle ABX\rangle \langle AB\, i-1\, i\rangle \langle AB\, i\; i+1\rangle \langle AB\, j-1\, j\rangle \langle AB\, j\; j+1\rangle}\,, \\ J_{i,j,k} \;&=\; \frac{\langle AB(i-1\;i\;i+1) \cap \Theta_{ijk}\rangle}{\langle ABX\rangle \langle AB\, i-1\, i\rangle \langle AB\, i\; i+1\rangle \langle AB\, j\; j+1\rangle \langle AB\, k\; k+1\rangle}\,. \end{aligned} \tag{4.30}$$

In these formulae (AB) define the integration point and $X = (CD)$ is an arbitrary bispinor of which the six-point integrand (4.29) is in fact independent. The $\cap$ symbols mean intersections in twistor geometry following the rules

$$\begin{aligned} \langle AB\,(i\,j\,k) \cap (l\,m\,n)\rangle \;&=\; \langle A\,i\,j\,k\rangle \langle B\,l\,m\,n\rangle - \langle A\,l\,m\,n\rangle \langle B\,i\,j\,k\rangle\,, \\ (i\,j\,k) \cap X \;&=\; D\,\langle C\,i\,j\,k\rangle - C\,\langle D\,i\,j\,k\rangle \end{aligned} \tag{4.31}$$

and the surface Θ_{ijk} is defined as

$$\Theta_{ijk} \;=\; \frac{1}{2}\left[(j\;j+1\,(i\;k\,k+1) \cap X) - (k\,k+1\,(i\;j\,j+1) \cap X)\right]\,. \tag{4.32}$$

Through the chain of back-substitutions the NMHV integrand (4.29) is reduced to twistor four-brackets as defined in (3.47) and R invariants.

It remains to compare formula (4.29) to our prediction for this integrand from the six-point correlator, namely $\widehat{R}^{(1)}_{6;1} + r$ (see (4.26)). So we wish to verify

$$\widehat{R}^{(1)}_{6;1} \;+\; r \;=\; \widehat{A}^{(1)}_{6;1}\,, \tag{4.33}$$

where the left-hand side is our prediction and the right hand side is the local twistor integrand (4.29). Since the $\rho_1 \dots \rho_6$ structure of the invariants is rigid and $\mathcal{N} = 4$ correlator and amplitude are cyclically invariant we will not do this for more than one Grassmann component; given the discussion in the preceding sections the obvious choice for us will be ρ_6^4. If random complex integers are chosen for the components of $Z_1, \ldots, Z_6, A, B, C, D$ (we limited the range to $[-100, 100]$ for real and imaginary parts) the evaluation of either side of (4.33) stays in the rational numbers which *Mathematica* can treat exactly; any disagreement would be noticed. The evaluation of the twistor integrand simply uses the determinant form of the $\langle ijkl \rangle$ four-bracket, while our x-space integrand can be calculated by matrix multiplication after gaining the x_i from the twistors by (3.45).

We have successfully run this check for hundreds of sample points confirming that the correspondence between correlation functions (as calculated by Lagrangian insertions in the $\mathcal{N} = 2$ formalism) and amplitudes holds at the loop integrand level for NMHV cases, too.

Last, according to [21]

$$\mathcal{A}_{5,1}^{(1)} = R_{135} \sum_{i=1}^{5} \left(J_{i,i+1,i+3} + J_{i,i+1} \right) , \tag{4.34}$$

$$\mathcal{A}_{5,0}^{(1)} = \sum_{i<j} J_{i,j} \tag{4.35}$$

from which we may check numerically that

$$\mathcal{A}_{5,1}^{(1)} + R_{135} \mathcal{A}_{5,0}^{(1)} = R_{135} \left(4\pi^2 \, x_{13}^2 x_{24}^2 \, g(1,2,3,4) + \text{cyclic} \right) \tag{4.36}$$

as stated in the main text.

5 Conclusions

We have illustrated with several examples that the tree-level $n+m$-point function of $\mathcal{N} = 4$ stress tensor multiplets generates all N^{k}MHV $(n+k)$-point $(m-k)$-loop amplitude integrands for $0 \leq k \leq m$ in the appropriate light-like limits. This extends the correlator/MHV amplitude duality discussed in [8, 16] to the *entire* super-amplitude in planar $\mathcal{N} = 4$ SYM. The feature that a given correlator can generate a variety of amplitudes has a close parallel in the super Wilson loop proposal of [13, 14].

We conjecture that the correlator/amplitude duality generally holds at the level of the *integrand*. As a highly non-trivial test we have used the new correspondence to construct the integrand of the NMHV six-point one-loop amplitude and confirmed its exact equality with the corresponding prediction of the BCFW recursion rules for the all-loops integrand [9] in local form [21]. To compare the two expressions we have substituted random generated complex rational numbers, which *Mathematica* can manipulate without numerical approximations.

The correlator computation relevant to the latter check was done entirely in the traditional x space variables, using only conformally covariant manipulations for the reduction and tensor decomposition of traces over $x^{\alpha\dot{\alpha}}$. The final x space formulae for the parity-odd sector of the integrand are strikingly simple; if a certain basis is used for the scalar integrals we find only coefficients $0, \pm 1$. We hope that this circle of ideas will be useful in other applications, too.

Note added

When this paper was ready for submission, two other publications on the duality between correlators and Wilson loops appeared [41], [42]. The former treats the duality in three dimensions, the latter proposes a twistor superspace version of it.

Acknowledgements

ES is grateful to Sergio Ferrara, Paul Howe and Raymond Stora and PH is grateful to Valya Khoze and Claude Duhr for a number of enlightening discussions. GK and ES acknowledge discussions with David Skinner and Simon Caron-Huot. BE and PH acknowledge support by STFC under the rolling grant ST/G000433/1. BE acknowledges hospitality at the ITP Leipzig during the final stages of this work.

A Partial non-renormalisation of the four-point correlator $G_{4;0}$

In the minimal on-shell $\mathcal{N}=4$ analytic superspace formalism [24] there is the equivalent of the x-space conformal inversion acting on the internal y variables. The stress-energy tensor multiplet has weight (-2) (to be identified with the $SU(4)$ harmonic charge $(+2)$ used in [17]) under this transformation. This defines it as the highest-weight state of the irrep $\mathbf{20'}$ of $SU(4)$. At the same time, it has conformal weight two. At $\rho = \bar{\rho} = 0$ the full structure of the four-point correlator is

$$G_{4;0}(x_1, \ldots, x_4) = \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} F_1 + \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2} F_2 + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2} F_3$$
$$+ \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} F_4 + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} F_5 + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} F_6 \,, \tag{A.1}$$

where $F_i(s, t; a)$ (with $i = 1, \ldots, 6$) are functions of the two independent conformal cross-ratios

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \,, \qquad t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,, \tag{A.2}$$

as well as of the 't Hooft coupling a. The six terms in (A.1) correspond to the six $SU(4)$ irreps in the tensor product $\mathbf{20'} \times \mathbf{20'} = \mathbf{1} + \mathbf{15} + \mathbf{20} + \mathbf{84} + \mathbf{105} + \mathbf{175}$. Each of them consists of a propagator structure and a conformally invariant function. The propagator structures are obtained by connecting the four points with free propagators (2.13) in all possible ways (Wick contractions). They have the required conformal weight two and internal charge two at each point. At tree level the first three terms are described by connected, the other three terms by disconnected graphs. The six coefficient functions $F_i(s,t;a) = C_i + aF_i^{(1)}(s,t) + a^2 F_i^{(2)}(s,t) + \ldots$ comprise tree-level constants and functions from loop corrections.

A very important property of the four-point correlator is that the six loop correction functions are not independent: they are all proportional to a single function of s,t. The loop correction to the correlator, i.e. the part excluding the tree-level contribution, takes the following factorised form

$$G_{4;0} - G_{4;0}^{(0)} \;=\; I(x_1,\ldots,x_4,y_1,\ldots,y_4)\, F(s,t;a)\,, \tag{A.3}$$

with the rational prefactor

$$\begin{aligned} I = {} & \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}(1-s-t) + \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}(t-s-1) \\ & + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2}(s-t-1) + \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} s + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} t\,. \end{aligned} \tag{A.4}$$

We stress that this result is valid to all orders in the coupling. This remarkable fact, known as "partial non-renormalisation" [20], can be explained in two equivalent ways:

One explanation is given by the superconformal Ward identities which the correlator has to satisfy. Apart from the simple fact that the odd variables of the supercorrelator $\langle \mathcal{T}_1 \ldots \mathcal{T}_4 \rangle$ can be gauged away by Q and $\bar{S}$ transformations, the various components in its Grassmann expansion satisfy differential constraints following from the full superconformal algebra with generators $Q, \bar{Q}, S$ and $\bar{S}$. Their general solution [20] involves an arbitrary two-variable function exactly as in (A.3), (A.4), and in addition a single variable function not shown in (A.3). The detailed analysis of the conformal partial wave expansion of the latter shows that it can only contribute at tree level, where it takes a fixed form [43, 44]. The loop corrections, i.e., the derivatives of the correlator with respect to the coupling, always take the form (A.3), (A.4). One can also see this directly in $\mathcal{N}$=4 analytic superspace in which the correlator becomes an expansion in $SL(2|2)$ invariants (ie characters or Schur polynomials labelled by $SL(2|2)$ representations). The renormalised two variable function comes directly from generic long (or typical) representations of $SL(2|2)$ whereas the protected one variable functions arise from short representations [45].

The alternative explanation [18] makes use of a Lagrangian insertion: In formula (2.15) we have stated that the $O(a^l)$ correction to G_n can be calculated using l insertions. If the integrand on the right-hand side is restricted to Born level this yields in fact exactly the $O(a^l)$ part of G_n, else all corrections at $O(a^m)$, $m \geq l$ are reproduced. The one-insertion scenario implies that the loop corrections to the four-point correlator $G_{4;0}$ are obtained from the nilpotent part $G_{5;1}$ of the five-point one. The latter is heavily restricted by $\mathcal{N} = 4$ conformal supersymmetry and has the following general form

$$G_{5;1} = P(x_1,\ldots,x_5;\rho_1,\ldots,\rho_5;y_1,\ldots,y_5)\, f(x_1,\ldots,x_5) \tag{A.5}$$

at $\bar{\rho}_i = 0$. Here $P(x, \rho, y)$ is a very specific nilpotent rational function of the space-time, odd and harmonic variables of Grassmann degree four, carrying the necessary conformal weights and internal charges at all five points. The only remaining freedom is in the arbitrary function $f(x)$. Further, the coefficient of the $(\rho_5)^4$ component of P turns out not to depend on x_5 and y_5,

$$P_{\rho_1=\ldots=\rho_4=0} = I(x_1, \ldots, x_4, y_1, \ldots, y_4)\ \rho_5^4\,, \tag{A.6}$$

with the same prefactor I as in (A.4). Integrating out the insertion point we find the factorised form (A.3) of the loop corrections, where the arbitrary two-variable function is given by $F(s,t) = \int d^4x_5\, f(x_1, \ldots, x_5)$.

Now, the practical question is how to compute the loop corrections. The absence of an off-shell formulation of $\mathcal{N} = 4$ SYM makes Feynman graph calculations with manifest $\mathcal{N} = 4$ supersymmetry impossible. We have to resort to component calculations (with no manifest supersymmetry) or to formulations in terms of $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superfields. The $\mathcal{N} = 2$ formulation has the advantage that it reproduces the phenomenon of partial renormalisation, for exactly the same reason as in the $\mathcal{N} = 4$ case. Here we just give the results of the one- and two-loop computations [18, 19, 32–34]:

$$\begin{aligned} F &= \frac{2N_c^2}{(4\pi^2)^4}\left[\frac{1}{4}a\,F^{(1)} + \frac{1}{16}a^2 F^{(2)} + O(a^3)\right]\,, \\ F^{(1)} &= x_{13}^2 x_{24}^2\, g(1,2,3,4)\,, \\ F^{(2)} &= x_{13}^2 x_{24}^2 \left[\frac{1}{2}(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)(g(1,2,3,4))^2\right. \end{aligned} \tag{A.7}$$

$$\begin{aligned} &+ x_{12}^2 h(1,2,3;1,2,4) + x_{23}^2 h(1,2,3;2,3,4) + x_{34}^2 h(1,3,4;2,3,4) \\ &\left. + x_{41}^2 h(1,2,4;1,3,4) + x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\right] \end{aligned} \tag{A.8}$$

where the off-shell one- and two-loop integrals are defined by

$$g(1,2,3,4) = \frac{1}{4\pi^2}\int \frac{d^4x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}\,, \tag{A.9}$$

$$h(1,2,3;1,2,4) = \frac{1}{(4\pi^2)^2}\int \frac{d^4x_0 d^4x_{0'}}{(x_{10}^2 x_{20}^2 x_{30}^2)x_{00'}^2(x_{10'}^2 x_{20'}^2 x_{40'}^2)} \tag{A.10}$$

in four dimensions.

Assembling this altogether gives the formulae quoted in (3.2-3.5). In particular the two-loop and one-loop squared pieces of $F^{(2)}$ reassemble into the suggestive form in (3.5).

B $\mathcal{N} = 2$ superfields

Real $\mathcal{N} = 2$ Minkowski space has the coordinates $x^{\alpha\dot{\alpha}}, \theta^{i\alpha}, \bar{\theta}^{i\dot{\alpha}}$ where $i \in \{1, 2\}$. The internal $SU(2)$ index i can be raised and lowered by $\epsilon^{ij}, \epsilon_{ij}$ like the Lorentz indices $\alpha, \dot{\alpha}$.

Harmonic superspace [26] has an additional coordinate $u = (u^+, u^-) \in SU(2)/U(1)$. Instead of choosing a coordinate representative of the coset one uses the entire matrix $u \in SU(2)$. This helps to preserve the manifest $SU(2)$ and to keep track of the local $U(1)$ charge. (In contrast, in the case of $\mathcal{N} = 4$ described below we prefer to work with coordinates on the harmonic coset.) An $SU(2)$ invariant combination of harmonics at two different points in harmonic space is given by

$$(12) \;=\; u^{+i}|_1 \epsilon_{ij} u^{+j}|_2 \,. \tag{B.1}$$

Analytic superspace has the coordinates $\{x,\ \theta^+, \bar\theta^+, u\}$ where

$$\theta^+ \;=\; \theta^i u_i^+ \,, \qquad \bar\theta^+ \;=\; \bar\theta^i u_i^+ \tag{B.2}$$

thus involving only one (plus projected) half of the odd coordinates. The $\mathcal{N} = 2$ matter multiplet (the hypermultiplet q^+) and the Yang-Mills multiplet (incorporated in the gauge prepotential V^{++}) can both be realised as unconstrained quantum fields on analytic superspace [26]:

$$q^+(x_\mathcal{A},\, \theta^+, \bar\theta^+, u) \,, \quad V^{++}(x_\mathcal{A},\, \theta^+, \bar\theta^+, u) \,, \qquad x_\mathcal{A} = x - 4\, i\, \theta^{(i} \bar\theta^{j)}\, u_i^+ u_j^- \,. \tag{B.3}$$

The field content of the $\mathcal{N} = 4$ super-Yang-Mills theory is equivalent to the physical fields of the two $\mathcal{N} = 2$ multiplets put together. The $\mathcal{N} = 4$ action is obtained when both fields transform in the adjoint representation of the gauge group:

$$S_{\mathcal{N}=4\,\mathrm{SYM}} \;=\; S_{\mathrm{HM/SYM}} + S_{\mathcal{N}=2\,\mathrm{SYM}} \tag{B.4}$$

with

$$\begin{aligned}
S_{\mathrm{HM/SYM}} \;&=\; -2 \int du d^4x_\mathcal{A} d^2\theta^+ d^2\bar\theta^+ \ \mathrm{Tr}\left(\tilde q^+ D^{++} q^+ + i\,\sqrt{2}\,\tilde q^+ [V^{++}, q^+]\right) &\text{(B.5)}\\
S_{\mathcal{N}=2\,\mathrm{SYM}} \;&=\; -\frac{1}{4g^2} \int d^4x_L d^4\theta \ \mathrm{Tr}(\widehat{W}^2_{\mathcal{N}=2}) \,, \qquad x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - 2i\theta^{i\alpha}\bar\theta_i^{\dot\alpha} &\text{(B.6)}\\
\widehat{W}_{\mathcal{N}=2} \;&=\; \frac{i}{4} u_i^+ u_j^+ \bar D^i_{\dot\alpha} \bar D^{j\dot\alpha} \sum_{r=1}^{\infty} \int du_1 \ldots du_r \, \frac{(-i\sqrt{2})^r V^{++}(u_1) \ldots V^{++}(u_r)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_r^+u^+)} \,.
\end{aligned}$$

In the definition of the field strength $\widehat{W}_{\mathcal{N}=2}$ the auxiliary harmonics $u_1, \ldots, u_r$ are integrated out. It is less obvious — but nonetheless true — that the field strength is also independent of the non-integrated harmonic variable u. $\widehat{W}_{\mathcal{N}=2}$ is in fact a chiral field depending on x_L, θ^i. Notice that in our convention V^{++} has been rescaled with the gauge coupling, $V \to \sqrt{2}\, g^{-1} V$ w.r.t. the definitions in [26], which has the effect that the coupling is present only in front of the SYM action (B.6) and that the physical scalar in V^{++} acquires a propagator with standard normalisation.

In this article we draw upon the Feynman graph calculations of [8,16,18] where the necessary Feynman rules and methods of calculation are explained in detail. We are interested in correlation functions of the operators

$$O \;=\; \mathrm{Tr}(q^+ q^+) \,, \qquad \tilde O \;=\; \mathrm{Tr}(\tilde q^+ \tilde q^+) \,, \qquad \widehat{O} \;=\; 2\,\mathrm{Tr}(\tilde q^+ q^+) \,,$$

like e.g.

$$\begin{aligned}\mathcal{G}_{n;0}(x_1,\ldots,x_n) &= \langle O(x_1)\,\tilde{O}(x_2)\,O(x_3)\ldots\tilde{O}(x_n)\rangle \\ &= \int \mathcal{D}\Phi\; e^{S_{\mathcal{N}=4\,\mathrm{SYM}}}\;\tilde{O}(x_1)\,O(x_2)\,\tilde{O}(x_3)\ldots O(x_n)\,.\end{aligned} \tag{B.7}$$

Differentiation of the path integral with respect to the coupling constant yields the identity

$$\begin{aligned}& g^2\frac{d}{dg^2}\,\mathcal{G}_{n;0}(x_1,\ldots,x_n) \\ &= \frac{1}{g^2}\int d^4x_0 d^4\theta_0 \int \mathcal{D}\Phi\; e^{S}\; O(x_1)\,\tilde{O}(x_2)\,O(x_3)\ldots\tilde{O}(x_n)\,\frac{1}{4g^2}\mathrm{tr}(\widehat{W}^2_{\mathcal{N}=2}) \\ &= \frac{1}{g^2}\int d^4x_0 d^4\theta_0\;\mathcal{G}_{n+1;1}(x_0;x_1,\ldots,x_6)\,.\end{aligned} \tag{B.8}$$

Equations (2.15), (2.16) are simply $\mathcal{N}$=4 analogues of this relation.

Restricted to the lowest order in the coupling constant (g^2 by the Feynman rules in [8, 16, 18]) this implies that the one-loop correction to the original n-point correlator is equal to the integral over the $(n+1)$-point function on the right hand side; we call this an "operator insertion". We write expressions corresponding to Euclidean Feynman rules, so for example the i in front of the action is absorbed by Wick rotation before the vertices are read off.

In the $\mathcal{N}=2$ formalism the left-handed odd variables θ^α carry R-charge (+1) and $\widehat{W}_{\mathcal{N}=2}$ is of charge (+2). Therefore the mixed correlator $\mathcal{G}_{n+1;1}(x_0;x_1,\ldots,x_n)$ must be of order $\theta^4 + O(\theta^5\bar\theta)$.

The insertion relation (B.8) is particularly simple to show starting from the form of the Yang-Mills action given in (B.6). On the other hand, the $\theta = 0$ term of the $\widehat{W}_{\mathcal{N}=2}$ multiplet is one of the (complex) physical scalars of the $\mathcal{N}=4$ SYM multiplet. With the given field rescaling we find $\widehat{W}_{\mathcal{N}=2} = g^{-1}\,\phi(x) + O(\theta)$. In the following section it is implied that the field redefinition by g has been undone, so $\widehat{W}_{\mathcal{N}=2,\,\mathrm{lin}} \to g\,W_{\mathcal{N}=2,\,\mathrm{lin}}$.

Last, in this appendix we have indicated the $U(1)$ charge assignments of the harmonics and of the fields to be in accord with the literature. In the rest of this work we simply write $q,\tilde{q}$ instead of $q^+,\tilde{q}^+$.

C Reduction $\mathcal{N}=4 \to \mathcal{N}=2$

Real $\mathcal{N}=4$ Minkowski space has the coordinates $x^{\alpha\dot\alpha},\theta^{\alpha A},\bar\theta^{\dot\alpha}_A$ where $A\in\{1,\ldots,4\}$ and $\alpha,\dot\alpha$ are the usual two-component indices. In order to make touch with [24] we rather complexify, tacitly keeping the notation $\theta,\bar\theta$ although the latter are not complex conjugates of each other in the following.

The $\mathcal{N}=4$ *analytic superspace* of [24] has additional coordinates ${y_{a'}}^a$ parametrising a coset of $GL(4)$:

$$\mathrm{Gr}(4,2) = \frac{GL(4,\mathcal{C})}{\mathcal{P}} = \begin{pmatrix} {\delta_b}^a & 0 \\ {y_{b'}}^a & {\delta_{b'}}^{a'}\end{pmatrix} = {g_B}^A \tag{C.1}$$

where $\mathcal{P}$ is the (parabolic) subgroup of upper triangular matrices with 2×2 blocks. We have split the indices as

$$A = (a, a')\,, \quad a \in \{1,2\}\,, \quad a' \in \{3,4\}\,. \tag{C.2}$$

We can use these to project onto one half of the Grassmann coordinates:

$$\rho^{\alpha a} = \theta^{\alpha\, a} + \theta^{\alpha a'}\, y_{a'}{}^{a}\,, \qquad \bar{\rho}_{a'}{}^{\dot\alpha} = y_{a'}{}^{a}\bar{\theta}^{\dot\alpha}_{a} + \bar{\theta}^{\dot\alpha}_{a'}\,. \tag{C.3}$$

The field strength multiplet

$$W^{[AB]} = \phi^{[AB]}(x) + \theta^{\alpha[A}\psi(x)^{B]}_{\alpha} + \theta^{[A}_{(\alpha}\theta^{B]}_{\beta)}F^{\alpha\beta} + O(\bar\theta) \tag{C.4}$$

can also be projected by the "harmonics"

$$W_{\mathcal{N}=4}(x_{\mathcal{A}}, \rho, \bar\rho, y) = \epsilon^{ab} g^{a}_{A} g^{b}_{B} W^{AB}\,. \tag{C.5}$$

We see that the θ dependence is reduced to ρ (similarly only $\bar\rho$ remains). The field strength multiplet thus lives on "analytic superspace" with the coordinates $x^{\alpha\dot\alpha}_{\mathcal{A}}, \rho^{\alpha a}, \bar\rho_{a'}{}^{\dot\alpha}, y_{a'}{}^{a}$. The change of basis $x \to x_{\mathcal{A}}$ is analogous to the $\mathcal{N}=2$ case. In particular, it involves $\bar\theta$ and so is irrelevant in the present context.

The $\mathcal{N}=2$ analytic superspace can be embedded into this larger space. In order to reduce the field strength multiplet to $\mathcal{N}=2$ pieces we need some of the $GL(4)$ raising operators, namely $D^{a}_{a'}$. They act only on the y variables according to

$$D^{a'}_{a}\, y_{b'}{}^{b} = \delta^{a'}_{b'}\,\delta^{b}_{a}\,. \tag{C.6}$$

We define

$$\begin{aligned} W_{\mathcal{N}=4} &\;\to\; q \\ D^{4}_{1}\, W_{\mathcal{N}=4} &\;\to\; \tilde{q} \\ D^{3}_{1}\, W_{\mathcal{N}=4} &\;\to\; W_{\mathcal{N}=2}\,, \end{aligned} \tag{C.7}$$

so that for instance (recall $\mathcal{T} = \mathrm{tr}(W^2_{\mathcal{N}=4})$)

$$\begin{aligned} &\frac{1}{8}\,(D^{4}_{1}|_2)^2\,(D^{4}_{1}|_4)^2\,D^{4}_{1}|_5\,(D^{3}_{1}|_6)^2\,\langle \mathcal{T}_1\mathcal{T}_2\mathcal{T}_3\mathcal{T}_4\mathcal{T}_5\mathcal{T}_6\rangle \\ &\to \langle O_1\,\tilde{O}_2\,O_3\,\tilde{O}_4\,\widehat{O}_5\,\mathrm{tr}(W^2_{\mathcal{N}=2})\rangle\,. \end{aligned} \tag{C.8}$$

Further, we define $\tilde{y}^{a'}_{a} = \epsilon_{ab}\,\epsilon^{a'b'}\,y^{b}_{b'}$, so for lowering and raising of the two-component flavour indices the same convention is used as in harmonic superspace. To add some detail:

$$\epsilon^{ab}\,\epsilon_{bc} = \delta^{a}_{c}\,, \qquad \epsilon_{a'b'}\,\epsilon^{b'c'} = \delta^{c'}_{a'}\,, \qquad \epsilon_{12} = \epsilon_{34} = 1\,. \tag{C.9}$$

The symbol y^2 denotes the determinant of the matrix y,

$$y^2 = -\frac{1}{2}\,\tilde{y}^{a}_{a'}\,y^{a}_{a'}\,. \tag{C.10}$$

On the level of the y, ρ variables the reduction to $\mathcal{N} = 2$ is accomplished by

$$\begin{aligned} y_3^2 &\to \mathbf{y}\,, \quad y_3^1, y_4^1, y_4^2 \to 0 \\ \theta^2, \theta^3 &\to \theta^i\,, \quad \theta^1, \theta^4 \to 0 \end{aligned} \tag{C.11}$$

so in particular

$$\rho^a \to \delta_2^a(\theta^2 + \theta^3\, y_3^2) = \delta_2^a\, \theta^i (1, \mathbf{y})_i = -\delta_2^a\, \theta^+ \tag{C.12}$$

where we identified $(1, \mathbf{y})_i = u_i^+$. It follows $(12) = u^{+i}|_1 u_i^+|_2 = \mathbf{y}_{12}$. Note that

$$y^2 \to 0\,, \qquad D_1^4\, y^2 \to \mathbf{y}\,, \qquad (D_1^4)^2\, y^2 \to 0 \tag{C.13}$$

as a consequence of the index contraction by the ϵ symbols. Last, if we define $(\rho^2)_{(\alpha\beta)} = \epsilon_{ba}\rho^a_\alpha \rho^b_\beta$ and similar for the $\mathcal{N} = 2$ variable θ^i_α:

$$(D_1^3)^2 \rho^4 = (D_1^3)^2 \frac{1}{12}\, (\rho^2)^{(\alpha\beta)}(\rho^2)_{(\alpha\beta)} = \frac{1}{6}\, (\theta^2)^{(\alpha\beta)}(\theta^2)_{(\alpha\beta)} = 2\, \theta^4 \tag{C.14}$$

D $\quad \mathcal{N} = 4$ correlators from $\mathcal{N} = 2$

As mentioned previously perturbative computations of correlation functions are most easily performed in $\mathcal{N}$=2 harmonic superspace and are not possible directly in $\mathcal{N}$=4 analytic superspace. But the amplitude/correlation function duality naturally relates superamplitudes to correlation functions in $\mathcal{N}$=4 analytic superspace. We here show how to reconstruct the full $\mathcal{N}$=4 (bosonic part of the) correlator from various permutations of an $\mathcal{N}$=2 correlator.

We will first switch off all superspace coordinates (corresponding to restricting ourselves to the MHV amplitudes). Of course in this paper we do not want to restrict ourselves to these cases, but they illustrate the procedure which we adapt in the main text to treat the various cases we are interested in. First we write down all allowed y structures. In the $\mathcal{N}$=4 case this means writing down all possible products of n y_{ij}^2 terms such that each index occurs exactly twice (this is simply because the R-charge of each operator, the energy momentum multiplet is two). Thus

$$\langle \mathcal{O}\mathcal{O} \dots \mathcal{O} \rangle = \sum_{\sigma \in S_n} y^2_{1\sigma(1)} y^2_{2\sigma(2)} \dots y_{n\sigma(n)} f_\sigma(x)\ , \tag{D.1}$$

where the sum is over all permutations of 1 to n (in fact "derangements" - a derangement being a permutation in which no element remains in its original position - a permutation in which at least one element i remained fixed would lead to $y_{ii}^2 = 0$). Note that different permutations may lead to the same y-structure. The most obvious example of this is that σ and σ^{-1} will always lead to the same y-structure so that $f_\sigma = f_{\sigma^{-1}}$. However one can also see that any cycle within a permutation may be replaced by its inverse to give the same y-structure, so if $\sigma = \mu\nu$ then $\sigma' = \mu^{-1}\nu$ also gives the same y-structure. Strictly speaking we should consider equivalence classes of all such related permutations but we won't worry too much about these details.

Such a correlator in $\mathcal{N}$=4 reduces to many different $\mathcal{N}$=2 correlators. They correspond to the projections of the $\mathcal{N} = 4$ scalar operator $\mathcal{O}$ onto $\mathcal{N} = 2$ operators made of hypermultiplet scalars, $\mathcal{O} \to O, \tilde{O}, \widehat{O}$ (see the definitions in (4.13)), as well as projections made of the $\mathcal{N} = 2$ SYM scalar. However we will show that the full $\mathcal{N}$=4 correlator can be reconstructed entirely from specific types of $\mathcal{N}$=2 hypermultiplet correlator. More precisely for any term in the full $\mathcal{N}$=4 correlator (D.1) (specified by a particular permutation σ) we identify a (not necessarily unique) $\mathcal{N}$=2 correlator which will give this term. The particular $\mathcal{N}$=2 correlator is determined as follows: write out the permutation σ as a product of disjoint cycles $\sigma = \sigma_1\sigma_2\ldots\sigma_m$. Then construct an $\mathcal{N}$=2 correlator as follows: put an operator $\mathcal{O}$ at the point given by the first element of σ_1, an operator $\tilde{O}$ at the point given by the second position, and so on, alternating between O and $\tilde{O}$. If the cycle has even length, then simply continue the procedure with the next cycle. If however the cycle has odd length we must put an operator $\widehat{O}$ at the point given by the last element of this odd cycle, then continue with the next cycle. The coefficient of the $\mathcal{N}$=4 y-structure in question f_σ can then be read off from the corresponding term in the $\mathcal{N}$=2 correlator.

The procedure is best illustrated with an example. Say we wish to determine the function $f_\sigma(x)$ in the $\mathcal{N}$=4, eight-point correlator given by $y_{15}^2 y_{14}^2 y_{45}^2 y_{26}^4 y_{37}^2 y_{38}^2 y_{78}^2 f_\sigma(x)$. The permutation in question here can be given as $\sigma = (154)(26)(378)$ (as mentioned above this is not unique, we could have chosen (145) as the first cycle instead for example). So according to the general procedure for determining an $\mathcal{N}$=2 correlator which will give this function, we put the operator O at points 1,2,3, $\tilde{O}$ at points 5,6,7 and $\widehat{O}$ at points 4,8 (corresponding to the last elements in the odd cycles). So in other words we consider the $\mathcal{N}$=2 correlator

$$\langle OOO\widehat{O}\tilde{O}\tilde{O}\tilde{O}\widehat{O}\rangle = \mathbf{y}_{51}\mathbf{y}_{41}\mathbf{y}_{54}\mathbf{y}_{62}^2\mathbf{y}_{37}\mathbf{y}_{38}\mathbf{y}_{87} f_\sigma(x) + \ldots \tag{D.2}$$

where we have only displayed the relevant term in this $\mathcal{N}$=2 correlator which we are interested in. The important point is that the $\mathcal{N}$=4 correlator reduces directly to this $\mathcal{N}$=2 correlator and the $\mathcal{N}$=4 y-structure reduces directly to this $\mathcal{N}$=2 y-structure, thus the functions $f_\sigma(x)$ are the same.[12]

We conclude that we can reproduce any term in the $\mathcal{N}$=4 correlator by considering appropriate $\mathcal{N}$=2 correlators. We need correlators with mostly O's and $\tilde{O}$'s, but we also may need a few correlators with $\widehat{O}$ operators. More precisely we need a $\widehat{O}$ for every odd cycle in the permutation σ, the rest of the operators in the correlator will be half O and half $\tilde{O}$.

So for example here we display all the types of $\mathcal{N}$=2 correlators needed to reconstruct

[12]In order to ensure we don't get a minus sign error, it is important that we always write the $\mathcal{N}$=2 terms as $\mathbf{y}_{ij}$ when i is associated with $\tilde{O}$ and j associated with O, rather than $\mathbf{y}_{ji}$.

fully the (bosonic) $\mathcal{N}$=4 correlator for $n = 3, 4, 5, 6, 7$

$$\begin{array}{lllll}
n=3 & \langle O\tilde{O}\hat{O}\rangle & \to & \langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle & \\
n=4 & \langle OO\tilde{O}\tilde{O}\rangle & \to & \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle & \\
n=5 & \langle OO\tilde{O}\tilde{O}\hat{O}\rangle & \to & \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle & \\
n=6 & \langle OOO\tilde{O}\tilde{O}\tilde{O}\rangle + \langle OO\tilde{O}\tilde{O}\hat{O}\hat{O}\rangle & \to & \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle & \\
n=7 & \langle OOO\tilde{O}\tilde{O}\tilde{O}\hat{O}\rangle & \to & \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle\ . & \qquad (D.3)
\end{array}$$

In particular we see that for $n = 6$ for the first time we need two different types of correlator, the second type, with two $\hat{O}$'s is needed to determine, $f_\sigma(x)$ whenever σ is a product of two three cycles, which the first type of correlator will miss.

It is interesting to count the number of different terms in the correlators in $\mathcal{N}$=4. The counting of the number of independent y-structures is equivalent to counting symmetric traceless $n \times n$ matrices A with positive integer entries, whose rows and columns add up to two. To see this imagine writing the correlator as $\langle \mathcal{O}\mathcal{O}\ldots\mathcal{O}\rangle = \sum_A \prod_{i,j=1}^n (y_{ij}^2)^{A_{ij}/2} f_A(x)$, where the sum runs over the set of such matrices A. The number of such structures for $n = 2, 3, 4, 5, 6, 7, 8$ is $1, 1, 6, 22, 130, 822, 6202$ and one can find more details and references for the counting of such objects here [46].

Finally in this paper we have been considering supercorrelators with odd coordinates turned on which complicates the analysis. However the above techniques *can* be used to obtain the component $\langle \mathcal{O}(1)\ldots\mathcal{O}(n)\mathcal{L}\ldots\mathcal{L}\rangle$ with all $\rho_i = 0\,,\ i = 1\ldots n$ from appropriate $\mathcal{N} = 2$ correlators, namely $\langle O(1)\ldots O(n)\mathcal{L}_{\mathcal{N}=2}\ldots\mathcal{L}_{\mathcal{N}=2}\rangle$. Essentially the Lagrangian components lift directly from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ and the rest lifts exactly as described above for the case with no Lagrangian insertions. It is this application which we make use of a number of times in this paper.

E Relations between different superspace variables

In this paper we make use of several different superspaces. The Grassmann odd variables we use are Nair's η and the momentum supertwistor variable χ (both familiar in the superamplitude context) and the analytic superspace odd variable ρ (useful for the correlation functions.) Furthermore all of these variables can be defined in terms of the standard $\mathcal{N}$=4 Minkoswski superspace variable θ which we have not made direct use of here. Nevertheless it is clear that the variables are not independent and we here give the relations between them, which are in fact crucial for understanding the duality.

Firstly the variables χ are defined in terms of θ as [29]

$$\chi_i^A = \lambda_i^\alpha \theta_{i\alpha}^A = \lambda_i^\alpha \theta_{i+1\,\alpha}^A\ . \qquad (E.1)$$

Secondly the variables $\rho^{\alpha a}$ are simply harmonic projections of θ given explicitly as

$$\rho_i^{\alpha a} = \theta_i^{\alpha a} + \theta_i^{\alpha a'} y_{i\,a'}{}^a\ . \qquad (E.2)$$

These two relations together yield a direct relation between χ and ρ which we have made repeated use of (2.12)

$$\chi_i \;=\; \langle i|(\rho_i - \rho_{i\,i+1}\, y_{i\,i+1}^{-1}\, y_i)\,, \qquad \chi'_i \;=\; \langle i|\rho_{i\,i+1}\, y_{i\,i+1}^{-1}\,, \qquad \langle i| \;=\; \epsilon_{\alpha\beta}\,\lambda_i^{\beta}\,. \tag{E.3}$$

Thirdly, a general formula relating the η to the χ variables was given in [29]:

$$\eta_i^A \;=\; \frac{\langle i-1\,i\rangle\,\chi_{i-1}^A + \langle i\,i+1\rangle\,\chi_{i-1}^A + \langle i+1\,i-1\rangle\,\chi_i^A}{\langle i-1\,i\rangle\langle i\,i+1\rangle}\,. \tag{E.4}$$

We already have seen the relation between χ and ρ (2.12)

$$\chi'_i \;=\; \langle i|\rho_{i\,i+1}\, y_{i\,i+1}^{-1}\,, \qquad \chi_i \;=\; \langle i|(\rho_i - \rho_{i\,i+1}\, y_{i\,i+1}^{-1}\, y_i)\,.$$

Substituting this in we obtain the desired relation between η and ρ

$$\begin{aligned} \eta'_i &= \frac{1}{\langle i-1\,i\rangle}\langle i-1|\,\sigma_{i-1\,i\,i+1} - \frac{1}{\langle i\,i+1\rangle}\langle i+1|\,\sigma_{i\,i+1\,i+2}\,, \\ \eta_i &= -\frac{1}{\langle i-1\,i\rangle}\langle i-1|\,\sigma_{i-1\,i\,i+1}\,y_i + \frac{1}{\langle i\,i+1\rangle}\langle i+1|\,\sigma_{i\,i+1\,i+2}\,y_{i+1}\,, \end{aligned} \tag{E.5}$$

where

$$\sigma_{ijk} \;=\; \rho_{ij}\,y_{ij}^{-1} \;-\; \rho_{jk}\,y_{jk}^{-1}\,. \tag{E.6}$$

One can also see this directly from the relation between both variables and the Minkowski superspace θ: Due to the matrix notation introduced in Appendix B we can drop the Lorentz and internal indices (writing θ' for $\theta^{a'}$) and rather give the variables a point label. We thus have

$$\rho_i \;=\; \theta_i \;+\; \theta'_i\,y_i\,. \tag{E.7}$$

In the light-cone limit differences of θ^A can be expressed in terms of η^A and bosonic spinors:

$$\theta_{i,i+1} \;=\; |i\rangle\,\eta_i\,, \qquad \theta'_{i,i+1} \;=\; |i\rangle\,\eta'_i \qquad x_{i,i+1} \;=\; |i\rangle\,[i|\,. \tag{E.8}$$

¿From the definition (E.7) we obtain

$$\theta_{12} \;=\; \rho_{12} - \theta'_1\,y_1 + \theta'_2\,y_2 \;=\; \rho_{12} - \theta'_1\,y_{12} - \theta'_{12}\,y_2\,. \tag{E.9}$$

Our goal is to write η_i, η'_i in terms of the analytic superspace variables ρ_i and y_i. Since we want linearity we unfortunately have to keep θ'_1 in the equation, while the isolated y_2 does not look critical.

The R invariants first occur at five points, which thus seems to be a natural and sufficiently non-trivial example. Proceeding like in (E.9) (which is repeated for completeness) we find the system

$$\begin{aligned} E_1: \quad \theta_{12} &= -\theta'_{12}\,y_2 + (-\theta'_1)\,y_{12} + \rho_{12}\,, \\ E_2: \quad \theta_{23} &= -\theta'_{23}\,y_3 + (\theta'_{12} - \theta'_1)\,y_{23} + \rho_{23}\,, \\ E_3: \quad \theta_{34} &= -\theta'_{34}\,y_4 + (\theta'_{12} + \theta'_{23} - \theta'_1)\,y_{34} + \rho_{34}\,, \\ E_4: \quad \theta_{45} &= -\theta'_{45}\,y_5 + (\theta'_{12} + \theta'_{23} + \theta'_{34} - \theta'_1)\,y_{45} + \rho_{45}\,. \end{aligned} \tag{E.10}$$

The $\theta_{51} = \ldots$ condition is not independent, of course. Putting in the light-cone variables the system becomes

$$\begin{aligned} E_1: \quad |1\rangle\,\eta_1 &= -|1\rangle\,\eta_1'\,y_2 + (-\theta_1')\,y_{12} + \rho_{12}\,, \\ E_2: \quad |2\rangle\,\eta_2 &= -|2\rangle\,\eta_2'\,y_3 + (|1\rangle\,\eta_1' - \theta_1')\,y_{23} + \rho_{23}\,, \\ E_3: \quad |3\rangle\,\eta_3 &= -|3\rangle\,\eta_3'\,y_4 + (|1\rangle\,\eta_1' + |2\rangle\,\eta_2' - \theta_1')\,y_{34} + \rho_{34}\,, \\ E_4: \quad |4\rangle\,\eta_4 &= -|4\rangle\,\eta_4'\,y_5 + (|1\rangle\,\eta_1' + |2\rangle\,\eta_2' + |3\rangle\,\eta_3' - \theta_1')\,y_{45} + \rho_{45}\,. \end{aligned} \tag{E.11}$$

Every equation E_i splits into two conditions because we can project with two different bosonic spinors. We label

$$E_{ia} = \langle i|\,E_i\,, \qquad E_{ib} = \langle i+1|\,E_i\,. \tag{E.12}$$

These are eight equations whereas we try to solve for four η_i and four η_i' and the two projections of θ_1', so a total of ten quantities. What we can additionally invoke is the conservation condition on the θ's. Splitting it into primed and un-primed halves and projecting with $\langle 4|$, $\langle 5|$ we find the four conditions

$$\begin{aligned} F_b: \quad \eta_4 &= -\frac{\langle 15\rangle}{\langle 45\rangle}\eta_1 - \frac{\langle 25\rangle}{\langle 45\rangle}\eta_2 - \frac{\langle 35\rangle}{\langle 45\rangle}\eta_3 \\ F_a: \quad \eta_4' &= -\frac{\langle 15\rangle}{\langle 45\rangle}\eta_1' - \frac{\langle 25\rangle}{\langle 45\rangle}\eta_2' - \frac{\langle 35\rangle}{\langle 45\rangle}\eta_3' \\ \eta_5 &= \frac{\langle 14\rangle}{\langle 45\rangle}\eta_1 + \frac{\langle 24\rangle}{\langle 45\rangle}\eta_2 + \frac{\langle 34\rangle}{\langle 45\rangle}\eta_3 \\ \eta_5' &= \frac{\langle 14\rangle}{\langle 45\rangle}\eta_1' + \frac{\langle 24\rangle}{\langle 45\rangle}\eta_2' + \frac{\langle 34\rangle}{\langle 45\rangle}\eta_3' \end{aligned} \tag{E.13}$$

The first two of these are the two missing conditions completing our system to a total of ten equations. The other two then simply yield η_5, η_5'. Solving the system is straightforward if cumbersome. We find

$$\langle 1|\theta_1' = \langle 1|\,\rho_{12}\,y_{12}^{-1}\,, \qquad \langle 5|\theta_1' = \langle 5|\,\rho_{51}\,y_{51}^{-1}\,. \tag{E.14}$$

Finally, in terms of the Q supersymmetric combination

$$\sigma_{512} = \rho_{51}\,y_{51}^{-1} - \rho_{12}\,y_{12}^{-1} \tag{E.15}$$

the solution for η, η' takes the simple form

$$\begin{aligned} \eta_1' &= \frac{1}{\langle 51\rangle}\langle 5|\,\sigma_{512} - \frac{1}{\langle 12\rangle}\langle 2|\,\sigma_{123}\,, \\ \eta_1 &= -\frac{1}{\langle 51\rangle}\langle 5|\,\sigma_{512}\,y_1 + \frac{1}{\langle 12\rangle}\langle 2|\,\sigma_{123}\,y_2 \end{aligned} \tag{E.16}$$

and cyclic. Note that the un-primed θ's are unambiguously determined, too, because $\theta_1 = \rho_1 - \theta_1' y_1$.

Our equations for η in terms of ρ carry over to the n-point case: Splitting ρ into θ, y makes (E.16) simplify to $\eta^A = \eta^A$ for any number of points, so this is a general solution. On the other hand, the number of odd degrees of freedom always matches between the ρ_i and the η_i, respectively.

F ρ_i^4 components of R invariants

A general formula for the R invariants [23] in terms of momentum supertwistor variables was given in [13]. The invariant is characterise by 5 labels $r, s-1, s, t-1, t$.

$$R_{r,s-1,s,t-1,t} = \quad \frac{\delta^4\big(\Sigma_{r\,s-1\,s\,t-1\,t}\big)}{\langle s-1\,s\,t-1\,t\rangle\langle s\,t-1\,t\,r\rangle\langle t-1\,t\,r\,s-1\rangle\langle t\,r\,s-1\,s\rangle\langle r\,s-1\,s\,t-1\rangle} \tag{F.1}$$

with

$$\Sigma_{r\,s-1\,s\,t-1\,t} \;=\; \langle s-1\,s\,t-1\,t\rangle\,\chi_r + (\text{cyclic})\,. \tag{F.2}$$

The twistor four-bracket was defined in equation (3.47) in Section 4.6. In the delta function in the numerator we cyclically shift the five arguments $\langle s\!-\!1\,s\,t\!-\!1\,t\rangle\,\chi_r \,\to\, \langle s\,t\!-\!1\,t\,r\rangle\,\chi_{s-1}$ etc. yielding a total of five terms. The evaluation of this visually somewhat stunning formula at a "Lagrangian point" $\rho_j = 0 : j \neq i$ is surprisingly easy because according to formula (2.12) only χ_{i-1}, χ_i are non-vanishing. First, we observe

$$\delta^{(4)}(\chi^A) \;=\; \chi^1\,\chi^2\,\chi^3\,\chi^4 \;=\; \frac{1}{4}\,\epsilon_{ab}\chi^a\chi^b\;\epsilon_{a'b'}\chi^{a'}\chi^{b'} \;=\; \frac{1}{4}\,(\chi)^2(\chi')^2\,. \tag{F.3}$$

Let us focus on the case R_{12345} at ρ_5^4. ¿From (2.12) we have

$$\chi'_4 \;=\; -\langle 4|\rho_5\,y_{45}^{-1}\,, \qquad \chi'_5 \;=\; \langle 5|\rho_5\,y_{51}^{-1} \tag{F.4}$$

and

$$\chi_4 \;=\; -\chi'_4\,y_4\,, \qquad \chi_5 \;=\; -\chi'_5\,y_1\,. \tag{F.5}$$

Then

$$\delta^{(4)}\big(X\,\chi_4^A + Y\,\chi_5^A\big) \;=\; \frac{1}{4}\,\big(X\,\chi_4 + Y\,\chi_5\big)^2\big(X\,\chi'_4 + Y\,\chi'_5\big)^2\,. \tag{F.6}$$

The second square on the right-hand side is equivalent to $\delta(X\,\chi'_4 + Y\,\chi'_5)$ so that we can rewrite the first square as $(X\chi'_4 y_{41})^2$. This in turn sends χ'_4 to zero in the second term, whence

$$\begin{aligned}\delta^4\big(X\,\chi_4^A + Y\,\chi_5^A\big) \;&=\; \frac{1}{4}\,(X\,\chi'_4\,y_{41})^2\,(Y\,\chi'_5)^2 \;=\; \frac{1}{4}\,X^2\,Y^2\,y_{14}^2\,(\chi'_4)^2(\chi'_5)^2 \\ &= \frac{1}{4}\,X^2\,Y^2\frac{y_{14}^2}{y_{45}^2\,y_{51}^2}\,(\langle 4|\rho_5)^2(\langle 5|\rho_5)^2 \;=\; X^2\,Y^2\,\langle 45\rangle^2\,\frac{y_{14}^2}{y_{45}^2\,y_{51}^2}\,\rho_5^4\end{aligned} \tag{F.7}$$

where we have put $\rho^4 \;=\; \frac{1}{12}\,(\rho^2)^{(\alpha\beta)}(\rho^2)_{(\alpha\beta)}$ as before.

The five-point case is somewhat degenerate because all five χ_i occur in a cyclic fashion in the numerator, whereby the labelling is arbitrary. To be definite let $r = 1$. We find immediately

$$R_{12345}|_{\rho_5^4} = \frac{\langle 45\rangle^2\langle 5123\rangle\langle 1234\rangle}{\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle}\frac{y_{14}^2}{y_{45}^2\,y_{51}^2}\,\rho_5^4 = \frac{x_{13}^2x_{24}^2}{x_{14}^2x_{25}^2x_{35}^2}\frac{y_{14}^2}{y_{45}^2\,y_{51}^2}\,\rho_5^4 \tag{F.8}$$

where (3.48) was used to translate to x space.

At six points or above, Lagrangian components of the R invariants may vanish: In some constellations only one or no non-vanishing χ will be amongst the five terms in the argument of the delta function. As in the main text, at six points we label the $R_{r,s-1,s,t-1,t}$ by the missing point, so e.g. $R_1 = R_{23456}$. For a start, we wish to rewrite $R_1|_{\rho_6^4}$ above in terms of x space variables. Using (F.7) we can immediately rewrite as follows:

$$R_1 = \frac{x_{35}^2}{x_{46}^2x_{36}^2}\frac{\langle 6234\rangle\langle 45\rangle}{\langle 4562\rangle\langle 34\rangle}\frac{y_{15}^2}{y_{16}^2y_{56}^2}\rho_6^4 = \frac{x_{35}^2}{x_{46}^2x_{36}^2}\frac{y_{15}^2}{y_{16}^2y_{56}^2}\rho_6^4 \times I \tag{F.9}$$

To do this note that $I\,x_{15}^2/x_{14}^2$ is conformally invariant:

$$\frac{x_{15}^2}{x_{14}^2}I = \frac{\langle 6234\rangle\langle 6145\rangle}{\langle 4562\rangle\langle 6134\rangle} = \frac{z_{51}z_{23}}{z_{13}z_{25}}\,. \tag{F.10}$$

Here we have substituted the momentum twistor four-brackets with the variables z_i as discussed previously in Section 3.5.2 in the main text. The one-dimensional cross-ratio can be rewritten in terms of the standard six-point cross-ratios:

$$u_1 = \frac{x_{31}^2x_{46}^2}{x_{36}^2x_{41}^2} = \frac{z_{12}z_{45}}{z_{14}z_{25}}\,, \qquad u_2 = \frac{x_{15}^2x_{24}^2}{x_{14}^2x_{25}^2} = \frac{z_{23}z_{56}}{z_{25}z_{36}}\,, \qquad u_3 = \frac{x_{26}^2x_{35}^2}{x_{25}^2x_{36}^2} = \frac{z_{34}z_{61}}{z_{36}z_{41}} \tag{F.11}$$

to give

$$\frac{x_{15}^2}{x_{14}^2}\,I = \frac{1-u_1+u_2-u_3+\sqrt{\Delta}}{2(1-u_3)}\,, \qquad \Delta = (1-u_1-u_2-u_3)^2 - 4\,u_1u_2u_3\,. \tag{F.12}$$

To obtain this expression, the simplest way is to use one-dimensional conformal invariance: Set, say $z_1 = 0, z_2 = \infty, z_3 = 1$ so that (F.11) determines u_1, u_2 and u_3 in terms of z_4, z_5, z_6 and then take the same limit in (F.10) replacing the z with the u. Conversely, in the z variables Δ becomes a perfect square so that it is easy to verify the last formula. Collecting terms we get

$$R_1|_{\rho_6^4} = \frac{x_{35}^2x_{14}^2}{x_{46}^2x_{36}^2x_{15}^2}\Big(\frac{1-u_1+u_2-u_3+\sqrt{\Delta}}{2(1-u_3)}\Big)\frac{y_{15}^2}{y_{16}^2y_{56}^2}\rho_6^4\,. \tag{F.13}$$

In a similar fashion we obtain

$$
\begin{aligned}
R_3|_{\rho_6^4} &= \frac{x_{35}^2 x_{14}^2}{x_{46}^2 x_{36}^2 x_{15}^2}\Big(\frac{1+u_1-u_2-u_3-\sqrt{\Delta}}{2u_3(1-u_3)}\Big)\frac{y_{15}^2}{y_{16}^2 y_{56}^2}\rho_6^4\,,\\
R_5|_{\rho_6^4} &= 0\,,\\
R_2|_{\rho_6^4} &= \frac{x_{35}^2 x_{14}^2}{x_{46}^2 x_{36}^2 x_{15}^2}\Big(\frac{1-u_1-u_2+u_3-\sqrt{\Delta}}{2u_3(1-u_1)}\Big)\frac{y_{15}^2}{y_{16}^2 y_{56}^2}\rho_6^4\,,\\
R_4|_{\rho_6^4} &= \frac{x_{35}^2 x_{14}^2}{x_{46}^2 x_{36}^2 x_{15}^2}\Big(\frac{u_1(1-u_1+u_2-u_3+\sqrt{\Delta})}{2u_3(1-u_1)}\Big)\frac{y_{15}^2}{y_{16}^2 y_{56}^2}\rho_6^4\,,\\
R_6|_{\rho_6^4} &= 0\,.
\end{aligned}
\tag{F.14}
$$

References

[1] L. Alday and J. Maldacena, *JHEP* **0706** (2007) 064 [arXiv:0705.0303 [hep-th]].

[2] J. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231 [arXiv:hep-th/9711200]; S. Gubser, I. Klebanov and A. Polyakov, *Phys. Lett.* **B428** (1998) 105 [arXiv:hep-th/9802109]; E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253 [arXiv:hep-th/9802150].

[3] G. P. Korchemsky, J. M. Drummond and E. Sokatchev, *Nucl. Phys.* **B795** (2008) 385 [arXiv:0707.0243 [hep-th]].

[4] A. Brandhuber, P. Heslop and G. Travaglini, *Nucl. Phys.* **B794** (2008) 231 [arXiv:0707.1153 [hep-th]].

[5] J. Drummond, J. Henn, G. Korchemsky and E. Sokatchev, *Nucl. Phys.* **B815** (2009) 142 [arXiv:0803.1466 [hep-th]].

[6] Z. Bern, L. Dixon, D. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, *Phys. Rev.* **D78** (2008) 045007 [arXiv:0803.1465 [hep-th]].

[7] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena and E. Sokatchev, arXiv:1007.3243 [hep-th].

[8] B. Eden, G. P. Korchemsky and E. Sokatchev, arXiv:1007.3246 [hep-th].

[9] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, JHEP **1101** (2011) 041 [arXiv:1008.2958 [hep-th]].

[10] A. Hodges, arXiv:0905.1473 [hep-th].

[11] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. **B715** (2005) 499-522. [hep-th/0412308]; R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94** (2005) 181602. [hep-th/0501052].

[12] R. H. Boels, JHEP **1011** (2010) 113. [arXiv:1008.3101 [hep-th]].

[13] L. J. Mason and D. Skinner, JHEP **1012** (2010) 018 [arXiv:1009.2225 [hep-th]].

[14] S. Caron-Huot, arXiv:1010.1167 [hep-th].

[15] A. V. Belitsky, G. P. Korchemsky and E. Sokatchev, arXiv:1103.3008 [hep-th].

[16] B. Eden, G. P. Korchemsky and E. Sokatchev, arXiv:1009.2488 [hep-th].

[17] B. Eden, P. Heslop, G. P. Korchemsky and E. Sokatchev, arXiv:1103.3714 [hep-th].

[18] B. Eden, C. Schubert and E. Sokatchev, Phys. Lett. **B482** (2000) 309-314. [hep-th/0003096].

[19] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev and P. West, Nucl. Phys. **B557** (1999) 355-379 [hep-th/9811172].

[20] B. Eden, A. C. Petkou, C. Schubert and E. Sokatchev, Nucl. Phys. **B607** (2001) 191-212 [hep-th/0009106].

[21] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo and J. Trnka, arXiv:1012.6032 [hep-th].

[22] V. P. Nair, Phys. Lett. **B214** (1988) 215.

[23] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. **B828** (2010) 317-374 [arXiv:0807.1095 [hep-th]].

[24] G. G. Hartwell and P. S. Howe, Int. J. Mod. Phys. **A10** (1995) 3901-3920 [hep-th/9412147]; Class. Quant. Grav. **12** (1995) 1823-1880.

[25] P. J. Heslop, P. S. Howe, Phys. Lett. **B516** (2001) 367-375. [hep-th/0106238]; P. J. Heslop, P. S. Howe, Nucl. Phys. **B626** (2002) 265-286. [hep-th/0107212]; P. J. Heslop, P. S. Howe, JHEP **0401** (2004) 058. [hep-th/0307210].

[26] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984) 469-498; A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **2** (1985) 601; Class. Quant. Grav. **2** (1985) 617.
A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, E. S. Sokatchev, "Harmonic superspace," Cambridge, UK: Univ. Pr. (2001) 306 p.

[27] B. Eden, P. S. Howe and P. C. West, Phys. Lett. **B463** (1999) 19-26 [hep-th/9905085].

[28] A. Brandhuber, P. Heslop, G. Travaglini, Phys. Rev. **D78** (2008) 125005. [arXiv:0807.4097 [hep-th]].

[29] A. Hodges, arXiv:0905.1473. N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP **1003** (2010) 020 [arXiv:0907.5418]. L. J. Mason, D. Skinner, JHEP **0911** (2009) 045. [arXiv:0909.0250 [hep-th]].

[30] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, "Magic identities for conformal four-point integrals," JHEP **0701** (2007) 064 [hep-th/0607160].

[31] P. S. Howe, C. Schubert, E. Sokatchev and P. West, Nucl. Phys. **B571** (2000) 71-90 [hep-th/9910011];

[32] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev and P. West, Phys. Lett. **B466** (1999) 20-26 [hep-th/9906051].

[33] F. Gonzalez-Rey, I. Y. Park and K. Schalm, Phys. Lett. **B448** (1999) 37-40 [hep-th/9811155].

[34] M. Bianchi, S. Kovacs, G. Rossi and Y. Stanev, Nucl. Phys. **B584** (2000) 216-232 [hep-th/0003203].

[35] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B **198** (1982) 474.

[36] Z. Bern, L. J. Dixon, V. A. Smirnov, Phys. Rev. **D72** (2005) 085001 [hep-th/0505205].

[37] Z. Bern, M. Czakon, D. A. Kosower, R. Roiban, V. A. Smirnov, Phys. Rev. Lett. **97** (2006) 181601 [hep-th/0604074]; F. Cachazo, M. Spradlin, A. Volovich, Phys. Rev. **D74** (2006) 045020. [hep-th/0602228].

[38] A. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Phys. Rev. Lett.* **105** (2010) 151605 [arXiv:1006.5703 [hep-th]].

[39] Z. Bern, L. J. Dixon, D. C. Dunbar, D. A. Kosower, Nucl. Phys. **B425** (1994) 217-260. [hep-ph/9403226].

[40] M. Czakon, Comput. Phys. Commun. **175** (2006) 559-571 [hep-ph/0511200].

[41] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, C. Ratti and A. Santambrogio, arXiv:1103.3675 [hep-th].

[42] T. Adamo, M. Bullimore, L. Mason and D. Skinner, arXiv:1103.4119 [hep-th].

[43] F. A. Dolan and H. Osborn, Nucl. Phys. B **629** (2002) 3 [arXiv:hep-th/0112251].

[44] F. A. Dolan, L. Gallot and E. Sokatchev, JHEP **0409** (2004) 056 [arXiv:hep-th/0405180].

[45] P. J. Heslop, P. S. Howe, JHEP **0301** (2003) 043. [hep-th/0211252].

[46] The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A002137