A NOTE ON A PERFECT EULER CUBOID.

Ruslan Sharipov

Abstract. The problem of constructing a perfect Euler cuboid is reduced to a single Diophantine equation of the degree 12.

1. Introduction.

An *Euler cuboid*, named after Leonhard Euler, is a rectangular parallelepiped whose edges and face diagonals all have integer lengths. A *perfect cuboid* is an Euler cuboid whose space diagonal is also of an integer length.

In 2005 Lasha Margishvili from the Georgian-American High School in Tbilisi won the Mu Alpha Theta Prize for the project entitled "Diophantine Rectangular Parallelepiped" (see http://www.mualphatheta.org/Science_Fair/...). He suggested a proof that a perfect Euler cuboid does not exist. However, by now his proof is not accepted by mathematical community. The problem of finding a perfect Euler cuboid is still considered as an unsolved problem. The history of this problem can be found in [1]. Here are some appropriate references: [2–35].

2. Passing to rational numbers.

Let $A_1B_1C_1D_1A_2B_2C_2D_2$ be a perfect Euler cuboid. Its edges are presented by positive integer numbers. We write this fact as

$$|A_1B_1| = a,\quad |A_1D_1| = b,\quad |A_1A_2| = c. \tag{2.1}$$

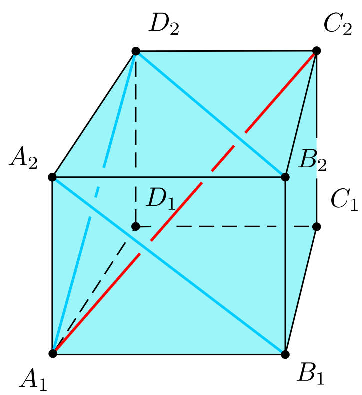

Fig. 2.1

Its face diagonals are also presented by positive integers (see Fig. 2.1):

$$|A_1D_2| = \alpha,\quad |A_2B_1| = \beta,\quad |B_2D_2| = \gamma. \tag{2.2}$$

And finally, the spacial diagonal of this cuboid is presented by a positive integer:

$$|A_1C_2| = d. \tag{2.3}$$

2000 *Mathematics Subject Classification.* 11D41, 11D72.

From (2.1), (2.2), (2.3) one easily derives a series of Diophantine equations for the integer numbers a, b, c, α, β, γ, and d:

$$\begin{aligned} &a^2+b^2=\gamma^2, && b^2+c^2=\alpha^2,\\ &c^2+a^2=\beta^2, && a^2+b^2+c^2=d^2. \end{aligned} \tag{2.4}$$

The main goal of this paper is to reduce the equations (2.4) to a single Diophantine equation for some other integer numbers.

Relying on the last equation (2.4), we introduce the following rational numbers:

$$x_1=\frac{a}{d}, \qquad x_2=\frac{b}{d}, \qquad x_3=\frac{c}{d}. \tag{2.5}$$

The numbers (2.5) are the components of a three-dimensional unit vector:

$$(x_1)^2+(x_2)^2+(x_3)^2=1. \tag{2.6}$$

From the first three equations (2.4) one easily derives the equations

$$\begin{aligned} &(x_1)^2+(x_2)^2=(d_3)^2,\\ &(x_2)^2+(x_3)^2=(d_1)^2,\\ &(x_3)^2+(x_1)^2=(d_2)^2, \end{aligned} \tag{2.7}$$

where the rational numbers d_1, $d_2\,d_3$ are given by the following fractions:

$$d_1=\frac{\alpha}{d}, \qquad d_2=\frac{\beta}{d}, \qquad d_3=\frac{\gamma}{d}. \tag{2.8}$$

The equations (2.6), (2.7), and (2.8) lead to the following theorem.

Theorem 2.1. *A perfect Euler cuboid does exist if and only if the equations (2.6) and (2.7) are solvable in positive rational numbers x_1, x_2, x_3 and d_1, d_2, d_3.*

Proof. The direct proposition of the theorem 2.1 is immediate from the formulas (2.4), (2.5), and (2.8). Conversely, assume that x_1, x_2, x_3 and d_1, d_2, d_3 are positive rational numbers obeying the equations (2.6) and (2.7). They a presented by some unique irreducible fractions with positive integer numerators and denominators:

$$x_1=\frac{\nu_1}{\delta_1}, \qquad x_2=\frac{\nu_2}{\delta_2}, \qquad x_3=\frac{\nu_3}{\delta_3}, \qquad d_1=\frac{\nu_4}{\delta_4}, \qquad d_2=\frac{\nu_5}{\delta_5}, \qquad d_3=\frac{\nu_6}{\delta_6}.$$

Let's denote through d the least common multiple of their denominators, i. e.

$$d=LCM(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6).$$

Then the following products are positive integer numbers:

$$\begin{aligned} &a=x_1\,d, && b=x_2\,d, && c=x_3\,d,\\ &\alpha=d_1\,d, && \beta=d_2\,d, && \gamma=d_3\,d. \end{aligned} \tag{2.9}$$

Applying (2.6) and (2.7) to (2.9), we derive the equations (2.4) for the integer numbers a, b, c, α, β, γ, and d. $\square$

3. A rational parametrization.

Combining (2.7) and (2.6), we derive the following equation for x_1 and d_1:

$$(x_1)^2 + (d_1)^2 = 1. \tag{3.1}$$

Rational solutions of the equation (3.1) are parametrized by a rational number u:

$$x_1 = \frac{2\,u}{1+u^2}, \qquad\qquad d_1 = \frac{1-u^2}{1+u^2}. \tag{3.2}$$

Since both x_1 and d_1 in (3.1) are positive, the parameter u satisfies the inequalities:

$$0 < u < 1. \tag{3.3}$$

The second equation in (2.7) is $(x_2)^2 + (x_3)^2 = (d_1)^2$. This equation can be written in a form quite similar to the equation (3.1):

$$\left(\frac{x_2}{d_1}\right)^2 + \left(\frac{x_3}{d_1}\right)^2 = 1. \tag{3.4}$$

Rational solutions of the equation (3.4) are parametrized by a rational number z:

$$\frac{x_2}{d_1} = \frac{2\,z}{1+z^2}, \qquad\qquad \frac{x_3}{d_1} = \frac{1-z^2}{1+z^2}. \tag{3.5}$$

Combining (3.2) and (3.5), we derive the formulas

$$\begin{aligned} x_2 &= \frac{2\,z\,(1-u^2)}{(1+u^2)\,(1+z^2)},\\ x_3 &= \frac{(1-u^2)\,(1-z^2)}{(1+u^2)\,(1+z^2)}. \end{aligned} \tag{3.6}$$

The parameter z in (3.5) and (3.6) obeys the inequalities similar to (3.3):

$$0 < z < 1. \tag{3.7}$$

Theorem 3.1. *The formulas (3.2) and (3.6) constitute a rational parametrization of the variables x_1, x_2, x_3 and d_1 by means of two parameters u and z obeying the inequalities (3.3) and (3.7). The equations*

$$(x_1)^2 + (x_2)^2 + (x_3)^2 = 1, \qquad\qquad (x_2)^2 + (x_3)^2 = (d_1)^2 \tag{3.8}$$

are fulfilled identically due to this parametrization.

The proof of the theorem 3.1 is pure calculations.

4. An extended parametrization.

Note that the equations (3.8) are two of the four equations (2.6) and (2.7) providing a perfect Euler cuboid. The other two equations are

$$(x_1)^2+(x_2)^2=(d_3)^2, \qquad\qquad (x_3)^2+(x_1)^2=(d_2)^2. \tag{4.1}$$

Let's substitute (3.2) and (3.6) into the first equation (4.1). As a result we get

$$(d_3)^2=\frac{4\,(u^2\,z^2+1)\,(u^2+z^2)}{(1+u^2)^2\,(1+z^2)^2}. \tag{4.2}$$

Similarly, substituting (3.2) and (3.6) into the second equation (4.1), we get

$$(d_2)^2=\frac{((1+u^2)\,(1+z^2)+2\,z(1-u^2))\,((1+u^2)\,(1+z^2)-2\,z(1-u^2))}{(1+u^2)^2\,(1+z^2)^2}. \tag{4.3}$$

Relying on (4.2), we define the following two quantities ξ and a:

$$\xi=u^2\,z^2+1, \qquad\qquad a=\frac{d_3\,(1+u^2)\,(1+z^2)}{2\,(u^2\,z^2+1)}. \tag{4.4}$$

Similarly, relying on (4.2), we define other two quantities ζ and b:

$$\begin{aligned} &\zeta=(1+u^2)\,(1+z^2)+2\,z(1-u^2),\\ &b=\frac{d_2\,(1+u^2)\,(1+z^2)}{(1+u^2)\,(1+z^2)+2\,z(1-u^2)}. \end{aligned} \tag{4.5}$$

The formulas (4.4) and (4.5) are consistent since the denominators of the fractions in them are positive. For a and b from (4.2), (4.3), (4.4), and (4.5), we derive.

$$a^2=\frac{u^2+z^2}{u^2\,z^2+1}, \tag{4.6}$$

$$b^2=\frac{(1+u^2)\,(1+z^2)-2\,z(1-u^2)}{(1+u^2)\,(1+z^2)+2\,z(1-u^2)}. \tag{4.7}$$

Since $d_2>0$ and $d_3>0$ (see (2.8), (2.2) and (2.3)), the quantities a and b are positive. Therefore, the formulas (4.6) and (4.7) define two positive functions

$$a=a(u,z), \qquad\qquad b=b(u,z). \tag{4.8}$$

The domain of the functions (4.8) is outlined by the inequalities (3.3) and (3.7):

$$D_{uz}=\{(u,z)\in\mathbb{R}^2\colon\ 0<u<1\ \text{ and }\ 0<z<1\}. \tag{4.9}$$

The functions (4.8) defined in the domain (4.9) constitute a mapping

$$f\colon\ D_{uz}\to\mathbb{R}^2. \tag{4.10}$$

Let's denote through D_{ab} the image of the domain D_{uz} under the mapping (4.10):

$$D_{ab} = \operatorname{Im} f = f(D_{uz}). \tag{4.11}$$

The domain (4.11) is shown in Fig. 4.1. It is an open triangle with one curvilinear side. The curvilinear side of the triangle D_{ab} is the graph of the function

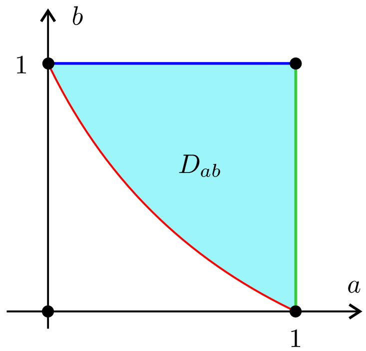

Fig 4.1

$$b(a) = -1 + \frac{2}{a+1}.$$

Using the formulas (4.6) and (4.7), one can prove that the mapping (4.10) sets up a bijective correspondence of the points of D_{uz} with the points of D_{ab}:

$$f\colon\ D_{uz} \to D_{ab}.$$

The inverse mapping

$$f^{-1}\colon\ D_{ab} \to D_{uz}$$

establishing the backward correspondence of the points of D_{ab} with those of D_{uz} is given by two algebraic functions

$$u = u(a,b), \qquad\qquad z = z(a,b). \tag{4.12}$$

Let's consider the second formula (4.4) and the second formula (4.5). We can write these two formulas in the following way:

$$\begin{aligned} d_2 &= \frac{(1+u^2)\,(1+z^2) + 2\,z(1-u^2)}{(1+u^2)\,(1+z^2)}\,b,\\ d_3 &= \frac{2\,(u^2\,z^2+1)}{(1+u^2)\,(1+z^2)}\,a. \end{aligned} \tag{4.13}$$

Substituting (4.12) into the formulas (3.2), (3.6), and (4.13), we can represent x_1, x_2, x_3 and d_1, d_2, d_3 as functions of two variables a and b:

$$\begin{aligned} &x_1 = x_1(a,b), && x_2 = x_2(a,b), && x_3 = x_3(a,b),\\ &d_1 = d_1(a,b), && d_2 = d_2(a,b), && d_3 = d_3(a,b). \end{aligned} \tag{4.14}$$

Definition 4.1. The functions (4.14) sharing the common domain D_{ab} constitute a parametrization for the problem of a perfect Euler cuboid. They extend the rational parametrization given by the functions (3.2) and (3.6).

Theorem 4.1. *The equations* (2.6) *and* (2.7) *providing a perfect Euler cuboid are fulfilled identically by the functions* (4.14).

The theorem 4.1 is analogous to the theorem 3.1. One can see that it is already proved by the above considerations.

5. The characteristic equation.

Unlike (3.2) and (3.6), the functions (4.14) are not explicit. Below we derive an algorithm for evaluating them. For this purpose let's return back to the formulas (4.4), (4.5), (4.6), and (4.7). From (4.4) and (4.6) we derive the equations

$$\xi=u^2\,z^2+1, \qquad\qquad \xi\,a^2=u^2+z^2. \tag{5.1}$$

Similarly, from (4.5) and (4.7) we derive the equations

$$\begin{aligned}\zeta&=(1+u^2)\,(1+z^2)+2\,z(1-u^2),\\ \zeta\,b^2&=(1+u^2)\,(1+z^2)-2\,z(1-u^2).\end{aligned} \tag{5.2}$$

Subtracting both equations (5.1) from each of the equations (5.2), we get

$$\begin{cases}\zeta-\xi\,(1+a^2)=2\,z\,(1-u^2),\\ \zeta\,b^2-\xi\,(1+a^2)=-2\,z\,(1-u^2).\end{cases} \tag{5.3}$$

The equations (5.3) constitute a system of two linear algebraic equations with respect to the variables ξ and ζ. Solving them, we derive

$$\xi=\frac{2\,z\,(1-u^2)\,(1+b^2)}{(1-b^2)\,(1+a^2)}, \qquad\qquad \zeta=\frac{4\,z\,(1-u^2)}{1-b^2}. \tag{5.4}$$

Now let's substitute θ for z^2 into (5.1). Then the equations (5.1) turn to a system of two linear algebraic equations with respect to the variables ξ and θ:

$$\begin{cases}\xi-u^2\,\theta=1,\\ \xi\,a^2-\theta=u^2.\end{cases} \tag{5.5}$$

Solving the system of linear equations (5.5), we obtain

$$\xi=\frac{(1-u^2)\,(1+u^2)}{1-a^2\,u^2}, \qquad\qquad \theta=\frac{a^2-u^2}{1-a^2\,u^2}. \tag{5.6}$$

In (5.4) and (5.6) we have two expressions for ξ. Equating them we derive the following expression for z expressing it through a, b and u:

$$z=\frac{(1+u^2)\,(1-b^2)\,(1+a^2)}{2\,(1+b^2)\,(1-a^2\,u^2)}. \tag{5.7}$$

Substituting (5.7) into the second equation (5.4), we derive:

$$\zeta=\frac{2\,(1+u^2)\,(1-u^2)\,(1+a^2)}{(1+b^2)\,(1-a^2\,u^2)}. \tag{5.8}$$

Note that the formulas (5.6), (5.7), and (5.8) are similar to each other. They express z, ξ, ζ, and θ through a, b, and u. But only two of the three variables a, b, and u are independent. Due to (4.12) the variable u is uniquely expressed through a and b within the domain D_{ab} shown in Fig. 4.1. In order to evaluate this expression let's recall that we have the following equation:

$$\theta = z^2. \tag{5.9}$$

Applying (5.6) and (5.8) to (5.9), we write (5.9) as

$$\frac{(1+u^2)^2\,(1-b^2)^2\,(1+a^2)^2}{4\,(1+b^2)^2\,(1-a^2\,u^2)^2} = \frac{a^2-u^2}{1-a^2\,u^2}. \tag{5.10}$$

The denominators of the fractions in the equation (5.10) are nonzero within the domain D_{ab}. For this reason it can be brought to a polynomial equation:

$$\begin{gathered} u^4\,a^4\,b^4 + (6\,a^4\,u^2\,b^4 - 2\,u^4\,a^4\,b^2 - 2\,u^4\,a^2\,b^4) + (4\,u^2\,b^4\,a^2 + \\ + 4\,a^4\,u^2\,b^2 - 12\,u^4\,a^2\,b^2 + u^4\,a^4 + u^4\,b^4 + a^4\,b^4) + (6\,a^4\,u^2 + 6\,u^2\,b^4 - \\ - 8\,a^2\,b^2\,u^2 - 2\,u^4\,a^2 - 2\,u^4\,b^2 - 2\,a^4\,b^2 - 2\,b^4\,a^2) + (u^4 + b^4 + \\ + a^4 + 4\,a^2\,u^2 + 4\,b^2\,u^2 - 12\,b^2\,a^2) + (6\,u^2 - 2\,a^2 - 2\,b^2) + 1 = 0. \end{gathered} \tag{5.11}$$

Theorem 5.1. *The equation (5.11) defines the function $u = u(a,b)$ from (4.12) in an implicit form. It is called the characteristic equation.*

Theorem 5.2. *A perfect Euler cuboid does exist if and only if the characteristic equation (5.11) has a rational solution such that $0 < u < 1$, while a and b are the coordinates of some point within the open domain D_{ab} shown in Fig. 4.1.*

The inhomogeneous polynomial equation (5.11) can be transformed to a homogeneous equation by adding one more variable c:

$$\begin{gathered} u^4\,a^4\,b^4 + 6\,a^4\,u^2\,b^4\,c^2 - 2\,u^4\,a^4\,b^2\,c^2 - 2\,u^4\,a^2\,b^4\,c^2 + 4\,u^2\,b^4\,a^2\,c^4 + \\ + 4\,a^4\,u^2\,b^2\,c^4 - 12\,u^4\,a^2\,b^2\,c^4 + u^4\,a^4\,c^4 + u^4\,b^4\,c^4 + a^4\,b^4\,c^4 + \\ + 6\,a^4\,u^2\,c^6 + 6\,u^2\,b^4\,c^6 - 8\,a^2\,b^2\,u^2\,c^6 - 2\,u^4\,a^2\,c^6 - 2\,u^4\,b^2\,c^6 - \\ - 2\,a^4\,b^2\,c^6 - 2\,b^4\,a^2\,c^6 + u^4\,c^8 + b^4\,c^8 + a^4\,c^8 + 4\,a^2\,u^2\,c^8 + \\ + 4\,b^2\,u^2\,c^8 - 12\,b^2\,a^2\,c^8 + 6\,u^2\,c^{10} - 2\,a^2\,c^{10} - 2\,b^2\,c^{10} + c^{12} = 0. \end{gathered} \tag{5.12}$$

Theorem 5.3. *A perfect Euler cuboid does exist if and only if the Diophantine equation (5.12) has a solution such that $c > 0$ and $0 < u/c < 1$, while a/c and b/c are the coordinates of some point within the open domain D_{ab} shown in Fig. 4.1.*

The theorems 5.2 and 5.3 constitute the main result of this paper. They can be used in numeric search for a perfect Euler cuboid.

References

1. Weisstein E. W., *Euler brick*, MathWorld, a Wolfram web resource http://mathworld.wolfram .com/EulerBrick.html.
2. Halcke P., *Deliciae mathematicae oder mathematisches Sinnen-Confect*, N. Sauer, Hamburg, Germany, 1719.
3. Saunderson N., *Elements of algebra,* Vol. 2, Cambridge Univ. Press, Cambridge, 1740.
4. Euler L., *Vollständige Anleitung zur Algebra*, Kayserliche Akademie der Wissenschaften, St. Petersburg, 1771.

5. Dickson L. E, *History of the theory of numbers,* Vol. 2*: Diophantine analysis*, Dover, New York, 2005.
6. Kraitchik M., *On certain rational cuboids*, Scripta Math. **11** (1945), 317-326.
7. Kraitchik M., *Théorie des Nombres,* Tome 3*, Analyse Diophantine et application aux cuboides rationelles*, Gauthier-Villars, Paris, 1947.
8. Kraitchik M., *Sur les cuboides rationelles*, Proc. Int. Congr. Math. **2** (1954), Amsterdam, 33-34.
9. Bromhead T. B., *On square sums of squares*, Math. Gazette **44** (1960), no. 349, 219-220.
10. Lal M., Blundon W. J., *Solutions of the Diophantine equations* $x^2 + y^2 2 = l^2$*,* $y^2 + z^2 = m^2$*,* $z^2 + x^2 = n^2$, Math. Comp. **20** (1966), 144-147.
11. Spohn W. G., *On the integral cuboid*, Amer. Math. Monthly **79** (1972), no. 1, 57-59.
12. Spohn W. G., *On the derived cuboid*, Canad. Math. Bull. **17** (1974), no. 4, 575-577.
13. Chein E. Z., *On the derived cuboid of an Eulerian triple*, Canad. Math. Bull. **20** (1977), no. 4, 509-510.
14. Leech J., *The rational cuboid revisited*, Amer. Math. Monthly **84** (1977), no. 7, 518–533; see also Erratum, Amer. Math. Monthly **85** (1978), 472.
15. Leech J., *Five tables relating to rational cuboids*, Math. Comp. **32** (1978), 657-659.
16. Spohn W. G., *Table of integral cuboids and their generators*, Math. Comp. **33** (1979), 428-429.
17. Lagrange J., *Sur le dérivé du cuboide Eulérien*, Canad. Math. Bull **22** (1979), no. 2, 239-241.
18. Leech J., *A remark on rational cuboids*, Canad. Math. Bull. **24** (1981), no. 3, 377–378.
19. Korec I., *Nonexistence of small perfect rational cuboid*, Acta Math. Univ. Comen. **42/43** (1983), 73-86.
20. Korec I., *Nonexistence of small perfect rational cuboid II*, Acta Math. Univ. Comen. **44/45** (1984), 39–48.
21. Wells D. G., *The Penguin dictionary of curious and interesting numbers*, Penguin publishers, London, 1986.
22. Bremner A., Guy R. K., *A dozen difficult Diophantine dilemmas*, Amer. Math. Monthly **95** (1988), no. 1, 31–36.
23. Bremner A., *The rational cuboid and a quartic surface*, Rocky Mountain J. Math. **18** (1988), no. 1, 105-121.
24. Colman W. J. A., *On certain semiperfect cuboids*, Fibonacci Quart. **26** (1988), no. 1, 54–57; see also *Some observations on the classical cuboid and its parametric solutions*, Fibonacci Quart. **26** (1988), no. 4, 338–343.
25. Korec I., *Lower bounds for perfect rational cuboids,*, Math. Slovaca **42** (1992), no. 5, 565–582.
26. Guy R. K., *Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square*, Unsolved Problems in Number Theory, 2nd ed., Springer-Verlag, New York, 1994, pp. 173–181.
27. Rathbun R. L., Granlund T., *The integer cuboid table with body, edge, and face type of solutions*, Math. Comp. **62** (1994), 441-442.
28. Van Luijk R., *On perfect cuboids,* Doctoraalscriptie, Mathematisch Instituut, Universiteit Utrecht, Utrecht, 2000.
29. Rathbun R. L., Granlund T., *The classical rational cuboid table of Maurice Kraitchik*, Math. Comp. **62** (1994), 442-443.
30. Peterson B. E., Jordan J. H., *Integer hexahedra equivalent to perfect boxes*, Amer. Math. Monthly **102** (1995), no. 1, 41–45.
31. Rathbun R. L., *The rational cuboid table of Maurice Kraitchik*, e-print math.HO/0111229 in Electronic Archive http://arXiv.org.
32. Hartshorne R., Van Luijk R., *Non-Euclidean Pythagorean triples, a problem of Euler, and rational points on K3 surfaces*, e-print math.NT/0606700 in Electronic Archive http://arXiv.org.
33. Waldschmidt M., *Open diophantine problems*, e-print math.NT/0312440 in Electronic Archive http://arXiv.org.
34. Ionascu E. J., Luca F., Stanica P., *Heron triangles with two fixed sides*, e-print math.NT/0608185 in Electronic Archive http://arXiv.org.
35. Sloan N. J. A, *Sequences A031173, A031174, and A031175*, On-line encyclopedia of integer sequences, OEIS Foundation Inc., Portland, USA.

Bashkir State University, 32 Zaki Validi street, 450074 Ufa, Russia
E-mail address: r-sharipov@mail.ru