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COMBINATORIAL MINORS OF MATRIX FUNCTIONS AND THEIR APPLICATIONS

VLADIMIR SHEVELEV

Abstract. For every matrix function we introduce its "combinatorial minors". As important examples of their applications, we give a recursive algorithms for solution of several general enumerative problems of permutations with restricted positions.

1. Introduction

Let $\mathcal{S}^{(n)}$ be symmetric group of permutations of numbers $\{1, ..., n\}$. Let $A = \{a_{ij}\}$ be square matrix of order n. Recall that permanent of A is defined by formula ([4])

$$per A = \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \tag{1.1}$$

If A is a $(0,1)$ matrix, then it defines a class $\mathcal{B} = \mathcal{B}(A)$ of permutations with restricted positions, such that the positions of its zeros are prohibited. Such class could be equivalently defined by a simple inequality: a permutation $\pi \in \mathcal{B}$ if and only if for its incidence matrix P we have $P \le A$. One of the most important application of $per A$ consists of the equality $|\mathcal{B}| = per A$. Thus $per A$ enumerates permutations with restricted positions of the class $\mathcal{B}(A)$.

Let $\gamma(\pi)$ be number of independent cycles of π, including cycles of length 1. Then the difference $d(\pi) = n - \gamma(\pi)$ is called *decrement* of π ([3]). Permutation π is called even (odd) if $d(\pi)$ is even (odd). Note that determinant of matrix A one can define by the formula

$$det A = \sum_{even} \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} - \sum_{odd} \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \tag{1.2}$$

Since, evidently, we also have

$$per A = \sum_{even} \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + \sum_{odd} \sum_{s \in \mathcal{S}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}, \tag{1.3}$$

then the numbers of even and odd permutations of class $\mathcal{B}(A)$ are given by vector

$$\left(\frac{1}{2}(per A + det A),\ \ \frac{1}{2}(per A - det A)\right). \tag{1.4}$$

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Note that, in the contrast to permanent, there exist methods of very fast calculation of $detA$. Therefore, the enumerative information given by (1.4) one can obtain approximately for the same time as the number $|\mathcal{B}|$ given by (1.1).

Let $m \geq 3$ and $0 \leq k < m$ be given integers. We say that a permutation π belongs to *class k modulo m* ($\pi \in \mathcal{S}_{k,m}^{(n)}$), if $d(\pi) \equiv k \pmod{m}$. Let now A be $(0,1)$ square matrix of order n. The first problem under our consideration is the enumeration of permutations $\pi \in \mathcal{B}$ of class k modulo m. It is clear that this problem is a natural generalization of problem of enumeration of even and odd permutation with restricted positions which is solved by (1.4). In order to solve this more general problem, put $\omega = e^{\frac{2\pi i}{m}}$ and introduce a new matrix function which we call *ω-permanent.*

Definition 1. *Let A be a square matrix of order n. We call ω-permanent of matrix A the following matrix function*

$$per_\omega A = \sum_{s \in \mathcal{S}_{0,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + \omega \sum_{s \in \mathcal{S}_{1,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} +$$

$$\omega^2 \sum_{s \in \mathcal{S}_{2,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)} + ... + \omega^{m-1} \sum_{s \in \mathcal{S}_{m-1,m}^{(n)}} \prod_{i=1}^{n} a_{i,s(i)}. \tag{1.5}$$

Note that, if $m = 1$, then $per_\omega A = perA$, and if $m = 2$, then $per_\omega A = detA$. In case $m \geq 3$, every sum in (1.5) essentially differs from permanent. Therefore, the known methods of evaluation of permanent ([1], ch. 7) are not applicable. However, using so-called "combinatorial miners", below we find an expansion $per_\omega A$ over the first row of matrix A. This allows to reduce a problem of order n to a few problems of order $n-1$.

The second problem under our consideration is another important problem of enumeration of full cycles with restricted positions. In connection of this problem, we introduce another new matrix function which we call *cyclic permanent.*

Definition 2. *Let A be square matrix of order n. The number*

$$Cycl(A) = \sum_{s} \prod_{i=1}^{n} a_{i,s(i)}, \tag{1.6}$$

where the summing is over all full cycles from $\mathcal{S}^{(n)}$, we call a cyclic permanent of A.

The third problem is a problems of enumeration of permutations with restricted positions with a restrictions on their cycle structure.

Denote $\gamma(s)$ the number of all cycles of permutation s including cycles of length 1. Recall ([1]) that the absolute value of Stirling number $S(n,k)$ of the first kind equals to number of permutations $s \in \mathcal{S}^{(n)}$ with $\gamma(s) = k$ ([1],[5]). A natural generalization of Stirling numbers of the first kind is the following matrix function.

Definition 3. *The matrix function*

$$S(A;n,k) = \sum_{s \in \mathcal{S}^{(n)}, \gamma(s)=k} \prod_{i=1}^{n} a_{i,s(i)}, \tag{1.7}$$

where n is order of square matrix A, we call Stirling function of index k.

Finally, recall ([5]) that a permutation $s \in \mathcal{S}^{(n)}$ with k_1 cycles of length 1, k_2 cycles of length 2, and so on, is said to be of cycle structure $\overline{k} = (k_1, k_2, ..., k_n)$. Denote $\nu(k_1, k_2, ..., k_n)$ the number of permutations of class $\overline{k} = (k_1, k_2, ..., k_n)$. Then polynomial

$$C(t_1, t_2, ..., t_n) = \sum \nu(k_1, k_2, ..., k_n) t_1^{k_1} t_2^{k_2} ... t_n^{k_n} \tag{1.8}$$

is called the cycle indicator of permutations of $\mathcal{S}^{(n)}$. A natural generalization of the cycle indicator of permutations with restricted positions is

$$C(A; t_1, t_2, ..., t_n) = \sum \nu(A; k_1, k_2, ..., k_n) t_1^{k_1} t_2^{k_2} ... t_n^{k_n} \tag{1.9}$$

where $\nu(A; k_1, k_2, ..., k_n)$ is the number of permutations of class $\mathcal{B}(A)$ with the cycle structure $\overline{k}$.

2. Observations in case $m = 2$ of ω-determinant

The case of determinant ($m = 2$) is a unique case when it is easy to obtain a required enumerative information formally given by formulas (1.4). For the passage to a general case it is important for us to understand how one can obtain such information from the definition (1.2) of determinant only. Essentially, the required information is contained in vector

$$\overline{det} A = (\sum_{even \ s \in \mathcal{S}} \prod_{i=1}^{n} a_{i,s(i)} \ , \ - \sum_{odd \ s \in \mathcal{S}} \prod_{i=1}^{n} a_{i,s(i)}), \tag{2.1}$$

and it immediately disappears if to use an identity of the form $1 - 1 = 0$. None of algorithms of fast calculation of determinant exists without such identity. On the other hand, in the Laplace algorithm of expansion of determinant over (the first) row such identity one can use only in the last step. Therefore, if *not to do* the last step, we can obtain the required enumerative information.

Example 1. *By the Laplace expansion, we have*

$$det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -(1-1) + (1-1) = -1 + 1 + 1 - 1 = 2 - 2$$

and, if not to do the useless (with the enumerative point of view) last step, then we have

$$\overline{det}\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} = (2, -2).$$

This means that there are two even and two odd permutations with the prohibited position $(1, 1)$.
Unfortunately, most likely, it is impossible to generalize for ω-permanent the Laplace expansion of determinant over the first row in its classic form

$$detA = \sum_{j=1}^{n}(-1)^{j-1}a_{1,j}M_{1j}, \tag{2.2}$$

where M_{1j} is minor of element a_{1j}, i.e., determinant of the complementary to a_{1j} submatrix A_{1j}.

Therefore, let us introduce a more suitable for our aims notion of a "combinatorial minor" of element a_{ij}. Let the complementary to a_{ij} submatrix A_{ij} have the following $n-1$ columns

$$c_1, c_2, ..., c_{j-1}, c_{j+1}, ..., c_n. \tag{2.3}$$

The first $j-1$ of these columns we change in the following cyclic order: $c_2, c_3, ..., c_{j-1}, c_1$. Then we obtain a new matrix $\overline{A}_{ij}$ with the columns

$$c_2, c_3, ..., c_{j-1}, c_1, c_{j+1}, ..., c_n. \tag{2.4}$$

Determinant of matrix $\overline{A}_{ij}$ we call *combinatorial minor* $(CM)_{ij}$ of element a_{ij} . It is easy to see that

$$(CM)_{i1} = M_{i1};\ \ (CM)_{ij} = (-1)^{j-2}M_{ij},\ \ j = 2, ..., n. \tag{2.5}$$

Therefore, e.g., expansion (2.2) one can rewrite in the form

$$detA = a_{1,1}(CM)_{11} - \sum_{j=2}^{n}a_{1,j}(CM)_{1j}. \tag{2.6}$$

In general, let us give a definition of combinatorial minors for arbitrary matrix function $X(A)$.

Definition 4. *Let X be matrix function defined on all square matrices of order $n \geq 3$. Let $A = \{a_{ij}\}$ be a square matrix of order n and A_{ij} be the complementary to a_{ij} submatrix with columns (2.3). Denote $\overline{A}_{ij}$ a new square matrix of order $n-1$ with columns (2.4). Then the number $X(\overline{A}_{ij})$ is called a combinatorial minor of a_{ij}.*

It appears that our observation (2.6) has a general character. So, in Sections 4 we give a generalization of the Laplace expansion of type (2.6) for $per_\omega A$, $Cycl(A)$, $S(A; n, k)$ and $C_n(t_1, t_2, ..., t_n)$.

3. Main lemma

Lemma 1. *Let* $\pi \in \mathcal{S}^{(n)}$ *with* $\pi(j)=1$ *and* $\sigma=\sigma_j(\pi),\ \ j\geq 2$, *such that*

$$\sigma(1)=\pi(2),\ \ \sigma(2)=\pi(3),\ \ ...,\ \ \sigma(j-2)=\pi(j-1),\ \ \sigma(j-1)=\pi(1),$$

$$\sigma(j)=\pi(j)=1,\ \ \sigma(j+1)=\pi(j+1),\ \ ...,\ \ \sigma(n)=\pi(n). \tag{3.1}$$

Let, further, $\pi^* \in \mathcal{S}^{(n-1)}$ *defined by the formula*

$$\pi^*(i)=\begin{cases}\sigma(i)-1, & if\ \ 1\leq i\leq j-1,\\ \sigma(i+1)-1, & if\ \ j\leq i\leq n-1.\end{cases} \tag{3.2}$$

Then permutations π *and* π^* *have the same number of cycles:*

$$\gamma(\pi)=\gamma(\pi^*). \tag{3.3}$$

Proof. From (3.1)-(3.2) we find

$$\pi^*(i)=\begin{cases}\pi(i+1)-1, & if\ \ i\neq j-1,\\ \pi(1)-1, & if\ \ i=j-1.\end{cases} \tag{3.4}$$

Consider a cycle of π containing number j. Let it has length $l\geq 2$, such that

$$\pi(j)=1,\ \ \pi(1)=k_1,\ \ \pi(k_1)=k_2,\ \ ...,\ \ \pi(k_{l-3})=k_{l-2},\ \ \pi(k_{l-2})=j.$$

Beginning with the equality $\pi(1)=k_1$, by (3.4), this means that

$$\pi^*(j-1)=k_1-1,\ \ \pi^*(k_1-1)=k_2-1,\ \ ...,$$
$$\pi^*(k_{l-3}-1)=k_{l-2}-1,\ \ \pi^*(k_{l-2}-1)=j-1.$$

Thus to cycle of π containing number j of length $l\geq 2$ corresponds a cycle of length $l-1$ of π^*. Quite analogously, we verify that to cycle of π not containing number j of length $l\geq 2$ corresponds a cycle of the same length of π^*. E.g., to cycle of length $l\geq 2$ of the form

$$\pi(j-1)=k_1,\ \ \pi(k_1)=k_2,\ \ \pi(k_2)=k_3,\ \ ...,\ \ \pi(k_{l-2})=k_{l-1},\ \ \pi(k_{l-1})=j-1$$

(beginning with the equality $\pi(k_1)=k_2$), corresponds the cycle of the same length

$$\pi^*(k_1-1)=k_2-1,\ \ \pi^*(k_2-1)=k_3-1,\ \ ...,$$
$$\pi^*(k_{l-2}-1)=k_{l-1}-1,\ \ \pi^*(k_{l-1}-1)=j-2$$

such that $\pi^*(j-2)=\pi(j-1)-1=k_1-1.$

■

Note that the structure of Lemma 1 completely corresponds to the procedure of creating the combinatorial minors.

4. Laplace expansions of type (2.6) of $per_\omega A$, $Cycl(A)$, $S(A;n,k)$ and $C(t_1,t_2,...,t_n)$

1) $per_\omega A$. Consider all permutations π with the condition $\pi(j)=1$. Let j corresponds to j-th column of matrix A. Then the considered permutations correspond to diagonals of matrix A having the common position $(1,j)$. If $j=1$, then, removing the first row and column, we diminish on 1 the number of cycles of every such permutation, but also we diminish on 1 the number of elements of permutations. Therefore, the decrement of permutations does not change. If $j\geq 2$, consider continuation of these diagonals in the matrix of combinatorial miner $\overline{A}_{1,j}$. Then, by Lemma 1, the number of cycles of every its diagonal does not change and, consequently, the decrement is diminished by 1. This means that we have the following expansion of $per_\omega A$ over the first row

$$det_\omega A = a_{1,1}(CM)_{11} + \omega\sum_{j=2}^{n} a_{1,j}(CM)_{1j}, \tag{4.1}$$

where $(CM)_{1j},\ \ j\geq 1$, are combinatorial minors of $per_\omega A$.

Note that, as for determinant (see Section 2), for the receiving the required enumerative information, we should prohibit to use the identities of type $1+\omega+...+\omega^{m-1}=0$.

2) $Cycl(A)$. For $n>1$, here we should ignore element a_{11}. Consider all full cycles π with the condition $\pi(j)=1$. Let j corresponds to j-th column of matrix A. Then the considered full cycles correspond to diagonals of matrix A having the common position $(1,j),\ \ j\geq 2$. Consider continuation of these diagonals in the matrix of combinatorial miner $\overline{A}_{1,j}$. Then, by Lemma 1, the number of cycles of every its diagonal does not change, i.e., they are full cycles of of order $n-1$. Therefore, we have the following expansion of $Cycle(A)$ over the first row of A

$$Cycle(A) = \sum_{j=2}^{n} a_{1,j}(CM)_{1j}, \tag{4.2}$$

where $(CM)_{1j},\ \ j\geq 1$, are combinatorial minors of $Cycle(A)$.

3) $S(A;n,k)$. From very close to 1) arguments, we have the following expansion of $S(A;n,k)$ over the first row of A

$$S(A;n,k) = a_{1,1}(CM)_{11}^{(k-1)} + \sum_{j=2}^{n} a_{1,j}(CM)_{1j}^{(k)}, \tag{4.3}$$

where $(CM)_{1j}^{(k)},\ \ j\geq 1$, are combinatorial minors of $S(A;n,k)$.

Note that close to (4.3) formula was found by the author in [6] but using much more complicated way.

4) $C(t_1, t_2, ..., t_n)$. We need lemma.

Lemma 2. *Let*

$$\{a_{1j}, a_{k_1 1}, a_{k_2 k_1}, ..., a_{k_r, k_{r-1}}, a_{j, k_r}\} \tag{4.4}$$

be a cycle. Then

$$\{a_{k_1 1}, a_{k_2 k_1}, ..., a_{k_r, k_{r-1}}, a_{j, k_r}\} \tag{4.5}$$

is a cycle with respect to the main diagonal of the matrix of combinatorial miner $\overline{A}_{1,j}$.

Proof. According to the construction of $\overline{A}_{1,j}$, the main its diagonal is

$$\{a_{22}, a_{33}, ..., a_{j-1j-1}, a_{j1}, a_{j+1j+1}, ..., a_{nn}\}. \tag{4.6}$$

With respect to this diagonal we have the following contour which shows that (4.5) is, indeed, a cycle.

$$\{a_{k_1 1} \to a_{j1} \to a_{jk_r} \to a_{k_r k_{r-1}} \to a_{kr-1k_{r-2}} \to ... \to a_{k_2 k_1} (\to a_{k_1 1})\}. \tag{4.7}$$

■

Quite analogously we can prove that to every another cycle of a diagonal containing element $a_1 j$ correspond the same cycle with respect to the main diagonal of the matrix $\overline{A}_{1,j}$.

Let A be $(0, 1)$ square matrix of order n. Denote $C^{(r)}(A; t_1, t_2, ..., t_n)$ a partial cyclic indicator of indicator (1.9) of permutations $\pi \in \mathcal{B}(A)$ for which $\{1, \pi(1), \pi(2), ..., \pi(r-1)\}$ is a cycle of length r. Then we have

$$\sum_{r=1}^{n} C^{(r)}(A; t_1, t_2, ..., t_n) = C(A; t_1, t_2, ..., t_n). \tag{4.8}$$

Therefore, it is sufficient to give an expansion of $C^{(r)}(A; t_1, t_2, ..., t_n)$, $r = 1, ..., n$. First of all, note that

$$C^{(1)}(A; t_1, t_2, ..., t_n) = a_{11} t_1 C(A; t_1, t_2, ..., t_{n-1}). \tag{4.9}$$

Furthermore, using Lemmas 1-2, we have

$$C^{(r)}(A; t_1, t_2, ..., t_n) = \frac{t_r}{t_{r-1}} \sum_{j=2}^{n} a_{1,j} (CM)_{1,j}, \tag{4.10}$$

where

$$(CM)_{1,j} = C^{(r-1)}(\overline{A}_{1,j}; t_1, t_2, ..., t_{n-1}), \quad j = 2, ..., n, \tag{4.11}$$

are the combinatorial miners of cyclic indicator of permutations with restricted positions.

Note that factor $\frac{t_r}{t_{r-1}}$ in (4.10) corresponds to the diminution of the length of cycle (4.5) with respect to length of cycle (4.4). Thus formulas (4.8)-(4.11) reduce calculation of cyclic indicator of n-permutations to $n-1$-permutations with a rather simple computer realization of this procedure. Note that similar but much more complicated procedure was indicated by the author in [7].

5. An example of enumerating the permutations of classes 0,1,2 modulo 3 with restricted positions

Let

$$A=\begin{pmatrix}1&1&1&1&0\\0&1&0&1&1\\1&0&1&1&1\\1&1&1&0&0\\1&1&1&1&1\end{pmatrix}.$$

Consider class $\mathcal{B}(A)$ of permutations with restricted positions and find the distribution of them over classes 0,1,2 modulo 3. We use ω-permanent with $\omega=e^{\frac{2\pi i}{3}}$ and its expansion over elements of the first row, given by (4.1). Recall that, for the receiving the required enumerative information, we cannot use identities of type $1+\omega+\omega^2=0$.

We have

$$per_\omega A = per_\omega\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&0&0\\1&1&1&1\end{pmatrix}+$$

$$\omega(\ per_\omega\begin{pmatrix}0&0&1&1\\1&1&1&1\\1&1&0&0\\1&1&1&1\end{pmatrix}+per_\omega\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&0&0\\1&1&1&1\end{pmatrix}+per_\omega\begin{pmatrix}1&0&0&1\\0&1&1&1\\1&1&1&0\\1&1&1&1\end{pmatrix})=$$

$$per_\omega\begin{pmatrix}1&1&1\\1&0&0\\1&1&1\end{pmatrix}+\omega(\ per_\omega\begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix}+per_\omega\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix})+$$

$$\omega^2(\ per_\omega\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}+per_\omega\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix})+\omega per_\omega\begin{pmatrix}1&1&1\\1&0&0\\1&1&1\end{pmatrix}+$$

$$\omega^2(\ per_\omega\begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix}+per_\omega\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix})+\omega per_\omega\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}+$$

$$\omega^2 per_\omega\begin{pmatrix}1&1&0\\1&1&1\\1&1&1\end{pmatrix}=(\omega+\omega^2)+(\omega+\omega^2+1)+(2\omega^2+1)+$$

$$(\omega^2+2+\omega)+(2+2\omega)+(\omega^2+1)+(\omega^2+1+\omega)+(2+\omega)+$$

$$(\omega+2\omega^2+1)+(\omega^2+2+\omega)=13+9\omega+10\omega^2.$$

Thus in $\mathcal{B}(A)$ we have 13 permutation of class 0; 9 permutations of class 1 and 10 ones from class 2 modulo 3.

■

6. On two sequences connected with $Cycle(A)$

In summer of 2010, the author published two sequences A179926 and A180026 in OEIS [8]. $a(n) := A179926(n)$ is defined as the number of permutations of the divisors of n of the form $d_1 = n,\ \ d_2,\ \ d_3,\ \ ...,\ \ d_{\tau(n)}$ such that $\frac{d_{i+1}}{d_i}$ is a prime or 1/prime for all i. Note that $a(n)$ is a function of exponents of prime power factorization of n only; moreover, it is invariant with respect to permutations of them. This sequence is equivalently defined as the number of ways, for a given finite multiset E, to get, beginning with E, all submultisets of E, if in every step we remove or join 1 element of E. Sequence A180026 differs from A179926 by an additional condition: $\frac{d_{\tau(n)}}{d_1}$ is a prime. In the equivalent formulation it corresponds to the condition that E is obtained from a submultiset in the last step, by joining 1 element.

Note that, it is easy to prove that, knowing any permissible permutation of divisors, say, $\delta_1 = n,\ \ \delta_2,\ \ ...,\ \ \delta_{\tau(n)}$ (such that $\frac{\delta_{i+1}}{\delta_i}$ is a prime or 1/prime), we can calculate $b(n) := A180026(n)$, using the following construction. Consider square (0,1) matrix $B = \{b_{ij}\}$ of order $\tau(n)$ in which $b_{ij} = 1$, if $\frac{\delta_i}{\delta_j}$ is prime or 1/prime, and $b_{ij} = 0$, otherwise. Then $b(n) = Cycle(B)$. In case of A179926, the construction is a little more complicated: $a(n) = Cycle(A)$, where A is obtained from B by the replacing the first its column by the column from 1's.

Note also that A. Heinz [2] proved that, in particular, $a(\Pi_{i=1}^{n} p_i)$, where p_i are distinct primes, equals to the number of Hamiltonian paths (or Gray codes) on n-cube with a marked starting node (see A003043 in [8]), while $b(\Pi_{i=1}^{n} p_i)$ equals to the number of directed Hamiltonian cycles on n-cube (see A003042 in [8]).

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Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. e-mail:shevelev@bgu.ac.il