Eulerian polynomials as moments, via exponential Riordan arrays

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Abstract

Using the theory of exponential Riordan arrays and orthogonal polynomials, we demonstrate that the "descending power" Eulerian polynomials, and their once shifted sequence, are moment sequences for simple families of orthogonal polynomials, which we characterize in terms of their three-term recurrence. We obtain the generating functions of the polynomial sequences in terms of continued fractions, and we also calculate their Hankel transforms.

1 Introduction

The Eulerian polynomials [9, 14, 17, 21]

$$P_n(x) = \sum_{k=0}^{n} W_{n,k} x^k$$

form the sequence $P_n(x)$ which begins

$$P_0(x) = 1, P_1(x) = 1, P_2(x) = 1 + x, P_3(x) = 1 + 4x + x^2, \ldots,$$

with the well-known triangle of Eulerian numbers [16]

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 11 & 11 & 1 & 0 & 0 & \ldots \\ 1 & 26 & 66 & 26 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

as coefficient array. These coefficients $W_{n,k}$ obey the recurrence [17]

$$W_{n,k} = (k+1)W_{n-1,k} + (n-k)W_{n-1,k-1}$$

with appropriate boundary conditions. The closed form expression

$$W_{n,k} = \sum_{i=0}^{n-k} (-1)^i \binom{n+1}{i} (n-k-i)^n$$

holds. The polynomials $P_n(x)$ were introduced by Euler [13] in the form

$$\sum_{k=0}^{\infty} (k+1)^n t^k = \frac{P_n(t)}{(1-t)^{n+1}}.$$

They have exponential generating function

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = \frac{(1-x)e^{(1-x)t}}{1-xe^{(1-x)t}}.$$

We have

$$P_n(x) = \sum_{k=0}^{n} A_{n,k} x^{n-k},$$

and hence we can regard them as "descending power" Eulerian polynomials.

In this note we show that the sequence of Eulerian polynomials $P_n(x)$ is the moment sequence of a family of orthogonal polynomials. In addition, we show that the sequence of shifted Eulerian polynomials $P_{n+1}(x)$ is similarly the moment sequence of a family of orthogonal polynomials. For this, we will require three results from the theory of exponential Riordan arrays (see Appendix for an introduction to exponential Riordan arrays). These are [5, 6]

1. The inverse of an exponential Riordan array $[g, f]$ is the coefficient array of a family of orthogonal polynomials if and only if the production matrix of $[g, f]$ is tri-diagonal;

2. If the production matrix of $[g, f]$ is tri-diagonal, then the elements of the first column of $[g, f]$ are the moments of the corresponding family of orthogonal polynomials;

3. The bivariate generating function of the production matrix of $[g, f]$ is given by

   $$e^{xy}(Z(x) + A(x)y)$$

   where

   $$A(x) = f'(\bar{f}(x)),$$

   and

   $$Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))},$$

   where $\bar{f}(x)$ is the compositional inverse (series reversion) of $f(x)$.

A quick introduction to exponential Riordan arrays can be found in the Appendix to this note. For general information on orthogonal polynomials and moments, see [8, 15, 28]. Continued fractions will be referred to in the sequel; [29] is a general reference, while [18, 19] discuss the connection of continued fractions to orthogonal polynomials, moments and Hankel transforms [20, 25]. We recall that for a given sequence a_n its Hankel transform is the sequence of determinants $h_n = |a_{i+j}|_{0\le i,j\le n}$. Many interesting examples of number triangles, including exponential Riordan arrays, can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences [26, 27]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix (Pascal's triangle) $\mathbf{B}$ with (n,k)-th element $\binom{n}{k}$ is A007318.

2 The Eulerian polynomials $P_n(x)$

We consider the sequence with e.g.f.

$$\frac{(\alpha-\beta)e^{(\alpha-\beta)t}}{\alpha-\beta e^{(\alpha-\beta)t}}.$$

This is the sequence that begins

$$1, \alpha, \alpha(\alpha+\beta), \alpha(\alpha^2+4\alpha\beta+\beta^2), \alpha(\alpha^3+11\alpha^2\beta+11\alpha\beta^2+\beta^3), \ldots.$$

Setting $\alpha=1$ and $\beta=x$ gives us the Eulerian polynomials $P_n(x)$. We have the

Proposition 1. *The production matrix of the exponential Riordan array*

$$\left[\frac{(\alpha-\beta)e^{(\alpha-\beta)t}}{\alpha-\beta e^{(\alpha-\beta)t}}, \frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta e^{(\alpha-\beta)t}}\right]$$

is tri-diagonal.

Proof. Writing the above exponential Riordan array as $[g,f]$, we have

$$f(t)=\frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta e^{(\alpha-\beta)t}}$$

and hence

$$f'(t)=\frac{e^{(\alpha+\beta)t}(\alpha-\beta)^2}{\beta e^{\alpha t}-\alpha e^{\beta t}},$$

and

$$\bar{f}(t)=\frac{1}{\alpha-\beta}\ln\left(\frac{\alpha t+1}{\beta t+1}\right).$$

Then

$$A(t)=f'(\bar{f}(t))=(\alpha t+1)(\beta t+1)=1+(\alpha+\beta)t+\alpha\beta t^2.$$

We have

$$g(t)=\frac{(\alpha-\beta)e^{(\alpha-\beta)t}}{\alpha-\beta e^{(\alpha-\beta)t}}$$

and hence

$$g'(t) = \frac{\alpha e^{(\alpha+\beta)t}(\alpha-\beta)^2}{(\beta e^{\alpha t} - \alpha e^{\beta t})^2},$$

and so

$$Z(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))} = \alpha(\beta t + 1) = \alpha + \alpha\beta t.$$

Thus the production matrix, which has bivariate g.f. given by

$$e^{ty}(\alpha + \alpha\beta t + (1 + (\alpha+\beta)t + \alpha\beta t^2)y),$$

is tri-diagonal. ☐

We note that the production matrix takes the form

$$\left(\begin{array}{ccccccc} \alpha & 1 & 0 & 0 & 0 & 0 & \ldots \\ \alpha\beta & 2\alpha+\beta & 1 & 0 & 0 & 0 & \ldots \\ 0 & 4\alpha\beta & 3\alpha+2\beta & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9\alpha\beta & 4\alpha+3\beta & 1 & 0 & \ldots \\ 0 & 0 & 0 & 16\alpha\beta & 5\alpha+4\beta & 1 & \ldots \\ 0 & 0 & 0 & 0 & 25\alpha\beta & 6\alpha+5\beta & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

For completeness, we note that while in the special case $\alpha = \beta$ the Riordan array is not obviously well-defined, the production matrix is, and it leads in this special case to the exponential Riordan array

$$\left[\frac{1}{1-\alpha t}, \frac{t}{1-\alpha t}\right]$$

which has general element $\binom{n}{k}\frac{n!}{k!}\alpha^{n-k}$. In the case $\alpha = \beta = 1$, we get the exponential Riordan array

$$\left[\frac{1}{1-t}, \frac{t}{1-t}\right]$$

whose inverse is the coefficient array of the Laguerre polynomials [3].

Returning now to the Eulerian polynomials, we set $\alpha = 1$ and $\beta = x$, to get

Theorem 2. *The Eulerian polynomials $P_n(x)$ are the moments of the family of orthogonal polynomials $Q_n(t)$ defined by $Q_0(t) = 1$, $Q_1(t) = t - 1$, and*

$$Q_n(t) = (t - ((n-1)x + n))Q_{n-1}(t) - (n-1)^2 x Q_{n-2}(t).$$

Proof. The initial polynomial terms of the sequence $Q_n(t)$ can be read from the elements of

$$\left[\frac{(1-x)e^{(1-x)t}}{1-xe^{(1-x)t}}, \frac{e^{(1-x)t}-1}{1-xe^{(1-x)t}}\right]^{-1} = \left[\frac{1}{1+t}, \frac{1}{1-x}\ln\left(\frac{1+t}{1+xt}\right)\right],$$

which begins

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 2 & -x-3 & 1 & 0 & \dots \\ -6 & 2x^2+5x+11 & -3(x+2) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Hence in particular $Q_0(t) = 1$ and $Q_1(t) = t - 1$. The three-term recurrence is derived from the production matrix, which in this case is

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ x & 2+x & 1 & 0 & 0 & 0 & \dots \\ 0 & 4x & 3+2x & 1 & 0 & 0 & \dots \\ 0 & 0 & 9x & 4+3x & 1 & 0 & \dots \\ 0 & 0 & 0 & 16x & 5+4x & 1 & \dots \\ 0 & 0 & 0 & 0 & 25x & 6+5x & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

□

Corollary 3. *The sequence of Eulerian polynomials $P_n(x)$ has ordinary generating function given by the continued fraction*

$$\cfrac{1}{1-t-\cfrac{xt^2}{1-(2+x)t-\cfrac{4xt^2}{1-(3+2x)t-\cfrac{9xt^2}{1-\cdots}}}}.$$

Corollary 4. *The Hankel transform of the sequence of Eulerian polynomials $P_n(x)$ is given by*

$$h_n = x^{\binom{n+1}{2}} \prod_{k=1}^{n} k!^2.$$

3 The shifted Eulerian polynomials $P_{n+1}(x)$

For the shifted Eulerian polynomials $P_{n+1}(x)$, we consider the exponential Riordan array

$$[g'(t), f(t)],$$

where

$$g'(t) = \frac{(\alpha-\beta)^2 e^{(\alpha+\beta)t}}{\beta e^{\alpha t} - \alpha e^{\beta t}},$$

where we retain the use of $g(t) = \frac{(\alpha-\beta)e^{(\alpha-\beta)t}}{\alpha-\beta e^{(\alpha-\beta)t}}$ from the previous section.

When $\alpha = 1$ and $\beta = x$, $g'(t)$ generates the shifted sequence $P_{n+1}(x)$. We then have

Proposition 5. *The production matrix of the exponential Riordan array*

$$\left[\frac{(\alpha-\beta)^2 e^{(\alpha+\beta)t}}{\beta e^{\alpha t}-\alpha e^{\beta t}}, \frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta e^{(\alpha-\beta)t}}\right]$$

is tri-diagonal.

Proof. As in the previous proposition, we obtain

$$A(t)=f'(\bar{f}(t))=(\alpha t+1)(\beta t+1)=1+(\alpha+\beta)t+\alpha\beta t^2,$$

where

$$\bar{f}(t)=\frac{1}{\alpha-\beta}\ln\left(\frac{\alpha t+1}{\beta t+1}\right).$$

Then

$$Z(t)=\frac{g''(\bar{f}(t))}{g'(\bar{f}(t))}=(\alpha+\beta)+2\alpha\beta t.$$

The bivariate generating function of the production matrix is then

$$e^{ty}((\alpha+\beta)+2\alpha\beta t+(1+(\alpha+\beta)t+\alpha\beta t^2)y),$$

and hence the production matrix is tri-diagonal. □

The production matrix in this case begins

$$\begin{pmatrix} \alpha+\beta & 1 & 0 & 0 & \dots \\ 2\alpha\beta & 2(\alpha+\beta) & 1 & 0 & \dots \\ 0 & 6\alpha\beta & 3(\alpha+\beta) & 1 & \dots \\ 0 & 0 & 12\alpha\beta & 4(\alpha+\beta) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the case $\alpha=\beta$, we obtain the exponential Riordan array

$$\left[\frac{1}{(1-\alpha t)^2}, \frac{t}{1-\alpha t}\right],$$

with (n,k)-th element $\binom{n+1}{k+1}\frac{n!}{k!}\alpha^{n-k}$. For $\alpha=\beta=1$ this gives us

$$\left[\frac{1}{(1-t)^2}, \frac{t}{1-t}\right],$$

which is A105278.

Specializing to the values $\alpha=1$, $\beta=x$, we get the

Theorem 6. *The shifted Eulerian polynomials $P_{n+1}(x)$ are the moments of the family of orthogonal polynomials $R_n(t)$ given by $R_0(t)=1$, $R_1(t)=t-x-1$, and for $n>1$,*

$$R_n(t)=(t-n(1+x))R_{n-1}(t)-n(n-1)xR_{n-2}(t).$$

Proof. The initial terms of the polynomial sequence $R_n(t)$ can be read from the elements of the inverse matrix

$$\left[\frac{(\alpha-\beta)^2 e^{(\alpha+\beta)t}}{\beta e^{\alpha t}-\alpha e^{\beta t}}, \frac{e^{(1-x)t}-1}{1-xe^{(1-x)t}}\right]^{-1} = \left[\frac{1}{(1+t)(1+tx)}, \frac{1}{1-x}\ln\left(\frac{1+t}{1+xt}\right)\right],$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -x-1 & 1 & 0 & 0 & \dots \\ 2x^2+2x+2 & -3(x+1) & 1 & 0 & \dots \\ -6(x^3+x^2+x+1) & 11x^2+14x+11 & -6(x+1) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The three-term recurrence is derived from the production matrix, which in this case is

$$\begin{pmatrix} 1+x & 1 & 0 & 0 & 0 & 0 & \dots \\ 2x & 2(1+x) & 1 & 0 & 0 & 0 & \dots \\ 0 & 6x & 3(1+x) & 1 & 0 & 0 & \dots \\ 0 & 0 & 12x & 4(1+x) & 1 & 0 & \dots \\ 0 & 0 & 0 & 20x & 5(1+x) & 1 & \dots \\ 0 & 0 & 0 & 0 & 30x & 6(1+x) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

Corollary 7. *The sequence of shifted Eulerian polynomials $P_{n+1}(x)$ has ordinary generating function given by the continued fraction*

$$\cfrac{1}{1-(1+x)t-\cfrac{2xt^2}{1-2(1+x)t-\cfrac{6xt^2}{1-3(1+x)t-\cfrac{12xt^2}{1-\cdots}}}}.$$

Corollary 8. *The Hankel transform of the shifted Eulerian polynomials $P_{n+1}(x)$ is given by*

$$h_n = (2x)^{\binom{n+1}{2}} \prod_{k=1}^{n} \binom{k+2}{2}^{n-k}.$$

4 The Eulerian number triangles

As with the Narayana numbers and their associated number triangles [7], we can distinguish between three distinct but related triangles of Eulerian numbers. Thus we have the triangle

A173018 [16, 17]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of Eulerian numbers $W_{n,k}$ that obey the recurrence

$$W_{n,k} = (k+1)W_{n-1,k} + (n-k)W_{n-1,k-1}$$

with appropriate boundary conditions, for which the closed form expression

$$W_{n,k} = \sum_{i=0}^{n-k} (-1)^i \binom{n+1}{i} (n-k-i)^n$$

holds. We have the reversal of this triangle, which is the triangle A123125 of the coefficients $A_{n,k}$ [1] where

$$A_{n,k} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n,$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 11 & 11 & 1 & 0 & \cdots \\ 0 & 1 & 26 & 66 & 26 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and finally we have the Pascal-like triangle of coefficients

$$\tilde{A}_{n,k} = A_{n+1,k+1} = \sum_{i=0}^{k+1} (-1)^i \binom{n+2}{i} (k-i)^{n+1},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdots \\ 1 & 57 & 302 & 302 & 57 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A008292.
We have

$$\tilde{A}_{n,k} = (n-k+1)\tilde{A}_{n-1}k - 1 + (k+1)\tilde{A}_{n-1}k,$$

with appropriate boundary conditions. As with the Narayana numbers, each of these triangles has signficant combinatorial applications and it is often important to distinguish one from the other.

Example 9. The sequence $a_n = \sum_{k=0}^{n} W_{n,k}2^k$ is the sequence A000670 of preferential arrangements, or rankings of competitors in a race, with ties [23]. The sequence

$$b_n = \sum_{k=0}^{n} A_{n,k}2^k = \sum_{k=0}^{n} W_{n,n-k}2^k$$

or A000629 is the sequence of rankings of competitors in a race, with ties and dropouts [22]. Note that from our results above, the sequence a_n has generating function given by

$$\cfrac{1}{1-x-\cfrac{2x^2}{1-4x-\cfrac{8x^2}{1-7x-\cfrac{18x^2}{1-\cdots}}}}.$$

The g.f. of the sequence a_{n+1} is given by

$$\cfrac{1}{1-3x-\cfrac{4x^2}{1-6x-\cfrac{12x^2}{1-9x-\cfrac{24x^2}{1-\cdots}}}}.$$

In this case it happens that b_n is the binomial transform of a_n, and hence [4] its g.f. has continued fraction expression

$$\cfrac{1}{1-2x-\cfrac{2x^2}{1-5x-\cfrac{8x^2}{1-8x-\cfrac{18x^2}{1-\cdots}}}}.$$

5 A related ODE

The form of $f(t)$ above is related to a simple ODE. This arises as follows. In order to have a tri-diagonal production matrix, we need to have an expression of the form

$$A(z) = f'(\bar{f}(z)) = 1 + \mu z + \nu z^2.$$

Now substituting $z = f(t)$ we obtain

$$f'(\bar{f}(f(t))) = 1 + \mu f(t) + \nu f(t)^2$$

or

$$f'(t) = 1 + \mu f(t) + \nu f(t)^2$$

or

$$\frac{dy}{dt} = 1 + \mu y + \nu y^2,$$

where $y = f(t)$. In the Eulerian case above, we have

$$\frac{dy}{dt} = (1 + \alpha y)(1 + \beta y),$$

with initial condition $y(0) = 0$. The form of $y = f(t)$ follows from this variant of the logistic equation.

6 Appendix: exponential Riordan array

The *exponential Riordan group* [2, 10, 12], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. We usually assume that

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(x) f(x)^i / i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, the binomial matrix $\mathbf{B}$ with (n, k)-th element $\binom{n}{k}$ is given by $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^x e^x = e^{2x}$, as expected (e^{2x} is the e.g.f. of 2^n).

To each exponential Riordan array $L = [g, f]$ is associated [11, 12] a matrix P called its *production* matrix, which has bivariate g.f. given by

$$e^{xy}(Z(x) + A(x)y)$$

where

$$A(x) = f'(\bar{f}(x)), \quad Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

We have

$$P = L^{-1}\bar{L}$$

where $\bar{L}$ [24, 29] is the matrix L with its top row removed.

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