FUN WITH VERY LARGE NUMBERS

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Dedicated to Professor Jon Borwein on the occasion of his sixtieth birthday.

Abstract. We give an example of a formula involving the sinc function that holds for every $N = 0, 1, 2, \ldots$, up to about $10^{102832732165}$, then fails for all larger N. We give another example that begins to fail after about $N \simeq \exp(\exp(\exp(\exp(\exp(\exp(e))))))$. This number is larger than the Skewes numbers.

Contents

1. Introduction

In a 1992 paper [6], Jon and Peter Borwein give examples of formulas that are "almost" true: that is, they are correct to anywhere from thousands to over 42 billion decimal places, but are not actually true. Here, we'll do something similar. We'll give examples of a formula involving the sinc function that holds for a ridiculously large number of values of $N = 0, 1, 2, \ldots$ before it begins to fail.

The sinc function is defined as: $\operatorname{sinc}(0) = 1$, and $\operatorname{sinc}(x) = \sin(x)/x$ if $x \neq 0$.

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Thanks to Professor Alessandro Languasco for his highly accurate computations of $M(100, 1)$.

The sinc function has many interesting properties. For example, there are the curious identities

$$\sum_{n=1}^{\infty} \operatorname{sinc}(n) = \sum_{n=1}^{\infty} \operatorname{sinc}(n)^2$$

and

$$\int_0^{\infty} \operatorname{sinc}(x)\,dx = \int_0^{\infty} \operatorname{sinc}(x)^2\,dx$$

Both sums equal $\pi/2 - 1/2$; both integrals equal $\pi/2$; see [2].

David and Jon Borwein and Bernard Mares [5] showed how to evaluate certain integrals involving products of sinc functions. A more recent paper [2] shows a connection between sums and integrals of products of sinc functions. That paper explains the curious identities above, and further explains why the integrals are 1/2 more than the sums.

Let $N > 0$ and a_0, a_1, $a_2, \ldots$ be positive numbers. Theorem 1 of [2] states, in part, that

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k n) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}(a_k x)\,dx \tag{1.1}$$

provided that

$$\sum_{k=0}^{N} a_k \le 2\pi. \tag{1.2}$$

If $N = 0$, the condition required for equality in (1.1) is simply that $a_0 < 2\pi$.

If a_k is a slowly divergent series, then the sum in (1.2) will exceed 2π only when N exceeds some very large number N_0. Therefore, equation (1.1) will be true for the many cases $N = 0, 1, 2, \ldots N_0$, and will fail for all $N > N_0$. We will show how to construct interesting examples with arbitrarily large N_0. We do this by finding infinite series that diverge very, very slowly.

For example, when the a_k are the reciprocals of primes of the form $10n + 9$, then we can estimate that $N_0 \simeq 10^{102832732165}$.

We could make estimates in a similar way if the sum in (1.2) is a convergent series whose sum exceeds 2π, but we will not consider that case here. Our results follow from standard theorems in number theory and numerical analysis.

We should also make clear that, even if we are able to get only a rough estimate for the value of N_0 such that equation (1.1) fails for $N > N_0$, this cutoff *is*, nevertheless, well-defined.

2. An Example With Odd Denominators

Example 1 (a) in [2] illustrates Theorem 1 of that paper using the sequence $a_k = 1/(2k + 1)$ for $k \ge 0$. In this case, one can calculate that

$$\sum_{k=0}^{N} a_k$$

does not exceed 2π until $N \ge 40249$. Therefore, equation (1.1) holds for every $N = 0, 1, 2, \ldots$, 40248, but fails for $N \ge 40249$. Interestingly, for this example, Crandall has shown [8, p. 24] that, at $N = 40249$, the left side minus the right side of (1) is positive but less than $10^{-226576}$. So, even if one did use a computer to check whether

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{2k+1}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{2k+1}\right) dx$$

all the way up to $N = 40249$, the left and right sides would appear to be the same at $N = 40249$ unless one had the foresight to perform the calculation to over 225,000 decimal places!

We all know that it is dangerous to rely on a formula merely because it is true for a few test cases, or a few hundred, or even a few thousand. However, everyone has done this at one time or another. This example shows how truly dangerous that practice can be!

3. More Examples With Denominators in Arithmetic Progressions

There's nothing special about the "2" in the denominators $2k + 1$ in the example above. What happens with $a_k = 1/(mk + 1)$ for other values of m?

As before, (1.1) will hold as long as (1.2) holds. For $m = 1, 2, 3$, and 4, one can add up terms of the series until the sum exceeds 2π. However, as m increases, the number of terms required to make the partial sum exceed 2π increases quite rapidly. For larger values of m, we can use the Euler-MacLaurin summation formula to accurately estimate the sum. This enables us to find the exact number of terms required to make the partial sum exceed 2π. The following version of the Euler-MacLaurin summation formula is based on taking $a = 0$, $b = n$ in [7, p. 309]:

$$\sum_{k=0}^{M} f(k) = \int_0^M f(x)\,dx + \frac{1}{2}(f(0) + f(M)) \tag{3.1}$$
$$+ \sum_{j=1}^{J} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(M) - f^{(2j-1)}(0)\right) + R.$$

B_k is the k^{th} Bernoulli number. The remainder R is bounded by

$$|R| \le |\frac{B_{2J+2}}{(2J+2)!} \cdot M \cdot f^{(2J+2)}(x_0)|,$$

where x_0 is some number between 0 and M, inclusive. Note that this gives us an approximation to the sum of the first $M + 1$ terms (not M terms) of the series.

A naive approach would be to apply Euler-Maclaurin summation to the function $f(x) = 1/(mx + 1)$. Then the derivative of order $(2J + 2)$ of $f(x)$ will be of the form $C/(mx + 1)^{2J+3}$. The error term R achieves its maximum value over $[0, M]$ at $x = 0$. This maximum value is not necessarily small. Often, it is too large for the formula to be useful.

For example, when we apply the Euler-Maclaurin formula with $J = 1$ to $f(x) = 1/(2x + 1)$, we get

$$\sum_{k=0}^{M} f(k) \simeq \frac{2}{3} + \frac{1}{2(2M+1)} - \frac{1}{6(2M+1)^2} + \frac{1}{2}\log(2M+1)$$

with an error term of

$$|R| \le \frac{8M}{15(2x+1)^5}.$$

However, watch what happens if we separately compute the sum of the first 100 terms of the series

$$s = \sum_{k=0}^{99} \frac{1}{2k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots \frac{1}{199} \simeq 3.28434218930163434565,$$

and then apply the Euler-Maclaurin formula to the "tail", $f(x) = 1/(2x+201)$. We get

$$\sum_{k=0}^{M} \frac{1}{2k+201} \simeq$$
$$\frac{302}{121203} + \frac{1}{2(2M+201)} - \frac{1}{6(2M+201)^2} + \frac{1}{2}\log(2M+201) - \frac{\log(201)}{2}.$$

The error term is now

$$|R| \le \frac{8M}{15(2x+201)^5}. \tag{3.2}$$

For a given M, this error term is much smaller at $x=0$ than was the previous expression for R.

So, in order to estimate the (largest) number of terms for which

$$\sum_{k \ge 0} \frac{1}{2k+1} < 2\pi,$$

we compute the value of M such that

$$\frac{302}{121203} + \frac{1}{2(2M+201)} - \frac{1}{6(2M+201)^2} + \frac{1}{2}\log(2M+201) - \frac{\log(201)}{2}$$

equals $2\pi - s \simeq 2.99884311787795213128$. The solution of this equation is $M \simeq 40148.81104$. But to be sure that $M = 40148$ is really the value we want, we must verify that:

- (1) using $M = 40148$, the estimated sum $< 2\pi$
- (2) the estimated sum plus the next term $> 2\pi$
- (3) the error term is less than 1/(last term)
- (4) the error term is less than 2π - (estimated sum)

Evaluated at $M = 40148$ and $x = 0$, error term (3.2) is about $6.52653 \cdot 10^{-8}$. This is less than any term in the series near this value of M. Moreover, the sum of the initial terms plus the estimated value of the next $M+1$ terms, differs from 2π by about $1.008 \cdot 10^{-5}$.

Therefore, it is safe to conclude that

$$\sum_{k=0}^{99} \frac{1}{2k+1} + \sum_{k=0}^{40148} \frac{1}{2k+201} = \sum_{k=0}^{40248} \frac{1}{2k+1} < 2\pi$$

and

$$\sum_{k=0}^{99} \frac{1}{2k+1} + \sum_{k=0}^{40149} \frac{1}{2k+201} = \sum_{k=0}^{40249} \frac{1}{2k+1} > 2\pi.$$

Therefore, the largest value of M such that

$$\sum_{k=0}^{M} \frac{1}{2k+1} < 2\pi$$

is $M = 40248$.

For $m = 1$ through $m = 20$, we compute the largest value of M such that

$$\sum_{k=0}^{M} \frac{1}{mk+1} < 2\pi.$$

We use the Euler-Maclaurin formula as in the previous example. For larger m, we need higher-order approximations in order to compute M with sufficient accuracy and to be assured that the error term is small enough. For each m, Table 1 shows the value of M, along with K, the number of initial terms, and the value of J that was used in the Euler-Maclaurin formula. With a sufficient number of initial terms and a modest value of J, we can keep the error small.

In all cases, the "tail" function $f(x)$, to which we apply the Euler-Maclaurin formula, is given by $f(x) = 1/(mx + Km + 1)$.

(We do not claim that the choice of either K or J is minimal or, in any sense, optimal. We could use fewer initial terms, but we would then need to increase J in order to keep the error term small enough. Making J larger increases the complexity of the equation involving M.)

As a check on this procedure, when $m = 1$, this technique successfully finds the 44-digit value of M obtained in [4] for which

$$\sum_{k=0}^{M} \frac{1}{k+1}$$

exceeds 100. See also [12]. This value can be calculated, for example, using $K = 10000$ and $J = 10$. We can also verify the first four values of M in the table by direct summation.

As a further check, if we double the number of initial terms and increase J by 1 or 2, then we get the same values of M shown in Table 1.

We conclude that, for each $m = 1, 2, \ldots, 10$, the equation

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{mk+1}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{mk+1}\right) dx$$

holds precisely for those $N = 0, 1, 2, \ldots M$, where M is the corresponding value in Table 1, and that it fails for larger N. Notice that, for $m = 20$, M exceeds 10^{45}. So, if we were testing "random" values of N, looking for a value that made (1.1) fail, we would be unlikely to find one.

The exact value of M corresponding to $m = 100$ can be obtained from the Euler-Maclaurin formula by taking $K = 50000$ and $J = 60$. The result is the 230-digit value of M shown below.

m	M	K	J
1	299	100	1
2	40248	100	1
3	6699356	1000	1
4	1199640415	1000	2
5	222209538173	1000	3
6	41928392459412	1000	3
7	8000791810720605	1000	4
8	1537961933686185453	1000	5
9	297136851932007766218	1000	6
10	57616626381701142703593	1000	7
11	11202463675353183172586907	1000	8
12	2182608774487516995740392959	1000	8
13	425930131275278060684396950683	1000	10
14	83225800344072649528263059652618	1000	10
15	16279019516889202909861702224716180	1000	11
16	3186898150182578894413446451622161442	1000	12
17	624331345650550634164994069452043597341	1000	13
18	122382651928233262387099042295616064808177	1000	14
19	24001760343280992647777613927571508451532532	1000	15
20	4709265577657827035628502288018792360631413283	1000	16

TABLE 1. The largest M for which $\sum_{k=0}^{M} \frac{1}{mk+1} < 2\pi$

$$
\begin{aligned}
M = & 1593063615376465609354995196169671319743497502894 0\\
& 858771929987635671621010359833817195983769138829 72\\
& 952853521684375899676769472229157697142575211889 27\\
& 151116548003599042566741587106668007049302125094 673\\
& 66576980776507184175875 5530945.
\end{aligned}
\tag{3.3}
$$

Therefore, we know that, for $a_k = 1/(100k + 1)$, equation (1.1) holds for $N \le M$, and fails for $N > M$.

4. An Example With Primes

Example (2) in [2] is even more striking. Here we pick a_k to be the reciprocals of the primes, with $a_0 = 1/2$, $a_1 = 1/3$, etc. The sum of the reciprocals of the primes

$$\sum_{k \ge 0} a_k = \sum_{k \ge 0} \frac{1}{p_k}$$

diverges. Then (1.1) becomes

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{n}{p_k}\right) = \int_0^{\infty} \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{x}{p_k}\right) \, dx \tag{4.1}$$

which will hold only as long as

$$\sum_{k=0}^{N} \frac{1}{p_k} < 2\pi.$$

So, how long *is* that? From analytic number theory (see, e.g, [9, p. 35] or [1, p. 156] [AP], we know that

$$\sum_{p \le x} \frac{1}{p} \simeq \log\log x + B + o(1),$$

where $B \simeq .26149\ldots$ is Mertens' constant. For computational purposes, we will simply drop the $o(1)$ term. In order to make this sum reach 2π, we must have

$$x \simeq \exp(\exp(2\pi - B)), \tag{4.2}$$

so that $x \simeq 10^{179}$. Then, the N where (4.1) ceases to hold is the number of primes up to that x, which, by the Prime Number Theorem [9, p. 10], is roughly $x/\log x \simeq 10^{176}$.

Without assuming the Riemann hypothesis, Crandall [8, Theorem 11] has shown that, at the first N for which (4.1) fails, the left side minus the right side is positive but less than $10^{-(10^{165})}$. Now, if both sides of (4.1) were on the order of $10^{-(10^{165})}$, then this small difference would not be very interesting. However, neither side of (4.1) is tiny. Crandall calculates [8, Theorem 4] that the right side exceeds $.686\pi$ for all N.

5. Some Examples With Primes in Arithmetic Progression

In the previous section, we saw that equation (4.1) holds "only" up to $N = N_0 \simeq 10^{176}$, and then fails for larger N. In this section, we will give examples where the corresponding N_0 is much larger. To do this, we will use a subset of the primes, the sum of whose reciprocals still diverges, but much more slowly than the sum of the reciprocals of all the primes. Specifically, we will use the set of primes in an arithmetic progression.

For positive integers r and s, let (r, s) be the greatest common divisor of r and s. Also, $\phi(m)$ denotes Euler's phi function, which is the number of positive integers $k \le m$ such that $(k, m) = 1$.

Dirichlet's theorem states that, if $(a, q) = 1$, then there are infinitely many primes in the arithmetic progression $qn + a$, where $n = 0, 1, 2, \ldots$. The number of primes $\le x$ in such an arithmetic progression is asymptotic to

$$\frac{x}{\phi(q) \log\log x}. \tag{5.1}$$

We also know that [1, p. 156], if $x > 2$,

$$\sum_{\substack{p \le x \\ p \equiv a \bmod q}} \frac{1}{p} = \frac{\log\log x}{\phi(q)} + M(q,a) + O\left(\frac{1}{\log x}\right), \tag{5.2}$$

where $M(q,a)$ is a constant that depends on q and a. The term inside the $O(...)$ will be small if x is large. However, the constant factor implied by the $O(...)$ is of unknown size, so we will not establish rigorous bounds in what follows. Our goal is only to obtain rough estimates, and we will assume that the $O(...)$ term is small enough to be neglected.

Given q and a with $(a,q) = 1$, if the sum up to x equals approximately 2π, then equation (5.2) tells us that

$$\log\log x \simeq \phi(q) \cdot (2\pi - M(q,a)),$$

so that

$$x \simeq \exp(\exp(\,\phi(q) \cdot (2\pi - M(q,a))\,)). \tag{5.3}$$

Compare (5.3) with (4.2). Notice that if $q \ge 3$, then $\phi(q) \ge 2$, which will have the effect of multiplying the "top level" exponent by some integer which is at least 2. This means that (5.3) will produce much larger values of x (and N_0) than we got from (4.2). Let's look at several examples where $M(q,a)$ is accurately known, so that we can make reasonable estimates.

Example A. Consider the primes in the arithmetic progression $3n+1$, that is, those primes that are $\equiv 1 \pmod 3$. Our sequence a_k will be the reciprocals of these primes: $a_0 = 1/7$, $a_1 = 1/13$, $a_2 = 1/19$ $a_3 = 1/31$, and so on. The sum of the a_k diverges very slowly. Here, $q = 3$ and $a = 1$. Languasco and Zaccagnini [11] computed the values of many $M(q,a)$ to over 100 decimals. We will use their value of $M(3,1)$ rounded to 10 decimals: $M(3,1) \simeq -0.3568904795$. (Using more decimals gives the same final result). Then, since $\phi(q) = 2$,

$$\log\log x = \phi(q) \cdot (2\pi - M(q,a)) \simeq 2 \cdot (2\pi + 0.3568904795) \simeq 13.2801515734$$

so

$$x \simeq \exp(\exp(13.2801515734)) \simeq \exp(585459.08163) \simeq 4.45176353778 \cdot 10^{254261}.$$

We now use Equation (5.1) to estimate N, the number of primes $\equiv 1 \pmod 3$ that are $\le x$. We get

$$N \simeq \frac{x}{\phi(q)\log x} \simeq \frac{4.45176353778 \cdot 10^{254261}}{2 \cdot 585459.08163} \simeq 3.8 \cdot 10^{254255}.$$

So, for this example, where the a_k in equation (1.1) are the reciprocals of the primes $\equiv 1 \pmod 3$, equation (1.1) holds for $N \le N_0 \simeq 10^{254255}$ and fails for larger N.

This number is so much larger than the value we obtained in Section 4, primarily because of the factor of $\phi(q) = 2$ in equation (5.3). We were also helped by the fact that $M(3,1)$ is negative. We should also emphasize that we make no rigorous claim about this value of N because we ignored the $O(...)$ term in equation (5.2).

10^{254255} is large, but we can do better! We'll simply choose an arithmetic progression with a value of q that has a larger $\phi(q)$. For our next examples, we'll look at the four residue classes (mod 10), that is, the primes in the four arithmetic progressions $10n+1$, $10n+3$, $10n+7$, and $10n+9$.

Example B. Consider the primes in the arithmetic progression $10n+1$. Our sequence a_k will be the reciprocals of these primes: $a_0 = 1/11$, $a_1 = 1/31$, $a_2 = 1/41$, and so on. Here, $q = 10$ and $a = 1$. Also, $\phi(q) = 4$; observe what a drastic effect this will have on the calculations below, compared to Example A. Again, we will use the value of $M(10, 1)$ from [11], rounded to 16 decimals: $M(10, 1) \simeq -0.2088344774302376$.

$$\log \log x = \phi(q) \cdot (2\pi - M(q, a)) \simeq 4 \cdot (2\pi + 0.2088344774302376) \simeq 25.9680791384$$

so

$$x \simeq \exp(\exp(25.9680791384)) \simeq \exp(1.89580417544 \cdot 10^{11}) \simeq 2.64164832039 \cdot 10^{82333729216}.$$

We now use Equation (5.1) estimate N, the number of primes $\equiv 1 \pmod{10}$ that are $\leq x$. We get

$$N \simeq \frac{x}{\phi(q) \log x} \simeq \frac{2.64164832039 \cdot 10^{82333729216}}{4 \cdot 1.89580417544 \cdot 10^{11}} \simeq 3.48 \cdot 10^{82333729204}.$$

So, for this example, where the a_k in equation (1.1) are the reciprocals of the primes in the arithmetic progression $10n + 1$, equation (1.1) holds until somewhere around $N \simeq 10^{82333729204}$ and fails for larger N.

Example C. Now take $q = 10$, $a = 3$. $M(10, 3) \simeq 0.1386504057476469$ and $\phi(q) = 4$. Then

$$\log \log x = \phi(q) \cdot (2\pi - M(q, a)) \simeq 4 \cdot (2\pi + 0.1386504057476469) \simeq 24.5781396057$$

so

$$x \simeq \exp(\exp(24.5781396057)) \simeq \exp(4.72226555917 \cdot 10^{10}) \simeq 1.89595583512 \cdot 10^{20508538744}.$$

Then

$$N \simeq \frac{x}{\phi(q) \log x} \simeq \frac{1.89595583512 \cdot 10^{20508538744}}{4 \cdot 4.72226555917 \cdot 10^{10}} \simeq 1.0 \cdot 10^{20508538733}.$$

With the $10n + 1$ and $10n + 3$ sequences, we get estimates near $10^{82333729204}$ and $10^{20508538733}$, respectively. The calculations were similar. The difference in results arises from the differing value of $M(q, a)$.

Example D. With the arithmetic progression $10n + 7$, the first few a_k values are: $a_0 = 1/7$, $a_1 = 1/17$, $a_2 = 1/37$, and $a_3 = 1/47$. Rounded to 16 decimals, $M(10, 7) = -0.1039035249178728$. Carrying out calculations similar to those above, we get

$$x \simeq 1.38984773649 \cdot 10^{54112058088}$$

and

$$N_0 \simeq 2.8 \cdot 10^{54112058076}.$$

Again, the fact that $M(10, 7) < 0$ helped make the final result, N_0, somewhat larger than that for the $10n + 3$ progression.

Example E. With the arithmetic progression $10n+9$, the first few a_k values are: $a_0 = 1/19$, $a_1 = 1/29$, $a_2 = 1/59$, and $a_3 = 1/49$. Rounded to 16 decimals, $M(10, 9) = -0.2644151905518937$. Carrying out calculations similar to those above, we get

$$\log \log x = \phi(q) \cdot (2\pi - M(q, a)) \simeq 4 \cdot (2\pi + 0.2644151905518937) \simeq 26.1904019909$$

so

$$x \simeq \exp(\exp(26.1904019909)) \simeq \exp(2.36781116183 \cdot 10^{11}) \simeq 9.98876322671 \cdot 10^{102832732176}.$$

Then

$$N_0 \simeq \frac{x}{\phi(q)\log x} \simeq \frac{9.98876322671 \cdot 10^{102832732176}}{4 \cdot 2.36781116183 \cdot 10^{11}} \simeq 1.05 \cdot 10^{102832732165}.$$

So, for this example, where the a_k in equation (1.1) are the reciprocals of the primes in the arithmetic progression $10n + 9$, equation (1.1) holds for $N < N_0 \simeq 10^{102832732165}$ and fails for larger N.

Example F.

Let's try one final example that has an even larger value of $\phi(q)$: $q = 100$, for which $\phi(q) = 40$. We'll take $a = 1$. The first five primes in the sequence $100n + 1$ are 101, 401, 601, 701, and 1201. The first few values of a_k are $a_0 = 1/101$, $a_1 = 1/401$, $a_2 = 1/601$, and $a_3 = 1/701$. Languasco and Zaccagnini [11] computed $M(100, 1)$ to 104 decimals, but, as we shall see, we end up with numbers on the order of 10^{109}, so 104 decimals appears to be not quite enough. Professor Languasco has kindly calculated and provided the following value, accurate to 136 decimals:

$$\begin{aligned} M(100,1) \simeq -0.&03273285064331009648655913209300480721164389442305\\ &808121239698784116683056664327790581593738706166\\ &3246914938921935479658943506066648789 2. \end{aligned}$$

Using this more accurate value, and computing x and N_0 as above, we get:

$$\log\log x = \phi(q) \cdot (2\pi - M(q,a)) \simeq 40 \cdot (2\pi + 0.0327328506...) \simeq 252.6367263129...$$

so

$$x \simeq \exp(\exp(252.6367263129...)) \simeq \exp(5.2328244314... \cdot 10^{109}) \simeq 9.1592327310 \cdot 10^{22725...82928},$$

where the last exponent on the right has 110 digits. Then

$$N_0 \simeq \frac{x}{\phi(q)\log x} \simeq \frac{9.1592327310 \cdot 10^{22725...82928}}{40 \cdot 5.2328244314... \cdot 10^{109}} \simeq 4.4 \cdot 10^{22725...82817},$$

where, again, the last exponent on the right has 110 digits. In case the reader is curious, the exact value of this integer is

22725867753590016842883928499103875597943173955147066 29
68535141240834265159795783322985106301427965854199828 17.

Since this is merely an approximation that came from ignoring the $O(...)$ term, the reader is advised against taking this too seriously. An integer with 110 digits is at least 10^{109}, so we can write

$$N_0 \simeq 10^{10^{109}}$$

Compare this with the value of M given in equation (3.3). There, M was the number of terms whose denominators are the arithmetic progression $100n + 1$, and for which the sum remained less than 2π. That M had 230 digits. When the terms are restricted to *primes* in the arithmetic progression $100n+1$, the corresponding number of terms has about 10^{109} *digits*. This is not entirely unexpected: the partial sums of the harmonic series increase as a log; the partial sums over primes increase as a log log.

6. Estimating the Sum of $1/p$ Over Primes in an Arithmetic Progression

Before generating more examples with even larger values of N_0, we will discuss equation (5.2) in more detail. Equation (5.2) gives an estimate of, but not rigorous bounds, for the size of

$$\sum_{\substack{p\le x\\ p\equiv a \bmod q}} \frac{1}{p}.$$

The reasons for this are twofold. First, the error term is $O(1/\log x)$, which means that the error is bounded by some multiple of $1/\log x$. However, we do not know what that multiple is. Second, for large q, it is hard to compute $M(q,a)$, so we do not know how large or small $M(q,a)$ can be. For q with $3\le q\le 300$, Languasco and Zaccagnini [11] use Dirichlet L-functions to compute $M(q,a)$ 20 decimals for all a with $1\le a<q$ and $(q,a)=1$. All of these numbers are available from Languasco's web page; see [11].

In this range of q, the largest M value is $M(269,2)\simeq .49776$.

For larger q, one can use the approximation

$$M(q,a)\simeq -\frac{\log\log x}{\phi(q)}+\sum_{\substack{p\le x\\ p\equiv a \bmod q}} \frac{1}{p}. \tag{6.1}$$

to get a non-rigorous estimate of $M(q,a)$. For $3\le q\le 300$, and summing up to $x=10^7$, approximation (6.1) gives values of $M(q,a)$ that agree to 4 or 5 decimals with those in file "matricesM.txt" on Languasco's web page.

For odd $q<10000$, $M(q,1)$ appears to be slightly negative; for example, $M(9999,1)\simeq -.0004$. Also, $M(q,2)$ appears to be slightly less than .5; for example, $M(9999,2)\simeq .49959$. In fact, this limited data for $q<10000$ suggests that $M(q,1)$ approaches 0, and $M(q,2)$ approaches $1/2$, as q approaches ∞.

In fact, this *is* what happens. The paper [11] by Languasco and Zaccagnini uses a result by Karl K. Norton to show that, if $1\le a<q$ and $(q,a)=1$, then as q approaches ∞, $M(q,a)$ approaches $1/a$ if a is prime, and approaches 0 otherwise.

7. Surpassing the Skewes Numbers

The logarithmic integral of x, $\mathrm{li}(x)$, is defined as

$$\mathrm{li}(x)=\lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon}\frac{1}{\log(t)}\,dt+\int_{1+\epsilon}^{x}\frac{1}{\log(t)}\,dt\right)$$

$\mathrm{li}(x)$ is a good approximation to $\pi(x)$, the number of primes $\le x$. For every value of x for which both $\pi(x)$ and $\mathrm{li}(x)$ have been computed, we observe that

$$\pi(x)<\mathrm{li}(x).$$

Nevertheless, Littlewood proved in 1914 that the difference $\pi(x) - \mathrm{li}(x)$ changes sign infinitely often. In 1933, Skewes [13] proved, assuming the Riemann Hypothesis, that there is an x less than the very large number

$$S_1 = e^{e^{e^{79}}} \simeq 10^{10^{10^{34}}}$$

such that $\pi(x) > \mathrm{li}(x)$.

Then, in 1955, Skewes [14] proved, this time without assuming the Riemann Hypothesis, that there is an x less than the much larger number

$$S_2 = e^{e^{e^{e^{7.705}}}} \simeq e^{e^{e^{2219}}} \simeq e^{e^{e^{79 \cdot 28}}} \simeq \left(e^{e^{e^{79}}}\right)^{28} = S_1^{28}$$

such that $\pi(x) > \mathrm{li}(x)$.

More recently, Bays and Hudson [3] use accurate values of the first one million pairs of complex zeros of the Riemann zeta function to show that there are values of x near 1.39822×10^{316} such that $\pi(x) > \mathrm{li}(x)$. In any case, in spite of numerical evidence seemingly to the contrary, we do know that there are values of x such that $\pi(x) > \mathrm{li}(x)$.

These Skewes numbers are much larger than most numbers that are in common use in mathematics.

Here we will present an example where equation (4.1) holds for $N = 0, 1, 2, \ldots N_0$, and fails for $N > N_0$, where N_0 is much greater than S_2. The key idea is to use the methods of the previous section, but with q such that $\phi(q)$ is very large.

Now let P be the largest known prime, which as of this writing is $2^{43112609} - 1$. This prime has 12,978,189 digits, so $P > 10^{10^7}$. We will take our arithmetic progression be the set of primes $\equiv 1 \pmod{P}$. (Since P is odd and we are interested only in primes in this arithmetic progression, this is equivalent to the set of primes in the progression $2P + 1$). Since P is prime, we have $\phi(P) = P - 1$. Equation (5.3) applies here. The calculations and Norton's result mentioned in the previous section suggest that $M(q, 1)$ may be close to 0. We will use $M(q, 1) = 0$ in our calculations.

With P being the largest known prime, we know from Dirichlet's theorem that the arithmetic progression $Pn + 1$ contains infinitely many primes. We don't know a single example of such a prime, but we do know that there are an infinite number of them!

Let the reciprocals of those primes be our a_k. The value of x for which the sum

$$\sum_{\substack{p \le x \\ p \equiv 1 \bmod P}} \frac{1}{p}.$$

surpasses 2π is approximately

$$x \simeq \exp(\exp(\,\phi(P) \cdot (2\pi - M(P, 1))\,)) \simeq \exp(\exp(2\pi(P - 1))) > \exp(\exp(\pi P)).$$

The last inequality above is not merely an approximation; it holds for $P > 2$. Therefore, the number of primes in the arithmetic progression $Pn + 1$ up to this x is about

$$N_0 \simeq \frac{x}{\phi(P) \log x} \simeq \frac{\exp(\exp(\pi P))}{(P - 1)\exp(\pi P)} = \frac{\exp(\exp(\pi P) - \pi P)}{P - 1}.$$

Regarding the numerator, it is easy to convince oneself that if P is large, then $e^{\pi P} - \pi P > e^{2P}$ (in fact, this holds if p is more than about .391). Therefore,

$$N_0 \simeq \frac{\exp(\exp(\pi P) - \pi P)}{P-1} > \frac{\exp(\exp(2P))}{P-1}$$

If $y > \log 2$, then $e^y > 2$ and, Multiplying each side by e^y, we get $e^{2y} > 2e^y$. Taking e to each side again, we get

$$\exp(\exp(2y)) > \exp(2\exp(y)) = \exp(\exp(y) + \exp(y)) = \exp(\exp(y)) \cdot \exp(\exp(y)).$$

So, if $P > 1$,

$$\frac{\exp(\exp(2P))}{P-1} = \frac{\exp(\exp(P))}{P-1} \cdot \exp(\exp(P)) > \exp(\exp(P)).$$

Therefore, we have the very rough estimate,

$$N_0 \simeq \exp(\exp(P)). \tag{7.1}$$

This all works because the factor of $P-1$ in the denominator is tiny compared to $\exp(P)$ and is even smaller when compared to the numerator. Likewise, $\exp(P)$ is small compared to $\exp(\exp(P))$. Our very rough approximation becomes

$$N_0 \simeq \exp(\exp(P)) \simeq \exp(\exp(10^{10^7})) = e^{e^{10^{10^7}}}. \tag{7.2}$$

A simple calculation shows that $\log\log\log(P) > 2.845 > e$, so that $P > e^{e^{e^e}}$. So, we can write N_0 as

$$N_0 \simeq \exp(\exp(P)) \simeq e^{e^{e^{e^{e^{e}}}}}.$$

How large is this N_0? N_0 and S_2, the larger of the two Skewes numbers, are far too large to calculate with directly, so we must use logarithms. In fact, we must use logarithms of logarithms to bring the numbers within range of most computers. Comparing $\log\log N_0$ and $\log\log S_2$, we get

$$\log\log N_0 \simeq 10^{10^7},$$

and

$$\log\log S_2 \simeq e^{e^{7.705}} \simeq 7.6 \cdot 10^{963} < 10^{1000} = 10^{10^3},$$

so N_0 is larger than S_2. How much larger? Because $7.705 \simeq e^{2.0419}$, we can write S_2 as a tower of height 6:

$$S_2 = e^{e^{e^{e^{7.705}}}} \simeq e^{e^{e^{e^{e^{2.0419}}}}}.$$

Above, we wrote N_0 as a tower of height 6:

$$N_0 \simeq e^{e^{e^{e^{e^{e}}}}}.$$

The only difference is that in N_0, the top-level exponent is e instead of 2.0419. But $e \simeq 2.0419 \cdot 1.3313$, so

$$N_0 \simeq S_2^{1.3313}.$$

8. An Even Larger Number

One can continue indefinitely playing this game: choose an arithmetic progression with an even more sparse distribution of primes; let a_k be the sequence of reciprocals of those primes, then estimate the point at which the sum of these reciprocals exceeds 2π, so that equation (1.1) holds for more and more values of N before it fails.

With that in mind, here is our final example.

As before, let P be the largest known prime, and let $q = P^P$. We know that arithmetic progression $qn + 1$ contains infinitely many primes for $n > 0$.

If p is any prime, then $\phi(p^k) = p^k - p^{k-1}$. Therefore, $\phi(q) = P^P - P^{P-1} = q - P^{P-1}$, which proportionally speaking, is very close to q. Again, let's take $M(q, 1)$ to be 0.

The value of x for which the sum

$$\sum_{\substack{p \le x \\ p \equiv 1 \bmod q}} \frac{1}{p}.$$

surpasses 2π is about

$$x \simeq \exp(\exp(\,\phi(q) \cdot (2\pi - M(P, 1))\,)) \simeq \exp(\exp(2\pi(q - P^{P-1}))) > \exp(\exp(\pi q)).$$

The inequality on the right holds for $P > 2$. Applying the same simplifying approximations in the previous section, the number of primes in this arithmetic progression up to this x is about

$$N_0 \simeq \frac{x}{\phi(q) \log x} = \frac{\exp(\exp(\pi q))}{(q - P^{P-1}) \exp(\pi q)} = \frac{\exp(\exp(\pi q) - \pi q)}{q - P^{P-1}}.$$

Just as in the previous section,

$$N_0 \simeq \frac{\exp(\exp(\pi q) - \pi q)}{q - P^{P-1}} > \frac{\exp(\exp(2q))}{q - P^{P-1}} = \frac{\exp(\exp(q))}{q - P^{P-1}} \cdot \exp(\exp(q)).$$

The fraction on the right is greater than 1. So, we will take as our approximation,

$$N_0 \simeq \exp(\exp(q)) = \exp(\exp(P^P))$$

where, approximately,

$$P^P \simeq (10^{10^7})^{10^{10^7}}$$

Just for fun, let's write this N_0 as a tower of e's. We have

$$\log\log\log N_0 \simeq \log P^P = P \log P \simeq (10^{10^7}) \cdot 10^7 \log 10 > (10^{10^7}) \cdot 10^7.$$

It follows that

$$\log\log\log\log\log N_0 > \log\log(10^{10^7} \cdot 10^7) \simeq 16.95 > e^e.$$

Therefore, we can write N_0 as, approximately,

$$\exp\exp\exp\exp\exp\exp(e) = e^{e^{e^{e^{e^{e^{e^{e}}}}}}}$$

9. How Small is the Left Side Minus the Right Side of Equation (1.1)?

When N is large enough that (1.1) does fail, the left side minus the right side is positive. As mentioned in Sections 2 and 4, Crandall has shown that this difference is surprisingly small when $a_k = 1/(2k+1)$, and when $a_k = 1/k^{th}$ prime. But how small are the differences for the a_k that we consider in Sections 3, 5, 7, and 8? One would expect that the differences would be much smaller. The author has not pursued these questions yet. Estimating how much smaller they are could make an interesting exercise for the reader.

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Languasco's web page at http://www.math.unipd.it/~languasc/Mertens-comput.html
has many computational results. File "Msumfinalresults.pdf" has $M(q,a)$ to 100 decimals for $3 \le q \le 100$.
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