SPECTRUM OF PERMANENT'S VALUES AND ITS EXTREMAL MAGNITUDES IN Λ_n^3 AND $\Lambda_n(\alpha,\beta,\gamma)$

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ABSTRACT. Let Λ_n^k denote the class of $(0,1)$ square matrices containing in each row and in each column exactly k 1's. The minimal value of k, for which the behavior of the permanent in Λ_n^k is not quite studied, is $k=3$. We give a simple algorithm for calculation upper magnitudes of permanent in Λ_n^3 and consider some extremal problems in a generalized class $\Lambda_n(\alpha,\beta,\gamma)$, the matrices of which contain in each row and in each column nonzero elements α,β,γ and $n-3$ zeros.

1. Introduction

Let Λ_n^k denote the class of $(0,1)$ square matrices containing in each row and in each column exactly k 1's. If $A\in\Lambda_n^3$, then matrix $k^{-1}A$ is doubly stochastic. Therefore, Λ_n^k-matrices are also called *doubly stochastic (0,1)-matrices* (cf. [11]). Furthermore, for a given real or complex numbers $\alpha,\beta,\dots\gamma$, denote $\Lambda_n(\alpha,\beta,\dots\gamma)$ the class of square matrices containing every number from $\{\alpha,\beta,\dots\gamma\}$ exactly one time in each row and in each column, such that the other elements are 0's.

Definition 1. *We call* p-spectrum *in* Λ_n^k *(denoting it* $ps[\Lambda_n^k]$*) the set of all the values which are taken by the permanent in* Λ_n^k.

Note that p-spectrum in Λ_n^1 trivially is $\{1\}$. It is known (cf. Tarakanov [25]) that

$$ps[\Lambda_n^2]=\{2,2^2,2^3,...,2^{\lfloor\frac{n}{2}\rfloor}\}.$$

But, for $k\geq 3$, p-spectrum of Λ_n^k, generally speaking, is unknown. Greenstein (cf. [11], point 8.4, Problem 3) put the problem of describing the p-spectrum in Λ_n^3. In this paper we find p-spectrum on symmetric matrices in Λ_n^3 with ones on the main diagonal and give an algorithm for calculation upper values of p-spectrum in Λ_n^3. We also obtain several results for a generalized class $\Lambda_n(\alpha,\beta,\gamma)$ with real nonzero numbers α,β,γ. Some results of the present paper were announced by the author in [21].

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2. What is known about Λ_n^3?

1) Explicit formula for $|\Lambda_n^3|$ (cf. Stanley [24], Ch.1)

$$|\Lambda_n^3| = 6^{-n} \sum_{k_1+k_2+k_3=n,\ \ k_i\geq 0} \frac{(-1)^{k_2} n!^2 (k_2+3k_3)! 2^{k_1} 3^{k_2}}{k_1! k_2! k_3!^2 6^{k_3}}. \tag{2.1}$$

2) Asymptotic formula for $|\Lambda_n^3|$ (cf. O'Neil [12])

$$|\Lambda_n^3| = \frac{(3n)!}{(36)^n} e^{-2}(1+O(n^{-1+\varepsilon})), \tag{2.2}$$

where $\varepsilon > 0$ is arbitrary small for sufficiently large n.

In addition, note that $|\Lambda_n(\alpha,\beta,\gamma)|$ with different α,β,γ is, evidently, the number of 3-rowed Latin rectangles of length n such that

$$|\Lambda_n(\alpha,\beta,\gamma)| = n! K_n, \tag{2.3}$$

where K_n is the number of reduced 3-rowed Latin rectangles with the first row $\{1,2,...,n\}$. It is known (Riordan [13], pp. 204-210) that

$$K_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} D_{n-k} D_k U_{n-2k}, \tag{2.4}$$

where D_n is subfactorial.

$$D_0 = 1,\ \ D_n = nD_{n-1} + (-1)^n,\ \ n \geq 1, \tag{2.5}$$

$\{U_n\}$ is sequence of Lucas numbers of the Ménage problem which is defined by Cayley recursion (cf. [13], p. 201)

$$U_0 = 1,\ \ U_1 = -1,\ \ U_2 = 0,$$

$$U_n = nU_{n-1} + \frac{n}{n-2} U_{n-2} + 4\frac{(-1)^n}{n-2},\ \ n \geq 3 \tag{2.6}$$

(see [23], sequences A102761, A000186).

Denote, furthermore, $\overline{\Lambda}_n^3$ the set of matrices in Λ_n^3 with 1's on the main diagonal. Note that

$$ps[\Lambda_n^3] = ps[\overline{\Lambda}_n^3]. \tag{2.7}$$

Indeed, it is well known that every Λ_n^3-matrix A has a diagonal of ones (i.e., a set of 1's no two in the same row or column). Let l be such a diagonal. There exists a permutation of rows and columns π such that $\pi(l)$ will be the main diagonal of $\pi(A)$. Nevertheless, $per(\pi(A)) = per A$ and (2.7) follows.

3) A known explicit formula for $|\overline{\Lambda}_n^3|$ (Shevelev [19]) has a close structure to (2.4):

$$|\overline{\Lambda}_n^3| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} S_{n-k} S_k U_{n-2k}, \tag{2.8}$$

where sequence $\{S_n\}$ is defined by recursion

$$S_0 = 1,\ \ S_1 = 0,\ \ S_n = (n-1)(S_{n-1} + \frac{1}{2}S_{n-2}),\ \ n \geq 2. \tag{2.9}$$

4) Asymptotic formula for $|\overline{\Lambda}_n^3|$ (Shevelev [19])

$$|\overline{\Lambda}_n^3| = C\sqrt{n}(\frac{n}{e})^{2n}(1 + O(n^{-1+\varepsilon}), \tag{2.10}$$

where

$$C = 2\sqrt{\pi e^{-5}} = 0.29098...$$

and $\varepsilon > 0$ is arbitrary small for sufficiently large n.

5) Denote $\widehat{\Lambda}_n^3$ the set of symmetric matrices in $\overline{\Lambda}_n^3$. P-spectrum on $\widehat{\Lambda}_n^3$ is given by the following theorem (Shevelev [16])

Theorem 1. *Let $R(n;3)$ denote the set of all partitions of n with parts more than or equal to 3. To every partition $r \in R(n;3) : n = n_1 + n_2 + ... + n_m,\ \ m = m(r)$, put in a correspondence the number*

$$H(r) = \prod_{i=1}^{m} a(n_i), \tag{2.11}$$

where sequence $\{a(n)\}$ is defined by the recursion

$$a(3) = 6,\ \ a(4) = 9,\ \ a(n) = a(n-1) + a(n-2) - 2,\ \ n \geq 5. \tag{2.12}$$

Then we have

$$ps[\widehat{\Lambda}_n^3] = \{H(r) : r \in R(n;3)\}. \tag{2.13}$$

6) The maximal value $M(n)$ of permanent in Λ_n^3 was found by Merriell [9].

Theorem 2. *If $n \equiv h \pmod 3$, $h = 0, 1, 2$, then*

$$M(n) = 6^{\frac{n-h}{3}}\lfloor(\frac{3}{2})^h\rfloor. \tag{2.14}$$

Note that, the case $h = 0$ of (2.14) easily follows from a general Minc-Bregman inequality for permanent of (0,1)-matrices (see [11], point 6.2, and [4]).

7) Put $M(n) = M^{(1)}(n)$. In case of $n \equiv 0 \pmod 3$, Bolshakov [3] showed that the second maximal $M^{(2)} < M^{(1)}(n)$ of permanent in Λ_n^3 (such that interval $(M^{(2)},\ M^{(1)})$ is free from values of permanent in Λ_n^3) equals to

$$M^{(2)}(n) = \begin{cases} 20, & if\ \ n = 6, \\ 120, & if\ \ n = 9 \\ \frac{9}{16}6^{\frac{n}{3}}, & if\ \ n \geq 12. \end{cases} \tag{2.15}$$

Note that both $M^{(1)}(n)$ and $M^{(2)}(n)$ are attained in $\widehat{\Lambda}_n^3$ (Shevelev [16]).

8) Denote $m(n)$ the minimal value of permanent in Λ_n^3. In 1979, Voorhoeve [26] obtain a beautiful lower estimate for $m(n)$:

$$m(n) \geq 6(\frac{4}{3})^{n-3}. \tag{2.16}$$

This estimate remains the best even after proof by Egorychev [7] and Falikman [8] the famous Van der Waerden conjectural lower estimate $perA \geq \frac{n!}{n^n}$ for every $n \times n$ doubly stochastic matrix A. Indeed, this estimate yields only $m(n) \geq 3^n \frac{n!}{n^n}$, such that (2.16) is stronger for $n \geq 4$.

9) Bolshakov [2] found p-spectrum in Λ_n^3 in cases $n \leq 8$. Namely, he added to the evident p-spectrums $ps[\Lambda_3^3] = \{6\}$ and $ps[\Lambda_4^3] = \{9\}$ also the following p-spectrums

$$ps[\Lambda_5^3] = \{12,\ \ 13\},\ \ ps[\Lambda_6^3] = \{17,\ \ 18,\ \ 20,\ \ 36\},$$

$$ps[\Lambda_7^3] = \{24,\ \ 25,\ \ 26,\ \ 27,\ \ 30,\ \ 31,\ \ 32,\ \ 54\}, \tag{2.17}$$

$$ps[\Lambda_8^3] = \{33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44, 45, 48, 49, 52, 72, 78, 81\}.$$

3. A generalization of Theorem 1 on matrices of class $\Lambda_n(\alpha,\beta,\gamma)$ with symmetric positions of elements

Denote $\overline{\Lambda}_n(\alpha,\beta,\gamma)$ the set of matrices in $\Lambda_n(\alpha,\beta,\gamma)$ with β's on the main diagonal. It is clear that, together with (2.3),

$$|\overline{\Lambda}_n(\alpha,\beta,\gamma)| = K_n. \tag{3.1}$$

Note that, as for sets Λ_n^3, $\overline{\Lambda}_n^3$, we have

$$ps[\Lambda_n(\alpha,\beta,\gamma)] = ps[\overline{\Lambda}_n(\alpha,\beta,\gamma)]. \tag{3.2}$$

Denote, furthermore, $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$ the set of matrices $M = \{m_{i,j}\}$ in $\overline{\Lambda}_n(\alpha,\beta,\gamma)$ with symmetric positions of elements: $m_{i,j} = \alpha$ if and only if $m_{j,i} = \gamma$.

P-spectrum on $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$ is given by the following theorem.

Theorem 3. *If to every partition $r \in R(n;3) : n = n_1 + n_2 + ... + n_m,\ \ m = m(r)$, corresponds the number*

$$H_{\alpha,\beta,\gamma}(r) = \prod_{i=1}^{m} a(n_i), \tag{3.3}$$

where sequence $\{a(n) = a(\alpha,\beta,\gamma;\ \ n\}$ is defined by the recursion

$$a(3) = \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma,$$

$$a(4) = \alpha^4 + \beta^4 + \gamma^4 + 4\alpha\beta^2\gamma + 2(\alpha\gamma)^2,$$

$$a(n) = \beta a(n-1) + \alpha\gamma a(n-2) +$$

$$\alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \beta - \alpha),\ \ n \geq 5, \tag{3.4}$$

then we have

$$ps[\widehat{\Lambda}_n(\alpha,\beta,\gamma)] = \{H_{\alpha,\beta,\gamma}(r) : r \in R(n;3)\}. \tag{3.5}$$

Proof. Let S_n be the symmetric permutation group of elements $1, ..., n$. Two positions $(i_1, j_1), (i_2, j_2)$ are called *independent* if $i_k \neq j_k$, k=1,2. We shall say that in the $n \times n$ matrix $M = \{m_{ij}\}$ a weight m_{ij} is appropriated to the position (i, j). Let $s \in S_n$ has not any cycle of length less than n. Consider a map

$$\sigma : (i, j) \mapsto (s^i(1),\ \ s^j(1)),$$

appropriating to the position $(s^i(1),\ \ s^j(1)$ the weight m_{ij}.

Lemma 1. 1) *the map* σ *is bijective;* 2) *if* E *is a set of pairwise independent positions, then* $\sigma(E)$ *is also a set of pairwise independent positions.*

Proof. a) Consider two distinct positions

$$(i_1, j_1),\ \ (i_2, j_2), \tag{3.6}$$

such that, at least, one of two inequalities holds

$$i_1 \neq i_2,\ \ j_1 \neq j_2. \tag{3.7}$$

Let $i_1 \neq i_2$ such that, say, $i_1 > i_2$. Show that $s^{i_1}(1) \neq s^{i_2}(1)$. Indeed, if to suppose that $s^{i_1}(1) = s^{i_2}(1)$, then $s^{i_1-i_2} = 1$, i.e., s has a cycle of length $i_1 - i_2 < n$ in spite of the condition. Conversely, if $s^{i_1}(1) \neq s^{i_2}(1)$, then $i_1 \neq i_2$, since s^{-1} has not any cycle of length less than n as well.
b) Let positions (3.6) be independent. The both of inequalities (3.7) hold and, as in a), we have $s^{i_1}(1) \neq s^{i_2}(1),\ \ s^{j_1}(1) \neq s^{j_2}(1)$, i.e. the positions $\sigma((i_1, j_1)), \sigma((i_1, j_1))$ are independent as well.
■

Lemma 2. *Let* $s \in S_n$ *have not any cycle of length less than* n. *Then* $(0, 1)$-*matrix* S *having 1's on only positions*

$$(s^1(1), s^2(1)), (s^2(1), s^3(1)), ..., (s^{n-1}(1), s^n(1)), (s^n(1), s^1(1))$$

is a incidence matrix of s.

Proof. Since s has not cycles of length less than n, then $\{s^1(1), ..., s^n(1)\}$ is a permutation of numbers $\{1, ..., n\}$. Thus the set of positions of 1's of matrix S coincides with the set of 1's of the incidence matrix of s : $(1, s(1)), ..., (n, s(n))$.
■

Let $P = P_n$ be $(0, 1)$-matrix with 1's on positions $(1, 2), (2, 3), ..., (n - 1, n), (n, 1)$ only.

Lemma 3. *Let $s \in S_n$ have not any cycle of length less than n. If S and S^{-1} are the incidence matrices of s and s^{-1}, then we have*

$$\sigma^{-1}(S) = P,\ \ \sigma^{-1}(S^{-1}) = P^{-1}. \tag{3.8}$$

Proof. Both of formulas follows from Lemma 2.

■

Noting that $\sigma(I) = I$, where I is the identity matrix, we conclude that

$$S^{-1} + I + S = \sigma(P^{-1} + I + P). \tag{3.9}$$

Moreover, since, by the bijection σ, to every diagonal (i.e., to every set of n pairwise independent positions) of the matrix $\alpha S^{-1} + \beta I + \gamma S$ corresponds one and only one diagonal of the matrix $\alpha P^{-1} + \beta I + \gamma P$ with the same products of weights, then we have

$$per(\alpha S^{-1} + \beta I + \gamma S) = per(\alpha P^{-1} + \beta I + \gamma P). \tag{3.10}$$

Note that from the definition it follows that, for every matrix $M \in \widehat{\Lambda}_n(\alpha,\beta,\gamma)$, we have a representation

$$M = \alpha S^{-1} + \beta I_n + \gamma S, \tag{3.11}$$

where S is the incidence matrix of a substitution s. In case when s has not any cycle of length less than n, the matrix M is completely indecomposable matrix in $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$. Thus, by (3.10), all completely indecomposable matrices of $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$ have the same permanent, equals to $per(\alpha P^{-1}+\beta I_n+\gamma P)$.

In general, a substitution s with the incidence matrix S in (3.11) cannot have cycles of length less than 3. Indeed, if for some i, we have either $s(i) = i$ or $s(s(i)) = i$, then in both cases $s(i) = s^{-1}(i)$ which means coincidence of positions 1's of the matrices S and S^{-1} in the i-th row.

Let $s \in S_n$ be an arbitrary substitution with cycles of length more than 2. Let

$$s = \prod_{j=1}^{r} s_j, \tag{3.12}$$

where $s_j \in S_{l_j},\ \ l_j \geq 3, \sum_{j=1}^r l_j = n$, be the decomposition of s in a product of cycles. Then the matrix $M = \alpha S^{-1} + \beta I_n + \gamma S$ is a direct sum of the matrices $M_j = \alpha S_{l_j}^{-1} + \beta I + \gamma S_{l_j}$ such that, by (3.10), $per M_j = per(\alpha P^{-1} + \beta I_{l_j} + \gamma P)$ and we have

$$per M = \prod_{j=1}^{r} per M_j = \prod_{j=1}^{r} per(\alpha P^{-1} + \beta I_{l_j} + \gamma P). \tag{3.13}$$

It is left to notice that Minc [10] found a recursion (3.4) for $per(\alpha I_n + \beta P + \gamma P^2)$ and, as well known, the multiplication an $n \times n$ matrix by P^{-1} does not change its permanent.

Therefore, $per(\alpha P^{-1}+\beta I_{l_j}+\gamma P)=per(\alpha I_n+\beta P+\gamma P^2)$.

■

Example 1. *Let us find* $ps[\widehat{\Lambda}_{11}(-1,3,2)]$.

We have the following partitions of 11 with the parts not less than 3:

$$11=8+3=7+4=6+5=3+4+4=3+3+5.$$

According to (3.4), for $a(n)=a(-1,3,2;\ n)$, we have $a(3)=16,\ \ a(4)=34$ and for $n\geq 5$,

$$a(n)=3a(n-1)-2a(n-2)+6(-1)^n.$$

Using induction, we find

$$a(n)=\begin{cases}2^{n+1}, & if\ \ n\ \ is\ \ odd,\\ 2^{n+1}+2, & if\ \ n\ \ is\ \ even.\end{cases}$$

Therefore,

$$ps[\widehat{\Lambda}_{11}(-1,3,2)]=$$

$$\{a(11),\ \ a(3)a(8),\ \ a(5)a(6),\ \ a^2(3)a(5),\ \ a(3)a^2(4)\}=$$
$$\{4096,\ \ 8224,\ \ 8320,\ \ 8704,\ \ 16384,\ \ 18496\}.$$

■

In the following examples we calculate p-spectrum for arbitrary n.

Example 2. *Let us find* $ps[\widehat{\Lambda}_n(-1,2,1)]$.

By induction, for $a(n)=a(-1,2,1;\ n)$, we have

$$a(n)=\begin{cases}2, & if\ \ n\ \ is\ \ odd,\\ 4, & if\ \ n\ \ is\ \ even.\end{cases}$$

Further, again using induction, one can find that, if n is even, then

$$ps[\widehat{\Lambda}_n(-1,2,1)]=\{4,\ \ 4^2,...,4^{\lfloor(\frac{n}{4})\rfloor}\}$$

and, if n is odd, then

$$ps[\widehat{\Lambda}_n(-1,2,1)]=\{2,\ \ 2\cdot 4,\ \ 2\cdot 4^2,...,2\cdot 4^{\lfloor(\frac{n-3}{4})\rfloor}\}.$$

■

Example 3. *Analogously, in case of* $\widehat{\Lambda}_n(-1,1,1)$, *for* $a(n)=a(-1,1,1;\ n)$, *we have*

$$a(n)=\begin{cases}4, \ if \ \ n\equiv 0 \pmod 6,\\ -2, \ if \ \ n\equiv 3 \pmod 6,\\ 1, \ otherwise.\end{cases}$$

and

$$ps[\widehat{\Lambda}_n(-1,1,1)]=$$
$$\begin{cases}\{1, \ -2, \ 4, ..., (-2)^{\lfloor(\frac{n-3}{3})\rfloor}\}, \ if \ \ n\equiv 1,2 \pmod 3,\\ \{1, \ -2, \ 4, ..., (-2)^{\lfloor(\frac{n-6}{3})\rfloor}, \ (-2)^{\lfloor(\frac{n}{3})\rfloor}\}, \ if \ \ n\equiv 0 \pmod 3.\end{cases}$$

It is interesting that, in case of n multiple of 3, the permanent omits the value $(-2)^{\lfloor(\frac{n-3}{3})\rfloor}$.

■

4. Merriell type theorems in a subclasses of $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$

Note that in class $\Lambda_n(\alpha,\beta,\gamma)$ the Minc-Bregman inequality and the Merriell theorem , generally speaking, do not hold even for positive α,β,γ. Nevertheles, some restrictions on α,β,γ allow to prove some analogs of the Merriell theorem. Recall that $M(n)$ (2.14) is attained in $\widehat{\Lambda}_n^3$. Denote $M_n(\alpha,\beta,\gamma)$ the maximal value of permanent in $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$.

Theorem 4. *Consider a class $\widehat{\Lambda}_n(\alpha,\beta,\gamma)$ with the numbers α,β,γ satisfying "triangle inequlities"*

$$0\le\alpha\le\beta+\gamma, \ \ 0\le\gamma\le\alpha+\beta, \tag{4.1}$$

and the following additional conditions

$$a^3(4)\le a^4(3), \ \ \alpha\gamma+\beta(a(3))^{\frac{1}{3}}\le(a(3))^{\frac{2}{3}}, \tag{4.2}$$

where sequence $\{a(n)\}$ is defined by recursion (3.4). Then, for n multiple of 3, we have

$$M_n(\alpha,\beta,\gamma)=(a(3))^{\frac{n}{3}}. \tag{4.3}$$

Proof. Note that conditions (4.1)-(4.2) are satisfied, e.g., in case $\alpha=\beta=\gamma=1$. Using induction, let us prove that

$$a(n)\le(a(3))^{\frac{n}{3}}. \tag{4.4}$$

Indeed, for $n=3$, this inequality is trivial, while, for $n=4$, it follows from the first condition (4.2). Let it hold for $n\le m-1$. Then, according to (3.4), we have

$$a(m)=\beta a(m-1)+\alpha\gamma a(m-2)+\alpha^{m-1}(\alpha-\beta-\gamma)+\gamma^{m-1}(\gamma-\alpha-\beta)\le$$
$$\beta(a(3))^{\frac{m-1}{3}}+\alpha\gamma(a(3))^{\frac{m-2}{3}}=$$
$$(a(3))^{\frac{m-2}{3}}(\beta(a(3))^{\frac{1}{3}}+\alpha\gamma)\le(a(3))^{\frac{m-2}{3}}(a(3))^{\frac{2}{3}}=(a(3))^{\frac{m}{3}}.$$

Note that, according Theorem 3, the equality in (4.4) holds in a direct sum of (3×3)-matrices of $\widehat{\Lambda}_3(\alpha,\beta,\gamma)$ which corresponds to the partition $n=3+3+...+3$. Let now $A\in\widehat{\Lambda}_n(\alpha,\beta,\gamma)$. By Theorem 3, there exists a partition of n with the parts not less than 3, $n=n_1+...+n_m$, such that

$$per A=\prod_{i=3}^{m}a(n_i)$$

and, in view of (4.4), we have

$$per A\leq\prod_{i=3}^{m}a(3)^{\frac{n_i}{3}}=(a(3))^{\frac{n}{3}}.$$

This proves (4.3).

■

Example 4. *Consider case $\beta=\gamma-\alpha$.*

Let us find the values of α, depending on the magnitude of γ, for which the conditions of Theorem 4 are satisfied. According to (3.4), we have

$$a(3)=\alpha^3+(\gamma-\alpha)^3+\gamma^3+3\alpha(\gamma-\alpha)\gamma=2\gamma^3, \tag{4.5}$$

$$a(4)=\alpha^4+(\gamma-\alpha)^4+\gamma^4+4\alpha(\gamma-\alpha)^2\gamma+2(\alpha\gamma)^2=2(\alpha^4+\gamma^4). \tag{4.6}$$

Thus the condition $a^3(4)\leq a^4(3)$ means that $\alpha^4+\gamma^4\leq 2^{\frac{1}{3}}\gamma^4$, or

$$0\leq\alpha\leq(2^{\frac{1}{3}}-1)^{\frac{1}{4}}\gamma=0.7140199...\gamma. \tag{4.7}$$

and it is easy to verify that the second condition in (4.2) is satisfied as well. As a collorary, we obtain the following result.

Theorem 5. *If (4.7) holds, then, for n multiple of 3, we have*

$$M_n(\alpha,\gamma-\alpha,\gamma)=2^{\frac{n}{3}}\gamma^n. \tag{4.8}$$

■

Simple forms of sequence $\{a(n)\}$ in Examples 1-2 allow to suppose that in case $\beta=\gamma-\alpha$ (or symmetrical case $\beta=\alpha-\gamma$) sequence $\{a(n)\}$ keeps a sufficiently simple form. We find this form in the following lemma.

Lemma 4. *If $\beta=\gamma-\alpha$, then sequence $\{a(n)\}$ which is defined by recursion (3.4) has the form*

$$a(n)=\begin{cases}2\gamma^n, & if\ \ n\ \ is\ \ odd,\\ 2(\alpha^n+\gamma^n), & if\ \ n\ \ is\ \ even.\end{cases} \tag{4.9}$$

Proof. Using induction with the base (4.5) -(4.6), suppose that (4.9) holds for $m\leq n$. Then, by (3.4), for even n, we have

$$a(n+1)=(\gamma-\alpha)a(n)+\alpha\gamma a(n-1)+2\alpha^n(\alpha-\gamma)=$$

$$2(\gamma-\alpha)(\alpha^n+\gamma^n)+2\alpha\gamma^n+2\alpha^n(\alpha-\gamma)=2\gamma^{n+1},$$

while, if n is odd, then we have

$$a(n+1)=2(\gamma-\alpha)\gamma^n+$$
$$2\alpha\gamma(\alpha^{n-1}+\gamma^{n-1})+2\alpha^n(\alpha-\gamma)=2(\alpha^{n+1}+\gamma^{n+1}).$$

■

Let now

$$\alpha\geq(2^{\frac{1}{3}}-1)^{\frac{1}{4}}\gamma=0.7140199...\gamma. \tag{4.10}$$

Theorem 6. *If (4.10) holds, then, for n multiple of 4, we have*

$$M_n(\alpha,\gamma-\alpha,\gamma)=(2(\alpha^4+\gamma^4))^{\frac{n}{4}}. \tag{4.11}$$

Proof. From (4.5), (4.6) and (4.10) we conclude that

$$a(3)\leq(a(4))^{\frac{3}{4}}. \tag{4.12}$$

Let us show that, for $n\geq3$,

$$a(n)\leq(a(4))^{\frac{n}{4}}. \tag{4.13}$$

For $n=4$, inequality (4.13) is trivial. For $n\geq5$, we have

$$\alpha^n+\gamma^n=\alpha^n(1+(\frac{\gamma}{\alpha})^n)\leq\alpha^n(1+(\frac{\gamma}{\alpha})^4)^{\frac{n}{4}}\leq(\alpha^4+\gamma^4)^{\frac{n}{4}}$$

and thus, using Lemma 4, we have

$$a(n)\leq2(\alpha^n+\gamma^n)<2^{\frac{n}{4}}(\alpha^4+\gamma^4)^{\frac{n}{4}}=(a(4))^{\frac{n}{4}},\ \ n\geq3.$$

Let now $A\in\widehat{\Lambda}_n(\alpha,\beta,\gamma)$. By Theorem 3, there exists a partition of n with the parts not less than 3, $n=n_1+...+n_m$, such that

$$per A=\prod_{i=3}^{m}a(n_i)$$

and, in view of (4.13), we have

$$per A\leq\prod_{i=3}^{m}a(4)^{\frac{n_i}{4}}=(a(3))^{\frac{n}{4}}$$

with the equality in a direct sum of (4×4)-matrices of $\widehat{\Lambda}_3(\alpha,\beta,\gamma)$ which corresponds to the partition $n=4+4+...+4$.

■

Note that, if $\alpha\neq(2^{\frac{1}{3}}-1)^{\frac{1}{4}}\gamma$, then in Theorem 5 we have only maximizing matrix (up to a permutation of the rows and columns) which corresponds to the partition $n=3+3+...+3$; in Theorem 6 we also have only maximizing matrix (up to a permutation of the rows and columns) which corresponds to the partition $n=4+4+...+4$. It is interesting that, only in case of the equality

$\alpha=\theta\gamma$, where $\theta=(2^{\frac{1}{3}}-1)^{\frac{1}{4}}$, the both of Theorems 5-6 are true for every n multiple of 12 with the equality of the maximums: $(2(\alpha^4+\gamma^4))^{\frac{n}{4}}=2^{\frac{n}{3}}\gamma^n$. Thus, up to a positive factor γ, *the class*

$$\widehat{\Lambda}_n(\theta,1-\theta,1),\ \ \theta=(2^{\frac{1}{3}}-1)^{\frac{1}{4}}, \tag{4.14}$$

possesses an interesting extremal property: it contains $\frac{n}{12}+1$ *maximizing matrices (up to a permutation of the rows and columns), instead of only maximizing matrix, if* $\theta\neq(2^{\frac{1}{3}}-1)^{\frac{1}{4}}$.

Indeed, the number of the maximizing matrices (up to a permutation rows and columns), is defined by the number of the following partitions of $n\equiv\pmod{12}$:

$$n=3+3+...+3,\ \ n=(4+4+4)+(3+...+3),...,$$
$$n=\underbrace{4+...+4}_{3i}+\underbrace{3+...+3}_{n-12i},\ \ i=0,1,...,\frac{n}{12}.$$

■

5. Estimate of cardinality of p-spectrum on circulants in Λ_n^3 and $\Lambda_n(\alpha,\beta,\gamma)$

Denote $\Delta_n^3\subset\Lambda_n^3$ the set of the circulants in Λ_n^3. Note that a circulant $A\in\Delta_n^3$ has a form $A=P^i+P^j+P^k,\ \ 0\le i<j<k\le n$, where $P=P_n$ is $(0,1)$-matrix with 1's on positions $(1,2),(2,3),...,(n-1,n),(n,1)$ only. Multiplicating A by P^{-i} we obtain circulant B of the form

$$B=I_n+P^r+P^s$$

with $per B=per A$. Since B is defined by a choice of two different values $0<r<k\le n$, then trivially

$$ps[\Delta_n^3]\le\binom{n}{2}<\frac{n^2}{2}.$$

Now we prove essentially more exact and practically unimprovable estimate.

Theorem 7. *We have*

$$ps[\Delta_n^3]\le\lfloor\frac{n^2+3}{12}\rfloor. \tag{5.1}$$

Proof. Let us back to the general form

$$A=P^i+P^j+P^k,\ \ 0\le i<j<k\le n.$$

Note that, A is defined by a choice of a vector (i,j,k), but its rotation, i.e., a passage to a vector of the form $(i+l,j+l,k+l)\pmod n$, does not change the magnitude of $per A$. Indeed it corresponds to the multiplication A by P^l, and our statement follows from the equality $per(P^lA)=per A$. Besides, its reflection relatively some diameter of the imaginary circumference of the rotation, by the symmetry, keeps magnitude of the permanent. Since geometrically three points on the imaginary circumference define a triangle, then our problem reduces to a triangle case of the following general problem, posed by Professor Richard H. Reis (South-East University of Massachusetts, USA) in a private communication to Hansraj Gupta in 1978):

" Let a circumference is split by the same n parts. It is required to find the number $R(n,k)$ of the incongruent convex k-gons, which could be obtaind by connection of some k from n dividing points. Two k-gons are considered congruent if they are coincided at the rotation of one relatively other along the circumference and (or) by reflection of one of the k-gons relatively some diameter."

In 1979, Gupta [6] gave a solution of the Reis problem in the form (for a short solution, see author's paper [22]):

$$R(n,k)=\frac{1}{2}\left(\binom{\lfloor(\frac{n-h_k}{2})\rfloor}{\lfloor(\frac{k}{2})\rfloor}+\frac{1}{k}\sum_{d|\,\gcd(k,n)}\varphi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1}\right). \tag{5.2}$$

If to denote $\Delta_n^k\subset\Lambda_n^k$ the set of the circulants in Λ_n^k, then from our arguments it follows that

$$ps[\Delta_n^k]\le R(n,k). \tag{5.3}$$

In case $k=3$, from (5.2)-(5.3) we find

$$ps[\Delta_n^3]\le\begin{cases}\frac{n^2}{12}, & if\;\; n\equiv 0 \pmod 6,\\ \frac{n^2-1}{12}, & if\;\; n\equiv 1,5 \pmod 6,\\ \frac{n^2-4}{12}, & if\;\; n\equiv 2,4 \pmod 6,\\ \frac{n^2+3}{12}, & if\;\; n\equiv 3 \pmod 6,\end{cases}$$

and (5.1) follows.

■

Example 5. *In case $n=5$ we have only two incongruent triangles corresponding to circulants I_5+P+P^2 and I_5+P+P^3.*

Nevertheless, the calculations give $per(I_5+P+P^2)=per(I_5+P+P^3)=13$. Thus $ps[\Delta_5^3]=\{13\}$, and $|ps[\Delta_5^3]|=1$.

Example 6. *In case $n=6$ we have three incongruent triangles corresponding to circulants $I_6+P+P^2,\;\; I_6+P+P^3$ and $I_6+P^2+P^4$.*

The calculations give $per(I_6+P+P^2)=20,\;\; per(I_6+P+P^3)=17$, while $per(I_6+P^2+P^4)=36$. Thus $ps[\Delta_6^3]=\{17,20,36\}$, and $|ps[\Delta_6^3]|=3$.

Note that a respectively large magnitude of $per(I_6+P^2+P^4)$ is explained by its decomposability in a direct product of circulants $(I_3+P+P^2)\otimes(I_3+P+P^2)$, such that $per(I_6+P^2+P^4)=(per(I_3+P+P^2))^2=6^2=36$.

It is clear that, in case of circulants in $\Lambda_n(\alpha,\beta,\gamma)$, the upper estimate (5.1) yields either the same estimate, if $\alpha=\beta=\gamma$, or $\lfloor\frac{n^2+3}{4}\rfloor$, if $\alpha=\beta\neq\gamma$ (and in the symmetric cases), or $\lfloor\frac{n^2+3}{2}\rfloor$, if α,β,γ are distinct numbers.

Add that a bijection indicated in [22] allows to apply formula (5.2) to enumerating the two-color bracelets of n beads, k of which are black and $n-k$ are white (see, e.g., the author's explicit formulas for sequences A032279-A032282, A005513-A005516 in [23]).

6. Algorithm of calculations of upper magnitudes of the permanent in $\widehat{\Lambda}_n^3$

Theorem 1 allows, using some additional arguments, to give an algorithm of calculations of upper magnitudes of the permanent in $\widehat{\Lambda}_n^3$. In connection with this, we need an important lemma for numbers (2.12).

Lemma 5. *For $n_1,n_2\geq 3$, we have*

$$a(n_1+n_2)\leq a(n_1)a(n_2)\ \ ,n_1,n_2\geq 3. \tag{6.1}$$

Proof. By usual way, from (2.12) we find

$$a(n)=\varphi^n+2+(-1)^n\varphi^{-n}, \tag{6.2}$$

where $\varphi=\frac{\sqrt{5}+1}{2}$ is the golden ratio.
Denote $\varepsilon(n)=(-1)^n\varphi^{-n}$. Since $n\geq 3$, then $|\varepsilon(n)|<0.24$, and, consequently, if $n=n_1+n_2$, then $(2+\varepsilon(n_1))(2+\varepsilon(n_2)>1.76^2>3$. Therefore, we have

$$a(n_1)a(n_2)=(\varphi^{n_1}+2+\varepsilon(n_1))(\varphi^{n_2}+2+\varepsilon(n_2))>$$

$$\varphi^{n_1+n_2}+3>\varphi^{n_1+n_2}+2+\varepsilon(n_1+n_2)=a(n_1+n_2). \tag{6.3}$$

■

Note that, actually, the difference between the hand sides of (6.3) more than $1.76(\varphi^{n_1}+\varphi^{n_2})$.

Let now $n\equiv j\pmod 3$, $j=0,1,2$, and $t\in\mathbf{N}$. Let $R(m;\nu)$ denote the set of all partitions of n with parts more than or equal to ν. For us an important role play cases $\nu=3,4$. To $r\in R(m;3)$, $\rho\in R(m;4)$ put in a correspondence the sets

$$H_{m;3}(r)=\{\Pi_{r_i\in r}a_{r_i}\};\ \ H_{m;4}(\rho)=\{\Pi_{\rho_i\in\rho}a_{\rho_i}\}. \tag{6.4}$$

In case $m=3$, when $\rho=\varnothing$, let us agree that $H_{3;4}$ is a singleton $\{6\}$.

Consider now the set $L_t^{(j)}=L_t^{(j)}(n)$:

$$(6.5)\qquad L_t^{(j)}=\bigcup_{i=1}^{4t+j}\{6^{\frac{n-j-3i}{3}}y: y\in H_{3i+j;4}(\rho), y\geq 9^{3t+j}6^{i-4t-j}\}.$$

Theorem 8. *(algorithm of calculation of upper magnitudes of the permanent in $\widehat{\Lambda}_n^3$). If $n\geq 4(3t+j)$, then the ordered over decrease set $L_t^{(j)}$ gives the $|L_t^{(j)}|$ upper magnitudes of the permanent in $\widehat{\Lambda}_n^3$.*

Proof. Note that the proof is the same for every value of j. Therefore, let us consider, say, $j=0$. From (6.1) it follows that, if $r\in R(n;3)$ contains λ_3 parts 3 and $\lambda_3\leq\frac{n}{3}-4t$, then, for $y\in H_{n-3\lambda_3;4}(\rho)$, we have

$$6^{\lambda_3}y\leq 6^{\frac{n}{3}-4t}9^{3t}.$$

This means that for the formation the list of all upper magnitudes of the permanent in $\widehat{\Lambda}_n^3$ in the condition $n\geq 12t$, which are bounded from below by $6^{\frac{n}{3}-4t}9^{3t}$, it is sufficient to consider only a part of the spectrum containing numbers $\{6^{\lambda_3}y\}$, where $y\in H_{n-3\lambda_3;4}(\rho)$ with the opposite condition $\lambda_3\geq\frac{n}{3}-4t$. From the equality $3\lambda_3+...+n\lambda_n=n$ with the condition $\lambda_3\geq\frac{n}{3}-4t$, we have

$$4\lambda_4+...+n\lambda_n\leq n-3(\frac{n}{3}-4t)=12t.$$

Since $12t$ does not depend on n, there is only a finite assembly of such partition for arbitrary n. This ensures a possibility of the realization of the algorithm.

For the considered $r\in R(n;3)$, for $\lambda_3\geq\frac{n}{3}-4t$, we have $H_{n;3}(r)=6^{\frac{n-m}{3}}$, where $y\in H_{m;4}(\rho)$, and m has the form $m=3i,\ 1\leq i\leq 4t$. Thus we should choose only $H_{n;3}(r)\geq 6^{\frac{n}{3}-4t}9^{3t}$, and this yields

$$y\geq 9^{3t}6^{\frac{n}{3}-4t}=9^{3t}6^{i-4t},\ \ i=1,2,...,4t.$$

■

In order to use Theorem 8 for calculation the upper magnitudes $M^{(1)}(n)>M^{(2)}(n)>...$ of the permanent in $\widehat{\Lambda}_n^3$, in case, say, $n\equiv 0 \pmod 3$,
1) we write a list of partition of numbers $3i,\ \ i=2,3,...,4t$ with the parts not less than 4.
2) The corresponding values of y we compare with $9^{3t}6^{i-4t}$ and keep only $y\geq 9^{3t}6^{i-4t}$.
3) After that we regulate over decrease numbers $\{y6^{\frac{n}{3}-i}\}$.

Below we give the first 10 upper magnitudes $\widehat{M}^{(1)}>\widehat{M}^{(2)}>...>\widehat{M}^{(10)}$, of the permanent in $\widehat{\Lambda}_n^3$ for $n\geq 24$, via numbers $\{a(n)\}$ (2.12).

$$(6.6)\qquad \widehat{M}^{(1)}=\begin{cases}a(3)^{\frac{n}{3}}=6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)a(3)^{\frac{n-4}{3}}=\frac{3}{2}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(4)^2a(3)^{\frac{n-8}{3}}=\frac{9}{4}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3.\end{cases}$$

Formula (6.6) shows that $M^{(1)}(n)$ is attained in $\widehat{\Lambda}_n^3$.

$$\widehat{M}^{(2)}=\begin{cases} a(4)^3a(3)^{\frac{n-12}{3}}=\frac{9}{16}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(7)a(3)^{\frac{n-7}{3}}=\frac{31}{36}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(5)a(3)^{\frac{n-5}{3}}=\frac{13}{6}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.7}$$

$$\widehat{M}^{(3)}=\begin{cases} a(6)a(3)^{\frac{n-6}{3}}=\frac{5}{9}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)^4a(3)^{\frac{n-16}{3}}=\frac{27}{32}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(8)a(3)^{\frac{n-8}{3}}=\frac{49}{36}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.8}$$

$$\widehat{M}^{(4)}=\begin{cases} a(4)a(5)a(3)^{\frac{n-9}{3}}=\frac{13}{24}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)a(6)a(3)^{\frac{n-10}{3}}=\frac{5}{6}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(4)a(7)a(3)^{\frac{n-11}{3}}=\frac{31}{24}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.9}$$

$$\widehat{M}^{(5)}=\begin{cases} a(9)a(3)^{\frac{n-9}{3}}=\frac{13}{36}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)^2a(5)a(3)^{\frac{n-13}{3}}=\frac{13}{16}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(4)^5a(3)^{\frac{n-20}{3}}=\frac{81}{64}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.10}$$

$$\widehat{M}^{(6)}=\begin{cases} a(4)a(8)a(3)^{\frac{n-12}{3}}=\frac{49}{144}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(5)^2)a(3)^{\frac{n-10}{3}}=\frac{169}{216}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(4)^2a(6)a(3)^{\frac{n-14}{3}}=\frac{5}{4}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.11}$$

$$\widehat{M}^{(7)}=\begin{cases} a(4)^2a(7)a(3)^{\frac{n-15}{3}}=\frac{31}{96}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(10)a(3)^{\frac{n-10}{3}}=\frac{125}{216}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(4)^3a(5)a(3)^{\frac{n-17}{3}}=\frac{39}{32}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.12}$$

$$\widehat{M}^{(8)}=\begin{cases} a(4)^6a(3)^{\frac{n-24}{3}}=\frac{81}{256}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)a(9)a(3)^{\frac{n-13}{3}}=\frac{13}{24}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(5)a(6)a(3)^{\frac{n-11}{3}}=\frac{65}{54}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.13}$$

$$\widehat{M}^{(9)}=\begin{cases} a(4)^3a(6)a(3)^{\frac{n-18}{3}}=\frac{5}{16}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(5)a(8)a(3)^{\frac{n-13}{3}}=\frac{637}{1296}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(5)^2a(4)a(3)^{\frac{n-14}{3}}=\frac{169}{144}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3;\end{cases} \tag{6.14}$$

$$\widehat{M}^{(10)}=\begin{cases} a(5)a(7)a(3)^{\frac{n-12}{3}}=\frac{403}{1296}6^{\frac{n}{3}},\ \ if\ \ n\equiv 0 \pmod 3,\\ a(4)^3a(7)a(3)^{\frac{n-19}{3}}=\frac{31}{64}6^{\frac{n-1}{3}},\ \ if\ \ n\equiv 1 \pmod 3,\\ a(11)a(3)^{\frac{n-11}{3}}=\frac{67}{72}6^{\frac{n-2}{3}},\ \ if\ \ n\equiv 2 \pmod 3.\end{cases} \tag{6.15}$$

7. Main conjectural inequality for maximum of permanent in completely indecomposable Λ_n^3-matrices

Denote $\Lambda_{n,1}^3$ the set of completely indecomposable matrices in Λ_n^3, i. e., the set of Λ_n^3-matrices containing no Λ_m^3-submatrices. Let $\mu_1(n)$ denote the maximum of permanent in $\Lambda^3_{n,1}$. Our very plausible conjecture which we call "*main conjectural inequality (MCI)*" is the following.

Conjecture 1. *(Cf. [21], pp. 165-166) For $n_1, n_2 \ge 3$, we have*

$$\mu_1(n_1+n_2) \le \mu_1(n_1)\mu_1(n_2). \tag{7.1}$$

In Lemma 5 we essentially proved that in subclass $\widehat{\Lambda}^3_n$ the MCI is valid.

Besides, in all known cases MCI holds. Moreover, as we shall see, our algorithm of calculation the consecutive upper magnitudes ($M = M_1 > M_2 > \ldots$) of permanent in Λ^3_n, which is based on MCI, reproduces all Merriell's and Bolshakov's results for M_1 and M_2. Note also that, for sufficiently large n, the number of consecutive upper magnitudes of permanent in Λ^3_n grows very quickly with every step of extension of the list of known p-specrums for small n. E.g., using the found by Bolshakov $ps[\Lambda^3_i],\ i \le 8$, we obtain, for sufficiently large n, 4, 7 and 11 upper values of $ps[\Lambda^3_n]$ in cases $n = 3k,\ 2k+1$ and $3k+2$ correspondingly. After calculation $ps[\Lambda^3_9]$, the number of upper values of, e.g., $ps[\Lambda^3_{3k}]$ increases more than thrice.

8. Algorithm of calculations of upper magnitudes of the permanent in Λ^3_n based on MCI

Let $n \equiv j \pmod 3),\ j = 0,1,2$, and $t \in \mathbf{N}$. Let $R(m;\nu)$ denote the set of all partitions of n with parts more than or equal to ν. For us an important role play cases $\nu = 3, 4$. To $r \in R(m;3),\ \ \rho \in R(m;4)$ put in a correspondence sets

$$\pi_{m;3}(r) = \{\Pi_{r_i \in r} x_{r_i}\};\ \ \pi_{m;4}(\rho) = \{\Pi_{\rho_i \in \rho} x_{\rho_i}\} \tag{8.1}$$

where x_s runs through all values of permanent in set $\Lambda^3_{s,1}$ of completely indecomposable matrices in Λ^3_s (in case $m = 3$, when $\rho = \varnothing$, let us agree that $\pi_{3;4}$ is a singleton $\{6\}$).

Consider now the set $E^{(j)}_t = E^{(j)}_t(n)$:

$$E^{(j)}_t = \bigcup_{i=1}^{4t+j} \{6^{\frac{n-j-3i}{3}} y : y \in \pi_{3i+j;4}(\rho), y \ge 9^{3t+j} 6^{i-4t-j}\}. \tag{8.2}$$

Theorem 9. *(algorithm of calculation of upper magnitudes of the permanent in Λ^3_n). If $n \ge 4(3t+j)$, then the ordered over decrease set $E^{(j)}_t$ gives the $|E^{(j)}_t|$ upper magnitudes of the permanent in Λ^3_n.*

Proof. We need three lemmas.

Lemma 6. *For $n \ge 4$, we have*

$$\mu_1(n) \le 3^{\frac{n}{2}}. \tag{8.3}$$

Proof. Let, firstly, $n \equiv 0 \pmod 4$. Note that $\mu_1(4) = D_4 = 9$. Using (7.1), we find

$$\mu_1(n) \le \mu_1(4)\mu_1(n-4) \le ... \le (\mu_1(4))^{\frac{n}{4}} = 3^{\frac{n}{2}}.$$

Let, furthermore, $n \equiv i \pmod 4$, $i = 1, 2, 3$. Note that, by (2.17), $\mu_1(5) \le 13 < 3^{2.5}$. Therefore, using (7.1), we have

$$\mu_1(n) \le (\mu_1(4))^{\frac{n-5i}{4}}(\mu_1(5))^i < 3^{\frac{n-5i}{2}}3^{2.5i} = 3^{\frac{n}{2}}.$$

■

Lemma 7. *Let* $n = 3\lambda_3+4\lambda_4+...+n\lambda_n$ *be a partition of* n *with the parts not less than* 3. *If* $\lambda_3 \le l$, *and* n *has the form* $n = 3l + 4m$, *then, for completely indecomposable matrices* $A_i \in \Lambda_i^3$, $i = 3, 4, ..., n$, *we have*

$$\prod_{i=3}^{n}(perA_i)^{\lambda_i} \le 6^l 9^m. \tag{8.4}$$

Proof. Using Lemma 6, we have

$$\prod_{i=3}^{n}(perA_i)^{\lambda_i} \le 6^{\lambda_3}\sqrt{3}^{4\lambda_4+...+n\lambda_n} \le 6^l\sqrt{3}^{n-\lambda_3} = 6\sqrt{3}^{4m} = 6^l 9^m.$$

■

Lemma 8. *Let* $n = 3\lambda_3 + 4\lambda_4 + ... + n\lambda_n$ *and*

$$\lambda_3 \le \frac{n-4j}{3} - 4t, \;\; n \ge 4(3t+j),$$

where t *is a nonnegative integer and* j *is the residue of* n *modulo* 3, $j = 0, 1, 2$, *then, for completely indecomposable matrices* $A_i \in \Lambda_i^3$, $i = 3, 4, ..., n$, *we have*

$$\prod_{i=3}^{n}(perA_i)^{\lambda_i} \le 6^{\frac{n-4j}{3}-4t}9^{3t+j}. \tag{8.5}$$

Proof. Put $l = \frac{n-4j}{3} - 4t$, $m = \frac{n-3l}{4} = 3t + j$. Now the lemma follows from Lemma 7.

■

It is left to note that, after these lemmas, the proof of Theorem 9 is the same as proof of Theorem 8.

■

Note that the using of this algorithm is based on the small elements of p-spectrum.

Consider, e.g., case $t = 0$, $j = 2$. According to (8.2), we have

$$E_0^{(2)} = \bigcup_{i=1}^{2}\{6^{\frac{n-2-3i}{3}}y : y \in \pi_{3i+j;4}(\rho), y \ge 81 \cdot 6^{i-2}\} =$$

$$\{6^{\frac{n-5}{3}} per A,\ \ A\in\Lambda_5^3 : 6 per A \geq 81\}\cup\{81\cdot 6^{\frac{n-8}{3}}\}. \tag{8.6}$$

Note that, the second set is a simpleton, since, by MCI, $\mu_1(8,3)\leq(\mu_1(4,3))^2 = 81$. Since, by (2.17), $M(5) = 13 < \frac{81}{6}$, then the first set in (8.6) is empty. Thus $E_0^{(2)} = E_0^{(2)}(n)$ is simpleton:

$$E_0^{(2)} = \{81\cdot 6^{\frac{n-8}{3}}\}$$

and we have

$$M^{(1)}(n) = 81\cdot 6^{\frac{n-8}{3}},\ \ n\geq 8,$$

which corresponds to Merriell's result in case $n\equiv 2 \pmod 3$.
Further research of the set (8.2), using (2.17), gives the following results:
1) $j = 0,\ \ n\geq 24.$

$$M^{(1)}(n) = 6^{\frac{n}{3}},\ \ M^{(2)}(n) = \frac{9}{16}6^{\frac{n}{3}},$$

$$M^{(3)}(n) = \frac{5}{9}6^{\frac{n}{3}},\ \ M^{(4)}(n) = \frac{13}{24}6^{\frac{n}{3}}. \tag{8.7}$$

The continuation of this list requires the knowing of $ps[\Lambda_9^3]$. Note that a more detailed analysis shows that after calculation $ps[\Lambda_9^3]$ in this case one can obtain the first $12 + |G|$ upper magnitudes of the permanent in Λ_n^3, where $G = ps[\Lambda_9^3]\cap([69,116]\setminus\{72,78,102,108\})$.
2) $j = 1,\ \ n\geq 28.$

$$M^{(1)}(n) = \frac{3}{2}6^{\frac{n-1}{3}},\ \ M^{(2)}(n) = \frac{8}{9}6^{\frac{n-1}{3}},$$

$$M^{(3)}(n) = \frac{31}{36}6^{\frac{n-1}{3}},\ \ M^{(4)}(n) = \frac{27}{32}6^{\frac{n-1}{3}},$$

$$M^{(5)}(n) = \frac{5}{6}6^{\frac{n-1}{3}},\ \ M^{(6)}(n) = \frac{13}{15}6^{\frac{n-1}{3}},\ \ M^{(7)}(n) = \frac{169}{216}6^{\frac{n-1}{3}}. \tag{8.8}$$

It is interesting that in this case $ps[\Lambda_9^3]$ is not used up to $M^{(7)}$, but the continuation of this list requires the knowing of $ps[\Lambda_{10}^3]$.
3) $j = 2,\ \ n\geq 32.$

$$M^{(1)}(n) = \frac{9}{4}6^{\frac{n-2}{3}},\ \ M^{(2)}(n) = \frac{13}{6}6^{\frac{n-2}{3}},$$

$$M^{(3)}(n) = 2\cdot 6^{\frac{n-2}{3}},\ \ M^{(4)}(n) = \frac{13}{9}6^{\frac{n-2}{3}},$$

$$M^{(5)}(n) = \frac{49}{36}6^{\frac{n-2}{3}},\ \ M^{(6)}(n) = \frac{4}{3}6^{\frac{n-2}{3}},$$

$$M^{(7)}(n) = \frac{31}{24}6^{\frac{n-2}{3}},\ \ M^{(8)}(n) = \frac{81}{64}6^{\frac{n-2}{3}},$$

$$M^{(9)}(n)=\frac{5}{4}6^{\frac{n-2}{3}},\ \ M^{(10)}(n)=\frac{11}{9}6^{\frac{n-2}{3}},\ \ M^{(11)}(n)=\frac{39}{32}6^{\frac{n-1}{3}}. \tag{8.9}$$

Note that the method not only gives a possibility to calculate the upper magnitudes of the permanent in Λ_n^3, but also indicates those direct products on which they are attained. E.g., in (8.9) M_9 is attained on direct products of some matrices $A_i\in\Lambda_i^3$:

$$A_8\otimes\underbrace{A_3\otimes...\otimes A_3}_{\frac{n-8}{3}};\ \ A_4\otimes A_7\otimes\underbrace{A_3\otimes...\otimes A_3}_{\frac{n-11}{3}};\ \ A_4\otimes A_4\otimes A_6\otimes\underbrace{A_3\otimes...\otimes A_3}_{\frac{n-14}{3}}.$$

Note also that the comparison of (8.7)-(8.9) with (6.6)-(6.15) shows that the following calculated $M^{(i)}$ are attained in $\widehat{\Lambda}_n^3,\ \ n\geq 32:$
in case $n\equiv 0 \mod 3$,

$$M^{(1)},M^{(2)},M^{(3)},M^{(4)};$$

in case $n\equiv 1 \mod 3$,

$$M^{(1)},M^{(3)},M^{(4)},M^{(5)},M^{(6)},M^{(7)}$$

(and is not attained $M^{(2)}$);
in case $n\equiv 2 \mod 3$,

$$M^{(1)},M^{(2)},M^{(5)},M^{(7)},M^{(8)},M^{(9)},M^{(11)}$$

(and are not attained $M^{(3)},M^{(4)},M^{(6)},M^{(10)}$).

9. Algorithm of a testing the parity of values of the permanent in Λ_n^3

It seems that, among all known methods of calculation of the permanent, only Ryser's method (cf. [11], Ch.7) could be used for a creating an algorithm of a testing the parity of values of the permanent. Let A be $n\times n$-matrix. Let A_r be a matrix which is obtained by changing some r columns of A by zero columns. Denote $S(A_r)$ the product of row sums of A_r. Then, by Ryser's formula, we have

$$per A=\sum S(A_0)-\sum S(A_1)+$$

$$\sum S(A_2)-...+(-1)^{n-1}\sum S(A_{n-1}). \tag{9.1}$$

Let now A have integer elements. Introduce the following matrix function

$$\Upsilon(A)=\begin{cases}1, & \textit{if all row sums of } A \textit{ are odd},\\ 0, & \textit{otherwise}.\end{cases} \tag{9.2}$$

From (9.1) we have

$$per A\equiv\sum\Upsilon(A_0)-\sum\Upsilon(A_1)+$$

$$\sum \Upsilon(A_2)+...+\sum \Upsilon(A_{n-1}) \pmod 2. \tag{9.3}$$

Using (9.3), let us create an algorithm of a search of the odd values of the permanent in Λ_n^3. Since, evidently, $perA \equiv detA \pmod 2$, then A should have pairwise distinct columns. Note that cases $n \equiv j \pmod 3$, $j=0,1,2$, are considered by the same way. Suppose, say, $n=3t$. According to (9.3), we are interested in only cases when after removing $r\geq 1$ columns of A, all row sums will be odd. Suppose that after removing r columns of A, we have that p sums remain to equal to 3 and $n-p$ sums equal to 1. This means that the total number of the removed 1's equals to $2(n-p)=6t-2p$. Since, removing a column, we remove three 1's, then the number of the removed columns equals to $r=2t-\frac{2p}{3}$. Thus $p=3m$ and $r=2(t-m)$, $m=0,1,...,t-1$. However, if $m=t-1$, then $r=2$. By the condition, these two columns are distinct, therefore, we conclude that at least one row sum equals to 2. The contradiction shows that the testing sequence is $r=4,6,...,2t$. In cases $n\equiv 1,2 \pmod 3$ we obtain the same testing sequence.

Example 7. *Let us check the parities of values of the permanent of circulants in* $\Delta_7^3 \subset \Lambda_7^3$.

In this case $t=\lfloor \frac{7}{3} \rfloor = 2$ and, therefore, the testing sequence contains only term $r=4$. Note that matrix A_r has all odd rows if and only if one row sum equals to 3 and each of 6 other row sums equals to 1. Indeed, let after the removing 4 columns of A, remain p sums equal to 3 and $7-p$ sums equal to 1. This means that the total number of the removed 1's equals to $2(7-p)$ and the number of the removed columns equals to $r=4=\frac{14-2p}{3}$, i.e., $p=1$. Moreover, since in a circulant all rows are congruent shifts of the first one, it is sufficient to consider the case when precisely the first row sum equals to 3 and others equal to 1 (the multiplication on 7 does not change the parity of the result). This opens a possibility of a momentary handy test on the parity every circulant of class Δ_7^3. This test consists of the removing all four columns beginning with 0. If now every rows $2,...,7$ has one 1, then the permanent is even; otherwise, it is odd. We check now directly that from $\binom{7}{3}=35$ circulants exactly 21 ones have odd permanent.

Remark 1. *In* 1967, *Ryser* [14] *did a conjecture that the number of the transversals of a latin square from elements* $1,...,n$ *(i.e., the number of subsets of its* n *pairwise distinct elements, none in the same row or column) has the same parity as* n. *If* n *is even, then the conjecture has been proved* *by Balasubramanian* [1]*. Besides, in* [1] *Balasubramanian did a conjecture for the parity of a sum of permanents, such that the truth of this conjecture yields Ryser's hypothesis for odd n. In the same year* (1990)*, using the result of Example* 7*, the author disproved Balasubramanian's conjecture (private communication to Brualdi). It is interesting that soon Parker (see*[5]*, p.*258*) indeed found several latin squares of order* 7 *with even number of transversals. Add that later (*[18]*) we found even an infinite set of counterexamples to the Balasubramanian conjecture.*

10. Open problems

1. To prove the MCI (Section 7).

2. (Cf.[17], pp.171-172). Consider class $\Lambda_n(1, 1+a, 1+b)$, where $0 \le a \le b < \frac{4e-9}{6}$. Since $\Lambda_n(1,1,1) = \Lambda_n^3$, then Voorhoeve's lower estimate for the permanent (2.16) trivially holds for matrices in $\Lambda_n(1, 1+a, 1+b)$. It is clear that, for $a > 0,\ b > 0$, it should exist an essentially stronger lower estimate. However, using Van der Waerden-Egorychev-Falikman theorem to class $\Lambda_n(\frac{1}{3+a+b}, \frac{1+a}{3+a+b}, \frac{1+b}{3+a+b})$ of doubly stochastic matrices, for the permanent of $\Lambda_n(1, 1+a, 1+b)$-matrices we obtain even weaker lower estimate of the order $C_1\sqrt{n}(\frac{3+a+b}{e})^n << C(\frac{4}{3})^n$. The problem is to find a stronger lower estimate for the permanent in $\Lambda_n(1, 1+a, 1+b)$.

3. (Cf.[17], pp.115-116). Let M be a circulant of order n with integer elements. We conjecture that, for every integer m, we have $per M \equiv (-1)^n per(mJ_n - M) \pmod{n}$, where $n \times n$-matrix J_n consists of 1's only. A special case of this conjecture, for $m = 1,\ M = I_n + P + ... + P^{k-1}$ in the equivalent terms was formulated by Yamamoto [27] and proved for $k \le 3$. The author [15] proved the truth of the conjecture in case $m = 1$ for arbitrary circulant M (including Yamamoto's conjecture for every k). In [17] the conjecture was proved for every m and prime n. The question is open in case of composite n even in case $k = 3$.

4. Two Latin rectangles let us call equivalent, if the sets of their elements in the corresponding columns are the same. Note that numbers $|\Lambda_n^3|$ one can treat as the numbers of equivalence classes of Latin triangles. Let $A = I_n + P + P^2$. In [20] the author proved that the cardinality of the corresponding equivalent class is $2^n + 6 + 2(-1)^n$. To find the cardinality of the equivalent class which is defined by matrix $I_n + P + P^3$.

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