Ramanujan Primes:
Bounds, Runs, Twins, and Gaps

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Abstract

The nth Ramanujan prime is the smallest positive integer R_n such that if $x \geq R_n$, then the interval $\left(\frac{1}{2}x, x\right]$ contains at least n primes. We sharpen Laishram's theorem that $R_n < p_{3n}$ by proving that the maximum of R_n/p_{3n} is $R_5/p_{15} = 41/47$. We give statistics on the length of the longest run of Ramanujan primes among all primes $p < 10^n$, for $n \leq 9$. We prove that if an upper twin prime is Ramanujan, then so is the lower; a table gives the number of twin primes below 10^n of three types. Finally, we relate runs of Ramanujan primes to prime gaps. Along the way we state several conjectures and open problems. The Appendix explains Noe's fast algorithm for computing $R_1, R_2, \ldots, R_n$.

1 Introduction

For $n \geq 1$, the nth *Ramanujan prime* is defined as the smallest positive integer R_n with the property that for any $x \geq R_n$, there are at least n primes p with $\frac{1}{2}x < p \leq x$. By its minimality, R_n is indeed a prime, and the interval $\left(\frac{1}{2}R_n, R_n\right]$ contains exactly n primes [10].

In 1919 Ramanujan proved a result which implies that R_n exists, and he gave the first five Ramanujan primes. (We formulate his result as a theorem and quote him.)

Theorem 1 (Ramanujan)**.** *"Let* $\pi(x)$ *denote the number of primes not exceeding* x*. Then* $\ldots \pi(x) - \pi\left(\frac{1}{2}x\right) \geq 1, 2, 3, 4, 5, \ldots,$ *if* $x \geq 2, 11, 17, 29, 41, \ldots,$ *respectively."*

Proof. This follows from properties of the Γ-function. See Ramanujan [4] for details, and Shapiro [8, Section 9.3B] for an exposition of Ramanujan's idea. □

The case $R_1 = 2$ is *Bertrand's Postulate*: for all $x \geq 2$, there exists a prime p with $\frac{1}{2}x < p \leq x$. For $n = 1, 2, 3, \ldots$, the nth Ramanujan prime [9, Sequence A104272] is

$$R_n = 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 181, 227, 229, 233, \ldots.$$

In the present paper, we report progress on three predictions [10, Conjectures 1, 2, 3] about Ramanujan primes: on bounds, runs, and twins.

In the next section, we sharpen Laishram's theorem that $R_n < p_{3n}$, where p_n denotes the nth prime. Namely, we prove the optimal bound that the maximum value of R_n/p_{3n} is $R_5/p_{15} = 41/47$. The proof uses another result of Laishram and a computation of the first 169350 Ramanujan primes by Noe's fast algorithm. Our first new conjecture follows.

In Section 3, we present statistics on the length of the longest run of Ramanujan primes among all primes $p < 10^n$, for $n \leq 9$. We pose an open problem on the unexpectedly long runs of non-Ramanujan primes, and make a new conjecture about both types of runs.

In Section 4, we prove that if the larger of two twin primes is Ramanujan, then its smaller twin is also Ramanujan, and we provide a table of data on the number of twins below 10^n, again for $n \leq 9$. We offer several new conjectures and open problems on twin primes.

In Section 5, we associate runs of odd Ramanujan primes to certain prime gaps.

The Appendix explains the algorithm for computing Ramanujan primes and includes a *Mathematica* program.

2 Bounds

Here are some estimates for the nth Ramanujan prime.

Theorem 2 (Sondow)**.** *The following inequalities hold:*

$$2n \log 2n < p_{2n} < R_n < 4n \log 4n < p_{4n} \qquad (n > 1). \tag{1}$$

Moreover, for every $\epsilon > 0$*, there exists* $N_0(\epsilon) > 0$ *such that*

$$R_n < (2 + \epsilon) n \log n \qquad (n \geq N_0(\epsilon)). \tag{2}$$

In particular, $R_n \sim p_{2n}$ *as* $n \to \infty$*.*

Proof. Inequalities of Rosser and Schoenfeld for $\pi(x)$, together with *Rosser's theorem* [6] that $p_n > n \log n$, lead to (1). The bound (2) follows from the Prime Number Theorem. For details, see Sondow [10]. ☐

A prediction [10, Conjecture 1] that (1) can be improved to $p_{2n} < R_n < p_{3n}$ has been proved by Laishram.

Theorem 3 (Laishram)**.** *For all $n \geq 1$, we have $R_n < p_{3n}$.*

Proof. Dusart's inequalities [2] for Chebychev's function

$$\theta(x) := \sum_{\text{prime } p \leq x} \log p \leq \pi(x) \log x$$

lead to an explicit value of $N_0(\epsilon)$ in (2), for each $\epsilon > 0$. For details, see Laishram [3]. ☐

Using one of those values and a fast algorithm for computing Ramanujan primes (see the Appendix), we sharpen Theorem 3 by giving an optimal upper bound on R_n/p_{3n}, namely, its maximum. (Notice that the rational numbers R_n/p_{3n} are all distinct, because the p_{3n} are distinct primes and $0 < R_n/p_{3n} < 1$. Thus the maximum occurs at only one value of n.)

Theorem 4. *The maximum value of R_n/p_{3n} is*

$$\max_{n\geq 1} \frac{R_n}{p_{3n}} = \frac{R_5}{p_{15}} = \frac{41}{47} = 0.8723\dots.$$

Proof. Since $41/47 > 0.8666\ldots = 13/15$, it suffices to show $R_n/p_{3n} < 13/15$ for $n \neq 5$.

Set $\epsilon = 3/5$ and substitute $2 + \epsilon = 13/5$ into (2). Using Rosser's theorem with $3n$ in place of n, we can write the result as

$$R_n < \frac{13}{15}\, 3n \log n < \frac{13}{15}\, p_{3n} \qquad (n \geq N_0(3/5)).$$

According to Laishram [3, Theorem 1], if $0 < \epsilon \leq 1.08$, then $N_0(\epsilon) = (2/\epsilon)^{c/\epsilon}$ in (2), where $c = c(\epsilon) = 6$ at $\epsilon = 0.6$. Hence $N_0(3/5) = (10/3)^{10} = 169350.87\ldots$, and so

$$\frac{R_n}{p_{3n}} < \frac{13}{15} \qquad (n > 169350).$$

To complete the proof, we compute the first 169350 Ramanujan primes and then check that $R_n/p_{3n} < 13/15$ when $5 \neq n \leq 169350$. ☐

Similarly, one can show that

$$\max_{n\neq 5} \frac{R_n}{p_{3n}} = \frac{R_{10}}{p_{30}} = \frac{97}{113} = 0.8584\dots,$$
$$\max_{n\neq 5 \text{ or } 10} \frac{R_n}{p_{3n}} = \frac{R_2}{p_6} = \frac{11}{13} = 0.8461\dots,$$

and so on down towards

$$\lim_{n\to\infty} \frac{R_n}{p_{3n}} = \frac{2}{3} = 0.666\dots.$$

We conclude this section with a related prediction.

Conjecture 1. *For $m = 1, 2, 3, \ldots$, let $N(m)$ be given by the following table.*

m	1	2	3	4	5	6	$7, 8, \ldots, 19$	$20, 21, \ldots$
$N(m)$	1	1245	189	189	85	85	10	2

Then we have

$$\pi(R_{mn}) \le m\pi(R_n) \qquad (n \ge N(m)).$$

Equivalently, if we define the function ρ by $\rho(n) := \pi(R_n)$, so that $R_n = p_{\rho(n)}$, then

$$\rho(mn) \le m\rho(n) \qquad (n \ge N(m)).$$

In the cases $m = 2, 3, \ldots, 20$, the statement has been verified for all n with $R_{mn} < 10^9$. The first few values of $\rho(n)$, for $n = 1, 2, 3, \ldots$, are [9, Sequence A179196]

$$\rho(n) = 1, 5, 7, 10, 13, 15, 17, 19, 20, 25, 26, 28, 31, 35, 36, 39, 41, 42, 49, 50, 51, 52, 53, \ldots.$$

Note that Theorems 2 and 3 imply $2n < \rho(n) < 3n$ for all $n > 1$, and $\rho(n) \sim 2n$ as $n \to \infty$. The latter yields $\rho(mn) \sim 2mn \sim m\rho(n)$ as $n \to \infty$, for any fixed $m \ge 1$.

3 Runs

Since $p_{2n} < R_n \sim p_{2n}$ as $n \to \infty$, the probability of a randomly chosen prime being Ramanujan is slightly less than 1/2, roughly speaking. More precisely, column 2 in Table 1 gives the probability P_n (rounded to 3 decimal places) that a prime $p < 10^n$ is a Ramanujan prime, for $n = 1, 2, \ldots, 9$.

Let us consider a coin-tossing model. Suppose that a biased coin has probability P of heads. According to Schilling [7], the expected length $\mathrm{E}L_N = \mathrm{E}L_N(P)$ of the longest run of heads in a sequence of N coin tosses is approximately equal to

$$\mathrm{E}L_N \approx \frac{\log N}{\log(1/P)} - \left(\frac{1}{2} - \frac{\log(1-P)+\gamma}{\log(1/P)}\right),$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. The variance $\mathrm{Var}L_N = \mathrm{Var}L_N(P)$ is close [7] to

$$\mathrm{Var}L_N \approx \frac{\pi^2}{6\log(1/P)^2} + \frac{1}{12}$$

"and is quite remarkable for the property that it is essentially constant with respect to" N.

For example, with a fair coin,

$$\mathrm{E}L_N \approx \frac{\log N}{\log 2} - \left(\frac{3}{2} - \frac{\gamma}{\log 2}\right) = \frac{\log N}{\log 2} - 0.667\ldots \qquad \left(P = \frac{1}{2}\right) \tag{3}$$

and

$$\mathrm{Var}L_N \approx \frac{\pi^2}{6(\log 2)^2} + \frac{1}{12} = 3.507\ldots \qquad \left(P = \frac{1}{2}\right). \tag{4}$$

Schilling points out that by (4) “the standard deviation of the longest run is approximately $(\mathrm{Var} L_N)^{1/2} \approx 1.873$, an amazingly small value. This implies that the length of the longest run is quite predictable indeed; normally it is within about two of its expectation.”

This is roughly true (replacing “two” with “seven”) of the longest run of Ramanujan primes in the sequence of prime numbers below 10^n (where $P = P_n \lessapprox 1/2$), at least for $n \leq 9$. But for non-Ramanujan primes (where $P = 1 - P_n \gtrapprox 1/2$), the actual length of the longest run is more than double the expected length, at least for $n = 6, 7, 8, 9$. (See Table 1, in which the two columns marked “Actual” are [9, Sequences A189993 and A189994].)

n	Probability P_n of a prime $p < 10^n$ being Ramanujan	Length of the longest run below 10^n of Ramanujan primes		non-Ramanujan primes	
		Expected	Actual	Expected	Actual
1	.250	1	1	2	3
2	.400	3	2	5	4
3	.429	6	5	8	7
4	.455	8	13	11	13
5	.465	11	13	14	20
6	.471	14	20	17	36
7	.476	17	21	20	47
8	.479	21	26	23	47
9	.482	24	31	26	65

Table 1: Length of the longest run of (non-)Ramanujan primes below 10^n.

Open Problem 1. *Explain the unexpectedly long runs of non-Ramanujan primes among primes $p < 10^n$, for $n \geq 6$.*

Formula (3) suggests the following predictions supported by Table 1. They strengthen an earlier prediction [10, Conjecture 2] that arbitrarily long runs of both types exist.

Conjecture 2. *We have*

$$\limsup_{N\to\infty} \frac{\text{length of the longest run of Ramanujan primes among primes} \leq p_N}{\log N/\log 2} \geq 1$$

and the same holds true if “Ramanujan” is replaced with “non-Ramanujan”.

For $n = 1, 2, \ldots$, the first run of n Ramanujan primes begins at

$$2, 67, 227, 227, 227, 2657, 2657, 2657, 2657, 2657, 2657, 2657, 2657, 562871, 793487, \ldots,$$

and the first run of n non-Ramanujan primes at

$$3, 3, 3, 73, 191, 191, 509, 2539, 2539, 5279, 9901, 9901, 9901, 11593, 11593, 55343, 55343, \ldots,$$

respectively [9, Sequences A174602 and A174641].

4 Twins

If $p_n + 2 = p_{n+1}$, then p_n and p_{n+1} are *twin primes*; the smallest are 3 and 5. If $R_n + 2 = R_{n+1}$, then R_n and R_{n+1} are *twin Ramanujan primes*; the smallest are 149 and 151.

Given primes p and $q > p$, a necessary condition for them to be twin Ramanujan primes is evidently that

$$\pi(p) - \pi\left(\frac{1}{2}p\right) + 1 = \pi(q) - \pi\left(\frac{1}{2}q\right). \tag{5}$$

To see that the condition is not sufficient, even when p and q are *consecutive primes* p_k and p_{k+1}, verify (5) for any one of the pairs

$$(p, q) = (p_k, p_{k+1}) = (\mathbf{11}, 13),\ (\mathbf{47}, 53),\ (\mathbf{67}, \mathbf{71}),\ (109, 113),\ (137, 139), \tag{6}$$

where Ramanujan primes are in **bold**.

It is less evident that (5) is a necessary condition for p and q even to be (ordinary) twin primes, but that is not hard to prove [10, Proposition 1].

Proposition 1. *If p and $q = p + 2$ are twin primes with $p > 5$, then (5) holds.*

The converse is false, even when p and q are consecutive primes both of which are Ramanujan, as the example $(p_{19}, p_{20}) = (\mathbf{67}, \mathbf{71}) = (R_8, R_9)$ shows.

As mentioned, each pair in (6) consists of consecutive primes $p < q$ satisfying (5). However, in no pair is q a Ramanujan prime but not p; in fact, such a pair cannot exist.

Proposition 2. (i). *If the larger of two twin primes is Ramanujan, then the smaller is also Ramanujan: they are twin Ramanujan primes.*
(ii). *More generally, given consecutive primes $(p, q) = (p_k, p_{k+1})$ satisfying (5), if $q = R_{n+1}$, then $p = R_n$.*

Proof. Part (i) is (vacuously) true for twin primes p and $q = p + 2$ with $p \leq 5$. For $p > 5$ it suffices, by Proposition 1, to prove part (ii).

Since $q = R_{n+1}$, we have $\pi(x) - \pi\left(\frac{1}{2}x\right) \geq n + 1$ when $x \geq q$, and (5) implies that $\pi(p) - \pi\left(\frac{1}{2}p\right) = n$. To prove that $p = R_n$, we have to show that $\pi(p-1) - \pi\left(\frac{1}{2}(p-1)\right) < n$, and that $\pi(x) - \pi(\frac{1}{2}x) \geq n$ for $x = p+1, p+2, \ldots, q-1$.

If ℓ is any prime, then $\pi(\ell - 1) + 1 = \pi(\ell)$ and $\pi\left(\frac{1}{2}(\ell - 1)\right) = \pi\left(\frac{1}{2}\ell\right)$, so that the quantity $\pi(y) - \pi\left(\frac{1}{2}y\right)$ increases by 1 from $y = \ell - 1$ to $y = \ell$. Taking $\ell = p$ or $\ell = q$, we infer that $\pi(\ell - 1) - \pi\left(\frac{1}{2}(\ell - 1)\right) = n - 1$ or n, respectively. As p and q are consecutive primes, it follows that $\pi(x) = \pi(q-1)$ and $\pi\left(\frac{1}{2}x\right) \leq \pi\left(\frac{1}{2}(q-1)\right)$, for $x = p+1, p+2, \ldots, q-1$, implying $\pi(x) - \pi(\frac{1}{2}x) \geq n$. This proves the required inequalities. □

Part (i) was conjectured by Noe [9, Sequence A173081].

Corollary 1. *If we denote*

$$\begin{aligned}\pi_{2,1}(x) &:= \#\{\textit{pairs of twin primes} \leq x : \textit{one or both are Ramanujan}\},\\ \pi_{2,2}(x) &:= \#\{\textit{pairs of twin primes} \leq x : \textit{both are Ramanujan}\},\end{aligned}$$

then for all $x \geq 0$ *we have the equalities*

$$\pi_{2,1}(x) = \#\{\textit{pairs of twin primes} \leq x \ : \textit{the smaller is Ramanujan}\},$$
$$\pi_{2,2}(x) = \#\{\textit{pairs of twin primes} \leq x \ : \textit{the larger is Ramanujan}\}.$$

Proof. By Proposition 2 part (i), given twin primes p and $p+2$, if $p+2 = R_{n+1}$, then $p = R_n$. The corollary follows. □

Table 2 gives some figures (see [9, Sequences A007508, A173081, A181678]) on

$$\pi_2(x) := \#\{\text{pairs of twin primes} \leq x\},$$

$\pi_{2,1}(x), \pi_{2,2}(x)$, and their ratios. Proposition 1 and Corollary 1 will help to explain why many values of the ratios are greater than might be expected a priori.

	$\pi_* = \pi_*(10^n)$					
n	π_2	$\pi_{2,1}$	$\pi_{2,2}$	$\pi_{2,1}/\pi_2$	$\pi_{2,2}/\pi_2$	$\pi_{2,2}/\pi_{2,1}$
1	2	0	0	0	0	–
2	8	6	0	.750	0	0
3	35	28	10	.800	.286	.357
4	205	167	73	.815	.356	.437
5	1224	694	508	.788	.415	.527
6	8169	6305	3468	.772	.425	.550
7	58980	45082	25629	.764	.434	.568
8	440312	335919	194614	.763	.442	.579
9	3424506	2605867	1537504	.761	.449	.590

Table 2: Counting three types of pairs of twin primes below 10^n.

The probability that two randomly chosen primes p and q are both Ramanujan is slightly less than $1/2 \times 1/2 = 1/4$, roughly speaking. The probability increases if p and q are twin primes, because then Proposition 1 guarantees that the necessary condition (5) holds.

For that reason, and based on the first 1000 Ramanujan primes, it was predicted [10, Conjecture 3] that more than 1/4 of the twin primes up to x are twin Ramanujan primes, if $x \geq 571$. This is borne out for $x = 10^n$, with $3 \leq n \leq 9$, by Table 2. It shows that the prediction can be improved to $\pi_{2,2}(x)/\pi_2(x) > 2/5$, for $x \geq 10^5$.

Corollary 1 implies that whether a twin prime pair is counted in $\pi_{2,1}(x)$ or $\pi_{2,2}(x)$ depends on only one of the two primes being Ramanujan. This suggests that the ratios $\pi_{2,1}(x)/\pi_2(x)$ and $\pi_{2,2}(x)/\pi_2(x)$ should approach 1/2 as x tends to infinity.

We conclude this section with these and other conjectures based on our results and on Table 2, as well as with two more open problems.

Conjecture 3. *For all* $x \geq 10^5$, *we have*

$$\frac{\pi_{2,1}(x)}{\pi_2(x)} < \frac{4}{5}, \qquad \frac{\pi_{2,2}(x)}{\pi_2(x)} > \frac{2}{5}, \qquad \frac{\pi_{2,2}(x)}{\pi_{2,1}(x)} > \frac{2/5}{4/5} = \frac{1}{2}.$$

Conjecture 4. *If* $\pi_2(x) \to \infty$ *as* $x \to \infty$*, then* $\pi_{2,1}(x) \sim \pi_{2,2}(x) \sim \frac{1}{2}\pi_2(x)$*.*

Recall Brun's famous theorem [1] that the series of reciprocals of the twin primes converges or is finite (unlike the series of reciprocals of all the primes, which Euler showed diverges). Its sum [9, Sequence A065421] is *Brun's constant* B_2,

$$B_2 := \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots \overset{?}{=} 1.9021605\ldots.$$

Here $\overset{?}{=}$ means that the value of B_2 is conditional "on heuristic considerations about the distribution of twin primes" (Ribenboim [5, p. 201]).

Open Problem 2. *Compute the analogous constant* $B_{2,1}$ *for twin primes at least one of which is Ramanujan,*

$$B_{2,1} := \left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\left(\frac{1}{29}+\frac{1}{31}\right)+\left(\frac{1}{41}+\frac{1}{43}\right)+\cdots.$$

The numbers $11, 17, 29, 41, \ldots$ [9, Sequence A178128] are the lesser of twin primes if at least one is Ramanujan. By Corollary 1, that is the same as the lesser of twin primes if it is Ramanujan.

Open Problem 3. *Compute the analogous constant* $B_{2,2}$ *for twin Ramanujan primes,*

$$B_{2,2} := \left(\frac{1}{149}+\frac{1}{151}\right)+\left(\frac{1}{179}+\frac{1}{181}\right)+\left(\frac{1}{227}+\frac{1}{229}\right)+\left(\frac{1}{239}+\frac{1}{241}\right)+\cdots.$$

The numbers $149, 179, 227, 239, \ldots$ [9, Sequence A178127] are the lesser of twin Ramanujan primes.

5 Prime gaps

Let us say that there is a *prime gap from* a *to* $b \geq a$ if none of the numbers $a, a+1, a+2, \ldots, b$ is prime. Given a run of r odd Ramanujan primes starting at p, we can associate to it a prime gap of length at least r starting at $\frac{1}{2}(p+1)$.

Proposition 3. (i). *If* $p = R_n$ *is odd, then the integer* $\frac{1}{2}(p+1)$ *is not prime.*
(ii). *More generally, given a run of* $r \geq 1$ *odd Ramanujan primes from* $p = R_n = p_k$ *to* $q = R_{n+r-1} = p_{k+r-1}$*, there is a prime gap from* $\frac{1}{2}(p+1)$ *to* $\frac{1}{2}(q+1)$*.*
(iii). *Parts* (i) *and* (ii) *are sharp in the sense that, for* certain *runs of Ramanujan primes* p *to* q*, both* $\frac{1}{2}(p+1)-1$ *and* $\frac{1}{2}(q+1)+1$ *are prime numbers.*
(iv). *But in the case* $r = 2$*, if* p *and* q *are* twin *Ramanujan primes, then the prime gap from* $\frac{1}{2}(p+1)$ *to* $\frac{1}{2}(q+1)$ *always lies in a longer prime gap of length* 5 *or more.*

Proof. (i). Since $p = R_n$ is odd, $\pi(p) = \pi(p+1)$, and the quantity $\pi(x) - \pi\big(\frac{1}{2}x\big)$ does not decrease from $x = p$ to $x = p+1$. Hence $\pi\big(\frac{1}{2}p\big) \geq \pi\big(\frac{1}{2}(p+1)\big)$, and so $\frac{1}{2}(p+1)$ is not prime.
(ii). By (i), the case $r = 1$ holds. Taking $r = 2$, let $p = R_n = p_k$ and $q = R_{n+1} = p_{k+1}$ be odd. By (i), neither $\frac{1}{2}(p+1)$ nor $\frac{1}{2}(q+1)$ is prime. If an integer i lies strictly between them, then the oddness of p and q implies $p + 1 < j := 2i - 1 < q - 1$. Since $p = p_k$ and $q = p_{k+1}$, we have $\pi(p) = k = \pi(j+1)$. As $p = R_n$ and $q = R_{n+1}$, it follows that $\pi(x) - \pi\big(\frac{1}{2}x\big)$ does not decrease from $x = p$ to $x = j+1$. Hence $\pi\big(\frac{1}{2}p\big) \geq \pi\left(\frac{1}{2}(j+1)\right) = \pi(i)$, and so i is also not prime. This proves (ii) for runs of length 2.

The general case follows easily by induction on r. Namely, given a run of length $r > 2$ from $R_n = p_k$ to $R_{n+r-1} = p_{k+r-1}$, break it into a run of length 2 from $R_n = p_k$ to $R_{n+1} = p_{k+1}$, concatenated with a run of length $r - 1$ from $R_{n+1} = p_{k+1}$ to $R_{n+r-1} = p_{k+r-1}$.
(iii). For $r = 1$, the composite number $\frac{1}{2}(R_2 + 1) = \frac{1}{2}(11 + 1) = 6$ lies between the primes 5 and 7. For an example with $r > 1$, take the run $(R_{293}, R_{294}) = (4919, 4931) = (p_{657}, p_{658})$ of length $r = 2$. It is associated to the prime gap from $\frac{1}{2}(R_{293}+1) = 2460$ to $\frac{1}{2}(R_{294}+1) = 2466$, which is bounded by the primes 2459 and 2467.
(iv). Since $p > 3$ and q are twin primes, $(p, q) = (6k-1, 6k+1)$ for some k. If $k = 2i$ is even, then $\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right) = (6i, 6i+1)$ lies in the prime gap from $6i$ to $6i+4$.

Now assume that $k = 2i+1$ is odd. Then $\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right) = (6i+3, 6i+4)$ will lie in a prime gap from $6i+2$ to $6i+6$, unless $6i+5$ is prime. But if $6i + 5 = \frac{1}{2}(q+3)$ were prime, then, since $q + 2 = 6k + 3$ is not prime, $\pi(x) - \pi\big(\frac{1}{2}x\big)$ would decrease from $x = q$ to $x = q+3$, contradicting the fact that q is a Ramanujan prime. This completes the proof. □

For part (iii), the first "sharp" example of a run of length $r = 1, 2, \ldots, 11$ begins at the Ramanujan prime

$$11, 4919, 1439, 7187, 37547, 210143, 3376943, 663563, 4429739, 17939627, 12034427,$$

respectively [9, Sequence A177804]. An example of part (iv) is the prime gap associated to the twin Ramanujan primes $R_{14} = 149$ and $R_{15} = 151$, which lies in the larger prime gap

$$74, \frac{1}{2}(R_{14} + 1) = 75, \frac{1}{2}(R_{15} + 1) = 76, 77, 78.$$

6 Appendix on the algorithm

To compute a range of Ramanujan primes R_i for $1 \leq i \leq n$, we perform simple calculations in each interval $(k/2, k]$ for $k = 1, 2, \ldots, p_{3n} - 1$. To facilitate the calculation, we use a counter s and a list L with n elements L_i. Initially, s and all L_i are set to zero. They are updated as each interval is processed.

After processing an interval, s will be equal to the number of primes in that interval, and each L_i will be equal either to the greatest index of the intervals so far processed that contain exactly i primes, or to zero if no interval having exactly i primes has yet been processed.

Having processed interval $k-1$, to find the number of primes in interval k we perform two operations: add 1 to s if k is prime, and subtract 1 from s if $k/2$ is prime. We then update the sth element of the list to $L_s = k$, because now k is the largest index of all intervals processed that contain exactly s primes.

After all intervals have been processed, the list R of Ramanujan primes is obtained by adding 1 to each element of the list L.

These ideas are captured in the following *Mathematica* program for finding the first 169350 Ramanujan primes.

```
nn = 169350;
L = Table[0, {nn}];
s = 0;
Do[
   If[PrimeQ[k], s++];
   If[PrimeQ[k/2], s--];
   If[s < nn, L[[s+1]] = k],
   {k, Prime[3*nn]-1}];
R = L + 1
```

Although it is adequate for computing a modest number of them, to compute many more requires a speedup of several orders of magnitude. That can be achieved by using a lower-level programming language and generating prime numbers via a sieve. With this speedup we computed all Ramanujan primes below 10^9 in less than three minutes on a 2.8 GHz Pentium 4 computer.

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