DERIVATION OF BELL POLYNOMIALS OF THE SECOND KIND

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Abstract

New methods for derivation of Bell polynomials of the second kind are presented. The methods are based on an ordinary generating function and its composita. The relation between a composita and a Bell polynomial is demonstrated. Main theorems are written and examples of Bell polynomials for trigonometric functions, polynomials, radicals, and Bernoulli functions are given.

1 Introduction

Bell polynomials are an important tool in solving various mathematical problems, among which is finding of higher derivatives of composite functions [1, 2, 3]. However, a general expression for Bell polynomials is rather difficult to derive. One of the main tools in computations of Bell polynomials is exponential generating functions [2, 3]. In this paper, it is proposed to use ordinary generating functions and their compositae [4] to derive expressions for Bell polynomials. Let us introduce the following notation. Let there be given a function $y(x)$ and an ordinary generating function $Y(x,z)=\sum_{n>0}\frac{y^{(n)}(x)}{n!}z^n$. By definition, the Bell polynomial of the second kind is written as

$$B_{n,k}(y^{(1)},y^{(2)},\dots y^{(n-k+1)})=\frac{1}{k!}\sum_{\pi_k\in C_n}\binom{n}{\lambda_1,\lambda_2,\cdots\lambda_k}y^{(\lambda_1)}y^{(\lambda_2)}\cdots y^{(\lambda_k)}$$

or as

$$B_{n,k}=\frac{n!}{k!}\sum_{\pi_k\in C_n}\frac{y^{(\lambda_1)}(x)}{\lambda_1!}\frac{y^{(\lambda_2)}(x)}{\lambda_2!}\cdots\frac{y^{(\lambda_k)}(x)}{\lambda_k!}$$

where $y^{(i)}$ is the i-th derivative of the function $y(x)$, C_n is the set of compositions of n, and π_k is the composition of n with k parts exactly $\{\lambda_1+\lambda_2+\dots\lambda_k=n\}$.

The polynomial $B_{n,k}$ has the form of a triangle in which the left part contains all derivatives of the function $y(x)$ and the right part contains $[y'(x)]^n$.

$$
\begin{array}{cccccccccc}
 & & & & y^{(1)} & & & & & \\
 & & & y^{(2)} & & [y^{(1)}]^2 & & & & \\
 & & y^{(3)} & & B_{3,2} & & [y^{(1)}]^3 & & & \\
 & y^{(4)} & & B_{4,2} & & B_{4,3} & & [y^{(1)}]^4 & & \\
\iddots & & \vdots & & \vdots & & \vdots & & \ddots & \\
y^{(n)} & B_{n,2} & & \cdots & & \cdots & & B_{n,n-1} & & [y^{(1)}]^n
\end{array}
$$

The generating function is $Y(x,z) = \sum_{n>0} \frac{y^{(n)}(x)}{n!} z^n = y(x+z) - y(x)$ [1, 2, 3]. Hence, we can introduce the composita of the generating function $Y(x,z)$ as [4, 5]

$$Y^{\Delta}(n,k,x) = \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x)}{\lambda_1!} \frac{y^{(\lambda_2)}(x)}{\lambda_2!} \cdots \frac{y^{(\lambda_k)}(x)}{\lambda_k!},$$

and the generating function for $Y^{\Delta}(n,k,x)$ will have the expression:

$$[Y(x,z)]^k = (y(x+z) - y(x))^k = \sum_{n \geq k} Y^{\Delta}(n,k,x) z^n$$

In view of the foregoing, we can write the relation for the Bell polynomial and composita of the ordinary generating function $Y(x,z)$:

$$B_{n,k} = \frac{n!}{k!} Y^{\Delta}(n,k,x). \tag{1}$$

Because there is a one-to-one relation between the composita and the Riordan array [5], the exponential Riordan array $(1, y(x))$ and the Bell polynomial $B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})$, where $y(x) = \sum_{n>0} y_n \frac{x^n}{n!}$, are equivalent.

2 Expressions for Bell polynomials based on the composita of a generating function $Y(\alpha, z)$

Let us consider the problem of finding the Bell polynomial $B_{n,k}$ as the problem of finding coefficients of an ordinary generating function $Y(\alpha,z)^k$. This is possible if we represent the generating function $Y(x,z)$ as $F(g(x), h(z))$; then we can use the apparatus of compositae introduced in [4, 5]. Let us consider the following examples:

Example 2.1. Let there be given a function $y(x)$ with two derivatives $y'(x)$ and $y''(x)$. Let us find an expression for the composita of this function. By definition,

$$Y^{\Delta}(n,k,x) = \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}}{\lambda_1!} \frac{y^{(\lambda_2)}}{\lambda_2!} \cdots \frac{y^{(\lambda_k)}}{\lambda_k!}.$$

Then, the generating function has the expression $y(x+z)-y(x)=y'(x)z+\frac{y''(x)}{2}z^2$. Hence, according to the formula of the composita for the polynomial $ax+bx^2$ [4], we obtain

$$Y^{\Delta}(n,k,x)=\binom{k}{n-k}[f'(x)]^{2k-n}\left(\frac{f''(x)}{2}\right)^{n-k}. \tag{2}$$

Thus, the Bell polynomial for the function with derivatives $y'(x)$ and $y''(x)$ is equal to

$$B_{n,k}=\frac{n!}{k!}\binom{k}{n-k}[f'(x)]^{2k-n}\left(\frac{f''(x)}{2}\right)^{n-k}. \tag{3}$$

Example 2.2. Let there be given a function $y(x)=x^m$, where $m>0$. The generating function is $Y(x,z)=(x+z)^m-x^m$. Let us find a composita of $Y(x,z)$; for this purpose, we are to find the coefficients:

$$x^{km}\left[\left(1+\frac{z}{x}\right)^m-1\right]^k=x^{km}\sum_{j=0}^{k}\binom{k}{j}\left(1+\frac{z}{x}\right)^{jm}(-1)^{k-j};$$

From whence, knowing that the coefficients for $\left(1+\frac{z}{x}\right)^{jm}$ are equal to $\binom{jm}{n}\frac{1}{x^n}$, we obtain the desired composita

$$Y^{\Delta}(n,k,x)=x^{km}\sum_{j=0}^{k}\binom{k}{j}\binom{jm}{n}x^{-n}(-1)^{k-j}.$$

Then the Bell polynomial is

$$B_{n,k}=\frac{n!}{k!}x^{km-n}\sum_{j=0}^{k}\binom{k}{j}\binom{jm}{n}(-1)^{k-j}.$$

Example 2.3. Let there be given a function $y(x)=x^{-m}$, where $m>0$. The generating function is $Y(x,z)=\frac{1}{(x+z)^m}-\frac{1}{x^m}$. Let us find a composita of $Y(x,z)$; for this purpose, we are to find the coefficients

$$Y(x,z)^k=\frac{1}{x^{mk}}\left[\frac{1}{\left(1+\frac{z}{x}\right)^m}-1\right]^k;$$

from whence it follows that the composita is equal to

$$\left(\sum_{j=1}^{k}\binom{k}{j}(-1)^{n+k-j}\binom{n+j\,m-1}{j\,m-1}\right)x^{-n-k\,m}.$$

Then the Bell polynomial is equal to

$$B_{n,k}=\frac{n!}{k!}\left(\sum_{j=1}^{k}\binom{k}{j}(-1)^{n+k-j}\binom{n+j\,m-1}{j\,m-1}\right)x^{-n-k\,m}.$$

Example 2.4. Let us consider the example of use of the composita for the generating function $f(z) = az + bz^2 + cz^3$:

$$F^{\Delta}(n,k) = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.$$

Substitution of $a = \frac{f'(x)}{1!}$, $b = \frac{f''(x)}{2!}$, $c = \frac{f'''(x)}{3!}$ gives the Bell polynomial:

$$B_{n,k} = \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} (f'(x))^{k-j} \left(\frac{f''(x)}{2}\right)^{2j+k-n} \left(\frac{f'''(x)}{6}\right)^{n-k-j}.$$

Let us consider the example $f(x) = x^3 + 2x$, $f'(x) = 3x^2 + 2$, $f''(x) = 6x$, $f'''(x) = 6$; then, $a = 3x^2 + 2$, $b = 3x$, $c = 1$. Then the Bell polynomial is

$$\frac{n!}{k!} \sum_{j=0}^{k} \binom{j}{n-k-j} \binom{k}{j} 3^{-n+k+2j} x^{-n+k+2j} \left(3x^2 + 2\right)^{k-j}$$

Presented below are the first terms of this polynomial

$$3x^2 + 2$$

$$6x,\ (3x^2 + 2)^2$$

$$6,\ 18x(3x^2 + 2),\ (3x^2 + 2)^3$$

$$0,\ 180x^2 + 48,\ 36x(3x^2 + 2)^2,\ (3x^2 + 2)^4$$

The same reasoning allows us to obtain Bell polynomials for functions whose generating functions $y(x+z) - y(x)$ are expressed in polynomials. Expressions for the compositae of polynomials and methods of their derivation are described in [4].

Example 2.5. Let us find a Bell polynomial for the function $\sin x$. For this purpose, we find the composita of the function $\sin(x+z) - \sin x$. Then

$$S(x,z) = \cos x \sin z + \sin x(\cos z - 1),$$

where $\sin z$ and $\cos z$ are generating functions, and $\sin x$ and $\cos x$ are coefficients. Hence the composita of the function $\cos x \sin z$ [4] is

$$F^{\Delta}(n,k,x) = (\cos x)^k \frac{(1 + (-1)^{n-k})}{2^k n!} \sum_{m=0}^{\frac{k}{2}} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m},$$

Now, let us write the coefficients $T_{n,k}$ for $\cos^k(z) = \sum_{n \geq 0} T_{n,k} z^n$.

$$T_{n,k} = \begin{cases} 1, & n = 0 \\ 0, & n - \text{odd}, \\ \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{i} \frac{(k-2i)^n}{(n)!} (-1)^{\frac{n}{2}}, & n - \text{even}. \end{cases}$$

Then we obtain the composita of the generating function $\sin x(\cos(z)-1)$

$$R^{\Delta}(n,k,x)=(\sin x)^k\frac{(-1)^n+1}{n!}\sum_{j=1}^{k}\frac{(-1)^{\frac{n}{2}+k-j}}{2^j}\binom{k}{j}\sum_{i=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}(j-2i)^n\binom{j}{i}$$

Next, from the theorem of the composita of the sum of generating functions [4], we obtain the desired composita

$$S^{\Delta}(n,k,x)=F^{\Delta}(n,k,x)+R^{\Delta}(n,k,x)+\sum_{j=1}^{k-1}\binom{k}{j}\sum_{i=j}^{n-k+j}F^{\Delta}(i,j,x)R^{\Delta}(n-i,k-j,x).$$

Presented below are the first terms of the Bell polynomial $B_{n,k}=\frac{n!}{k!}S^{\Delta}(n,k,x)$ for the function $\sin x$:

$$\cos x$$

$$-\sin x,\ \cos^2 x$$

$$-\cos x,\ -3\cos x\sin x,\ \cos^3 x$$

$$\sin x,\ 3\sin^2 x-4\cos^2 x,\ -6\cos^2 x\sin x,\ \cos^4 x$$

$$\cos x,\ 15\cos x\sin x,\ 15\cos x\sin^2 x-10\cos^3 x,\ -10\cos^3 x\sin x,\ \cos^5 x$$

Now the derivative $f_1^{(4)}(x)$ for the function $f_1(x)=e^{\sin x}$ is expressed as

$$f_1^{(4)}(x)=e^{\sin x}\left(\sin x+3\sin^2 x-4\cos^2 x-6\cos^2 x\sin x+\cos^4 x\right).$$

The derivative $f_2^{(5)}(x)$ for $f_2(x)=\sin^3 x$ is expressed as

$$f_2^{(5)}(x)=3\sin^2 x(\cos x)+6\sin x(15\cos x\sin x)+6(15\cos x\sin^2 x-10\cos^3 x)=$$

$$=183\sin^2 x\cos x-60\cos^3 x.$$

In the same way, we can find a Bell polynomial for the function $\cos x$; for this purpose, we are to find the composita of the generating function:

$$C(x,z)=\cos x(\cos z-1)-\sin x\sin z.$$

Example 2.6. Let us consider the function $y(x)=\sqrt[3]{x}$. The generating function is $Y(x,z)=\sqrt[3]{x+z}-\sqrt[3]{x}$. Hence

$$Y(x,z)^m=(-1)^m(\sqrt[3]{x})^m\left[1-\sqrt[3]{\left(1+\frac{z}{x}\right)}\right]^m.$$

Given the composita of the generating function $1-\sqrt[3]{1-z}$ [5], we obtain the desired composita

$$Y^{\Delta}(n,m,x)=\begin{cases}(\sqrt[3]{x})^m(\frac{1}{3})^n, & n=m,\\ (\sqrt[3]{x})^m\frac{m}{n}\sum\limits_{k=1}^{n-m}\binom{k}{n-m-k}3^{-2n+m+k}(-1)^k\binom{n+k-1}{n-1}x^{-n}, & n>m.\end{cases}$$

3 Method based on operations on compositae $Y^{\Delta}(n,k,x)$

Let us consider peculiarities of the generating function $Y(x,z)=\sum_{n>0}\frac{y^{(n)}(x)}{n!}z^n=y(x+z)-y(x)$. For this purpose, we prove the following theorem.

Theorem 3.1. Let there be given a composition $f(x)=g(y(x))$ and functions $g(x)$, $y(x)$ with an infinite number of derivatives in the general case. Then the generating functions $F(x,z)=\sum_{n\geqslant 0}\frac{f^{(n)}(x)}{n!}z^n$, $Y(x,z)=\sum_{n\geqslant 1}\frac{y^{(n)}(x)}{n!}z^n$ и $G(x,z)=\sum_{n\geqslant 0}\frac{g^{(n)}(x)}{n!}z^n$ form the composition

$$F(x,z)=G(y,Y(x,z)).$$

Proof. Let us write the known Faa di Bruno formula [1, 2, 3]:

$$f^{(n)}(x)=\sum_{k=1}^{n}g^{(k)}(y)\frac{n!}{k!}\sum_{\pi_k\in C_n}\frac{y^{(\lambda_1)}(x)}{\lambda_1!}\frac{y^{(\lambda_2)}(x)}{\lambda_2!}\cdots\frac{y^{(\lambda_k)}(x)}{\lambda_k!}.$$

Hence

$$\frac{f^{(n)}(x)}{n!}=\sum_{k=1}^{n}\frac{g^{(k)}(y)}{k!}\sum_{\pi_k\in C_n}\frac{y^{(\lambda_1)}(x)}{\lambda_1!}\frac{y^{(\lambda_2)}(x)}{\lambda_2!}\cdots\frac{y^{(\lambda_k)}(x)}{\lambda_k!} \qquad (4)$$

Thus, we obtain the formula for the composition of ordinary generating functions [4]. It is evident that the nonzero term of $F(x,z)$ is equal to $g(y(x))$. ☐

The peculiarity here is that in the operation of the composition of generating functions, the argument x in $G(x,z)$ is replaced by $y(x)$. Let us turn to the problem of finding compositae of the generating functions $(y(x+z)-y(x))$ using the operations of summation, product, and composition.

Theorem 3.2. Let there be generating functions $F(x,z)=f(x+z)-f(x)=\sum\limits_{n>0}\frac{f^{(n)}(x)}{n!}z^n$, $G(x,z)=g(x+z)-g(x)=\sum\limits_{n>0}g^{(n)}(x)z^n$ and their compositae $F^{\Delta}(n,k,x)$, $G^{\Delta}(n,k,x)$. Then the generating function $A(x,z)=F(x,z)+G(x,z)$ has the composita

$$A^{\Delta}(n,k,x)=F^{\Delta}(n,k,x)+\sum_{j=1}^{k-1}\binom{k}{j}\sum_{i=j}^{n-k+j}F^{\Delta}(i,j,x)G^{\Delta}(n-i,k-j,x)+G^{\Delta}(n,k,x).$$

Proof. without proof ☐

Example 3.3. Let there be $f(x)=x^2$, $F(x,z)=2xz+z^2$, a composita $F^{\Delta}(n,k,x)=\binom{k}{n-k}(2x)^{2k-n}$ and $g(x)=\ln(x)$, $G(x,z)=\ln(x+z)-\ln(x)=\ln(1+\frac{z}{x})$, and a composita $G^{\Delta}(n,k,x)=\frac{k!}{n!}\left[{n \atop k}\right]x^{-n}$. Then for the function $a(x)=x^2+\ln(x)$, the Bell polynomial is

$$B_{n,k}=\frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=j}^{n-k+j}\frac{j!}{i!}\left[{i \atop j}\right]\binom{k-j}{n-i-k+j}2^{2(k-j)-n+i}x^{2(k-j)-n}.$$

Now let us turn to finding of the composita $Y^{\Delta}(n,k,x)$ of the function $y(x)=f(x)g(x)$ expressed as the product of the functions $f(x)$ and $g(x)$. Let us prove the following theorem.

Theorem 3.4. Let there be a function $a(x) = f(x)g(x)$; then the composita of the function $Y(x,z) = f(x+z)g(x+z) - f(x)g(x)$ is equal to

$$Y^{\Delta}(n,k,x) = \sum_{j=0}^{k} \binom{k}{j} \left(\sum_{i=0}^{n} F(i,j,x) G(n-i,j,x) \right) [f(x)g(x)]^{k-j} (-1)^{k-j}.$$

where $F(n,k)$ are coefficients of the generating function $[f(x+z)]^k$, and $G(n,k)$ – $[g(x+z)]^k$.

Proof. Here we have the second peculiarity: it is necessary to take into account the rule of finding a derivative of the product. According to the Leibniz rule, we can write

$$\frac{y^{(n)}}{n!} = \sum_{i=0}^{n} \frac{f^{(i)}}{i!} \frac{g^{(n-i)}}{(n-i)!}.$$

Hence

$$y(x+z) = f(x+z)g(x+z)$$

Now let us find coefficients for the expression $[f(x+z)g(x+z) - f(x)g(x)]^k$. By removing the brackets and substituting the expression for the coefficients of the generating functions $f(x+z)$ and $g(x+z)$, we obtain the desired formula. □

Given the composita of the generating function $f(x+z) - f(x)$ – $F^{\Delta}(n,k,x)$, the coefficients of the generating function $[f(x+z)]$ are calculated by the formula:

$$F(n,k,x) = \sum_{j=0}^{k} \binom{k}{j} F^{\Delta}(n,j,x) f(x)^{k-j}.$$

Example 3.5. Let there be a function x^{ax} (see the example in [3]). Let us find an expression for the n-th derivative of this function. Let us write it in the form $\exp(ax\ln(x))$. For this purpose, we find a composita of the function $(x+z)\ln(x+z) - x\ln(x)$ and expressions for coefficients of the generating functions $(x+z)^k$ and $\ln(x+z)^k$. For the first function, $F(n,k) = \binom{k}{n} x^{k-n}$; for the second function, $G(n,k) = \sum_{j=0}^{k} \binom{k}{j} \frac{j!}{n!} \left[{n \atop j} \right] x^{-n} \ln(x)^{k-j}$. Then the composita of the function $(x+z)\ln(x+z) - x\ln(x)$ is

$$A^{\Delta}(n,k,x) = x^{k-n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\sum_{i=0}^{n} \binom{j}{i} \frac{1}{(n-i)!} \sum_{m=0}^{j} m! \binom{j}{m} \left[{n-i \atop m} \right] (\ln x)^{k-m} \right).$$

From this it follows that the composita of the function $ax\ln x$ is equal to $a^k A^{\Delta}(n,k,x)$. Presented below are the first terms of this composita.

$$a(\ln x + 1)$$

$$\frac{a}{2\,x}, \quad a^2(\ln x + 1)^2$$

$$-\frac{a}{6x^2},\quad \frac{a^2\,\ln x + a^2}{x},\quad a^3(\ln x+1)^3$$

$$\frac{a}{12\,x^3},\quad -\frac{4\,a^2\,\ln x + a^2}{12\,x^2},\quad \frac{3\,a^3\,\ln^2 x + 6\,a^3\,\ln x + 3\,a^3}{2\,x},\quad a^4(\ln x+1)^4$$

Hence the expression for the n-th derivative of the generating function x^{ax} has the form:

$$[x^{ax}]^{(n)} = x^{ax}\sum_{k=1}^{n}\frac{n!}{k!}a^k x^{k-n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(\sum_{i=0}^{n}\binom{j}{i}\sum_{m=0}^{j}\frac{m!}{(n-i)!}\binom{j}{m}\left[{n-i \atop m}\right](\ln x)^{k-m}\right).$$

Now let us consider the operation of product of compositae. For this purpose, we prove the following theorem.

Theorem 3.6. Let there be functions $f(x)$, $g(x)$ and compositae of the generating functions $F^{\Delta}(n,k,x)$ for $f(x+z)-f(x)$ and $G^{\Delta}(n,k,x)$ for $g(x+z)-g(x)$. Then for the composition of the functions $y(x)=g(f(x))$, the composita of the generating function $Y(x,z)=g(f(x+z))-g(f(x))$ is

$$Y^{\Delta}(n,m,x)=\sum_{k=m}^{n}F^{\Delta}(n,k,x)G^{\Delta}(k,m,f(x)).$$

Proof.

$$[f(x+z)-f(x)]^m=\sum_{n\geq m}F^{\Delta}(n,m,x)z^n$$

From formula (4) we have

$$Y^{\Delta}(n,m,x)=\sum_{k=m}^{n}F^{\Delta}(n,k,x)G^{\Delta}(k,m,f(x)).$$

Given the expression for the coefficients $Y(n,k,x)$ of the generating function $y(x+z)^k$, the expression for the coefficients of the composition of the generating functions $a(x+z) = y(f(x+z)$ has the form:

$$A(n,m,x)=\begin{cases} y(f(x))^m, & n=0\\ \sum_{k=1}^{n}F^{\Delta}(n,k,x)Y(k,m,x), & n>0.\end{cases}$$

□

Note that this theorem holds true for Bell polynomials as well [3], because

$$B_{n,m}(x)=\sum_{k=m}^{n}\frac{n!}{k!}F^{\Delta}(n,k,x)\frac{k!}{m!}G^{\Delta}(k,m,f(x))=\frac{n!}{m!}\sum_{k=m}^{n}F^{\Delta}(n,k,x)G^{\Delta}(k,m,f(x)).$$

Example 3.7. Let us find a composita of the function $f(x)=\frac{1}{x}$. The generating function for the composita is $F(x,z)=\frac{1}{x+z}-\frac{1}{x}=\frac{1}{x}\frac{-\frac{z}{x}}{1+\frac{z}{x}}$. Hence

$$F^{\Delta}(n,k,x)=\binom{n-1}{k-1}(-1)^n x^{-n-k}.$$

Now let us write the composition $a(x)=\frac{1}{\ln(x)}$. The composita for the generating function $\ln(x+z)-\ln(x)$ is $\frac{k!}{n!}\left[{n \atop k}\right]x^{-n}$. From this it follows that the desired composita is equal to

$$A^{\Delta}(n,m)=\sum_{k=m}^{n}\frac{k!}{n!}\left[{n \atop k}\right]x^{-n}\binom{k-1}{m-1}(-1)^k(\ln(x))^{-n-k},$$

and the Bell polynomial is

$$B_{n,k}=m!\sum_{k=m}^{n}k!\left[{n \atop k}\right]x^{-n}\binom{k-1}{m-1}(-1)^k(\ln(x))^{-n-k}.$$

Example 3.8. Let us find a Bell polynomial for the function $a(x)=\frac{1}{1-x-x^2}$, the function $a(x)=g(f(x))$, where $g(x)=\frac{1}{1-x}$, $f(x)=x+x^2$. The composita of the function $f(x)$ is equal to $F^{\Delta}(n,k,x)=\binom{k}{n-k}(2x+1)^{2k-n}$ (see example No. 2.2). The composita of the function $g(x)=\frac{1}{1-x}$ is equal to $F^{\Delta}(n,k,x)=\binom{n-1}{k-1}(1-x)^{-k-n}$. Using theorem 3.6, we obtain the desired Bell polynomial:

$$B_{n,m}=\frac{n!}{m!}\sum_{k=m}^{n}\binom{k-1}{m-1}\binom{k}{n-k}(2x+1)^{2k-n}(1-x-x^2)^{-m-k}.$$

$$\frac{2\,x+1}{\left(-x^2-x+1\right)^2}$$

$$\frac{2}{\left(-x^2-x+1\right)^2}+\frac{2\left(2\,x+1\right)^2}{\left(-x^2-x+1\right)^3},\ \frac{\left(2\,x+1\right)^2}{\left(-x^2-x+1\right)^4}$$

$$\frac{12\ (2\,x+1)}{\left(-x^2-x+1\right)^3}+\frac{6\left(2\,x+1\right)^3}{\left(-x^2-x+1\right)^4},\ \frac{6\ (2\,x+1)}{\left(-x^2-x+1\right)^4}+\frac{6\left(2\,x+1\right)^3}{\left(-x^2-x+1\right)^5},\ \frac{\left(2\,x+1\right)^3}{\left(-x^2-x+1\right)^6}$$

Example 3.9. Let us find a Bell polynomial for the function $\tan(x)$. For this purpose, we represent the generating function as

$$A(x,z)=\tan(x+z)-\tan(x)=\frac{\tan(x)+\tan(z)}{1-\tan(x)\tan(z)}-\tan(x)=\frac{\tan(z)\sec(x)^2}{1-\tan(x)\tan(z)}.$$

Hence $A(x,z)=f(x,\tan(z))$, where $f(x,z)=\frac{\sec(x)^2 z}{1-\tan(x)z}$. Then the composita of $f(x,z)$ is equal to

$$F^{\Delta}(n,k,x)=\binom{n-1}{k-1}\tan(x)^{n-k}\sec(x)^{2k}.$$

The composita of the generating function $\tan(z)$ is

$$G^{\Delta}(n,k)=\frac{1+(-1)^{n-k}}{n!}\sum_{j=k}^{n}2^{n-j-1}\left\{{n \atop j}\right\}j!(-1)^{\frac{n+k}{2}+j}\binom{j-1}{k-1}.$$

Using the theorem of product of compositae [5], we obtain the composita of the desired function:

$$G^{\Delta}(n,m)=\sum_{k=m}^{n}G^{\Delta}(n,k)F^{\Delta}(k,m)=$$

$$=\sum_{k=m}^{n}\frac{1+(-1)^{n-k}}{n!}\sum_{j=k}^{n}2^{n-j-1}\left\{{n \atop j}\right\}j!(-1)^{\frac{n+k}{2}+j}\binom{j-1}{k-1}\binom{k-1}{m-1}\tan(x)^{k-m}\sec(x)^{2m}.$$

Hence the Bell polynomial is equal to

$$B_{n,m}=\frac{\sec(x)^{2m}}{m!}\sum_{k=m}^{n}\frac{1+(-1)^{n-k}}{2}\sum_{j=k}^{n}2^{n-j}\left\{{n \atop j}\right\}j!(-1)^{\frac{n+k}{2}+j}\binom{j-1}{k-1}\binom{k-1}{m-1}\tan(x)^{k-m}.$$

$$\sec(x)^2$$

$$2\sec(x)^2\tan(x),\ \sec(x)^4$$

$$6\sec(x)^2\tan(x)^2+2\sec(x)^2,\ 6\sec(x)^4\tan(x),\sec(x)^6]$$

$$24\sec(x)^2\tan(x)^3+16\sec(x)^2\tan(x),36\sec(x)^4\tan(x)^2+8\sec(x)^4,12\sec(x)^6\tan(x),\sec(x)^8$$

Given the composita of the function $\tan(x)$, we can obtain the composita of $\cot(x)$ by representing $\cot(x)=\frac{1}{\tan(x)}$ (see example 3.7).

Example 3.10. Let us derive a Bell polynomial for the function $\arctan(x)$. For this purpose, we write the generating function

$$A(x,z)=\arctan(x+z)-\arctan(x)=\arctan\left(\frac{z}{1+x^2+xz}\right).$$

Let us find a composita of the function $\frac{z}{1+x^2+xz}$. We represent it as

$$f(x,z)=\frac{1}{(1+x^2)}\frac{z}{1+\frac{xz}{1+x^2}}.$$

Hence, the composita of the function $f(x,z)$ is equal to

$$F^{\Delta}(n,k)=\binom{n-1}{k-1}(-1)^{n-k}\frac{x^{n-k}}{(1+x^2)^n}.$$

Given the composita of the generating function $\arctan(z)$ [5]

$$\frac{\left((-1)^{\frac{3n+k}{2}}+(-1)^{\frac{n-k}{2}}\right)k!}{2^{k+1}}\sum_{j=k}^{n}\frac{2^j}{j!}\binom{n-1}{j-1}\left[{j \atop k}\right],$$

we obtain the composita of the desired generating function $A(x,z)$:

$$A^{\Delta}(n,m)=\sum_{k=m}^{n}\binom{n-1}{k-1}\frac{(-x)^{n-k}}{(1+x^2)^n}\frac{\left((-1)^{\frac{3k+m}{2}}+(-1)^{\frac{k-m}{2}}\right)m!}{2^{m+1}}\sum_{j=m}^{k}\frac{2^j}{j!}\binom{k-1}{j-1}\left[{j \atop m}\right].$$

Hence the desired Bell polynomial is equal to

$$B_{n,m}=n!\sum_{k=m}^{n}\binom{n-1}{k-1}\frac{(-x)^{n-k}}{(1+x^2)^n}\frac{\left((-1)^{\frac{3k+m}{2}}+(-1)^{\frac{k-m}{2}}\right)}{2^{m+1}}\sum_{j=m}^{k}\frac{2^j}{j!}\binom{k-1}{j-1}\left[{j \atop m}\right].$$

Presented below are the first terms of the Bell polynomial for the function $\arctan(x)$

$$\frac{1}{x^2+1}$$

$$-\frac{2\,x}{\left(x^2+1\right)^2},\ \frac{1}{\left(x^2+1\right)^2}$$

$$6\left(\frac{x^2}{\left(x^2+1\right)^3}-\frac{1}{3\,\left(x^2+1\right)^3}\right),\ -\frac{6\,x}{\left(x^2+1\right)^3},\ \frac{1}{\left(x^2+1\right)^3}$$

$$24\left(\frac{x}{\left(x^2+1\right)^4}-\frac{x^3}{\left(x^2+1\right)^4}\right),\ 12\left(\frac{3\,x^2}{\left(x^2+1\right)^4}-\frac{2}{3\,\left(x^2+1\right)^4}\right),\ -\frac{12\,x}{\left(x^2+1\right)^4},\ \frac{1}{\left(x^2+1\right)^4}$$

Example 3.11. Let us find a Bell polynomial for the function $a(x)=\frac{x}{\sqrt{1-x^2}}$. For this purpose, we represent this function in the form $g(h(g(f(x)))=\frac{1}{\sqrt{\frac{1}{x^2}-1}}$. Let us write the compositae for the functions $f(x)=x^2$ and $g(x)=\frac{1}{x}$

$$F^{\Delta}(n,k,x)=\binom{k}{n-k}(2x)^{2k-n}$$

$$G^{\Delta}(n,k,x)=\binom{n-1}{k-1}(-1)^n x^{-n-k}.$$

Hence the composita of $g(f(x)=\frac{1}{x^2}$ is equal to

$$x^{-n-2k}\sum_{k=m}^{n}\binom{k}{n-k}2^{2k-n}\binom{k-1}{m-1}(-1)^k.$$

Now let us find a composita of the function $\frac{1}{\sqrt{x}}$. For this purpose, we also use the composition of the functions $g(h(x))$. Let us derive a composita for the function $\sqrt{x}$. For this purpose, we write the generating function $\sqrt{x+z}$

$$H(x,z)=\sqrt{x+z}-\sqrt{x}=-\sqrt{x}2\frac{(1-\sqrt{1-4\frac{z}{4x}})}{2}.$$

Note that the generating function in the brackets is the generating function for Catalan numbers [5]. Given the composita of the function, we obtain the composita for $H(x,z)$

$$H^{\Delta}(n,k,x) = \frac{k}{n}\binom{2n-k-1}{n-1}(-1)^{n-k}(\sqrt{x})^k 2^k 4^{-n}.$$

Hence the composita of the function $\frac{1}{\sqrt{x}}$ is equal to

$$(-1)^n(\sqrt{x})^m 4^{-n}\sum_{k=m}^{n}\frac{k}{n}\binom{2n-k-1}{n-1}2^k\binom{k-1}{m-1}.$$

This result was obtained by L. Comtet [3]. Now from theorem 3.6, we obtain the composita of the function $a(x) = \frac{x}{\sqrt{1-x^2}}$

$$x^{m-n}\sum_{k=m}^{n}\frac{(-1)^k}{k4^k}\sum_{j=m}^{k} j2^j\binom{j-1}{m-1}\binom{2\,k-j-1}{k-1}\sum_{i=k}^{n}(-1)^i\binom{i-1}{k-1}\binom{i}{n-i}2^{2i-n}(1-x^2)^{-\frac{m}{2}-k}$$

Example 3.12. Let us find a Bell polynomial for the generating function of Bernoulli numbers $a(x) = \frac{x}{e^x-1}$. For this purpose, we write the expressions for the coefficients of the generating functions $F(x,z) = (x+z)^k$ and $G(x,z) = \left(\frac{1}{x+z-1}\right)^k$. Hence

$$F(n,k,x) = \binom{k}{n}x^{k-n}.$$

$$G(n,k,x) = \binom{n+k-1}{k-1}(x-1)^{-n-k}(-1)^n.$$

Using formula (4) for the composition of the generating functions $g(x+z)^m$ and e^{x+z}, we obtain expressions for the coefficients of $h(x+z) = [\frac{1}{e^{(x+z)}-1}]^m$

$$H(n,m,x) = \begin{cases} \frac{1}{(e^x-1)^m}, & n=0 \\ \frac{1}{n!}\sum_{k=0}^{m}(-1)^k\ k!\binom{m+k-1}{m-1}\left\{ {n \atop k} \right\}(e^x-1)^{-m-k}\ e^{k\,x}, & n>0. \end{cases}$$

From theorem 3.4 we obtain the composita of the product $x\frac{1}{e^x-1}$

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{x}{e^x-1}\right)^{m-j}\sum_{i=0}^{n}H(i,j,x)\binom{j}{n-i}x^{-n+j+i}.$$

Then the Bell polynomial for the generating function of Bernoulli numbers has the form:

$$B_{n,m} = \frac{n!}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{x}{e^x-1}\right)^{m-j}\sum_{i=0}^{n}H(i,j,x)\binom{j}{n-i}x^{-n+j+i}.$$

4 Bell polynomials of inverse functions

Theorem 4.1. Let there be given a function $f(x)$ and its composita $F^{\Delta}(n,m,x)$. For the composita $Y^{\Delta}(n,m,x)$ of the inverse function $f^{-1}(x) = y(x)$, the following recurrent expressions hold true:

$$Y_1^{\Delta}(n,m,x) = \begin{cases} \frac{1}{F^{\Delta}(m,m,y(x))} & n = m, \\ -\frac{1}{F^{\Delta}(m,m,g(x))} \sum\limits_{k=m+1}^{n} Y^{\Delta}(n,k,x) F^{\Delta}(k,m,y(x)) & n > 0. \end{cases} \quad (5)$$

$$Y_2^{\Delta}(n,m,f(x)) = \begin{cases} \frac{1}{F^{\Delta}(n,n,x)} & n = m, \\ -\frac{1}{F^{\Delta}(n,n,x)} \sum\limits_{k=m}^{n-1} F^{\Delta}(n,k,x) Y^{\Delta}(k,m,f(x)) & n > 0. \end{cases} \quad (6)$$

Proof. For self-inverse functions, the condition

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

is fulfilled. Hence from theorem 3.6, we can write

$$\sum_{k=m}^{n} Y^{\Delta}(n,k,x) F^{\Delta}(k,m,y(x)) = \sum_{k=m}^{n} F^{\Delta}(n,k,x) Y^{\Delta}(k,m,f(x)) = \delta(n,m).$$

Simple transformations give us formulae (5, 6). □

Now from formula (1) we can write the Bell polynomial of the inverse function

$$B_{n,m} = \frac{n!}{m!} Y_1^{\Delta}(n,m,x) = \frac{n!}{m!} Y_2^{\Delta}(n,m,f(x)).$$

Example 4.2. Let us consider a simple example. Let there be a function $f(x) = x^2$, its composita $F^{\Delta}(n,k,x) = \binom{k}{n-k}(2x)^{2k-n}$, and inverse function $g(x) = \sqrt{x}$. Let us find an expression for the Bell polynomial of the function $\sqrt{x}$, given the composita of the function $f(x) = x^2$. In view of expression (5), we obtain

$$Z_1^{\Delta}(n,m,x) = \begin{cases} \frac{1}{(2\sqrt{x})^m}, & m = n \\ -\frac{1}{2^m \sqrt{x}^m} \sum_{k=m+1}^{n} Z_1^{\Delta}(n,k,x) \binom{m}{k-m} (2\sqrt{x})^{2m-k}, & n > m. \end{cases}$$

In view of expression (6), we derive

$$Z_2^{\Delta}(n,m,x) = \begin{cases} \frac{1}{(2x)^n}, & m = n \\ -\frac{1}{2^n x^n} \sum_{k=m}^{n-1} \binom{k}{n-k} (2x)^{2k-n} Z_2^{\Delta}(k,m,x), & n > m. \end{cases}$$

Hence the Bell polynomial for the function $\sqrt{x}$ is equal to

$$B_{n,m} = \frac{n!}{m!} Z_1^{\Delta}(n,m,x) = \frac{n!}{m!} Z_2^{\Delta}(n,m,\sqrt{x}).$$

Example 4.3. Let there be a function $f(x)=x\exp(x)$ and Lambert function $W(x)$. Let us find an expression for the n-derivative of the function $W(f(x))$. From theorem 3.4 the composita of the function $f(x)$ is equal to

$$F^{\Delta}(n,k,x)=e^{k\,x}\sum_{i=0}^{n}\frac{k^{n-i}\binom{k}{i}x^{k-i}}{(n-i)!}$$

and the Bell polynomial is equal to

$$B_{n,k}=\frac{n!}{k!}e^{k\,x}\sum_{i=0}^{n}\frac{k^{n-i}\binom{k}{i}x^{k-i}}{(n-i)!}.$$

Hence from theorem 4 and in view of the fact that these are self-inverse functions we obtain

$$W^{(n)}(f(x))=\begin{cases}\frac{1}{B_{1,1}} & n=1,\\ -\sum_{k=1}^{n-1}B_{n,k}W^{(k)}(f(x)), & n>1.\end{cases}$$

Now we can write

$$W^{(n)}=\begin{cases}\frac{1}{1+x}e^{-x} & n=1,\\ -\frac{e^{-n\,x}}{(x+1)^n}n!\sum\limits_{m=1}^{n-1}e^{m\,x}\frac{W^{(m)}}{m!}\sum\limits_{j=1}^{m}(-1)^{j-m}\binom{m}{j}\sum\limits_{i=0}^{n}\frac{j^{n-i}\binom{j}{i}x^{m-i}}{(n-i)!} & n>1\end{cases}$$

Presented below are the first terms for the derivative

$$\frac{1}{1+x}e^{-x}$$

$$\frac{-x-2}{(1+x)^3}e^{-2x}$$

$$\frac{(2\,x^2+8\,x+9)}{(1+x)^5e^{-3x}}$$

$$\frac{(-6\,x^3-36\,x^2-79\,x-64)}{(1+x)^7}e^{-4x}$$

$$\frac{24\,x^4+192\,x^3+622\,x^2+974\,x+625}{(1+x)^9}e^{-5x};$$

from whence we can obtain an expression for coefficients of the sequence A042977 [6].

5 Conclusion

For derivation of the Bell polynomial of the second kind for the generating function $Y(x,z)=y(x+z)-y(x)$, it is necessary to use the composita of the generating function that can be obtained:

1) directly from the expression $Y(x,z)$ through transformations;
2) from theorem (3.1–4.1).
Next, using formula (1), the desired polynomial is derived. The numerous examples considered in the paper convincingly prove the efficiency of the proposed methods.

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