THE SKEW DIAGRAM POSET AND COMPONENTS OF SKEW CHARACTERS

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ABSTRACT. We investigate the poset of skew diagrams ordered by adding or forming the union of skew diagrams. We will show that a skew diagram which has at least n convex corners to the upper left and also to the lower right is larger than the skew diagram consisting of n disconnected single boxes. Using this property, we obtain lower bounds for the number of components, constituents and pairs of components which differ by one box in a given skew character.

1. INTRODUCTION AND NOTATION

Characters of the symmetric group are being investigated since the beginning of the 20th century. Skew characters of the symmetric group decompose in the same way as skew Schur functions and their decomposition corresponds to the decomposition of products of Schubert classes (see [Gut1]) as well as the decomposition of the restriction of irreducible affine Hecke algebras to the Iwahori-Hecke algebras (see [Ram]).

We introduce a new poset on the set of skew diagrams (Section 2) which allows us to obtain results about skew characters (Section 3).

We mostly follow the standard notation in [Sag] or [Sta]. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a weakly decreasing sequence of non-negative integers where only finitely many of the λ_i are positive. We regard two partitions as the same if they differ only by the number of trailing zeros and call the positive λ_i the parts of λ. The length is the number of positive parts and we write $l(\lambda) = l$ for the length and $|\lambda| = \sum_i \lambda_i$ for the sum of the parts. With a partition λ we associate a diagram, which we also denote by λ, containing λ_i left-justified boxes in the i-th row and we use matrix style coordinates to refer to the boxes.

We write $dp(\lambda) = n$ if the partition λ has n different parts. Furthermore we set

$$\delta_n = (n, n-1, n-2, \ldots, 2, 1).$$

The conjugate λ^c of λ is the diagram which has λ_i boxes in the i-th column.

For $\mu \subseteq \lambda$ we define the skew diagram λ/μ as the difference of the diagrams λ and μ defined as the difference of the set of the boxes. Rotation of λ/μ by $180°$ yields a skew diagram $(\lambda/\mu)°$ which is well defined up to translation.

A skew tableau T is a skew diagram in which positive integers are written into the boxes. A semistandard tableau of shape λ/μ is a filling of λ/μ with positive integers such that the entries weakly increase amongst the rows from left to right and strictly increase amongst the columns from top to bottom. The content of a semistandard tableau T is $\nu = (\nu_1, \ldots)$ if the number of occurrences of the entry i

in T is ν_i. The reverse row word of a tableau T is the sequence obtained by reading the entries of T from right to left and top to bottom starting at the first row. Such a sequence is said to be a lattice word if for all $i, n \geq 1$ the number of occurrences of i among the first n terms is at least the number of occurrences of $i+1$ among these terms. The Littlewood-Richardson (LR) coefficient $c(\lambda; \mu, \nu)$ equals the number of semistandard tableaux of shape λ/μ with content ν such that the reverse row word is a lattice word. We will call those tableaux LR tableaux. The LR coefficients play an important role in different contexts (see [Sag] or [Sta] for further details).

A standard Young tableaux of shape λ is a filling of λ with the numbers $1, \ldots, |\lambda|$ such that the entries increase in each row from left to right and in each column from top to bottom. The number of standard Young tableaux of shape λ is denoted by f^λ which is given by the well known hook length formula

$$f^\lambda = \frac{|\lambda|!}{\prod(\text{hook length})}.$$

Obviously the number of standard Young tableaux with n boxes f_n is given by $f_n = \sum_{\lambda \vdash n} f^\lambda$. Notice furthermore that f_n is also the number of involutions in the symmetric group S_n plus 1.

The irreducible characters $[\lambda]$ of the symmetric group S_n are naturally labeled by partitions $\lambda \vdash n$. The skew character $[\lambda/\mu]$ corresponding to a skew diagram λ/μ is defined by the LR coefficients

$$[\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)[\nu].$$

The translation symmetry gives $[\lambda/\mu] = [\alpha/\beta]$ if the skew diagrams of λ/μ and α/β are the same up to translation while rotation symmetry gives $[(\lambda/\mu)^\circ] = [\lambda/\mu]$. The conjugation symmetry $c(\lambda^c; \mu^c, \nu^c) = c(\lambda; \mu, \nu)$ is also well known and furthermore we have $c(\lambda; \mu, \nu) = c(\lambda; \nu, \mu)$.

A basic skew diagram λ/μ is a skew diagram which satisfies $\mu_i < \lambda_i$ and $\mu_i \leq \lambda_{i+1}$ for each $1 \leq i \leq l(\lambda)$. This means that λ/μ doesn't contain empty rows or column in λ/μ. Empty rows or columns of a skew diagram don't influence the filling and so deleting empty rows or columns doesn't change the skew character or LR fillings.

Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty sub-diagrams of a skew diagram $\mathcal{D}$ such that the union of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{D}$. Then we say that the skew diagram $\mathcal{D}$ is *disconnected* or *decays* into the skew diagrams $\mathcal{A}$ and $\mathcal{B}$ if no box of $\mathcal{A}$ (viewed as boxes in $\mathcal{D}$) is in the same row or column as a box of $\mathcal{B}$. Notice, that this also covers the case, when $\mathcal{B}$ again decays into two subdiagrams and $\mathcal{A}$ is between those two. We write $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$ if up to translation $\mathcal{D}$ decays into $\mathcal{A}$ and $\mathcal{B}$ and normally write $\mathcal{A}$ and $\mathcal{B}$ as basic skew diagrams. A skew diagram is connected if it does not decay. If $\mathcal{D} = \mathcal{A} \otimes \mathcal{B} = \mathcal{C}$ then by translation symmetry $[\mathcal{D}] = [\mathcal{C}]$, so reordering $\mathcal{A}, \mathcal{B}$ doesn't change the skew character.

A skew character whose skew diagram $\mathcal{D}$ decays into the skew diagrams $\mathcal{A}, \mathcal{B}$ is equivalent to the product of the characters of the disconnected diagrams induced to a larger symmetric group. We have

$$[\mathcal{D}] = ([\mathcal{A}] \times [\mathcal{B}]) \uparrow_{S_n \times S_m}^{S_{n+m}} =: [\mathcal{A}] \otimes [\mathcal{B}]$$

with $|\mathcal{A}| = n, |\mathcal{B}| = m$. If $\mathcal{D} = \lambda/\mu$ and $\mathcal{A}, \mathcal{B}$ are proper partitions α, β then we have

$$[\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)[\nu] = \sum_\nu c(\nu; \alpha, \beta)[\nu] = [\alpha] \otimes [\beta].$$

2. THE POSET OF SKEW DIAGRAMS

For a partition λ we can define a path starting to the right at the lower left corner of λ and following the shape of λ to the upper right corner, ending with an upward going segment. We write this path as a sequence of v's and h's denoting either a vertical or horizontal step.
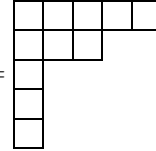
For example the path corresponding to the partition $\lambda = (5, 3, 1, 1, 1) =$

is given by the sequence
$$s = (hvvvhhvhhv).$$

For a skew diagram λ/μ we define two paths, the outer and inner path. The outer path, whose sequence we denote by $o(\lambda/\mu)$ or simply o, is the path of λ. The inner path, whose sequence we denote by $i(\lambda/\mu)$ or i, starts in the lower left corner of λ/μ upwards to the lower left corner of μ (there is no upward step if $l(\lambda) = l(\mu)$) follows the path of μ and ends with steps to the right at the upper right, provided $\mu_1 < \lambda_1$.

So for $\lambda/\mu = (5, 3, 1, 1, 1)/(5, 2, 1)$ (see Figure 1)
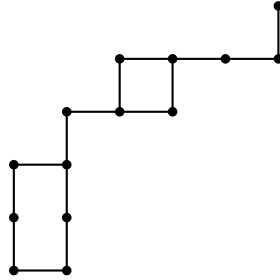
Figure 1. $\lambda/\mu = (5, 3, 1, 1, 1)/(5, 2, 1)$

we have the paths $o = s$ as above and $i = (vvhvhvhhhv)$.

Let λ/μ be a skew diagram. Then both the o and i sequence have $\lambda_1 + l(\lambda)$ entries. Furthermore, both sequences have λ_1 entries h and $l(\lambda)$ entries v. For all $j \leq \lambda_1 + l(\lambda)$ the number of entries h among the first j entries of i is at most the number of entries h among the first j entries of o. Otherwise the partition μ wouldn't be contained in λ. Furthermore, if those numbers are equal then the o and i path touch each other after j steps. From this follows, that for a basic skew diagram there is no $j < \lambda_1 + l(\lambda)$ such that the number of entries h among the first j entries of i and o are the same and both sequences continue with the same entry in the $j + 1$'st position.

The sum $\mu + \nu = \lambda$ of two partitions μ, ν is defined by $\lambda_i = \mu_i + \nu_i$. The partition $\mu \cup \nu$ contains the parts of both μ and ν. These operations are conjugate to another
$$(\mu + \nu)^c = \mu^c \cup \nu^c$$
and do not commute
$$(\lambda \cup \mu) + \nu \neq (\lambda + \nu) \cup \mu.$$

For example, we have

Note that $\mu \cup \nu = \mu \cup \nu_1 \cup \nu_2 \cup \cdots \cup \nu_{l(\nu)}$. Because of this we sometimes say that for $\mu + \nu$ we insert the columns of ν into μ and for $\mu \cup \nu$ that we insert the rows of ν into μ. Note that this $+$ and $\cup$ introduce a partial order on the set of partitions and we say that a partition λ is larger than λ' if λ can be obtained from λ' by repeatedly using the operations $+, \cup$ with arbitrary partitions in any order. This should not be confused with the lexicographic order. Note that the sequences of μ and $\mu + (1^n)$ differ by one h if $n \leq l(\mu)$. By symmetry the sequences of μ and $\mu \cup (n)$ differ by one v if $n \leq \mu_1$. To be more precise, the sequence of $\mu + (1^n)$ ($n \leq l(\mu)$) is obtained from μ by inserting an h such that there are exactly n entries v to the right of the new h which also means that there are exactly $l(\mu) - n$ entries v to the left.

For two skew diagrams $\mathcal{A} = \lambda/\mu, \mathcal{B} = \lambda'/\mu'$ we define the operations $\mathcal{A} + \mathcal{B} = \alpha/\beta$ and $\mathcal{A} \cup \mathcal{B} = \alpha'/\beta'$ by $\alpha = \lambda + \lambda', \beta = \mu + \mu'$ and $\alpha' = \lambda \cup \lambda', \beta' = \mu \cup \mu'$, respectively. Clearly $\mathcal{A} + \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ are then again skew diagrams. Usually we regard two skew diagrams as the same if they contain boxes in the same position but for this definition the underlying partitions λ, λ', μ and μ' are important because different choices for λ and μ would lead to different α/β. For example, we have $(2,1)/(1^2) = $ ☐. But if we add in both cases (1^2) we would get $(3,2)/(1^2) = $ $\neq$ $= (3,1)/(1)$. However, this will never cause any problem because in general we assume that the skew diagrams are basic.

On the set of basic skew diagrams we define a partial order as follows. Let $\mathcal{A}, \mathcal{B}$ be skew diagrams, then we say that $\mathcal{A}$ is greater or equal to $\mathcal{B}$ if there exists $n \in \mathbb{N}$ and for $1 \leq i \leq n$ it is $\circ^i \in \{+, \cup\}$ and $\mathcal{C}^i$ a skew diagram so that we have

$$\mathcal{A} = \left(\cdots \left((\mathcal{B} \circ^1 \mathcal{C}^1) \circ^2 \mathcal{C}^2 \right) \cdots \right) \circ^n \mathcal{C}^n.$$

Notice that it is not enough that λ or μ are larger than α or β, respectively, for λ/μ to be larger than α/β. For example, (2) is clearly larger than (1) but $(3,2)/(2) = $ is not larger than $(3,2)/(1) = $.
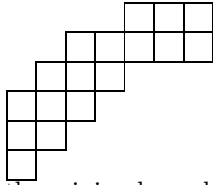
What are the covering relations? Let α/β and λ/μ be basic skew diagrams, then λ/μ covers α/β, $\alpha/\beta < \cdot \lambda/\mu$, if either

- $\lambda/\mu = \alpha/\beta + (1^x)/(1^y)$ with $0 \leq y \leq x \leq l(\alpha)$ or
- $\lambda/\mu = \alpha/\beta \cup (x)/(y)$ with $0 \leq y \leq x \leq \alpha_1$.
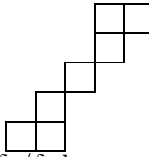
Note that we assumed that both λ/μ and α/β are basic. If $\lambda/\mu = \alpha/\beta + (1^x)/(1^y)$ with $0 \leq y \leq x \leq l(\alpha)$ but λ/μ is not basic, then λ/μ does not cover α/β.
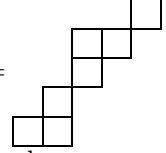
Note that λ/μ has exactly one non-empty row or column more than α/β if λ/μ covers α/β. So the above partial order is a graded partial order with ranking function $\rho(\lambda/\mu) = \lambda_1 + l(\lambda)$ the number of non-empty rows and columns of the basic skew diagram λ/μ.

Definition 2.1. Let λ/μ be a basic skew diagram, and let n be minimal with $dp(\mu) + 1, dp(\lambda) \geq n$. We then say, that λ/μ has δ value n and write $\delta(\lambda/\mu) = n$.

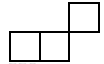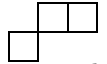For example $\lambda/\mu =$ [diagram] has δ value 4. Note, that the δ value of a given skew diagram is the minimal number of convex corners of either the inner or outer path. Furthermore, the skew diagram consisting of n disconnected boxes [box] has δ value n, so $\delta(\delta_n/\delta_{n-1}) = n$. We would like to show that λ/μ with $\delta(\lambda/\mu) = n$ is larger than δ_n/δ_{n-1}, but this is false.

Take for example the skew diagram $\lambda/\mu =$ [diagram] with $\delta(\lambda/\mu) = 4$. It is easy to see, that λ/μ can not be obtained from δ_4/δ_3 by repeatedly applying $+$ and $\cup$ to δ_4/δ_3 and so λ/μ is not larger than δ_4/δ_3. On the other hand $\alpha/\beta =$ [diagram] is obtained by reordering the disconnected components of λ/μ and we have

$$\left(\text{[diagram]} + (1^4)/(1^1) \right) \cup (2)/(2) = \text{[diagram]} \cup (2)/(2) = \text{[diagram]} = \alpha/\beta.$$

To fix this, we will now introduce an equivalence relation on the set of basic skew diagrams, so that $\overline{\lambda/\mu} = \overline{\alpha/\beta}$ if λ/μ and α/β are the same up to translation of the skew diagrams into which λ/μ and α/β may decompose. For example $\lambda/\mu =$ [diagram] and $\alpha/\beta =$ [diagram] both decompose into [box] $\otimes$ [boxes] and so $\overline{\lambda/\mu} = \overline{\alpha/\beta}$. We may now define a partial order on the set of these equivalence classes by giving the cover relations and assume transitivity. Let $\overline{\lambda/\mu}$ cover $\overline{\alpha/\beta}$ if there is a skew diagram $\mathcal{A} \in \overline{\lambda/\mu}$ which covers a skew diagram $\mathcal{B} \in \overline{\alpha/\beta}$. All skew diagrams in the same equivalence class have the same number of non-empty rows and columns. From this follows that also the poset of equivalence classes is graded with grading function $\rho(\overline{\lambda/\mu}) = \lambda_1 + l(\lambda)$. Notice that $\overline{\lambda/\mu} > \overline{\alpha/\beta}$ does not require the existence of skew diagrams $\mathcal{A} \in \overline{\lambda/\mu}, \mathcal{B} \in \overline{\alpha/\beta}$ with $\mathcal{A} > \mathcal{B}$. Setting $\delta(\overline{\lambda/\mu}) = \delta(\lambda/\mu)$ is well defined.

Lemma 2.2. *Let $\lambda/\mu = (1) \otimes \alpha/\beta$ with $\delta(\alpha/\beta) = n$ and assume that $\overline{\alpha/\beta} \geq \overline{\delta_n/\delta_{n-1}}$.*
Then $\overline{\lambda/\mu} \geq \overline{\delta_{n+1}/\delta_n}$.

Proof. We may assume, that λ/μ has in the lower left corner the single box (1) and atop to the right the skew diagram α/β.

Since $\overline{\alpha/\beta} \geq \overline{\delta_n/\delta_{n-1}}$ there is a sequence of covering skew diagrams from δ_n/δ_{n-1} to α/β. So we can choose $\circ^i \in \{+, \cup\}$ and $\mathcal{A}^i \in \{(1^{a+b})/(1^b), (a+b)/(b)\}$ such that with $\mathcal{B}^0 = \delta_n/\delta_{n-1}, \mathcal{B}^m = \alpha/\beta$ and $\mathcal{C}^i = \mathcal{B}^{i-1} \circ^i \mathcal{A}^i$ with $\overline{\mathcal{C}^i} = \overline{\mathcal{B}^i}$ we have $\overline{\mathcal{B}^i} <_{\cdot} \overline{\mathcal{B}^{i+1}}$.

Now, set $\widetilde{\mathcal{A}} = (a+b+1)/(b+1)$ if $\mathcal{A} = (a+b)/(b)$ and $\widetilde{\mathcal{A}} = \mathcal{A}$ otherwise.

Let $\widetilde{\mathcal{B}^0} = \delta_{n+1}/\delta_n, \widetilde{\mathcal{C}^m} = \lambda/\mu$ and let $\widetilde{\mathcal{B}^i} = (1) \otimes \mathcal{B}^i$ and $\widetilde{\mathcal{C}^i} = (1) \otimes \mathcal{C}^i$ such that both $\widetilde{\mathcal{B}^i}, \widetilde{\mathcal{C}^i}$ contain a single disconnected box in the lower left corner. We then have $\widetilde{\mathcal{C}^i} = \widetilde{\mathcal{B}^{i-1}} \circ^i \widetilde{\mathcal{A}^i}$ with $\overline{\widetilde{\mathcal{C}^i}} = \overline{\widetilde{\mathcal{B}^i}}$ and so $\overline{\widetilde{\mathcal{B}^i}} <_{\cdot} \overline{\widetilde{\mathcal{B}^{i+1}}}$. $\qquad\square$

Theorem 2.3. *Let λ/μ be a basic skew diagram with $\delta(\lambda/\mu) = n$.*
Then $\overline{\lambda/\mu} \geq \overline{\delta_n/\delta_{n-1}}$.

Proof. We will give a procedure to reduce λ/μ by one rank without changing the δ value. Repeatedly applying this procedure will result in a minimal skew diagram with fixed δ value and we will see, that we can always reduce the rank by one without changing the δ value, unless $\lambda/\mu = \delta_n/\delta_{n-1}$. This shows, that $\overline{\delta_n/\delta_{n-1}}$ is the unique minimal element with $\delta(\overline{\lambda/\mu}) = n$ and that all $\overline{\lambda/\mu}$ with $\delta(\overline{\lambda/\mu}) = n$ are larger than $\overline{\delta_n/\delta_{n-1}}$.

To give this procedure we will call a pair (X_i, X_o) where X_i denotes the ith step of the inner sequence and X_o denotes the oth step of the outer sequence a *removable pairing* if both are either h's or v's and if either the inner h is weakly atop the outer h or the inner v is weakly to the left of the outer v. If $X_i = X_o = h$ we will call this an h pairing and if both are v's we will call it an v pairing.

If α/β is obtained from λ/μ by removing a removable pairing then $\lambda/\mu = \alpha/\beta + (1^{a+b})/(1^a)$ in case of h pairings and $\lambda/\mu = \alpha/\beta \cup (a+b)/(a)$ in case of v pairings. In both cases $\lambda/\mu \geq \alpha/\beta$.

Because of Lemma 2.2 we may assume, that λ/μ does not decay into $(1) \otimes \mathcal{A}$ with $\mathcal{A}$ some arbitrary skew diagram. If λ/μ would decay in this way, then $\delta(\mathcal{A}) = n-1$ and if we prove that $\overline{\mathcal{A}} \geq \overline{\delta_{n-1}/\delta_{n-2}}$ then by Lemma 2.2 $\overline{\lambda/\mu} \geq \overline{\delta_n/\delta_{n-1}}$.

We have the following possibilities for the skew diagram. We may assume that in each case none of the previous case applied.

(1) $\lambda/\mu = \delta_n/\delta_{n-1}$ (or to be precise $\lambda/\mu = \emptyset$, because we assumed $\lambda/\mu \neq (1) \otimes \mathcal{A}$). Then there is nothing to prove.

(2) Suppose there is a removable h pairing (h_i, h_o) such that both h_i and h_o are next to another h in the inner and outer sequence, respectively, and removing it reduces the rank by one. Then the inner and outer ways are as follows

$$i: \qquad \ldots \ldots h h_i \ldots$$
$$o: \qquad \ldots h h_o \ldots \ldots .$$

Then we can remove this pair and reduce the rank by one without changing the δ value. The same applies to v pairings instead of h pairings.

(3) Suppose there is a removable h pairing (h_i, h_o) such that both h_i and h_o are next to another h in the inner and outer sequence, respectively, but removing this pairing gives an α/β which has rank more than one less than λ/μ. So the inner and outer way are as follows

$$i: \qquad \ldots \ldots h h_i \ldots$$
$$o: \qquad \ldots h h_o \ldots \ldots .$$

Let (h_i, h_o) be the pairing in λ/μ such that $i - o$ is minimal of all pairings we could choose. Because λ/μ is basic and the h pairing can be removed it is $i > o$. Since λ/μ doesn't cover α/β it follows that α/β can't be basic. So there has to be a k such that in α/β the inner sequence $i_1 \ldots i_k X \ldots$ has in the first k positions the same number of h's (and v's) as the outer sequence $o_1 \ldots \widehat{h_o} \ldots o_{k+1} X \ldots$ has in the first k positions and both continue with the same step $X \in \{h, v\}$, where $\widehat{h_o}$ means, that h_o was removed.

So for λ/μ we have

$\lambda/\mu:$ $i:$ $\ldots\ldots\ldots.i_k\ldots h_i\ldots$

                     $o:$ $\ldots h_o \ldots o_k \ldots\ldots$

while for α/β we have

$\alpha/\beta:$ $i:$ $\ldots\ldots\ldots.i_k X \ldots$

                     $o:$ $\ldots \widehat{h_o} \ldots o_{k+1} X \ldots.$

Let k be minimal with this property.

Since k is minimal, we have $i_k \neq o_{k+1}$. But $i_k = h$ and $o_{k+1} = v$ is not possible, because λ/μ is basic and a skew diagram.

So we have $i_k = h$ and $o_{k+1} = v$.

If $X = h$ (so $i_{k+1} = h$) then the pairing (i_{k+1}, h_o) would be a removable pairing and by choice $k + 1 - o < i - o$ which contradicts the minimality of $i - o$. The pairing (i_{k+1}, h_o) would also be removable using (2) because the minimality of k assures there cannot appear non-basic configurations between the positions o and $k + 1$.
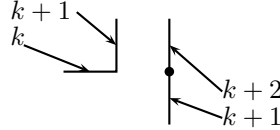
So we have $X = v$ (see Figure 2).

Figure 2. after $X = v$ determined

If $i_{k+2} = v$ we could remove (i_{k+1}, o_{k+2}). This would remove one row and in this situation reduce the rank by only 1 without changing the δ value.

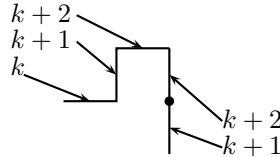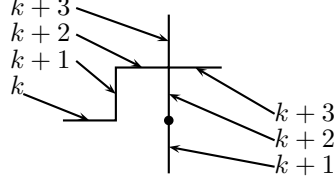So we have $i_{k+2} = h$ (see Figure 3).
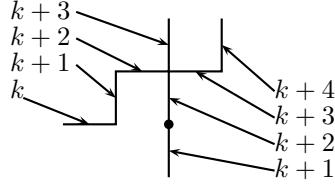

Figure 3. $i_{k+2} = h$

Since λ/μ is basic and the inner and outer path meet after the $k + 2$nd step, it follows that $i_{k+3} \neq o_{k+3}$ and therefore we have $i_{k+3} = v$ and $o_{k+3} = h$ (see Figure 4).

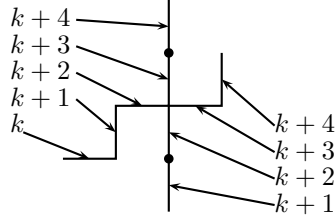**Figure 4.** $i_{k+3} = v$ and $o_{k+3} = h$

If we would have $o_{k+4} = h$, we could remove (h_i, o_{k+4}) which contradicts the minimality of $i - o$.

So we have $o_{k+4} = v$ (see Figure 5).

Figure 5. $o_{k+4} = v$

But if now $i_{k+4} = h$ this would contradict $\lambda/\mu \neq (1) \otimes \mathcal{A}$ so $i_{k+4} = v$ and we have the situation as in Figure 6.

Figure 6. $i_{k+4} = v$

But now (i_{k+3}, o_{k+2}) is a removable v pairing and removing it changes the rank by one without altering the δ value.

The same applies to v pairings instead of h pairings. This means, that it is not possible to have only removable h or v pairing whose removal would, without altering the δ value, reduce the rank by more than one.
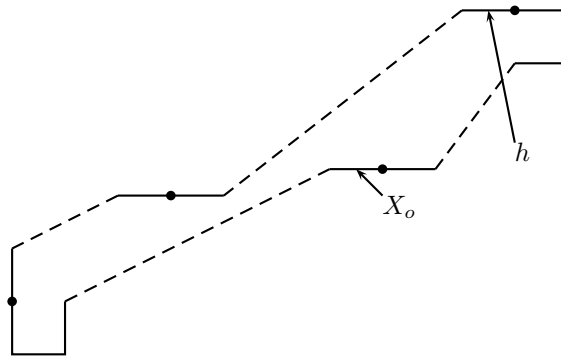
(4) Suppose now that there are only non removable pairings (X_i, X_o) with $X = X_i = X_o \in \{h, v\}$ such that X_i is next to another X in the inner sequence and X_o is next to another X in the outer sequence. Suppose this is an h pairing. For the skew diagram this means, that the outer h is in a higher position than the inner h (see Figure 7).
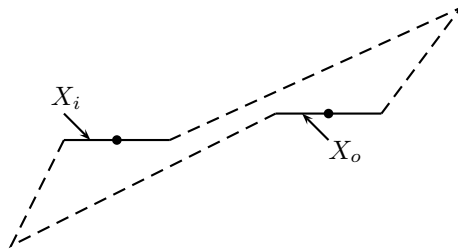
**Figure 7.** λ/μ has only non removable h pairings

Since λ/μ is basic, the outer sequence starts with an h and because there are no removable pairings it continues with an v. Because λ/μ doesn't decay into a single box and another skew diagram the inner sequence has to start with vv. If now the outer sequence would contain a subsequence vv this would give a removable pairing, so the outer sequence does not contain a subsequence vv. Because λ/μ is basic, the outer sequence ends with an v and because it doesn't contain the subsequence vv it ends with hv. Since λ/μ doesn't decay into a single box and another skew diagram the inner sequence has to end with hh. This h in the inner sequence together with X_o from the outer sequence form a removable pairing (see Figure 8).

Figure 8. The removable pairing

(5) So we may now assume that there are no pairings (X_i, X_o) with $X = X_i = X_o \in \{h, v\}$ such that X_i is next to another X in the inner sequence and X_o is next to another X in the outer sequence. By rotation symmetry we may assume that λ has strictly more different parts than μ (otherwise exchange the inner and outer sequence). Since there exists none of the above pairings and $dp(\lambda) > dp(\mu)$ it follows that $\lambda = \delta_m$ for some m and, furthermore, that we have for the inner sequence either $i:\quad \dots hh_i \dots$ or $i:\quad \dots vv_i \dots$.

Suppose we are in the first case that we have $i : \quad \ldots hh_i \ldots$ (see Figure 9).
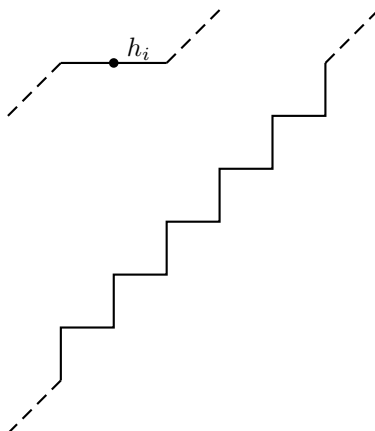
Figure 9. h_i next to another h

Then we can remove the column containing h_i and by doing so reduce the rank by one without changing the δ value (see Figure 10).
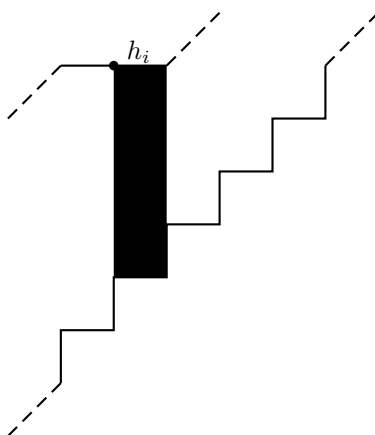
Figure 10. The removable pairing

This finishes the proof. $\qquad\square$

3. Application to skew characters:
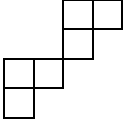### Lower bounds for the number of components, constituents and pairs of components which differ by one box

In this section we are interested in skew characters and so do not strictly distinguish between the skew diagrams and equivalence classes of skew diagrams up to translation.

Definition 3.1. We say that a skew diagram $\mathcal{A}$ or skew character $[\mathcal{A}] = [\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)$ is of cc-type (a, b) if $[\mathcal{A}]$ has $a = \sum_{c(\lambda; \mu, \nu) \neq 0} 1$ components and $b = \sum_\nu c(\lambda; \mu, \nu)$ constituents. We then also write $cc(\mathcal{A}) = (a, b)$ or $cc([\mathcal{A}]) = (a, b)$. Note that always $a \leq b$ so there is no way of confusing the order. Furthermore we say that $\mathcal{A}$ with $cc(\mathcal{A}) = (a, b)$ has cc-type at least (c, d) if $a \geq c$ and $b \geq d$.

For example, the skew character corresponding to $(2, 1) \otimes (2, 1) =$ is

$$[(2, 1)] \otimes [(2, 1)] = [4, 2] + [4, 1^2] + [3^2] + 2[3, 2, 1] + [3, 1^3] + [2^3] + [2^2, 1^2]$$

and so $cc((2, 1) \otimes (2, 1)) = cc([(4, 3, 2, 1)/(2^2)]) = (7, 8)$.

For the following proofs we use the following lemma which we proved in [Gut2] and is a generalization of a lemma in [Gut1].

Lemma 3.2 (Lemma 3.1, [Gut2])**.** *Let* $\lambda, \mu, \nu, \lambda', \mu', \nu'$ *be partitions with* $c(\lambda'; \mu', \nu') \neq 0$.

Then

$$c(\lambda; \mu, \nu) \leq c(\lambda + \lambda'; \mu + \mu', \nu + \nu')$$

and by conjugation

$$c(\lambda; \mu, \nu) \leq c(\lambda \cup \lambda'; \mu \cup \mu', \nu \cup \nu').$$

Remark 3.3. Note that $\lambda^1 + \nu \neq \lambda^2 + \nu$ for $\lambda^1 \neq \lambda^2$ so this lemma tells us that adding a skew diagram $\mathcal{B}$ to a skew diagram $\mathcal{A}$ weakly increases the number of components and constituents of $[\mathcal{A} + \mathcal{B}]$ compared to $[\mathcal{A}]$ (or $[\mathcal{B}]$). By conjugation the same applies to the row wise addition of two skew diagrams $\mathcal{A} \cup \mathcal{B}$. This allows us to consider small examples of $[\mathcal{A}]$ to give a lower bound on the number of components and constituents of larger $[\mathcal{A}']$ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by successively adding, column or row wise, $\mathcal{B}^i$ for some skew diagrams $\mathcal{B}^i$.

We will now introduce a partial order on the set of skew characters by giving the cover relations. Let χ and ψ be skew characters, then we say that χ covers ψ if there exists skew diagrams λ/μ and α/β with $\chi = [\lambda/\mu]$ and $\psi = [\alpha/\beta]$ such that λ/μ covers α/β. Since the number of non empty rows and columns of a skew diagram is fixed for a given skew character this partial order of skew characters is also graded with ranking function $\rho([\lambda/\mu]) = \lambda_1 + l(\lambda)$ for basic skew diagrams λ/μ. Note that this partial order is compatible with the partial order on the set of equivalence classes of skew diagrams of Section 2.

Theorem 3.4. *Let* λ/μ *be a basic skew diagram with* $\delta(\lambda/\mu) = n$. *Then* $cc(\lambda/\mu)$ *is at least* (p_n, f_n) *where* p_n *is the number of partitions of* n *and* f_n *the number of standard Young tableaux with* n *boxes.*

Proof. Let $\delta_n = (n, n - 1, n - 2, \ldots, 2, 1)$ then as an easy consequence of the LR rule we have

$$[\delta_n/\delta_{n-1}] = [\underbrace{(1) \otimes (1) \otimes \cdots \otimes (1)}_{n\text{-times}}] = [1]^n = \sum_{\lambda \vdash n} f^\lambda [\lambda]$$

where f^λ is the number of standard Young tableaux of shape λ. So we have $cc(\delta_n/\delta_{n-1}) = (p_n, f_n)$. Since $\delta(\lambda/\mu) = n$ λ/μ is larger than δ_n/δ_{n-1} by Theorem 2.3 and so $cc(\lambda/\mu)$ is at least (p_n, f_n). $\square$

We will use the following notation in the remaining part of this chapter.

Definition 3.5. We let $\bar{p}_n$ denote the number of partitions of n with two different kinds of 1's and 2's. For the partitions of 2 with two different kinds of 1's and 2's see Example 3.7.

Let g_n denote the number of unordered pairs (ν^1, ν^2) of partitions of n with $|\nu^1 \cap \nu^2| = n - 1$. So g_n counts the pairs of partitions of n which differ only by one box.

Lemma 3.6. *Then $\bar{p}_n = g_{n+2}$ for all n.*

Proof. We give a bijection of partitions of n with two different kinds of 1's and 2's to pairs (ν^1, ν^2) of partitions of $n+2$ which differ only by one box. We may assume that ν^1 is lexicographically larger than ν^2.

Suppose the two kinds of 1's are the usual 1 and the other be $1'$ and the two kinds of 2's are 2 and $2'$. Let $\bar{\lambda}$ be such a partition of n and let λ denote the partition formed by the usual parts of $\bar{\lambda}$. Furthermore, let n_1 denote the number of $1'$ in $\bar{\lambda}$ and n_2 denote the number of $2'$ in $\bar{\lambda}$. So $\bar{\lambda} = \lambda \cup (2'^{n_2}, 1'^{n_1})$.

For a partition $\bar{\lambda}$ now define the bijection by setting

$$\nu^1 = \lambda \cup (n_1 + n_2 + 2, n_2), \qquad \nu^2 = \lambda \cup (n_1 + n_2 + 1, n_2 + 1).$$

Now obviously ν^1 is lexicographically larger than ν^2 and both partitions differ only by one box. Furthermore, different $\bar{\lambda}$ correspond to different triples (λ, n_1, n_2) and so give different pairs (ν^1, ν^2).

Finally the inverse map is obtained as follows. If ν^1 and ν^2 differ by only one box (and ν^1 is lexicographically larger than ν^2), then ν^2 is obtained from ν^1 by removing a box in one row and placing it in a lower row. Let all the other rows form λ then the two rows which are different are of the form $(a + 1)$ and (b) in ν^1 and (a) and $(b+1)$ in ν^2 for $a \geq b \geq 0$. Now $a + 1 > b + 1$ since otherwise $\nu^1 = \nu^2$. So to exclude this case we may instead assume that the rows are $(c + 2)$ and (b) in ν^1 and $(c + 1)$ and $(b + 1)$ in ν^2 for $c \geq b \geq 0$. Setting $n_1 = c - b$ and $n_2 = b$ gives the inverse map. $\qquad\square$

Example 3.7. We have $\bar{p}_2 = 5$ and there is the following correspondence given by the above bijection.

$\bar{\lambda}$	λ	ν^1	ν^2
2	2		
$2'$	0		
1^2	1^2		
$1, 1'$	1		
$1'^2$	0		
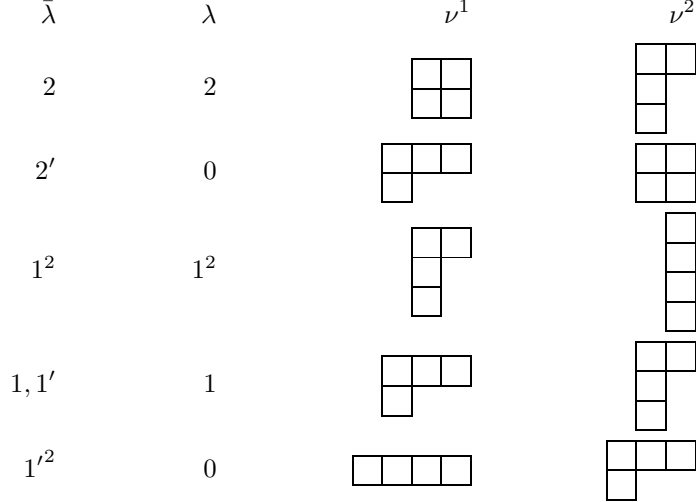
Remark 3.8. Lemma 3.6 is useful because one sees directly that the generating function for $\bar{p}_n$ is given by

$$\sum_{i \geq 0} \bar{p}_i x^i = \frac{1}{(1-x)(1-x^2)} \prod_{i \geq 1} \frac{1}{1-x^i}.$$

In the following theorem the condition $\delta(\lambda/\mu) \geq 2$ only makes sure that λ/μ is neither a partition nor a rotated partition but constrains λ/μ not in any other way. The case that λ/μ is a partition α or rotated partition α° is uninteresting for the theorem because then $[\lambda/\mu] = [\alpha]$ is irreducible.

Theorem 3.9. *Let λ/μ be a basic skew diagram with $\delta(\lambda/\mu) = n \geq 2$.*

Then $[\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)[\nu]$ contains at least g_n pairs of characters $([\nu^1], [\nu^2])$ whose corresponding diagrams differ only by one box, i.e. there are ν^1, ν^2 with $\left|\nu^1 \cap \nu^2\right| = \left|\nu^1\right| - 1 = \left|\nu^2\right| - 1$ and $c(\lambda; \mu, \nu^1), c(\lambda; \mu, \nu^2) \neq 0$ (with g_n as in Lemma 3.6).

Furthermore, if $\lambda = (\lambda_1, \ldots, \lambda_l), \mu = (\mu_1, \ldots, \mu_m)$ with $\lambda_l, \mu_m \geq 1$ set $\mathcal{A} = (\lambda_1 - 2, \lambda_l - 1)/(\mu_1 - 1)$ and $\mathcal{B} = (\lambda_2, \lambda_3, \ldots \lambda_{l-1})/(\mu_2, \mu_3, \ldots, \mu_m)$ with $[\mathcal{A}]$ having a components and $[\mathcal{B}]$ having b components. Then there are at least $\max(a, b)$ of those pairs ν^1, ν^2.

Proof. We first show there are at least $\max(a, b)$ pairs ν^1, ν^2.

We can deduce this part of the theorem from the fact that $[(2, 1)/(1)] = [2] + [1^2]$ contains two characters whose corresponding diagrams differ only by one box.

We explicitly show how to obtain λ/μ from $(2, 1)/(1)$.

The skew diagram $(\lambda_1, \lambda_l)/(\mu_1)$ is larger than $(2, 1)/(1)$

$$(\lambda_1, \lambda_l) = (2, 1) + (\lambda_1 - 2, \lambda_l - 1), \qquad (\mu_1) = (1) + (\mu_1 - 1)$$

and $\mathcal{A} = (\lambda_1 - 2, \lambda_l - 1)/(\mu_1 - 1)$ is a skew diagram. Let α be a partition such that $[\alpha]$ appears in $[\mathcal{A}]$, so $c((\lambda_1 - 2, \lambda_l - 1); (\mu_1 - 1), \alpha) \neq 0$.

Then by Lemma 3.2 $[\alpha + (1^2)]$ and $[\alpha + (2)]$ both appear in $[(\lambda_1, \lambda_l)/(\mu_1)]$ and, furthermore, $\alpha + (1^2) \cap \alpha + (2) = \alpha + (1)$ so $\alpha + (1^2)$ and $\alpha + (2)$ differ by only one box.

Now λ/μ is larger than $(\lambda_1, \lambda_l)/(\mu_1)$

$$\lambda = (\lambda_1, \lambda_l) \cup (\lambda_2, \lambda_3, \ldots, \lambda_{l-1}), \qquad \mu = (\mu_1) \cup (\mu_2, \mu_3, \ldots, \mu_m)$$

and $\mathcal{B} = (\lambda_2, \lambda_3, \ldots \lambda_{l-1})/(\mu_2, \mu_3, \ldots, \mu_m)$ is a skew diagram. Let β be a partition such that $[\beta]$ appears in $[\mathcal{B}]$.

Then by Lemma 3.2 $[(\alpha + (1^2)) \cup \beta]$ and $[(\alpha + (2)) \cup \beta]$ both appear in $[\lambda/\mu]$ and

$$\left((\alpha + (1^2)) \cup \beta\right) \cap \left((\alpha + (2)) \cup \beta\right) = (\alpha + (1)) \cup \beta$$

so $\nu^1 = (\alpha + (1^2)) \cup \beta$ and $\nu^2 = (\alpha + (2)) \cup \beta$ differ only by one box.

Furthermore, notice that a different choice for α or β yields a different pair ν^1, ν^2. This proves that there are at least $\max(a, b)$ pairs ν^1, ν^2.

Now we will prove that there are also at least g_n pairs ν^1, ν^2.

As mentioned above, as an easy consequence of the LR rule we have

$$[\delta_n/\delta_{n-1}] = [\underbrace{(1) \otimes (1) \otimes \cdots \otimes (1)}_{n\text{-times}}] = [1]^n = \sum_{\lambda \vdash n} f^\lambda [\lambda]$$

where f^λ is the number of standard Young tableaux of shape λ, in particular, all irreducible characters of S_n appear in $[\delta_n/\delta_{n-1}]$. So by definition of g_n $[\delta_n/\delta_{n-1}]$ contains g_n characters $[\alpha], [\beta]$ whose corresponding diagrams differ only by one box.

By Theorem 2.3 λ/μ is larger than δ_n/δ_{n-1}, so there exist skew diagrams $\mathcal{B}^i$ such that λ/μ is obtained from δ_n/δ_{n-1} by using the operations $+, \cup$ together with the $\mathcal{B}^i$. Let $\circ^i$ be either $+$ or $\cup$ then

$$\lambda/\mu = ((\delta_n/\delta_{n-1} \circ^1 \mathcal{B}^1) \circ^2 \mathcal{B}^2) \cdots \circ^j \mathcal{B}^j.$$

Choose $[\alpha^i]$ contained in $[\mathcal{B}^i]$ and $[\bar{\nu}^1], [\bar{\nu}^2]$ contained in $[\delta_n/\delta_{n-1}]$ with $\left|\bar{\nu}^1 \cap \bar{\nu}^2\right| = n - 1$. Set

$$\nu^1 = ((\bar{\nu}^1 \circ^1 \alpha^1) \circ^2 \alpha^2) \cdots \circ^j \alpha^j, \qquad \nu^2 = ((\bar{\nu}^2 \circ^1 \alpha^1) \circ^2 \alpha^2) \cdots \circ^j \alpha^j$$

then by Lemma 3.2 both $[\nu^1], [\nu^2]$ appear in $[\lambda/\mu]$ and, furthermore, $\left|\nu^1 \cap \nu^2\right| = \left|\nu^1\right| - 1$. Finally a different choice of $\bar{\nu}^1, \bar{\nu}^2$ gives different ν^1, ν^2 (for fixed $(\alpha^i, \circ^i)$) and there are by definition g_n choices for $\bar{\nu}^1, \bar{\nu}^2$. $\qquad \square$

Remark 3.10. In the On-Line Encyclopedia of Integer Sequences [OEIS] $g_n = \bar{p}_{n-2}$ has the id: A000097, p_n has the id: A000041 and f_n has the id: A000085. Their first terms are

n :	1	2	3	4	5	6	7	8	9	10	11	12	13
g_n :	0	1	2	5	9	17	28	47	73	114	170	253	365
p_n :	1	2	3	5	7	11	15	22	30	42	56	77	101
f_n :	1	2	4	10	26	76	232	764	2620	9496	35696	140152	568504

Lemma 3.11. *Let α, β be partitions with $dp(\alpha) \geq dp(\beta) = n$. Then $[\alpha] \otimes [\beta]$ has cc-type at least (p_{n+1}, f_{n+1}) and contains g_{n+1} pairs of components $([\nu^1], [\nu^2])$ such that their corresponding partitions differ only by one box.*

Proof. This follows directly from the previous theorems by setting $\lambda/\mu = \alpha \otimes \beta^\circ$ because then $dp(\lambda) = dp(\alpha) + 1, dp(\mu) = dp(\beta)$. $\qquad \square$

Lemma 3.12. *Let λ/μ be a skew diagram with $|\lambda/\mu| = n$.*
  Then $[\lambda/\mu]$ contains at most
  - *g_n pairs $([\nu^1], [\nu^2])$ such that $\left|\nu^1 \cap \nu^2\right| = n - 1$,*
  - *p_n components,*

- $\min(f_n, p_n f^\mu, p_n f^{\bar{\lambda}})$ *constituents (with* $\bar{\lambda} = (\lambda_1 - \lambda_l, \lambda_1 - \lambda_{l-1}, \ldots, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2, 0))$.

Proof. The first two statements are trivial, because there are not more irreducible characters of S_n.

For the third statement notice, that λ/μ is smaller than δ_n/δ_{n-1} which gives by Lemma 3.2 $c(\lambda; \mu, \nu) \leq c(\delta_n; \delta_{n-1}, \nu) = f^\nu$. Since the LR coefficient is symmetric in μ and ν we also have $c(\lambda; \mu, \nu) \leq f^\mu$ and by rotation symmetry $c(\lambda; \mu, \nu) \leq f^{\bar{\lambda}}$.

So for the number of constituents of $[\lambda/\mu]$

$$\sum_\nu c(\lambda; \mu, \nu) \leq \sum_\nu f^\nu = f_n,$$

$$\sum_\nu c(\lambda; \mu, \nu) = \sum_{\nu \vdash n} c(\lambda; \mu, \nu) \leq \sum_{\nu \vdash n} f^\mu = p_n f^\mu,$$

$$\sum_\nu c(\lambda; \mu, \nu) = \sum_{\nu \vdash n} c(\lambda; \mu, \nu) \leq \sum_{\nu \vdash n} f^{\bar{\lambda}} = p_n f^{\bar{\lambda}}.$$

Notice that all three bounds are reached for $\lambda/\mu = \delta_n/\delta_{n-1}$.   □

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