Inverses of Motzkin and Schröder Paths

Heinrich Niederhausen
Florida Atlantic University, Boca Raton

May 2011

Abstract

We suggest three applications for the inverses: For the inverse Motzkin matrix we look at Hankel determinants, and counting the paths inside a horizontal band, and for the inverse Schröder matrix we look at the paths inside the same band, but ending on the top side of the band.

1 Introduction

We adopt the convention that lattice paths without restrictions are called "*Grand*"; the *Grand Catalan numbers* (step set $\{\nearrow, \searrow\}$) are the number of paths from the origin, taking only $\nearrow$ and $\searrow$ steps, and ending on the x-axis at $(2n, 0)$. The Grand Catalan numbers are the *Central Binomial coefficients*, $\binom{2n}{n}$, with generating function $1/\sqrt{1-4t^2} = \sum_{n\geq 0}\binom{2n}{n}t^{2n}$. The *wheighted Grand Motzkin* numbers G_n take steps from $\{\nearrow, \searrow, \longrightarrow\}$, and end on the x-axis in $(n, 0)$. The horizontal steps get the weight ω. Their generating function is

$$g(t) := \sum_{n\geq 0} G_n t^n = 1/\sqrt{(1-\omega t)^2 - 4t^2}, \tag{1}$$

and it is seen immediately that for $\omega = 0$ the Grand Catalan numbers are recovered. If $\omega = 2$, the $1/\sqrt{(1-2t)^2 - 4t^2} = 1/\sqrt{1-4t}$ is again a generating function for the Grand Catalan numbers, but we get $\sum_{n\geq 0}\binom{2n}{n}t^n$. The *general* Grand Motzkin numbers $G(n, j)$ enumerate all paths to (n, j), and the first few are given in the following table.

$\uparrow j$						
7						
6						
5						1
4					1	5ω
3				1	4ω	$5+10\omega^2$
2			1	$3\omega\searrow$	$4+6\omega^2$	$20\omega+10\omega^3$
1		1	2ω	$3+3\omega^2 \rightarrow$	$12\omega+4\omega^3$	$10+30\omega^2+5\omega^4$
0	1	ω	$2+\omega^2$	$6\omega+\omega^3 \nearrow$	$6+12\omega^2+\omega^4$	$30\omega+20\omega^3+\omega^5$
-1		1	2ω	$3+3\omega^2$	$12\omega+4\omega^3$	$10+30\omega^2+5\omega^4$
-2			1	3ω	$4+6\omega^2$	$20+10\omega^3$
-3				1	4ω	$5+10\omega^2$
-4					1	5ω
-5						1
-6						
$n \rightarrow$	0	1	2	3	4	5

The general Grand Motzkin numbers (G_n is given in row 0)

The lower half of the table is the mirror image of the top half; if we write the table in matrix form, $G(n,j)$ stands in row n and column j, and we obtain a *Riordan matrix* G, because $G(n+1,j+1) = G(n,j) + \omega G(n,j+1) + G(n,j+2)$ (see Rogers [9], and [6]). It follows that

$$\begin{aligned}\sum_{n\geq j} G(n,j)\,t^n &= \frac{1}{\sqrt{(1-\omega t)^2 - 4t^2}}\left(\frac{1}{2t}\left(1 - t\omega - \sqrt{(\omega t - 1)^2 - 4t^2}\right)\right)^j \\ &= g(t)\left(\frac{1}{2t}\left(1 - \omega t - 1/g(t)\right)\right)^j\end{aligned}$$

$\uparrow n$				
0	1			
1	ω	1		
2	$2+\omega^2$	2ω	1	
3	$6\omega+\omega^3$	$3+3\omega^2$	3ω	1
4	$6+12\omega^2+\omega^4$	$12\omega+4\omega^3$	$4+6\omega^2$	4ω
5	$30\omega+20\omega^3+\omega^5$	$10+30\omega^2+5\omega^4$	$20\omega+10\omega^3$	$5+10\omega^2$
6	$20+90\omega^2+30\omega^4+\omega^6$	$60\omega+60\omega^3+6\omega^5$	$15+60\omega^2+15\omega^4$	$30\omega+20\omega^3$
$j \rightarrow$	0	1	2	3

The Riordan matrix $G = (G(n,j))_{\substack{n=0,\ldots \\ j=0,\ldots,n}}$ (G_n is given in column 0)

If we *restrict* the $\left\{\nearrow, \searrow, \xrightarrow{\omega}\right\}$-paths to the first quadrant, they become *Motzkin paths* $M(n,j)$. We will look at the inverse $(m_{i,j})$ of the matrix M, and find it useful in some applications (see also A. Ralston and P. Rabinowitz, 1978 [8, p. 256]). Especially, the *bounded* Motzkin numbers $M_{n;w}^{(k)}$, the number of Motzkin paths staying strictly below the parallel to the x-axis at height k, have a generating function expressed by the inverse $(m_{i,j})$, through the *inverse Motzkin polynomial* $m_k(t) = \sum_{i=0}^{k} m_{k,i} t^{k-i}$,

$$\sum_{n \geq 0} M_{n;\omega}^{(k)} t^n = \frac{m_{k-1}(t)}{m_k(t)}$$

(see (8). That makes us wonder if paths with different lengths of the horizontal steps $(w, 0)$ have similar properties. In the case of $w = 2$ (*Schröder paths*) and $\omega = 1$ we have a result, $\mathcal{S}^{(k)}(t) :=$

$$\sum_{n \geq 0} S_n^{(k)} t^n = \frac{(1-t)\sum_{i=0}^{(k-2)/2} t^{2i} (-1)^i s_{k-2-2i}(t) + (k \bmod 2)(-1)^{(k-1)/2} t^{k-1}}{(1-t)\sum_{i=0}^{(k-1)/2} t^{2i} (-1)^i s_{k-1-2i}(t) + ((k-1) \bmod 2)(-1)^{k/2} t^k}$$

where the Motzkin terms (M and m) are replaced by the corresponding Schröder terms (S and s), and $s_i(t)$ is the *inverse Schröder polynomial.* Perhaps more interesting is the generating function identity described in Theorem 3,

$$t^{-k} \mathcal{S}^{(k)}(t)\, s_{k-1}(t) = t^{-k} \mathcal{S}^{(k)}(t, k-1)$$

(as power series) where $S^{(k)}(t, k-1)$ is the generating function of the bounded Schröder number ending on $y = k - 1$, just below the upper boundary.

2 Motzkin Numbers

Leaving the Grand Motzkin numbers behind, we introduce the restriction of counting only paths that do not go below the x-axis. A *general weighted Motzkin path* is counted by the recursion

$$M(n, m; \omega) = M(n-1, m+1; \omega) + \omega M(n-1, m; \omega) + M(n-1, m-1; \omega)$$

for $m \geq 0$, and $M(n, m; \omega) = 0$ if $m < 0$. The numbers $M(n, m; \omega)$ are weighted counts of all such path from $(0, 0)$ to (n, m), and we give the special name $M_{n;\omega}$ to the Motzkin numbers $M(n, 0; \omega)$. These numbers (with weight $\omega = 1$) have been studied by Th. Motzkin in 1946 [7].

$\uparrow m$						
7						
6						
5						1
4					1	5ω
3				1	4ω	$4+10\omega^2$
2			1	3ω $\searrow$	$3+6\omega^2$	$15\omega+10\omega^3$
1		1	2ω	$2+3\omega^2 \rightarrow$	$8\omega+4\omega^3$	$5+20\omega^2+5\omega^4$
0	1	ω	$1+\omega^2$	$3\omega+\omega^3 \nearrow$	$2+6\omega^2+\omega^4$	$10\omega+10\omega^3+\omega^5$
	0	1	2	3	4	5
$M_{n;\omega}$ is given in row 0.						

The above table shows that for $\omega = 1$ the original Motzkin numbers are $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ (sequence A001006 in the On-Line Encyclopedia of Integer Sequences (OEIS)).

It is well-known that the general ω-weighted Motzkin numbers have the generating function

$$\mu(t; j, \omega) := \sum_{n \geq 0} M(n+j, j; \omega) t^n = \left(\frac{1 - \omega t - \sqrt{(1-\omega t)^2 - 4t^2}}{2t^2} \right)^{j+1}$$

thus

$$\mu(t) := \sum_{n \geq 0} M_{n;\omega} t^n = \sum_{n \geq 0} M(n, 0; \omega) t^n = \frac{1 - \omega t - \sqrt{(1-\omega t)^2 - 4t^2}}{2t^2} \tag{2}$$

is the generating function of the Motzkin numbers, satisfying the quadratic equation [1]

$$\mu(t) = 1 + \omega t \mu(t) + t^2 \mu(t)^2 \tag{3}$$

Hence

$$M_{n+2;\omega} - \omega M_{n+1;\omega} = \sum_{i=0}^{n} M_{i;\omega} M_{n-i;\omega}$$

a well-known identity, combinatorially shown by using the *"First Return Decomposition"*. The generating function (in t^2) of the Catalan numbers C_n is easily obtained by setting $\omega = 0$ in (2), but it also follows from $\omega = 2$

$$\frac{1 - 2t - \sqrt{(1-2t)^2 - 4t^2}}{2t^2} = \frac{1 - 2t - \sqrt{1-4t}}{2t^2} = \sum_{n \geq 1} C_n t^{n-1}$$

(in t). Or we can choose $\omega = 1$ and get

$$(1+t)\sum_{n\geq 1} C_n \left(\frac{t}{1+t}\right)^{n-1} = \sum_{n\geq 0} M_{n;1} t^n$$

$$M_{n;1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} C_{k+1}$$

For general ω follows from (2) the explicit expression

$$M_{n;\omega} = \sum_{k=0}^{n/2} \binom{n}{2k} \frac{\omega^{n-2k}}{2k+1} \tbinom{2k+1}{k}.$$

3 The Inverse

Define $\phi(t)$ such that $t/\phi(t)$ is the compositional inverse of $t\mu(t)$ thus

$$\phi(t\mu(t)) = \mu(t) = 1 + \omega t\mu(t) + t^2\mu(t)^2$$

by (3), and therefore

$$\phi(t) = 1 + \omega t + t^2$$

This simple form of the inverse is the reason for many special results for Motzkin numbers. Note that

$$1/\phi(t) = \left(1 + \omega t + t^2\right)^{-1} = \sum_{n\geq 0} U_n(-\omega/2)\, t^n$$

the generating function of the *Chebychef polynomials* of the second kind.

Because of the inverse relationship between $t\mu(t)$ and $t/\phi(t)$ we have that the matrix inverse of $(M(i,j;\omega))_{n\times n}$ equals $(m_{i,j})_{n\times n}$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 \\ 9 & 12 & 9 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ -1 & 2 & 3 & -4 & 1 \end{pmatrix} = (m_{i,j})_{4\times 4}$$

Inverse Motzkin matrix when $\omega = 1$

where

$$\sum_{i\geq 0} m_{i,j} t^i = t^j \phi(t)^{-j-1}$$

Note that $(m_{i,j})$ is also a *Riordan matrix*. The above generating function for $m_{i,j}$ implies that

$$m_{i,j} = \left[t^i\right] \frac{1}{1+\omega t + t^2} \left(\frac{t}{1+\omega t + t^2}\right)^j = \left[t^{i-j}\right] \left(1 + \omega t + t^2\right)^{-j-1} = C_{i-j}^{j+1}(-\omega/2).$$

The polynomials $C_n^{\lambda}(x) = \sum_{k=0}^{n/2} \binom{n-k+\lambda-1}{n-k} \binom{n-k}{n-2k} (-1)^k (2x)^{n-2k}$ are the *Gegenbauer polynomials*, and therefore

$$m_{i,j} = \sum_{l=0}^{(i-j)/2} \binom{i-l}{i-j-l} \binom{i-j-l}{l} (-1)^l (-\omega)^{i-j-2l} \tag{4}$$

The recurrence relation for the (orthogonal) Gegenbauer polynomials

$$2x(n+\lambda) C_n^{\lambda}(x) = (n+2\lambda-1) C_{n-1}^{\lambda} + (n+1) C_{n+1}^{\lambda}(x)$$

gives us immediately a recurrence for the inverse numbers $m_{i,j}$, $0 \leq j \leq i-1$,

$$(i-j) m_{i,j} = -\omega i m_{i-1,j} - (i+j) m_{i-2,j}$$

with initial values $m_{i,j} = \delta_{i,j}$ for $j \geq i$.

We need later in the paper the following Motzkin ploynomial

$$\begin{aligned} \sum_{j=0}^{k} m_{k,j} t^{k-j} &= \sum_{j=0}^{k} C_j^{k-j+1)} (-\omega/2) t^j \\ &= \sum_{l=0}^{k/2} \sum_{j=0}^{k-2l} \binom{k-l}{k-j-l} \binom{k-j-l}{k-j-2l} (-1)^l (-\omega)^{k-j-2l} t^{k-j} \\ &= \sum_{l=0}^{k/2} \binom{k-l}{l} (-1)^l t^{2l} (1-\omega t)^{k-2l} \end{aligned} \tag{5}$$

From

$$\left((M(i,j))_{0 \leq i,j \leq n} \right)^{-1} = (m_{i,j})_{0 \leq i,j \leq n}$$

follows

$$\sum_{k=0}^{n} M(k,i;w) m_{k,j} = \delta_{i,j}.$$

However, in the case of Motzkin matrices more than this simple linear algebra result holds.

Lemma 1 *For all nonnegative integers i and i holds*

$$M(i,j;\omega) = \sum_{k=0}^{j} m_{j,k} M_{i+k;\omega}$$

and

$$m_{i,j} = \sum_{k=0}^{i-j} m_{i+1,j+1+k} M_{k;\omega}$$

The proof can be done via generating functions. Note that

$$\sum_{n\geq 0}\sum_{j\geq 0} x^j t^n M(n,j;\omega) = \frac{\mu(t)}{1-xt\mu(t)} = \frac{1}{1+\omega x+x^2-x/t}\left(\mu(t)-\frac{x}{t}\right)$$

and

$$\sum_{j\geq 0} x^j \sum_{i\geq j} m_{i,j} t^i = \sum_{j\geq 0} x^j t^j \phi(t)^{j+1} = \frac{\phi(t)}{1-xt\phi(t)} = \frac{1}{1/\phi(t)-xt} = \frac{1}{1+\omega t+t^2-xt}.$$

Replace t by x and x by $1/t$ in the above generating function for the inverse $m_{i,j}$ to get the Laurent series

$$\sum_{j\geq 0} t^{-j} \sum_{i\geq j} m_{i,j} x^i = \frac{1}{1+\omega x+x^2-x/t}$$

hence

$$\sum_{n\geq 0}\sum_{j\geq 0} x^j t^n M(n,j;\omega) = \left(\mu(t)-\frac{x}{t}\right)\sum_{j\geq 0} t^{-j} \sum_{i\geq j} m_{i,j} x^i$$

Now both sides must be power series in x and t. This condition gives the Lemma. The Lemma also has the

Corollary 2

$$\sum_{k=0}^{j} m_{j,k} M_{i+k,w} = \delta_{i,j} \textit{ for } 0\leq i\leq j \tag{6}$$

because $M(i,j;\omega) = \delta_{i,j}$ for all $0\leq i\leq j$.

4 Two applications of the inverse Motzkin matrix

The Lemma says that

$$(m_{i,j})_{0\leq i,j\leq n} (M_{i+j;\omega})_{0\leq i,j\leq n} = (M(i,j;\omega))_{0\leq i,j\leq n}$$

which gives a direct way of calculcating the first *Hankel determinant*

$$\det (M_{i+j;\omega})_{0\leq i,j\leq n} = \frac{1}{\det(m_{i,j})}\det(M(i,j;w)) = 1 \tag{7}$$

However, subsequent Hankel determinants are more complicated; we want to show a way how to calculate a determinant proposed by Cameron and Yip [2]. For a broader theory of Hankel determinants in lattice path enumeration see [3].

4.1 The Hankel determinant $\left|\alpha M_{i+j;\omega}+\beta M_{i+j+1;\omega}\right|_{0\le i,j\le n-1}$

The Hankel determinant of $(\alpha M_{i+j;\omega}+\beta M_{i+j+1;\omega})_{0\le i,j\le n-1}$ equals for $\omega=1$

$$= \begin{vmatrix} \alpha+2\beta & 2\alpha+4\beta & 4\alpha+7\beta & \dots & \alpha M_{n-1;1}+\beta M_{n;1} \\ 2\alpha+4\beta & 4\alpha+7\beta & 7\alpha+9\beta & & \\ 4\alpha+7\beta & 7\alpha+9\beta & 4\alpha+7\beta & \vdots & \vdots \\ 7\alpha+9\beta & 9\alpha+21\beta & 9\alpha+21\beta & & \\ \vdots & \vdots & \vdots & & \\ \alpha M_{n-1;1}+\beta M_{n;1} & \alpha M_{n;1}+\beta M_{n+1;1} & \alpha M_{n+1;1}+\beta M_{n+2;1} & & \alpha M_{2n-2;1}+\beta M_{2n,n;1} \end{vmatrix}$$

$$= \left|(M_{i+j;1})_{0\le i,j\le n-1}\right| \begin{vmatrix} \alpha & 0 & 0 & \dots & -\beta m_n(0) \\ \beta & \alpha & 0 & & -\beta m_n(1) \\ 0 & \beta & \alpha & & -\beta m_n(2) \\ & & & \vdots & \\ 0 & 0 & 0 & \alpha & -\beta m_n(n-2) \\ 0 & 0 & 0 & \beta & \alpha-\beta m_n(n-1) \end{vmatrix}$$

because the last column in the matrix on the right when multiplied with the i-th row of the matrix on the left gives $\alpha M_{i+n-1;\omega}-\beta\sum_{k=0}^{n-1}m_{n,k}M_{i+k;\omega} = \alpha M_{i+n-1;\omega}+\beta M_{i+n;\omega}-\beta\delta_{i,n}$ by Corollary 2. Now

$$\begin{vmatrix} \alpha & 0 & 0 & \dots & -\beta m_{n,0} \\ \beta & \alpha & 0 & & -\beta m_{n,1} \\ 0 & \beta & \alpha & \dots & -\beta m_{n,2} \\ & & & \vdots & \\ 0 & 0 & \dots & \alpha & -\beta m_{n,n-2} \\ 0 & 0 & \dots & \beta & \alpha-\beta m_{n,n-1} \end{vmatrix}$$

$$= \alpha^{-\binom{n}{2}} \begin{vmatrix} \alpha & 0 & 0 & \dots & -\beta m_{n,0} \\ \alpha\beta & \alpha^2 & 0 & & -\alpha\beta m_{n,1} \\ 0 & \alpha^2\beta & \alpha^3 & \dots & -\alpha^2\beta m_{n,2} \\ & & & \vdots & \\ 0 & 0 & \dots & \alpha^{n-1} & -\alpha^{n-2}\beta m_{n,n-2} \\ 0 & 0 & \dots & \alpha^{n-1}\beta & \alpha^{n-1}-\alpha^{n-1}\beta m_{n,n-1} \end{vmatrix}$$

$$= \alpha^{-\binom{n}{2}} \begin{vmatrix} \alpha & 0 & 0 & \dots & -\beta m_{n,0} \\ 0 & \alpha^2 & 0 & & \beta^2 m_{n,0}-\alpha\beta m_{n,1} \\ 0 & 0 & \alpha^3 & \dots & -\beta^3 m_{n,0}+\alpha\beta^2 m_{n,1}-\alpha^2\beta m_{n,2} \\ & & & \vdots & \\ 0 & 0 & \dots & \alpha^{n-1} & -\sum_{i=0}^{n-2}(-1)^{n-2-i}\beta^{n-1-i}\alpha^i m_{n,i} \\ 0 & 0 & \dots & 0 & \alpha^n-\sum_{i=0}^{n-1}(-1)^{n-1-i}\beta^{n-i}\alpha^i m_{n,i} \end{vmatrix}$$

Therefore $\det\left(\left(\alpha M_{i+j;\omega}+\beta M_{i+j+1;\omega}\right)_{0\le i,j\le n-1}\right)=\alpha^n-\sum_{i=0}^{n-1}(-1)^{n-1-i}\beta^{n-i}\alpha^i m_{n,i}=\sum_{i=0}^{n}(-\beta)^{n-i}\alpha^i m_{n,i}=\sum_{i=0}^{n}(-1)^{n-i}\beta^{n-i}\alpha^i P_{n-i}^{(-i-1)}(-\omega/2)$. This can be written explicitly as $\det\left(\left(\alpha M_{i+j;\omega}+\beta M_{i+j+1;\omega}\right)_{0\le i,j\le n-1}\right)=$

$$
\begin{aligned}
&(-\beta)^n\sum_{k=0}^{n}(-\alpha/\beta)^k m_{n,k}\\
&=(-\beta)^n U_n\left(\frac{-\alpha/\beta-\omega}{2}\right)=(-\beta)^n\sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k(-\alpha/\beta-\omega)^{n-2k}\\
&=\sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k\beta^{2k}(\alpha+\beta\omega)^{n-2k}\\
&=\frac{2^{-n-1}}{\sqrt{(\alpha+\omega\beta)^2-4\beta^2}}\times\\
&\times\left(\left(\sqrt{(\alpha+\omega\beta)^2-4\beta^2}+\alpha+\omega\beta\right)^{n+1}+\left(\sqrt{(\alpha+\beta\omega)^2-4\beta^2}-\alpha-\beta\omega\right)^{n+1}\right)
\end{aligned}
$$

If $\alpha=\beta=1$, then $\det\left(\left(M_{i+j;\omega}+M_{i+j+1;\omega}\right)_{0\le i,j\le n-1}\right)=$

$$
\begin{aligned}
&\frac{1}{2^{n+1}\sqrt{(\omega+1)^2-4}}\left(\left(1+\omega+\sqrt{(\omega+1)^2-4}\right)^{n+1}-\left(1+\omega-\sqrt{(\omega+1)^2-4}\right)^{n+1}\right)\\
&=\sum_{k=0}^{n}(-1)^{n-k}\binom{k}{n-k}(\omega+1)^{2k-n}
\end{aligned}
$$

which approaches $n+1$ if $\omega\to 1$. In the case of Dyck path, we obtain $\delta_{0,n}$ for this determinat of the sum of matrices. If $\beta=1$ and $\alpha=0$, then the determinant is the second Hankel determinant of the Motzkin numbers,

$$\det\left(\left(M_{i+j+1;\omega}\right)_{0\le i,j\le n-1}\right)=\sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k\omega^{n-2k}$$

If $\alpha=1$ and $\beta=0$ then $\det\left(M_{i+j;\omega}\right)_{0\le i,j\le n-1}=1$, independent of ω (see (7). The same approach also shows the recursion

$$|M_{i+j+2;\omega}|_{0\le i,j\le n-1}=|M_{i+j+2;\omega}|_{0\le i,j\le n-2}+|M_{i+j+1;\omega}|^2_{0\le i,j\le n-1}$$

4.2 Motzkin in a band

The number of Motzkin paths staying strictly below the line $y = k$ for $k > 0$ is known to have the generating function [4, Proposition 12]

$$\sum_{n\geq 0} M_n^{(k)} t^n = \mu(t) \frac{1 - (t\mu(t))^{2k}}{1 - (t\mu(t))^{2(k+1)}} = \frac{1}{t} \frac{\left(\frac{1}{t\mu}\right)^k - (t\mu)^k}{\left(\frac{1}{t\mu}\right)^{k+1} - (t\mu)^{k+1}}$$

k										
k	0	0	0	0	0	0	0	0	0	0
3				1	4	14	44	133	392	1140
2			1	3	9	25	69	189	518	1422
1		1	2	5	12	30	76	196	512	1353
0	1	1	2	4	9	21	51	127	323	835
-1	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	n

$M_n^{(4)}$ is given in row 0.

From $\mu(t)(1-\omega t) - 1 = t^2\mu(t)^2$ (see (3)) follows

$$\mu_{1,2}(t) = \left(1 - \omega t \pm \sqrt{(1-\omega t)^2 - 4t^2}\right) / \left(2t^2\right)$$
$$t\mu_{1,2}(t) = \left(1 - \omega t \pm \sqrt{(1-\omega t)^2 - 4t^2}\right) / (2t)$$

thus

$$\mu_1 + \mu_2 = (1-\omega t)/t^2 \text{ and } \mu_1\mu_2 = 1/t^2$$

Hence

$$\sum_{n\geq 0} M_{n;\omega}^{(k)} t^n = \frac{1}{t} \frac{(t\mu_1)^k - (t\mu_2)^k}{(t\mu_1)^{k+1} - (t\mu_2)^{k+1}} = \frac{1}{t} \frac{(t\mu_2)^{-k} - (t\mu_1)^{-k}}{(t\mu_2)^{-k-1} - (t\mu_1)^{-k-1}}$$
$$= \frac{\sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-1-j}{j} t^{2j} (1-\omega t)^{k-1-2j}}{\sum_{j=0}^{k/2} (-1)^j \binom{k-j}{j} t^{2j} (1-\omega t)^{k-2j}}$$
$$= \frac{\sum_{i=0}^{k-1} m_{k-1,i} t^{k-1-i}}{\sum_{i=0}^{k} m_{k,i} t^{k-i}}$$

(see (5)). The OEIS lists many special cases for k; here are a few, with $\omega = 1$.

1. $\sum_{n\geq 0} M_{n;1}^{(1)} t^n \frac{1}{1-t} \iff 1, 1, 1, 1, \ldots$
2. $\sum_{n\geq 0} M_{n;1}^{(2)} t^n = \frac{1-t}{(1-t)^2 - t^2} = 1 + t + 2t^2 + 4t^3 + 8t^4 + 16t^5 \ldots$ thus $1, 1, 2, 4, 8, 16, 32, 64, \ldots$, the powers of 2.
3. $\sum_{n\geq 0} M_{n;1}^{(3)} t^n = \frac{2t-1}{(1-t)(t^2+2t-1)}$ thus $1, 1, 2, 4, 9, 21, 50, 120, \ldots$ (A171842)

4. $\sum_{n\geq 0} M_{n;1}^{(4)} t^n = \left(1-3t+t^2+t^3\right) / \left(1-4t+3t^2+2t^3-t^4\right)$, thus $1,1,2,4,9,21,51,127,322,826,\cdots$: (A005207), generating function by Alois P. Heinz.

The special form of the generating function

$$\sum_{n\geq 0} M_{n;\omega}^{(k)} t^n = \frac{\sum_{i=0}^{k-1} m_{k-1,i} t^{k-1-i}}{\sum_{i=0}^{k} m_{k,i} t^{k-i}} \tag{8}$$

works with weight ω, for all $k = 1, 2, \ldots$. It is equivalent to the recursion $\sum_{j=0}^{k} M_{n-j}^{(k)} m_{k,k-j} = 0$ for all $n \geq k$, with initial values $\sum_{j=0}^{n} M_{n-j}^{(k)} m_{k,k-j} = m_{k-1,k-1-n}$ for all $n = 0, \ldots, k-1$.

5 Horizontal steps of length w

A "natural" generalization of Motzkin paths is a lattice path W that takes horizontal steps of some positive length w, weighted by ω. We would like to see similar results as (8) in such cases. However, we have a result only for the case $w = 2$, the Schröder paths.

$\uparrow m$									1	0
7								1	0	8
6							1	0	7	7ω
5						1	0	6	6ω	27
4					1	0	5	5ω	20	35ω
3				1	0	4	4ω	14	24ω	$48+10\omega^2$
2			1	0	3	3ω	9	15ω	$28+6\omega^2$	63ω
1		1	0	2	2ω	5	8ω	$14+3\omega^2$	30ω	$42+20\omega^2$
0	1	0	1	ω	2	3ω	$5+\omega^2$	10ω	$14+6\omega^2$	$35\omega+\omega^3$
	0	1	2	3	4	5	6	7	8	$n\rightarrow$

$w = 3$ (W_n is given in row 0)

5.1 The recursion for W

Let us consider the step set $\{\nearrow, \searrow, \longrightarrow^w\}$, where $\rightarrow^w =: (w, 0)$, for any positive integer w. Denote the number of paths from $(0,0)$ to (n,j) by $W(n,j;\omega)$, where the horizontal steps are weighted by ω. We get the recursion

$$\begin{aligned} W(n,j;\omega) &= W(n-1,j+1;\omega) + W(n-1,j-1;\omega) + \omega W(n-w,j;\omega) \\ W(n,j;\omega) &= 0 \text{ for } j < n \\ W_{n;\omega} &= W(n,0;\omega). \end{aligned}$$

The generating function is well known,

$$\sum_{n\geq 0} W_{n;\omega} t^n = \frac{1-\omega t^w - \sqrt{\left(1-\omega t^w\right)^2 - 4t^2}}{2t^2} =: \mu_w(t;\omega) \tag{9}$$

The recursion can be reformulated as

$W(n,j;\omega) = W(n+1,j-1;\omega) - W(n,j-2;\omega) - \omega W(n+1-w,j-1;\omega)$ for $m \geq n$

We find the generating function identity $\sum_{i\geq 0} W(i,j;\omega)\, t^i =$

$$\sum_{i\geq 0} W(i+1,j-1;\omega)\, t^i - \sum_{i\geq 0} W(i,j-2;\omega)\, t^i$$
$$-\,\omega \sum_{i\geq w-1} W(i+1-w,j-1;\omega)\, t^i$$
$$= \sum_{i\geq 0} W(i+1,j-1;\omega)\, t^i - \omega\left(\sum_{i\geq -1} W(i+1,j-1;\omega)\, t^{i+1+w-1}\right)$$
$$-\sum_{i\geq 0} W(i,j-2;\omega)\, t^i$$
$$= t^{-1}\sum_{i\geq 0} W(i+1,j-1;\omega)\, t^{i+1} - \omega t^{w-1}\left(\sum_{i\geq -1} W(i+1,j-1;\omega)\, t^{i+1}\right)$$
$$-\sum_{i\geq 0} W(i,j-2;\omega)\, t^i$$
$$= \left(t^{-1} - \omega t^{w-1}\right)\left(\sum_{i\geq 0} W(i,j-1;\omega)\, t^i - \delta_{j,1}\right) - \sum_{i\geq 0} W(i,j-2;\omega)\, t^i$$

Let $\mathcal{W}(t,j;\omega) = \sum_{i\geq 0} W(i,j;\omega)\, t^i$. In this notation,

$$\mathcal{W}(t,j;\omega) = \frac{1-\omega t^w}{t}\mathcal{W}(t,j-1;\omega) - \mathcal{W}(t,j-2;\omega) \text{ for } j>1 \tag{10}$$
$$\mathcal{W}(t,1;\omega) = \frac{1}{t}\left((1-\omega t^w)\,\mathcal{W}(t,0;\omega) - 1\right)$$

For example,

$\mathcal{W}(t,2;\omega) = \frac{(1-\omega t^w)}{t}\mathcal{W}(t,1;\omega) - \mathcal{W}(t,0;\omega) = \frac{(1-\omega t^w)}{t}\frac{1}{t}\left((1-\omega t^w)\,\mathcal{W}(t,0;\omega)-1\right) - \mathcal{W}(t,0;\omega)$

$= \left(\frac{(1-\omega t^w)^2}{t^2} - 1\right)\mathcal{W}(t,0;\omega) - \frac{(1-\omega t^w)}{t^2}$, and $\mathcal{W}(t,0;\omega) = \mu_w(t;\omega)$ is given in (9).

$\mathcal{W}(t,3;\omega) = \frac{(1-\omega t^w)}{t}\mathcal{W}(t,2;\omega) - \mathcal{W}(t,1;\omega)$

$= \frac{(1-\omega t^w)}{t}\left(\left(\frac{(1-\omega t^w)^2}{t^2} - 1\right)\mathcal{W}(t,0;\omega) - \frac{(1-\omega t^w)}{t^2}\right) - \frac{1}{t}\left((1-\omega t^w)\,\mathcal{W}(t,0;\omega) - 1\right)$

$= \left(\frac{(1-\omega t^w)^2}{t^2} - 2\right)\left(\frac{(1-\omega t^w)}{t}\mu_w(t;\omega)\right) + \frac{1}{t} - \frac{(1-\omega t^w)^2}{t^3}$

We find an explicit expression for $\mathcal{W}(t,j;\omega)$ in the next section.

5.2 Solution to Recursion for $\mathcal{W}$ and $\mathcal{W}^{(k)}$

The linear recursion (10) is called Fibonacci-like. It is of the form

$$\sigma_n = u\sigma_{n-1} + v\sigma_{n-2}$$

with $u = \frac{1-\omega t^w}{t}$ and $v = -1$, for $n > 1$. We know the inital values σ_0 and $\sigma_1 = u\sigma_0 - 1/t$.

Hence $\sigma_n = [\tau^n] \frac{\sigma_0 + (\sigma_1 - u\sigma_0)\tau}{1-u\tau-v\tau^2} = [\tau^n] \frac{\sigma_0 - \tau/t}{1-u\tau+\tau^2}$ in this case, or $\sigma_n =$

$$[\tau^n] \left(\sigma_0 - \frac{\tau}{t}\right) \sum_{i=0}^{\infty} \binom{i}{j} (-1)^j \, u^{i-j} \tau^{i-j+2j} \tag{11}$$

$$= \sigma_0 \sum_{j=0}^{n} \tbinom{n-j}{j} (-1)^j \left(\frac{1-\omega t^w}{t}\right)^{n-2j} - \frac{1}{t} \sum_{j=0}^{n-1} \tbinom{n-1-j}{j} (-1)^j \left(\frac{1-\omega t^w}{t}\right)^{n-1-2j}$$

Let us define

$$p_n(t) := \sum_{j=0}^{n} \tbinom{n-j}{j} \left(\frac{1-\omega t^w}{t}\right)^{n-2j} (-1)^j \tag{12}$$

Hence

$$\mathcal{W}(t,j;\omega) = \left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2 - 4t^2}}{2t^2}\right) p_j(t) - p_{j-1}(t)/t \tag{13}$$

where $p_j = 0$ for all $j < 0$.

The generating function $\mathcal{W}^{(k)}(t,j;\omega) = \sum_{n\geq 0}^{(k)} W^{(k)}(n,j;\omega)\, t^n$ is generating the case where the lattice paths stay strictly below $y = k$; the numbers $W^{(k)}(n,j;\omega)$ are the number of paths with ω-weighted horizontal steps of length w, and diagonal up and down steps, that do not reach the line $y = k$, and stay above the x-axis. That means, $0 \leq j < k$. We also know $\mathcal{W}^{(k)}(t,0;\omega)$

$$= \sum_{n\geq 0} W_n^{(k)} t^n = \mu_w(t;\omega) \frac{1 - (t\mu_w(t;\omega))^{2k}}{1 - (t\mu_w(t;\omega))^{2(k+1)}}$$

$$= \frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2 - 4t^2}}{2t^2} \frac{1 - \left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2 - 4t^2}}{2t}\right)^{2k}}{1 - \left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2 - 4t^2}}{2t}\right)^{2(k+1)}}$$

The recursion is the same as for $\mathcal{W}(t,j;\omega)$. Only the initial values have changed (see $\mathcal{W}^{(k)}(t,0;\omega)$ above).

We get

$$\mathcal{W}^{(k)}(t,j;\omega) = \left(\mu_w(t;\omega) \frac{1 - (t\mu_w(t;\omega))^{2k}}{1 - (t\mu_w(t;\omega))^{2(k+1)}}\right) p_j(t) - p_{j-1}(t)$$

and $\sum_{n\geq 0} W_n^{(k)} t^n$

$$= \frac{1}{2t^2} \frac{\left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2-4t^2}}{2t}\right)^{-k} - \left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2-4t^2}}{2t}\right)^{k}}{\left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2-4t^2}}{2t}\right)^{-(k+1)} - \left(\frac{1-\omega t^w - \sqrt{(1-\omega t^w)^2-4t^2}}{2t}\right)^{(k+1)}}$$

$$= 2\frac{\sum_{i=0}^{(k-1)/2} (-1)^i 2^{2i} t^{2i} (1-\omega t^w)^{k-2i} \sum_{j=0}^{\infty} \binom{j+i}{i}\binom{k}{2j+2i+1}}{\sum_{i=0}^{(k+1)/2} (-1)^i 2^{2i} t^{2i} (1-\omega t^w)^{k+1-2i} \sum_{j=0}^{\infty} \binom{j+i}{i}\binom{k+1}{2j+1+2i}}$$

$$= \frac{\sum_{i=0}^{k/2} (-1)^i t^{2i} (1-\omega t^w)^{k-2i} \binom{k-i-1}{i}}{\sum_{i=0}^{(k+1)/2} (-1)^i t^{2i} (1-\omega t^w)^{k+1-2i} \binom{k-i}{i}} = \frac{p_{k-1}(t)}{t p_k(t)} \tag{14}$$

where $p_n(t)$ is given in (12).

6 Schröder numbers

If $w = 2$, then every horizontal steps gains two units. We denote the number of paths to (n, j) by $S(n, j; \omega)$, the general weighted Schröder numbers.

$j\uparrow$					1	0	$5+5\omega$
3				1	0	$4+4\omega$	0
2			1	0	$3+3\omega$	0	$9+15\omega+6\omega^2$
1		1	0	$2+2\omega$	0	$5+8\omega+3\omega^2$	0
0	1	0	$1+\omega$	0	$2+3\omega+\omega^2$	0	$5+10\omega+6\omega^2+\omega^3$
$n\rightarrow$	0	1	2	3	4	5	6

The general weighted Schröder numbers $S(n, j; \omega)$. The numbers $S_{n;\omega}$ are in row 0.

The following matrix contains the "*compressed*" Schröder numbers by removing the zeroes and shifting all entries into the empty places. This is the same effect as replacing t^2 in $\sum_{n=0}^{\infty} S(n, j; \omega) t^j$ by t.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 22 & 16 & 6 & 1 & 0 \\ 90 & 68 & 30 & 8 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -2 & 8 & -6 & 1 & 0 \\ 2 & -12 & 18 & -8 & 1 \end{pmatrix}$$

compressed Schröder numbers ($\omega = 1$) inverse compressed Schröder numbers

The power series $\sum_{n=0}^{\infty} S(n, j; \omega) t^j$ is given in (9). For the compressed Schröder numbers this equation says

$$\mathcal{S}(t, j; \omega) = \sum_{n=0}^{\infty} S(n, j; \omega) t^j = \left(\frac{1-\omega t - \sqrt{(1-\omega t)^2 - 4t}}{2t}\right) p_j(t) - p_{j-1}(t)/t$$

where

$$p_n(t) = t^{-n} \sum_{j=0}^{n} \binom{n-j}{j} t^j (1-\omega t)^{n-2j} (-1)^j \tag{15}$$

All references to Schröder numbers will from now on mean the compressed Schröder numbers. Note that

$$\mathcal{S}^{(k)}(t;\omega) = \sum_{n\geq 0} S_n^{(k)} t^n = \frac{p_{k-1}(t)}{t p_k(t)} \tag{16}$$

by (14).

6.1 Inverse Schröder numbers

From (9) we see that

$$\mu_s(t) = 1 + \omega t^2 \mu_s(t) + t^2 \mu_s(t)^2 .$$

Hence

$$\begin{aligned} \phi(t\mu_s(t)) &= \mu_s(t) = 1 + \omega t^2 \mu_s(t) + t^2 \mu_s(t)^2 \\ \phi(t) &= 1 + \frac{\omega t^2}{\phi(t)} + t^2 \end{aligned}$$

thus $\phi(t) = \frac{1}{2} + \frac{1}{2}t^2 + \frac{1}{2}\sqrt{(1+t^2)^2 + 4t^2\omega}$, a power series in t^2. We let $\xi = t^2$ and get

$$\begin{aligned} \phi(\xi) &= \frac{1}{2} + \frac{1}{2}\xi + \frac{1}{2}\sqrt{(1+\xi)^2 + 4\xi\omega} \\ \mu_s(\xi) &= 1 + \omega\xi\mu_s(t) + \xi\mu_s(\xi)^2 \\ &= \frac{1 - \omega\xi - \sqrt{(1-\omega\xi)^2 - 4\xi}}{2\xi} \end{aligned}$$

Lagrange inversion tells us that for all $0 \leq i \leq k$ holds

$$(i+1)\left[\mu_s^{-k-1}\right]_{k-i} = (k+1)\left[\phi^{-i-1}\right]_{k-i} = (k+1)\, s_{k,i}$$

and therefore

$$s_{k,j} = \left[\mu_s^{-k-1}\right]_{k-j} = \frac{j+1}{k+1}\left[t^{k-j}\right]\left(\frac{1-\omega t-\sqrt{(1-\omega t)^2-4t}}{2t}\right)^{-k-1}$$

$$= \left[t^{k-j}\right]\frac{j+1}{k+1}\left(\frac{2t\left(1-\omega t+\sqrt{(1-\omega t)^2-4t}\right)}{4t}\right)^{k+1}$$

$$= \left[t^{k-j}\right]\frac{j+1}{k+1}\left(\frac{1}{2}(1-\omega t)\left(1+\sqrt{1-\frac{4t}{(1-\omega t)^2}}\right)\right)^{k+1}$$

$$= \frac{j+1}{k+1}(-1)^{k-j}\sum_{m=0}^{k-j}2^{2m-k-1}\binom{k+1-2m}{k-j-m}\omega^{k-j-m}\sum_{l=0}^{k+1}\binom{k+1}{l}\binom{l/2}{m}$$

Now $\sum_{l=0}^{k+1}\binom{k+1}{l}\binom{\frac{1}{2}l}{m} = \frac{k+1}{k-2m+1}\binom{k-m}{m}2^{k+1-2m}$ for $0 \le m$ (see Gould [5, (3.163)], who attributes the formula to Carlitz). Therefore

$$s_{k,j} = (-1)^{k-j}\sum_{m=0}^{k-j}\binom{k+1-2m}{k-j-m}\frac{j+1}{k-m+1}\binom{k-m+1}{m}\omega^{k-j-m},$$

the compressed weighted *inverse Schröder numbers*. We need the following polynomials: $\sum_{k\ge 0}s_{n,k}t^{n-k}$

$$= \sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{k+1}{n-m+1}\binom{n-m+1}{m}\binom{n+1-2m}{n-k-m}t^{n-k}(-1)^{n-k}\omega^{n-k-m}$$

$$= t^n\sum_{m=0}^{n}\frac{1}{n-m+1}\binom{n-m+1}{m}\omega^{-m}\sum_{k=0}^{\infty}(k+1)\binom{n+1-2m}{n-k-m}(-\omega)^{n-k}t^{-k}$$

$$= \sum_{m=0}^{n}\frac{1}{n-m+1}\binom{n-m+1}{m}(-1)^{m+1}(1-\omega t)^{n-2m}t^m(tm\omega+m-n-1)$$

Hence

$$s_n(t) = \sum_{k\ge 0}s_{n,k}t^{n-k} \tag{17}$$

$$= \sum_{m=0}^{n}\left(\frac{t\omega m}{n-m+1}-1\right)\binom{n-m+1}{m}(-1)^{m+1}(1-\omega t)^{n-2m}t^m$$

1	0	0	0	0
−2	1	0	0	0
2	−4	1	0	0
−2	8	−6	1	0
2	−12	18	−8	1

The compressed inverse $(s_{n,k})$ for $\omega = 1$

This matrix is A080246 in the OEIS. At the same reference we find the generating function of the k-th column,

$$\sum_{n\geq k} s_{n,k} t^n = \left(\frac{1-t}{1+t}\right)^k .$$

Also,

$$\sum_{n\geq 0}\sum_{k=0}^{n} s_{n,k} t^{n-k} = \sum_{k=0}^{\infty} t^{-k}\left(\frac{1-t}{1+t}\right)^k = \frac{t\,(1+t)}{2t+t^2-1}.$$

Example: $\sum_{k=0}^{n} s_{4,k} t^{4-k} = s_4(t) = 1 - 8t + 18t^2 - 12t^3 + 2t^4$.

6.2 Delannoy numbers

The numbers $D(n,k) = \sum_{l=0}^{n}\binom{k}{l}\binom{n+k-l}{k}\omega^l$ are the Delannoy numbers; the numbers $D(n, n+j)$ are counting all weighted Grand Schröder paths to $(2n+j, j)$. Hence they satisfy the recursion

$$D(n,n+j) = \omega D(n-1,n-1+j) + D(n,n+j-1) + D(n-1,n+j) \quad (18)$$

$j\uparrow$							1	0
5						1	0	$7+6\omega$
4					1	0	$6+5\omega$	0
3				1	0	$5+4\omega$	0	$21+30\omega+10\omega^2$
2			1	0	$4+3\omega$	0	$15+20\omega+6\omega^2$	0
1		1	0	$3+2\omega$	0	$10+12\omega+3\omega^2$	0	129
0	1	0	$2+\omega$	0	$6+6\omega+\omega^2$	0	63	0
$n\rightarrow$	0	1	2	3	4	5	6	7

Uncompressed Grand Schröder numbers ($\omega = 1$)

The generating function

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{k}{l}\binom{n+k-l}{n-l}\omega^l t^n = \frac{1}{1-t}\left(\frac{1+t\omega}{1-t}\right)^k$$

shows that $D(n,k)$ is a Sheffer polynomial of degree n in k. The *Delannoy polynomial* is of the form

$$\begin{aligned} d_k(t) &= \sum_{j=0}^{k} D(k-j,j)\,t^j = \sum_{j=0}^{k}\sum_{l=0}^{k-j}\binom{j}{l}\binom{k-l}{j}\omega^l t^j \\ &= \sum_{l=0}^{k}\binom{k-l}{l}\omega^l t^l \sum_{j=0}^{k-2l}\binom{k-2l}{j} t^j = \sum_{l=0}^{k}\binom{k-l}{l}\omega^l t^l\,(1+t)^{k-2l} \end{aligned}$$

and has generating function

$$\sum_{k=0}^{\infty} d_k\left(t\right) x^k = \frac{1}{1-x-t\left(x+\omega x^2\right)}.$$

From (18) follows for $\omega = 1$ that

$$d_{k-1}\left(t\right) = td_{k-1}\left(t\right) + td_{k-2}\left(t\right) + d_k\left(t\right)$$

Also for $\omega = 1$ holds

$$p_k\left(t\right) = t^{-k} \sum_{l=0}^{k} \tbinom{k-l}{l} t^l \left(1-t\right)^{k-2l} \left(-1\right)^l = t^{-k} d_k\left(-t\right)$$

(see (15)). Hence

$$\mathcal{S}^{(k)}\left(t;1\right) = \sum_{n\geq 0} S_{n;1}^{(k)} t^n = \frac{p_{k-1}\left(t\right)}{tp_k\left(t\right)} = \frac{d_{k-1}\left(-t\right)}{d_k\left(-t\right)} \tag{19}$$

by (16). This shows an intimate connection between the generating function of the Schröder numbers in a band and the Delannoy polynomials, when $\omega = 1$. The Delannoy polynomials at negative argument, $d_k\left(-t\right)$, satisfy for $\omega = 1$ the same recursion as $d_k\left(t\right)$,

$$d_{k-1}\left(-t\right) = td_{k-1}\left(-t\right) + td_{k-2}\left(-t\right) + d_k\left(-t\right).$$

This follows again from (18).

Another connection exists with the inverse polynomial $s_n\left(t\right)$; from (17) $s_n\left(t\right) = \sum_{k\geq 0} s_{n,k} t^{n-k} =$

$$t\omega \sum_{m=1}^{n} \binom{n-m}{m-1} \left(-1\right)^{m+1} \left(1-\omega t\right)^{n-2m} t^m - \sum_{m=0}^{n} \binom{n-m+1}{m} \left(-1\right)^{m+1} \left(1-\omega t\right)^{n-2m} t^m$$

follows for $\omega = 1$

$$s_n\left(t\right) = \frac{t^2}{1-t} d_{n-1}\left(-t\right) + d_{n+1}\left(-t\right) / \left(1-t\right). \tag{20}$$

and vice-versa,

$$\begin{aligned} d_{n+1}\left(-t\right) &= \left(1-t\right) s_n\left(t\right) - t^2 d_{n-1}\left(-t\right) \\ &= \left(1-t\right) \sum_{i=0}^{n/2} t^{2i} \left(-1\right)^i s_{n-2i}\left(t\right) + \left(n \bmod 2\right) \left(-1\right)^{(n+1)/2} t^{n+1} \end{aligned}$$

Hence the generating function of the bounded Schröder numbers can for $\omega = 1$ be written as

$$\mathcal{S}^{(k)}\left(t;1\right) = \sum_{n\geq 0} S_{n;1}^{(k)} t^n = \frac{\left(1-t\right) \sum_{i=0}^{(k-2)/2} t^{2i} \left(-1\right)^i s_{k-2-2i}\left(t\right) + \left(k \bmod 2\right) \left(-1\right)^{(k-1)/2} t^{k-1}}{\left(1-t\right) \sum_{i=0}^{(k-1)/2} t^{2i} \left(-1\right)^i s_{k-1-2i}\left(t\right) + \left(\left(k-1\right) \bmod 2\right) \left(-1\right)^{k/2} t^k}$$

7 Schröder in a Band

From (19) follows for $\omega = 1$ the generating function of the (compressed) bounded (by k) Schröder numbers,

$$\mathcal{S}^{(k)}(t;1) = \frac{d_{k-1}(-t)}{d_k(-t)} \tag{21}$$

Example: $\mathcal{S}^{(4)}(t;1) = \frac{d_3(-t)}{d_4(-t)} = \frac{1-5t+5t^2-t^3}{1-7t+13t^2-7t^3+t^4}$
$= 1 + 2t + 6t^2 + 22t^3 + 89t^4 + 377t^5 + 1630t^6 + 7110t^7 + 31\,130t^8 + 136\,513t^9 + 599\,041t^{10} + 2629\,418t^{11} + 11\,542\,854t^{12} + 50\,674\,318t^{13} + 222\,470\,009t^{14} + 976\,694\,489t^{15} + 4287\,928\,678t^{16} + O\left(t^{17}\right)$

$\uparrow m$									
k=4									
3				1	7	36	168	756	3353
2			1	6	29	132	588	2597	11 430
1		1	4	16	67	288	1253	5480	24 020
0	1	2	6	22	89	377	1630	7110	31130
	0	1	2	3	4	5	6	7	$\to n$

The compressed bounded ($k = 4$) Schröder numbers ($\omega = 1$)

From the recursion (18) follows that

$$d_{k-1}(-t) = td_{k-1}(-t) + td_{k-2}(-t) + d_k(-t) \text{ and therefor}$$
$$d_{k-1}(-t) = \frac{t}{1-t}d_{k-2}(-t) + \frac{1}{1-t}d_k(-t)$$

Hence

$$d_{k-1}(-t) - td_{k-2}(-t) = \frac{t^2}{1-t}d_{k-2}(-t) + \frac{1}{1-t}d_k(-t) = s_{k-1}(t)$$

(see (20)) and

$$\frac{d_{k-1}(-t)}{d_k(-t)}\left(d_{k-1}(-t) - td_{k-2}(-t)\right) - \frac{d_{k-1}(-t)}{d_k(-t)}s_{k-1}(t) = 0$$

Therefore

$$\frac{d_{k-1}(-t)}{d_k(-t)}s_{k-1}(t) = \mathcal{S}^{(k)}(t, k-1; 1) - \left(d_{k-2}(-t) - td_{k-3}(-t)\right)$$

(see (21).

Theorem 3 *The power series part of $t^{-k}\mathcal{S}^{(k)}(t;1)\,s_{k-1}(t)$ equals $t^{-k}S^{(k)}(t, k-1; 1)$.*

Example: (a) $t^{-4}\mathcal{S}^{(4)}(t;1)\,s_3(t) = \frac{(t-1)\left(2t^3-8t^2+6t-1\right)\left(1-4t+t^2\right)}{(1-7t+13t^2-7t^3+t^4)t^4}$
$= (t^{-4} - 4t^{-3} + 2t^{-2}) + 1 + 7t + 36t^2 + 168t^3 + 756t^4 + 3353t^5 + 14\,783t^6 +$

$65\,016t^7 + 285\,648t^8 + 1254\,456t^9$
$+5508\,097t^{10} + 24\,183\,271t^{11} + 106\,173\,180t^{12} + O\left(t^{13}\right)$

(b) $t^{-4}\mathcal{S}^{(4)}\left(t, 4-1; 1\right) = t^{-4}\frac{1-5t+5t^2-t^3}{1-7t+13t^2-7t^3+t^4}\left(1-6t+8t^2-2t^3\right) - \frac{2t^2+1-4t}{t^4} =$

$1+7t+36t^2+168t^3+756t^4+3353t^5+14\,783t^6+65\,016t^7+285\,648t^8+1254\,456t^9$
$+5508\,097t^{10} + 24\,183\,271t^{11} + 106\,173\,180t^{12} + O\left(t^{13}\right)$

References

[1] Bernhart, F. R. (1999) Catalan, Motzkin, and Riordan numbers. *Discrete Math.* **204**, 73 – 112.

[2] Cameron, N. T. and Yip, A. C. (2010) Hankel Determinants of Sums of Consecutive Motzkin Numbers. *Linear Algebra and its Appl.* **434**, 712–722 .

[3] J. Cigler and C. Krattenrhaler (2011) Some determinants of path generating functions, *Adv. Appl. Math.* **46**, 144-174

[4] Flajolet, P. (1980), Combinatorial aspects of continued fractions. *Discrete Math.* **32** , 125 –161.

[5] Gould, H.W. (1972) *Combinatorial Identities*, Morgantown, W. Va.

[6] Merlini, D.,Rogers, D. G., Sprugnoli, R., and Verri, M.C. (1997). On some alternative characterizations of Riordan arrays. *Canadian J. Math.* **49**, 301-320.

[7] Motzkin, T. (1948) Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. *Bull. Amer. Math. Soc.* **54**, Number 4 , 352-360.

[8] Ralston, A. and Rabinowitz, P. (1978), *A First Course in Numerical Analysis,* Mcgraw-Hill, ISBN 0070511586

[9] Rogers, D.G. (1978). Pascal triangles, Catalan numbers and renewal arrays, *Discrete Math.* **22**, 301-310.

[10] Schröder, E. (1870) Vier kombinatorische Probleme. *Z. Math. Phys.* **15**, 361 – 376.