The leading root of the partial theta function

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Abstract

I study the leading root $x_0(y)$ of the partial theta function $\Theta_0(x, y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2}$, considered as a formal power series. I prove that all the coefficients of $-x_0(y)$ are strictly positive. Indeed, I prove the stronger results that all the coefficients of $-1/x_0(y)$ after the constant term 1 are strictly negative, and all the coefficients of $1/x_0(y)^2$ after the constant term 1 are strictly negative except for the vanishing coefficient of y^3.

Key Words: Partial theta function, Rogers–Ramanujan function, q-series, formal power series, root, implicit function theorem.

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1 Introduction

Consider a formal power series of the form

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2} \tag{1.1}$$

where the coefficients $(\alpha_n)_{n=0}^{\infty}$ belong to a commutative ring-with-identity-element R and we impose the normalization $\alpha_0 = \alpha_1 = 1$. We can regard f as a formal power series in y whose coefficients are polynomials in x, i.e. $f \in R[x][[y]]$. Then, for any formal power series $X(y)$ with coefficients in R, the composition $f(X(y), y)$ makes sense as a formal power series in y. In particular, it is easy to see — either by the implicit function theorem for formal power series [18, p. A.IV.37] [44, Proposition 3.1] or by a direct inductive argument — that there exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$, which I call the "leading root" of f. Since $x_0(y)$ obviously has constant term -1, it is convenient to write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

Among the interesting series $f(x,y)$ of this type are the "partial theta function" [9, Chapter 13] [10, Chapter 6]

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2} \tag{1.2}$$

and the "deformed exponential function" [34, 33, 32, 45, 46, 47, 48]

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2} \;. \tag{1.3}$$

More generally one can consider the rescaled three-variable Rogers–Ramanujan function [47]

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \,\cdots\, (1+q+\ldots+q^{n-1})} \;, \tag{1.4}$$

which reduces to the foregoing when $q = 0$ and $q = 1$, respectively.

I have recently discovered empirically that the power series $\xi_0(y)$ has all nonnegative (in fact strictly positive) coefficients in the first two cases, and more generally in the third case whenever $q > -1$. More precisely, I have verified this for Θ_0 and F through orders y^{6999} and y^{899}, respectively, using a formula [47] that relates $\xi_0(y)$ to the series expansion of $\log f(x,y)$. For $\widetilde{R}$, I have proven [47] that $\xi_0(y,q)$ has the form

$$\xi_0(y,q) \;=\; 1 + \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)} \, y^n \tag{1.5}$$

where

$$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor} \tag{1.6}$$

and $P_n(q)$ is a self-inversive polynomial in q with integer coefficients; and I have verified for $n \le 349$ that $P_n(q)$ has *two* interesting positivity properties:

(a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.

(b) $P_n(q) > 0$ for $q > -1$.

Of course, I conjecture that these properties hold for all n, but I have (as yet) no proof.

The main purpose of this paper is to give a simple proof of the coefficientwise positivity of $\xi_0(y)$ in the case of the partial theta function (1.2):

Theorem 1.1. *For the partial theta function (1.2), the formal power series*

$$\xi_0(y) \;=\; 1+y+2y^2+4y^3+9y^4+21y^5+52y^6+133y^7+351y^8+948y^9+2610y^{10}+\ldots \tag{1.7}$$

has strictly positive coefficients.

In fact, with a bit more work one can prove a pair of successively stronger results:

Theorem 1.2. *For the partial theta function (1.2), the formal power series*

$$1/\xi_0(y) \;=\; 1-y-y^2-y^3-2y^4-4y^5-10y^6-25y^7-66y^8-178y^9-490y^{10}-\ldots \tag{1.8}$$

has strictly negative coefficients after the constant term 1.

Theorem 1.3. *For the partial theta function (1.2), the formal power series*

$$1/\xi_0(y)^2 \;=\; 1-2y-y^2 \qquad -y^4-2y^5-7y^6-18y^7-50y^8-138y^9-386y^{10}-\ldots \tag{1.9}$$

has strictly negative coefficients after the constant term 1 except for the vanishing coefficient of y^3.

For further discussion of the relationship between these results, see Section 7.

In addition, I have discovered empirically a vast strengthening of Theorems 1.1 and 1.2. Please note first that any power series $g(y) = 1 + \sum_{n=1}^{\infty} a_n y^n \in \mathbb{Z}[[y]]$ can be written uniquely as an infinite product $g(y) = \prod_{m=1}^{\infty} (1-y^m)^{-c_m}$ with coefficients $c_m \in \mathbb{Z}$.[1] We then have:

Conjecture 1.4. *For the partial theta function (1.2), when the formal power series $\xi_0(y)$ is written in the form $\xi_0(y) = \prod\limits_{m=1}^{\infty} (1-y^m)^{-c_m}$, the coefficient sequence*

$$(c_m)_{m=1}^{\infty} \;=\; 1, 1, 2, 4, 10, 23, 61, 157, 426, 1163, 3253, 9172, 26236, 75634, \ldots \tag{1.10}$$

is strictly positive ($c_m > 0$), increasing ($\Delta c \ge 0$), strictly convex ($\Delta^2 c > 0$), and satisfies $\Delta^k c \ge 0$ for $k = 3, 4$. [By contrast, the sequence $\Delta^5 c$ starts with -3.]

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[1] See e.g. [6, Theorem 10.3]. Some authors [16, 35] [43, pp. 20–21] call $(a_n)_{n=1}^{\infty}$ the *Euler transform* of $(c_m)_{m=1}^{\infty}$, and $(c_m)_{m=1}^{\infty}$ the *inverse Euler transform* of $(a_n)_{n=1}^{\infty}$. However, this should not be confused with an unrelated (and more widely used) "Euler transformation" of sequences, involving binomial coefficients.

Conjecture 1.5. *For the partial theta function (1.2), when the formal power series $\xi_0(y)$ is written in the form $2 - 1/\xi_0(y) = \prod\limits_{m=1}^{\infty} (1-y^m)^{-c'_m}$, the coefficient sequence*

$$(c'_m)_{m=1}^{\infty} \;=\; 1, 0, 0, 1, 2, 6, 15, 40, 110, 303, 853, 2419, 6950, 20110, \ldots \tag{1.11}$$

is nonnegative and convex.

I have verified these conjectures through order y^{6999}, but I have no idea how to prove them. Perhaps one should try to find a combinatorial interpretation of the coefficients (c_m) and (c'_m).

The series $\xi_0(y)$ appears to possess one further striking property, which I have again verified through order y^{6999}:

Conjecture 1.6. *For the partial theta function (1.2), the coefficient sequence of $\xi_0(y) = \sum_{n=0}^{\infty} a_n y^n$ is log convex, i.e. $a_{n-1}a_{n+1} \ge a_n^2$ for all $n \ge 1$.*

A classic theorem of Kaluza [27] relates Conjecture 1.6 to Theorem 1.2: namely, if the coefficient sequence $(a_n)_{n=0}^{\infty}$ of a formal power series f is strictly positive and log convex, then $1/f$ has nonpositive coefficients after the constant term; and if in addition $a_0 a_2 > a_1^2$, then $1/f$ has strictly negative coefficients after the constant term.[2] But it is easily seen that the converse does not hold.[3] So Conjecture 1.6, if true, is a strengthening of Theorem 1.2.

The plan of this paper is as follows: I begin (Section 2) by recalling two identities for the partial theta function, which will play a central role in the proofs of Theorems 1.1–1.3. I then give, in successive sections, the proofs of Theorem 1.1–1.3 (Sections 3–5). Next I state and prove some identities for the three-variable Rogers–Ramanujan function (1.4) that may turn out to be useful in proving the conjectures concerning its leading root (Section 6). Finally, I place Theorems 1.1–1.3 in a more general context [42] and mention some stronger properties possessed by the power series $\xi_0(y)$ for the cases (1.2)–(1.4) that appear empirically to be true (Section 7).

A MATHEMATICA file `partialtheta_xi0.m` containing the series $\xi_0(y)$ for the partial theta function through order y^{6999} is available as an ancillary file with the preprint version of this paper at `arXiv.org`.

2 Identities for the partial theta function

In this section we recall a pair of identities for the partial theta function (1.2) that will serve as the foundation for our proofs of Theorems 1.1–1.3. We use the standard notation $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ and $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

[2] The assertion about strict negativity is not explicitly stated by Kaluza [27], but it follows easily from his proof. See also [30, Lemma 2.2].

[3] For instance, let $f(y) = 1/(1-y-cy^2)$; then $1/f$ has nonpositive coefficients after the constant term whenever $c \ge 0$; but the coefficients of f are log convex only when $c = 0$.

Lemma 2.1. *The partial theta function (1.2) satisfies*

$$\Theta_0(x,y) \;=\; (y;y)_\infty \, (-x;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n \, (-x;y)_n} \tag{2.1}$$

$$\Theta_0(x,y) \;=\; (-x;y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n \, y^{n^2}}{(y;y)_n \, (-x;y)_n} \tag{2.2}$$

as formal power series and as analytic functions on $(x,y) \in \mathbb{C} \times \mathbb{D}$.[4]

In order to make this paper self-contained for readers who (like myself!) are not experts in q-series, we provide here an easy proof of (2.1) that uses nothing more than Euler's first and second identities [22, eqs. (1.3.15) and (1.3.16)]

$$\frac{1}{(t;q)_\infty} \;=\; \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \tag{2.3}$$

$$(t;q)_\infty \;=\; \sum_{n=0}^{\infty} \frac{(-t)^n \, q^{n(n-1)/2}}{(q;q)_n} \tag{2.4}$$

valid for $(t,q) \in \mathbb{D} \times \mathbb{D}$ and $(t,q) \in \mathbb{C} \times \mathbb{D}$, respectively.

Proof of (2.1) [19, 2]. Write

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2} \, \frac{(y;y)_\infty}{(y;y)_n \, (y^{n+1};y)_\infty} \tag{2.5}$$

and insert Euler's first identity for $1/(y^{n+1};y)_\infty$: we obtain

$$\begin{aligned}
\Theta_0(x,y) \;&=\; (y;y)_\infty \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(y;y)_n} \sum_{k=0}^{\infty} \frac{y^{(n+1)k}}{(y;y)_k} && \text{(2.6a)} \\
&=\; (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k} \sum_{n=0}^{\infty} \frac{(xy^k)^n \, y^{n(n-1)/2}}{(y;y)_n} && \text{(2.6b)} \\
&=\; (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k} \, (-xy^k;y)_\infty \quad \text{by Euler's second identity} && \text{(2.6c)} \\
&=\; (y;y)_\infty \, (-x;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k \, (-x;y)_k} \,. && \text{(2.6d)}
\end{aligned}$$

□

[4] Here $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. The right-hand sides of (2.1) and (2.2) have removable singularities at $x = -y^{-k}$ $(k = 0, 1, 2, \ldots)$. To see that these singularities are indeed removable, just rewrite $(-x;y)_\infty/(-x;y)_n$ as $(-xy^n;y)_\infty$.

And here is an easy proof of both (2.1) and (2.2) that uses only Heine's first and second transformations [22, eqs. (1.4.1) and (1.4.5)]

$$
{}_2\phi_1(a,b;c;q,z) \;=\; \frac{(b;q)_\infty\,(az;q)_\infty}{(c;q)_\infty\,(z;q)_\infty}\;{}_2\phi_1(c/b,z;az;q,b) \tag{2.7}
$$

$$
{}_2\phi_1(a,b;c;q,z) \;=\; \frac{(c/a;q)_\infty\,(az;q)_\infty}{(c;q)_\infty\,(z;q)_\infty}\;{}_2\phi_1(abz/c,a;az;q,c/a) \tag{2.8}
$$

for the basic hypergeometric function

$$
{}_2\phi_1(a,b;c;q,z) \;=\; \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(q;q)_n\,(c;q)_n}\, z^n \;. \tag{2.9}
$$

Here (2.7) is valid when $|q|<1$, $|z|<1$ and $0<|b|<1$, while (2.8) is valid when $|q|<1$, $|z|<1$ and $0<|c|<|a|$.

Proof of (2.1) and (2.2) [11]. In (2.7) and (2.8), set $b=q$ and $z=-x/a$, then take $a\to\infty$ and $c\to 0$; we obtain (2.1) and (2.2) with y renamed as q. $\square$

Remarks. Identity (2.1) goes back to Heine in 1847 [24, bottom p. 306], who derived it (as here) as a limiting case of his fundamental transformation (2.7).[5] In the modern literature it can be found in Fine [21, eq. (7.32)].

I don't know who first found identity (2.2); I would be grateful to any reader who can supply a reference. I first learned (2.2) from the paper of Andrews and Warnaar [11, eq. (2.1)], but it is surely much older.

The elementary proof of (2.1) given here is in essence that given recently by Chen and Xia [19, eq. (2.10)] and Alladi [2, p. second proof of (1.6)].[6] Our proof of (2.1) and (2.2) using Heine's transformations follows Andrews and Warnaar [11, eq. (2.1)][7], but at least for (2.1) the argument goes back to Heine himself [24, p. 306]. Note also that if one takes this latter proof of (2.1) and inserts in it the standard *proof* of Heine's first transformation [22, sec. 1.4], one obtains the elementary proof of (2.1).

A combinatorial proof of (2.1) was given recently by Yee [53, Theorem 2.1], and combinatorial proofs of both (2.1) and the equality (2.1)=(2.2) were given recently by Kim [29, Section 2].

Many generalizations of (2.1)/(2.2), with additional parameters, are known. For instance, (2.1)/(2.2) can be extended from the partial theta function to more gen-

[5]Heine makes the change of variables $x=-zq$ and $y=q^2$. The formula in [24, bottom p. 306] has a typographical error in which the factor y^n $(=q^{2n})$ in the numerator of the right-hand side is inadvertently omitted. The correct formula can be found in the 1878 edition of Heine's book [25, p. 107].

[6]Alladi's eq. (1.6) is equivalent to our (2.1) under the substitutions $x=-aq$ and $y=q^2$.

[7]See also Andrews [7, proof of Theorem 1] for this proof of (2.1).

eral basic hypergeometric functions ${}_1\phi_1$.[8] Another generalization of (2.1) appears in Ramanujan's lost notebook [40, p. 40] [10, Entry 6.3.1]; it was proven by Andrews [4, Section 4] and recently re-proven combinatorially by Kim [29, Section 4]. An even more general formula was proven subsequently by Andrews [5, Section 3], with a later simplification and further generalization by R.P. Agarwal [1]; see also [10, Sections 6.2 and 6.3]. A formula generalizing the equality (2.1)=(2.2) appears in Ramanujan's lost notebook [40, p. 40] [10, Entry 1.6.7] and has an easy q-series proof [10, p. 27]; a combinatorial proof was recently given by Kim [29, Section 4].

A very beautiful formula for the sum of two partial theta functions, which generalizes both (2.1) and the Jacobi triple product identity, was found by Warnaar [51, Theorem 1.5]. A closely related identity for the product of two partial theta functions, which also generalizes (2.1), was found by Andrews and Warnaar [11, Theorem 1.1] and recently re-proven combinatorially by Kim [29, Section 3]; see also [10, Section 6.6].

Finally, Andrews [8, Theorem 5] has recently proven a finite-sum generalization of (2.1):

$$\sum_{n=0}^{N} \frac{x^n\, y^{n(n-1)/2}}{(y;y)_{N-n}} \;=\; (-x;y)_N \sum_{n=0}^{N} \frac{y^n}{(y;y)_n\,(-x;y)_n}\,. \tag{2.10}$$

Likewise, by using [8, Corollary 3] with $\alpha = q$, $\tau = -x/\beta$ and taking $\beta \to \infty$ and $\gamma \to 0$, one can derive a finite-sum generalization of (2.2):

$$\sum_{n=0}^{N} \frac{x^n\, y^{n(n-1)/2}}{(y;y)_{N-n}} \;=\; (-x;y)_N \sum_{n=0}^{N} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\,(-x;y)_n\,(y;y)_{N-n}}\,. \tag{2.11}$$

See also [41] for a combinatorial proof of the finite Heine transformation that underlies (2.10) and (2.11).

3 Proof of Theorem 1.1

The proof of Theorem 1.1 can be based on either (2.1) or (2.2). For concreteness let us use (2.1), which we rewrite as

$$\Theta_0(x,y) \;=\; (y;y)_\infty\,(-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\,(-xy;y)_{n-1}}\right]. \tag{3.1}$$

[8] For the case of (2.2), this generalization can be found in papers of Bhargava and Adiga [17] and Srivastava [49]. A special case of this generalization can be found in Ramanujan's second notebook [39, Entry 9 in Chapter 16] [14, p. 18] and again in a page published with the lost notebook [40, p. 362] [10, Entry 1.6.1]. A combinatorial proof of this special case was recently given by Berndt, Kim and Yee [15, Theorem 5.6].

So $\Theta_0(-\xi_0(y), y) = 0$ is equivalent to

$$\begin{aligned}
\xi_0(y) \;&=\; 1 + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n \, (y\xi_0(y);y)_{n-1}} && (3.2a) \\
&=\; 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n} (1-y^j) \prod\limits_{j=1}^{n-1} [1-y^j \xi_0(y)]} \;. && (3.2b)
\end{aligned}$$

This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:

Proposition 3.1. *Define the map* $\mathcal{F}\colon\ \mathbb{Z}[[y]] \to \mathbb{Z}[[y]]$ *by*

$$(\mathcal{F}\xi)(y) \;=\; 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n} (1-y^j) \prod\limits_{j=1}^{n-1} [1-y^j \xi(y)]} \;, \tag{3.3}$$

and define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[[y]]$ *by* $\xi_0^{(0)} = 1$ *and* $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$*. Then*

$$\xi_0^{(0)} \;\preceq\; \xi_0^{(1)} \;\preceq\; \xi_0^{(2)} \;\preceq\; \ldots \;\preceq\; \xi_0 \tag{3.4}$$

(where $f \preceq g$ *denotes* $[y^n]f(y) \le [y^n]g(y)$ *for all* n*) and*

$$\xi_0^{(k)}(y) \;=\; \xi_0(y) \,+\, O(y^{3k+1}) \;. \tag{3.5}$$

In particular, $\lim\limits_{k\to\infty} \xi_0^{(k)}(y) = \xi_0(y)$ *in the sense of convergence of formal power series (i.e. every coefficient eventually stabilizes at its limit), and* $\xi_0(y)$ *has strictly positive coefficients.*

Proof. If $f(y)$ and $g(y)$ are formal power series satisfying $0 \preceq f \preceq g$, then it is easy to see that $\prod\limits_{j=1}^{n-1} [1-y^j f(y)]^{-1} \preceq \prod\limits_{j=1}^{n-1} [1-y^j g(y)]^{-1}$ and hence $0 \preceq \mathcal{F}f \preceq \mathcal{F}g$. Applying this repeatedly to the obvious inequality $0 \preceq \xi_0^{(0)} \preceq \xi_0^{(1)}$, we obtain $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \ldots$.

Likewise, if $f(y)$ and $g(y)$ are formal power series satisfying $f(y) - g(y) = O(y^\ell)$ for some $\ell \ge 0$, then it is not hard to see that $(\mathcal{F}f)(y) - (\mathcal{F}g)(y) = O(y^{\ell+3})$ [coming from the $n=2$ term in (3.3) and the $j=1$ factor in the second product]. Applying this repeatedly to the obvious fact $\xi_0^{(1)}(y) - \xi_0^{(0)}(y) = O(y)$, we obtain $\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) = O(y^{3k+1})$. It follows that $\xi_0^{(k)}(y)$ converges as $k \to \infty$ (in the topology of formal power series) to a limiting series $\xi_0^{(\infty)}(y)$, and that this limiting series satisfies $\mathcal{F}\xi_0^{(\infty)} = \xi_0^{(\infty)}$. But this means, by (3.1)/(3.2), that $\xi_0^{(\infty)}(y) = \xi_0(y)$. It also follows that $\xi_0^{(k)}(y) = \xi_0(y) + O(y^{3k+1})$.

Since $\xi_0^{(1)}(y)$ manifestly has strictly positive coefficients, it follows from (3.4) that $\xi_0(y)$ also has strictly positive coefficients. $\square$

Remarks. 1. By a slightly more refined version of the same argument, one can prove inductively that

$$\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) \;=\; y^{3k+1} + (4k+2)y^{3k+2} + (4k+1)(2k+3)y^{3k+3} + O(y^{3k+4}) \quad (3.6)$$

for $k \geq 1$, and hence that

$$\xi_0(y) - \xi_0^{(k)}(y) \;=\; y^{3k+1} \;+\; (4k+2)y^{3k+2} \;+\; (4k+1)(2k+3)y^{3k+3} \;+\; O(y^{3k+4}) \quad (3.7)$$

for $k \geq 1$.

2. The series

$$\xi_0^{(1)}(y) \;=\; 1 + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\,(y;y)_{n-1}} \;=\; 1 + \frac{1-\Theta_0(-y,y)}{(y;y)_\infty^2} \quad (3.8a)$$

$$= \; 1 + y + 2y^2 + 4y^3 + 8y^4 + 15y^5 + 27y^6 + 47y^7 + 79y^8 + \ldots \quad (3.8b)$$

enumerates weakly unimodal sequences of positive integers (also called "stacks" or "stack polyominoes") by total weight [12, 52] [50, Section 2.5] [35, sequence A001523]. It would be interesting to seek combinatorial interpretations of $\xi_0^{(k)}(y)$ for $k \geq 2$, or at least of $\xi_0(y)$.[9]

3. Empirically I have observed that the $\xi_0^{(k)}$ obey inequalities stronger than (3.4), namely $\xi_0^{(k)}/\xi_0^{(k-1)} \succeq 1$ for $k \geq 1$. I have verified this through order y^{500} for $1 \leq k \leq 20$, but I do not see how to prove it. If true, this exhibits $\xi_0(y)$ as an infinite product of nonnegative series $\xi_0^{(k)}(y)/\xi_0^{(k-1)}(y)$, reminiscent of but different from Conjecture 1.4.

4. The recursion $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$ could alternatively have been started with $\xi_0^{(0)} = 0$ instead of $\xi_0^{(0)} = 1$. The only difference is that we would then have $\xi_0^{(k)}(y) - \xi_0(y) = O(y^{3k})$ instead of $O(y^{3k+1})$. In this case

$$\xi_0^{(1)}(y) \;=\; \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n} \;=\; \frac{1}{(y;y)_\infty} \;=\; \sum_{n=0}^{\infty} p(n)\, y^n \quad (3.9a)$$

$$= \; 1 + y + 2y^2 + 3y^3 + 5y^4 + 7y^5 + 11y^6 + 15y^7 + 22y^8 + \ldots \quad (3.9b)$$

is the generating function for all partitions of the integer n. Perhaps $\xi_0^{(k)}(y)$ for $k \geq 2$ have a simpler interpretation with this choice of $\xi_0^{(0)}$.[10]

Furthermore, with this choice of $\xi_0^{(0)}$ we have empirically not only $\xi_0^{(k)}/\xi_0^{(k-1)} \succeq 1$ for $k \geq 2$, but in fact $\xi_0^{(k)}(y)/\xi_0^{(k-1)}(y) = \prod_{m=1}^{\infty}(1-y^m)^{-c_m^{(k)}}$ with nonnegative coefficients $c_m^{(k)}$. I have verified this through order y^{500} for $2 \leq k \leq 20$. If true, this implies Conjecture 1.4.

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[9]**Note added:** Thomas Prellberg [37] has recently found a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$ in terms of rooted trees enriched by stack polyominoes, using results from [38] and [13, Chapter 3].

[10]**Note added:** Thomas Prellberg [37] has found a combinatorial interpretation of $\xi_0^{(k)}(y)$ also for this choice of $\xi_0^{(0)}$.

5. If we use (2.2) instead of (2.1), then we are led to the recursion based on the map $\mathcal{G}\colon\ \mathbb{Z}[[y]] \to \mathbb{Z}[[y]]$ defined by

$$(\mathcal{G}\xi)(y) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{\xi(y)^n \, y^{n^2}}{\prod\limits_{j=1}^{n} (1-y^j) \, \prod\limits_{j=1}^{n-1} [1-y^j \xi(y)]} \;. \tag{3.10}$$

Using $\xi_0^{(0)} = 1$, we have for this map the slower convergence $\xi_0^{(k)}(y) - \xi_0(y) = O(y^k)$ [coming from the $\xi(y)^n$ factor in the numerator of the $n=1$ term in (3.10)]. In this case the series

$$\xi_0^{(1)}(y) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{y^{n^2}}{(y;y)_n \, (y;y)_{n-1}} \;=\; 1 \,+\, \frac{1-\Theta_0(-y,y)}{(y;y)_\infty} \tag{3.11a}$$

$$\;=\; 1 + y + y^2 + y^3 + 2y^4 + 3y^5 + 5y^6 + 7y^7 + 10y^8 + \ldots \tag{3.11b}$$

enumerates n-stacks with strictly receding walls [12, 52] [35, sequence A001522]. Once again we have empirically $\xi_0^{(k)}/\xi_0^{(k-1)} \succeq 1$ for $k \ge 1$; I have verified this through order y^{2000} for $1 \le k \le 20$. Furthermore, for this map taking $\xi_0^{(0)} = 0$ yields $\xi_0^{(1)} = 1$, so we obtain the *same* sequence (shifted by one) with both initial conditions.[11] $\square$

It is useful to abstract what we have done here (see [47] for details and extensions). Consider a formal power series (with coefficients in a commutative ring-with-identity-element R)

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n \tag{3.12}$$

where

(a) $a_0(0) = a_1(0) = 1$;

(b) $a_n(0) = 0$ for $n \ge 2$; and

(c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$.

Then it is easy to see that there exists a unique formal power series $\xi_0(y)$ with coefficients in R satisfying $f(-\xi_0(y), y) = 0$, and it has constant term 1. Let us rearrange $f(-\xi_0(y), y) = 0$ as

$$\xi_0(y) \;=\; 1 \,+\, \sum_{n=0}^{\infty} (-1)^n \, \widehat{a}_n(y) \, \xi_0(y)^n \;, \tag{3.13}$$

where $\widehat{a}_n(y)$ is defined by

$$\widehat{a}_n(y) \;=\; \begin{cases} a_n(y) - 1 & \text{for } n = 0,1 \\ a_n(y) & \text{for } n \ge 2 \end{cases} \tag{3.14}$$

[11] **Note added:** Thomas Prellberg [37] has found a combinatorial interpretation also for these $\xi_0^{(k)}(y)$.

Now suppose that the ring R carries a partial order compatible with the ring structure (typically we will have $R = \mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$) and that

$$(-1)^n \widehat{a}_n(y) \;\succeq\; 0 \qquad \text{for all } n \geq 0 \,, \tag{3.15}$$

where $f(y) \succeq 0$ means that f has all nonnegative coefficients. Then the recursion argument used in Proposition 3.1, applied to (3.13), shows that $\xi_0(y) \succeq 1 + \sum_{n=0}^{\infty} (-1)^n \, \widehat{a}_n(y)$. The case treated here was

$$f(x,y) \;=\; \frac{\Theta_0(x,y)}{(y;y)_\infty \, (-xy;y)_\infty} \;=\; 1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n \, (-xy;y)_{n-1}} \;. \tag{3.16}$$

The value of the identity (2.1) or (2.2) for our purposes is that powers of x on the left-hand side are transformed into powers of $-x$ on the right-hand side, so that (3.15) holds for the latter.

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2 on the strict negativity of the coefficients of $\xi_0(y)^{-1}$ after the constant term 1. It is convenient to state and prove first an abstract result of this form [47]; then we verify the hypotheses of this abstract result in our specific case.

Proposition 4.1. *Consider a formal power series (with coefficients in a partially ordered commutative ring R)*

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y) \, x^n \tag{4.1}$$

where

(a) $a_0(0) = a_1(0) = 1$*;*

(b) $a_n(0) = 0$ *for* $n \geq 2$*; and*

(c) $a_n(y) = O(y^{\nu_n})$ *with* $\lim_{n\to\infty} \nu_n = \infty$.

Let $\xi_0(y)$ be the unique power series satisfying $f(-\xi_0(y), y) = 0$. Suppose that

$$1 \;-\; \frac{a_1(y)}{a_0(y)} \;\succeq\; 0 \tag{4.2}$$

and that

$$(-1)^n \, \frac{a_n(y)}{a_0(y)} \;\succeq\; 0 \qquad \textit{for all } n \geq 2 \,. \tag{4.3}$$

Then

$$\xi_0(y)^{-1} \;\preceq\; \frac{a_1(y)}{a_0(y)} \;-\; \sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_0(y)} \;\preceq\; 1 \,. \tag{4.4}$$

Proof. Start from the equation $\sum_{n=0}^{\infty}(-1)^n a_n(y)\, \xi_0(y)^n = 0$, divide by $a_0(y)\xi_0(y)$, and bring $\xi_0(y)^{-1}$ to the left-hand side: we have

$$\xi_0(y)^{-1} \;=\; \frac{a_1(y)}{a_0(y)} \;-\; \sum_{n=2}^{\infty}(-1)^n\, \frac{a_n(y)}{a_0(y)}\, \xi_0(y)^{n-1}\;. \tag{4.5}$$

Now write $\xi_0(y)^{-1} = 1 - \psi(y)$: we obtain

$$\psi(y) \;=\; 1 \;-\; \frac{a_1(y)}{a_0(y)} \;+\; \sum_{n=2}^{\infty}(-1)^n\, \frac{a_n(y)}{a_0(y)}\, [1-\psi(y)]^{-(n-1)}\;. \tag{4.6}$$

By hypothesis (4.6) is of the form

$$\psi(y) \;=\; b_1(y) \;+\; \sum_{n=2}^{\infty} b_n(y)\, [1-\psi(y)]^{-(n-1)} \tag{4.7}$$

where $b_n(y) \succeq 0$ and $b_n(y) = O(y)$ for all $n \ge 1$. An iterative argument as in the proof of Proposition 3.1 then proves that $\psi(y) \succeq 0$ and in fact

$$\psi(y) \;\succeq\; 1 \;-\; \frac{a_1(y)}{a_0(y)} \;+\; \sum_{n=2}^{\infty}(-1)^n\, \frac{a_n(y)}{a_0(y)}\;. \tag{4.8}$$

□

Proof of Theorem 1.2. This time we find it convenient to use (2.2) instead of (2.1). We therefore apply Proposition 4.1 to the power series

$$f(x,y) \;=\; \frac{\Theta_0(x,y)}{(-xy;y)_\infty} \;=\; 1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-xy;y)_{n-1}} \tag{4.9a}$$

$$\;=\; 1 + x - \frac{xy}{1-y} + \sum_{n=2}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-xy;y)_{n-1}}\;. \tag{4.9b}$$

The first three terms in (4.9b) give $a_0(y) = 1$ and $a_1(y) = 1 - y/(1-y)$, so that $1 - a_1(y)/a_0(y) = y/(1-y) \succeq 0$. On the other hand, the final sum in (4.9b) is manifestly a power series with nonnegative coefficients in $-x$ and y, which proves that $(-1)^m a_m(y) \succeq 0$ for all $m \ge 2$. □

Remarks. 1. We can obtain an explicit formula for the coefficients $a_m(y)$ by inserting into (4.9b) the expansion [3, Theorem 3.3]

$$\frac{1}{(-xy;y)_{n-1}} \;=\; \sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_y (-xy)^k \qquad \text{for } n \ge 2 \tag{4.10}$$

where the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \;=\; \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \qquad \text{for } 0 \le k \le n\;. \tag{4.11}$$

This yields

$$f(x,y) \;=\; 1 + x - \frac{xy}{1-y} + \sum_{n=2}^{\infty}\sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_y (-xy)^k\, \frac{(-x)^n\, y^{n^2}}{(y;y)_n}\;. \tag{4.12}$$

Extracting the coefficient of x^m for $m = n+k \ge 2$, we have

$$(-1)^m a_m(y) \;=\; y^m \sum_{n=2}^{m} \begin{bmatrix} m-2 \\ m-n \end{bmatrix}_y \frac{y^{n(n-1)}}{(y;y)_n}\;. \tag{4.13}$$

Since the q-binomial coefficients are polynomials in q with nonnegative integer coefficients [3, Theorem 3.2 or 3.6], we see once again that $(-1)^m a_m(y) \succeq 0$ for all $m \ge 2$. We also see from (4.13) that $a_m(y)$ is a rational function of the form $a_m(y) = P_m(y)/(y;y)_m$ where $P_m(y)$ is a polynomial with integer coefficients.

2. It would be interesting to seek a combinatorial interpretation of the coefficients of $1-1/\xi_0(y)$, analogously to what Prellberg [37] has done for $\xi_0(y)$ [see footnotes 9–11 above].

5 Proof of Theorem 1.3

Next we prove Theorem 1.3. It is convenient once again to state and prove first an abstract result [47], and then verify the hypotheses of this abstract result in our specific case.

Proposition 5.1. *Consider a formal power series $f(x,y)$ satisfying all the hypotheses of Proposition 4.1. Then*

$$\xi_0(y)^{-2} \;\preceq\; \left(\frac{a_1(y)}{a_0(y)}\right)^2 \;-\; 2\sum_{n=2}^{\infty}(-1)^n\, \frac{a_n(y)}{a_0(y)} \left(\frac{a_0(y)}{a_1(y)}\right)^{n-2}\;. \tag{5.1}$$

Proof. Divide both sides of (4.5) by $\xi_0(y)$ and then insert (4.5) in the first term on the right-hand side: we obtain

$$\xi_0(y)^{-2} \;=\; \frac{a_1(y)}{a_0(y)}\,\xi_0(y)^{-1} \;-\; \sum_{n=2}^{\infty}(-1)^n\,\frac{a_n(y)}{a_0(y)}\,\xi_0(y)^{n-2} \tag{5.2a}$$

$$\;=\; \left(\frac{a_1(y)}{a_0(y)}\right)^2 \;-\; \sum_{n=2}^{\infty}(-1)^n\,\frac{a_n(y)}{a_0(y)}\left[1 + \frac{a_1(y)}{a_0(y)}\,\xi_0(y)\right]\xi_0(y)^{n-2}\;. \tag{5.2b}$$

Now, by hypothesis we have $(-1)^n a_n(y)/a_0(y) \succeq 0$ for all $n \geq 2$. By Proposition 4.1 we have $\xi_0(y)^{-1} \preceq a_1(y)/a_0(y) \preceq 1$, hence $\xi_0(y)^{n-2} \succeq [a_0(y)/a_1(y)]^{n-2} \succeq 1$ for all $n \geq 2$. Finally, multiplying (4.5) by $\xi_0(y)$ and rearranging gives

$$\frac{a_1(y)}{a_0(y)}\, \xi_0(y) \;=\; 1 + \sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_0(y)}\, \xi_0(y)^n \;\succeq\; 1 \,. \tag{5.3}$$

Inserting these facts into (5.2b) proves (5.1). □

Proof of Theorem 1.3. We again use (2.2) and thus apply Proposition 5.1 to the power series (4.9b). While proving Theorem 1.2 we showed that $a_0(y) = 1$, $a_1(y) = 1 - y/(1-y) \preceq 1$ and $(-1)^n a_n(y) \succeq 0$ for all $n \geq 2$, so all the hypotheses of Proposition 5.1 are satisfied. Furthermore, from either (4.9b) or (4.13) it is easy to see that

$$(-1)^n a_n(y) \;\succeq\; \frac{y^{n+2}}{(1-y)(1-y^2)} \;\succeq\; \frac{y^{n+2}}{1-y} \tag{5.4}$$

for all $n \geq 2$.[12] From (5.1) we then have

$$\begin{aligned}
\xi_0(y)^{-2} \;&\preceq\; \left(\frac{a_1(y)}{a_0(y)}\right)^2 - 2\sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_0(y)} \left(\frac{a_0(y)}{a_1(y)}\right)^{n-2} && (5.5a)\\
&\preceq\; \left(\frac{a_1(y)}{a_0(y)}\right)^2 - 2\sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_0(y)} && (5.5b)\\
&\preceq\; \left(\frac{1-2y}{1-y}\right)^2 - \frac{2y^4}{(1-y)^2} && (5.5c)\\
&=\; 1 - 2y - y^2 - \sum_{n=4}^{\infty} (n-3)\, y^n \,, && (5.5d)
\end{aligned}$$

which proves Theorem 1.3. □

Remarks. 1. If we use (4.13) and expand the right-hand side of (5.1), we obtain

$$\xi_0(y)^{-2} \;\preceq\; 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 49y^8 - 130y^9 - 343y^{10} - \ldots \,, \tag{5.6}$$

which differs from the exact $\xi_0(y)^{-2}$ starting at order y^8. The difference at order y^8 arises from a contribution to $\xi_0(y)$ that is proportional to $a_2(y)^2$. The full structure of the contributions to $\xi_0(y)$ and its powers can be read off the explicit implicit function formula [44]: see [47] for details.

2. It would be interesting to seek a combinatorial interpretation of the coefficients of $1 - 1/\xi_0(y)^2$, analogously to what Prellberg [37] has done for $\xi_0(y)$ [see footnotes 9–11 above].

[12] It suffices to take the term $n = 2$ in (4.9b) or (4.13), using the fact that all other terms are $\succeq 0$.

6 Identities for $R(x, y, q)$

In this section we obtain some simple identities for the three-variable Rogers–Ramanujan function [47]

$$R(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(q;q)_n} \;. \tag{6.1}$$

The basic principle is in fact more general, and applies to an arbitrary power series of the form

$$F(x,q) \;=\; \sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} \;. \tag{6.2}$$

Lemma 6.1. *For arbitrary coefficients $(a_n)_{n=0}^{\infty}$ and an arbitrary constant α, we have*

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} \;=\; \frac{1}{(\alpha;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-\alpha)^\ell\, q^{\ell(\ell-1)/2}}{(q;q)_\ell} \sum_{n=0}^{\infty} a_n\, (q^\ell x)^n \tag{6.3}$$

as formal power series.

Proof. Write

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} \;=\; \sum_{n=0}^{\infty} a_n x^n\, \frac{(\alpha q^n;q)_\infty}{(\alpha;q)_\infty} \tag{6.4}$$

and substitute Euler's second identity (2.4) for $(\alpha q^n;q)_\infty$, yielding

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} \;=\; \frac{1}{(\alpha;q)_\infty} \sum_{n=0}^{\infty} a_n x^n \sum_{\ell=0}^{\infty} \frac{(-\alpha q^n)^\ell\, q^{\ell(\ell-1)/2}}{(q;q)_\ell} \tag{6.5a}$$

$$\;=\; \frac{1}{(\alpha;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-\alpha)^\ell\, q^{\ell(\ell-1)/2}}{(q;q)_\ell} \sum_{n=0}^{\infty} a_n\, (q^\ell x)^n \;. \tag{6.5b}$$

□

Specializing to $a_n = y^{n(n-1)/2}$ and $\alpha = q$, we obtain a simple identity that expresses $R(x,y,q)$ in terms of the partial theta function:

Corollary 6.2. *The three-variable Rogers–Ramanujan function (6.1) satisfies*

$$R(x,y,q) \;=\; \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\, q^{\ell(\ell+1)/2}}{(q;q)_\ell}\, \Theta_0(xq^\ell, y) \tag{6.6}$$

as formal power series and as analytic functions on $(x,y,q) \in \mathbb{C} \times \mathbb{D} \times \mathbb{D}$.

From Corollary 6.2 we can obtain a pair of identities for $R(x,y,q)$ that generalize (2.1)/(2.2) and reduce to them when $q = 0$:

Corollary 6.3. *We have*

$$R(x,y,q) \;=\; \frac{(y;y)_\infty}{(q;q)_\infty} \sum_{\ell=0}^\infty \frac{(-1)^\ell \, q^{\ell(\ell+1)/2}}{(q;q)_\ell} \, (-xq^\ell;y)_\infty \sum_{n=0}^\infty \frac{y^n}{(y;y)_n \, (-xq^\ell;y)_n} \qquad (6.7)$$

$$R(x,y,q) \;=\; \frac{1}{(q;q)_\infty} \sum_{\ell=0}^\infty \frac{(-1)^\ell \, q^{\ell(\ell+1)/2}}{(q;q)_\ell} \, (-xq^\ell;y)_\infty \sum_{n=0}^\infty \frac{(-xq^\ell)^n \, y^{n^2}}{(y;y)_n \, (-xq^\ell;y)_n} \qquad (6.8)$$

as formal power series and as analytic functions on $(x,y,q) \in \mathbb{C} \times \mathbb{D} \times \mathbb{D}$.

Proof. Just substitute (2.1)/(2.2) into (6.6). $\square$

The function $\widetilde{R}$ defined in (1.4) is simply the rescaled version of R normalized to have $\alpha_0 = \alpha_1 = 1$:

$$\widetilde{R}(x,y,q) \;=\; R((1-q)x,y,q) \;=\; \sum_{n=0}^\infty \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \,\cdots\, (1+q+\ldots+q^{n-1})} \,. \qquad (6.9)$$

Unfortunately, I do not see how to imitate the proof of Theorem 1.1/Proposition 3.1 when $-1 < q < 0$ or $0 < q \le 1$. But perhaps I am missing something.

7 Discussion

The positivity results stated in Theorems 1.1–1.3 can be better understood by placing them in the following general context [42]: For $\alpha \in \mathbb{R} \setminus \{0\}$, let us define the class $\mathcal{S}_\alpha$ to consist of those formal power series $f(y)$ with real coefficients and constant term 1 for which the series

$$\frac{f(y)^\alpha - 1}{\alpha} \;=\; \sum_{m=1}^\infty b_m(\alpha) \, y^m \qquad (7.1)$$

has all nonnegative coefficients. The class $\mathcal{S}_0$ consists of those f for which the formal power series

$$\log f(y) \;=\; \sum_{m=1}^\infty b_m(0) \, y^m \qquad (7.2)$$

has all nonnegative coefficients. The containment relations between the classes $\mathcal{S}_\alpha$ are given by the following fairly easy result [42]:

Proposition 7.1. *Let* $\alpha, \beta \in \mathbb{R}$. *Then* $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ *if and only if either*

(a) $\alpha \le 0$ *and* $\beta \ge \alpha$, *or*

(b) $\alpha > 0$ *and* $\beta \in \{\alpha, 2\alpha, 3\alpha, \ldots\}$.

Moreover, the containment is strict whenever $\alpha \neq \beta$.

For the partial theta function (1.2), Theorem 1.1 states that $\xi_0 \in \mathcal{S}_1$; Theorem 1.2 states the stronger result that $\xi_0 \in \mathcal{S}_{-1}$ (and hence that $\xi_0 \in \mathcal{S}_\alpha$ for all $\alpha \ge -1$); and Theorem 1.3 states the yet stronger result that $\xi_0 \in \mathcal{S}_{-2}$ (and hence that $\xi_0 \in \mathcal{S}_\alpha$ for all $\alpha \ge -2$). This is best possible, since from

$$\frac{\xi_0(y)^\alpha - 1}{\alpha} \;=\; y \,+\, \frac{\alpha+3}{2} y^2 \,+\, \frac{(\alpha+2)(\alpha+7)}{6} y^3 \,+\, O(y^4) \tag{7.3}$$

we see immediately that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -2$.

For the deformed exponential function (1.3), I conjecture that $\xi_0 \in \mathcal{S}_{-1}$ (see also [44, Example 4.3]), and I have verified this through order y^{899}. It follows from the asymptotics of $\xi_0(y)$ as $y \uparrow 1$ [45] that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -1$.

For the function $\widetilde{R}$ defined in (1.4), I conjecture that $\xi_0 \in \mathcal{S}_{-1}$ for all $q > -1$, and I have verified this through order y^{349}. More strongly, I conjecture that for $q > -1$ there is a function $\alpha_\star(q)$ such that $\xi_0(y;q) \in \mathcal{S}_\alpha$ if and only if $\alpha \ge \alpha_\star(q)$, and having the following properties:

(a) $\alpha_\star(q) = -3$ for $-1 < q \le -1/2$.

(b) $\alpha_\star(q)$ is strictly increasing on $-1/2 \le q \le 1$ and strictly decreasing on $q \ge 1$.

(c) $\alpha_\star(0) = -2$.

(d) $\alpha_\star(1) = -1$.

(e) $\alpha_\star(q) = \alpha_\star(1/q)$ for $q > 0$.

Since

$$\frac{\xi_0(y,q)^\alpha - 1}{\alpha} \;=\; \frac{y}{1+q} \,+\, \frac{\alpha+3}{2}\, \frac{y^2}{(1+q)^2} \,+\, O(y^3)\;, \tag{7.4}$$

we see immediately that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -3$. Figure 1 shows numerical computations of the largest real root of $b_m(\alpha)$ [cf. (7.1)], as a function of $q \in (-1,2]$, for $2 \le m \le 50$. The upper envelope of these curves should be $\alpha_\star(q)$. The simple conjecture $\alpha_\star(q) \le -2+q$ (shown as a dashed black line) barely fails in the range $0 < q \lesssim 0.145103$ because of the coefficient of y^3, and in the range $0.378619 \lesssim q \lesssim 0.660551$ because of the coefficient of y^5; but it appears to hold for $-1 < q \le 0$. Indeed, for $-1 < q \le 0$ it appears that $b_m(\alpha) \ge 0$ whenever $\alpha \ge -3$ and $m \ne 3$.

Finally, though in this paper I have treated $\xi_0(y)$ as a formal power series, it is not difficult to show [45, 48], using Rouché's theorem, that $\xi_0(y)$ is in fact convergent for $|y| < \delta_1 \approx 0.2247945929$, where δ_1 is the positive root of $\sum_{\ell=-1}^\infty \delta^{\ell^2/2} = 2$. (This proof applies to both Θ_0 and F, and more generally to $\widetilde{R}$ for all $q \ge 0$.) Then the coefficientwise positivity established in Theorem 1.1 implies, by Pringsheim's theorem, that the first singularity of $\xi_0(y)$ for the partial theta function lies on the positive real axis, namely at the point $y = y_{01}^\star$ where the leading root $x_0(y)$ collides with the next root $x_1(y)$: this is the solution of the system

$$\Theta_0(x,y) \;=\; 0 \text{ and } \frac{\partial\Theta_0(x,y)}{\partial x} \;=\; 0 \tag{7.5}$$

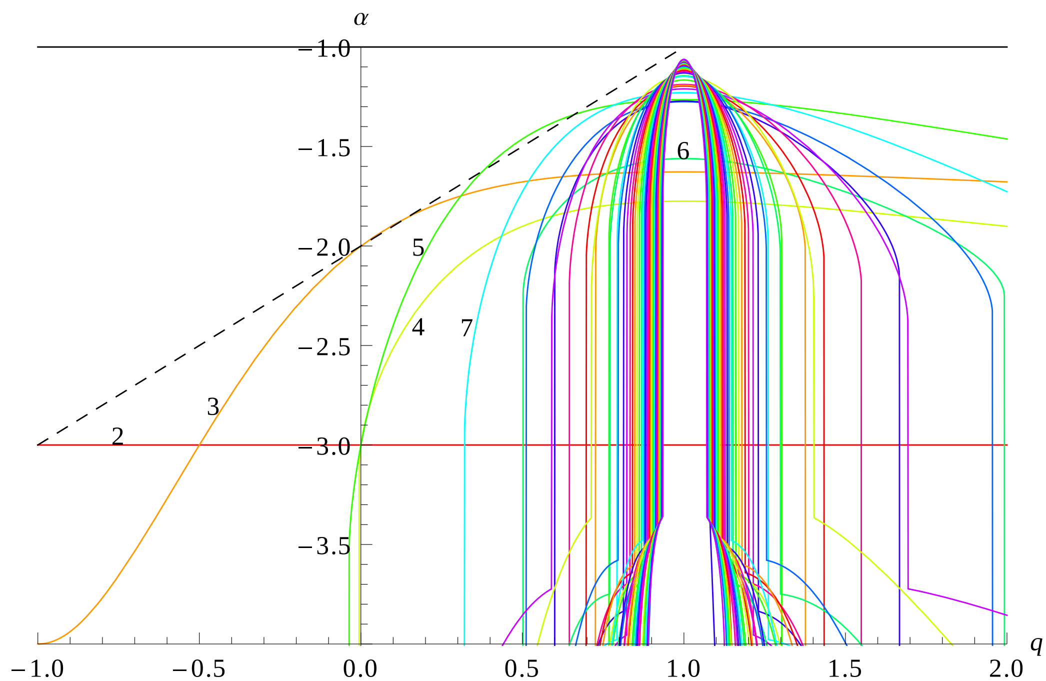

Figure 1: Largest real root of $b_m(\alpha)$ as a function of q for $2 \leq m \leq 50$. The curves corresponding to $m \leq 7$ are labeled. The dashed black line is $\alpha = -2 + q$.

and lies at $(x, y) = (x^\star_{01}, y^\star_{01}) \approx (-2.3203769443, 0.3092493386)$.[13] Similarly, for the deformed exponential function (1.3) it is known [34, 33, 32] that $\xi_0(y)$ is analytic in a complex neighborhood of the real interval $0 < y < 1$; therefore, if the coefficients are indeed nonnegative, Pringsheim's theorem implies the striking fact that $\xi_0(y)$ is analytic in the whole unit disc $|y| < 1$.

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[13] For more information concerning the real roots of the partial theta function and related polynomials, see [23, p. 100] [26, pp. 330–331] [36, vol. 1, Part II, Problem 200, pp. 143 and 345–346, and vol. 2, Part IV, Problem 176, pp. 66 and 245–246] [31] [7, Sections 2 and 3] [20, Example 4.10] [28, Theorem 4].

work was completed.

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