Counting Self-Dual Interval Orders

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Abstract

In this paper, we present a new method to derive formulas for the generating functions of interval orders, counted with respect to their size, magnitude, and number of minimal and maximal elements. Our method allows us not only to generalize previous results on refined enumeration of general interval orders, but also to enumerate self-dual interval orders with respect to analogous statistics.

Using the newly derived generating function formulas, we are able to prove a bijective relationship between self-dual interval orders and upper-triangular matrices with no zero rows. Previously, a similar bijective relationship has been established between general interval orders and upper-triangular matrices with no zero rows and columns.
Key words: interval orders, $(\mathbf{2}+\mathbf{2})$-free posets, self-dual posets

1 Introduction

The aim of this paper is to enumerate interval orders (also known as $(\mathbf{2}+\mathbf{2})$-free posets) with respect to several natural poset statistics, including the size, the magnitude, and the number of minimal and maximal elements. We are mostly motivated by the generating function formulas recently obtained by Bousquet-Mélou et al. [3], Kitaev and Remmel [17], and Dukes et al. [7].

Although the formulas derived in this paper provide a common generalization of these previous results, the method we use is different. The previous results were derived using a recursive bijection between interval orders and ascent sequences, due to Bousquet-Mélou et al. [3]. In this paper, we instead use an encoding of interval orders by upper-triangular matrices without zero rows and zero columns (we call such matrices *Fishburn matrices*). Our approach, which builds upon previous work of Haxell et al. [14], is considerably simpler than the approach based on ascent sequences. More importantly, our approach

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allows to easily capture the notion of poset duality, which corresponds to transposition of Fishburn matrices. Consequently, we are able to adapt our method to the problem of enumerating self-dual interval orders, for which no explicit enumeration has been known before. As a by-product, we establish a bijective correspondence between self-dual interval orders and upper-triangular integer matrices with no zero rows, in which several natural poset statistics map into natural matrix statistics.

Basic Notions

All the posets considered in this paper are assumed to be finite. We also assume that the posets are *unlabeled*, that is, isomorphic posets are taken to be identical. Let P be a poset with a strict order relation $\prec$. A *strict down-set* of an element $y \in P$ is the set $D(y)$ of all the elements of P that are smaller than y, i.e., $D(y) = \{x \in P;\ x \prec y\}$. Similarly, the *strict up-set* of y, denoted by $U(y)$, is the set $\{x \in P;\ x \succ y\}$. Note that y is a minimal element of P if and only if $D(y)$ is empty, and y is a maximal element if and only if $U(y)$ is empty.

For a poset P, the following conditions are known to be equivalent [2, 10]:

- P is $(\mathbf{2}+\mathbf{2})$-*free*, that is, P does not have an induced subposet isomorphic to the disjoint union of two chains of length two.
- P has an *interval representation*, that is, to each element $x \in P$ we may associate a real closed interval $[l_x, r_x]$, in such a way that $x \prec y$ if and only if $r_x < l_y$.
- For any two elements $x, y \in P$, the strict down-sets $D(x)$ and $D(y)$ are comparable by inclusion, i.e., $D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$.
- For any two elements $x, y \in P$, the strict up-sets $U(x)$ and $U(y)$ are comparable by inclusion.

The posets that satisfy these properties are known as *interval orders* or as $(\mathbf{2}+\mathbf{2})$-*free posets*. Let us review some of their basic properties. For a more thorough exposition, the reader is referred to Fishburn's monograph [13].

Let P be an interval order. Two elements x and y of P are *indistinguishable* if $U(x) = U(y)$ and $D(x) = D(y)$. This is an equivalence relation on P. If no two distinct elements of P are indistinguishable, then P is said to be *primitive*. Every interval order P can be uniquely obtained from a primitive interval order P' by simultaneously replacing each element of P' by a positive number of 'duplicates'. Thus, the enumeration of primitive interval orders is a key step in the enumeration of general interval orders.

Since any two strict down-sets in P are comparable by inclusion, it is possible to arrange all the distinct strict down-sets into an increasing chain

$$D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_m,$$

where m is the number of distinct strict down-sets determined by elements of P. An element $x \in P$ is said to have *level* i, if $D(x) = D_i$. Note that D_1 is always the empty set, and the elements of level 1 are exactly the minimal elements of P. Following Fishburn [11, 12], we call the number m of distinct strict down-sets the *magnitude* of P. It turns out that m is also equal to the number of distinct strict up-sets, and we can order the strict up-sets of P into a decreasing chain

$$U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_m,$$

and we say that x has *up-level* i if $U(x) = U_i$. The maximal elements of P are precisely the elements of up-level m, and we have $U_m = \emptyset$. It can be shown [12] that an element of level i has an up-level greater than or equal to i. An interval representation of P can be obtained by mapping an element x with level i and up-level j to the (possibly degenerate) interval $[i, j]$. This is the unique representation of P by intervals with endpoints belonging to the set $[m] = \{1, 2, \ldots, m\}$, and in particular, there is no interval representation of P with fewer than m distinct endpoints.

The *dual* of a poset P is the poset $\overline{P}$ with the same elements as P and an order relation $\overline{\prec}$ defined by $x \overline{\prec} y \iff y \prec x$. A poset is *self-dual* if it is isomorphic to its dual. The dual of an interval order P of magnitude m is again an interval order of magnitude m, and an element of level i and up-level j in P has the level $m+1-j$ and up-level $m+1-i$ in $\overline{P}$.

Throughout this paper, an important part will be played by a bijective correspondence between interval orders and a certain kind of integer matrices, which we will call Fishburn matrices. We will state the key properties of the correspondence without proof; more details can be found, e.g., in the work of Fishburn [11, 13], where these matrices are called 'characteristic matrices'.

A *Fishburn matrix* is an upper-triangular square matrix M of nonnegative integers with the property that every row and every column contains a nonzero entry. A Fishburn matrix is called *primitive* if all its entries are equal to 0 or 1. We will assume throughout this paper that each matrix has its rows numbered from top to bottom, and columns numbered left-to-right, starting with row and column number one. We let M_{ij} denote the entry of M in row i and column j.

An interval order P of magnitude m corresponds to an $m \times m$ Fishburn matrix M with M_{ij} being equal to the number of elements of P that have level i and up-level j. Conversely, given an $m \times m$ Fishburn matrix M, we may recover the corresponding interval order P by taking the collection of intervals that contains precisely M_{ij} copies of the interval $[i, j]$, and taking this to be the interval representation of P.

This correspondence is a bijection between Fishburn matrices and interval orders. In fact, in this correspondence, each nonzero entry M_{ij} of M can be associated with a set of M_{ij} indistinguishable elements of P. Note that the sum of the i-th row of M is equal to the number of elements of level i in P, and similarly for column-sums and up-levels.

Primitive interval orders correspond to primitive Fishburn matrices. If the order P is mapped to a matrix M, then the dual order $\overline{P}$ is mapped to the matrix $\overline{M}$ obtained from M by transposition along the diagonal running from bottom-left to top-right. If a matrix M is equal to $\overline{M}$, we call it *self-dual*. Of course, self-dual matrices are representing precisely the self-dual interval orders.

Previous work and our results

Interval orders are equinumerous with several other combinatorial structures. Apart from the correspondence between interval orders and Fishburn matrices, there are also bijections mapping interval orders to ascent sequences [3], Stoimenow matchings [20], certain classes of pattern-avoiding permutations [3, 19], or special kinds of inversion tables [18]. Some of these combinatorial structures have been studied independently even before their relationship to interval orders was discovered. Thus, some results on the enumeration of interval orders were first derived in different contexts, and some of them were in fact derived several times under different guises.

The concept of interval order has been introduced by Fishburn [10] in 1970. In 1976, Andresen and Kjeldsen [1], motivated by a counting problem related to subgraphs of transitively oriented tournaments, introduced (under different terminology) the problem of enumerating Fishburn matrices. They studied, among other problems, the number of primitive Fishburn matrices with respect to their dimension and the number of elements in the first row, which in poset terminology corresponds to the number of primitive interval orders of a given magnitude and number of minimal elements (but not the number of all elements). Andresen and Kjeldsen obtained asymptotic bounds for the number of these matrices, as well as recurrence formulas that allowed them to compute several exact initial values. At the time of their writing, the connection between Fishburn matrices and interval orders was not known, and it appears that their results went unnoticed by later works on interval orders.

In 1987, Haxell, McDonald and Thomason [14] provided an efficient way to compute the number of interval orders, using a recurrence derived using Fishburn matrices, which were already known to be equinumerous with interval orders, thanks to the work of Fishburn [13].

In 1998, Stoimenow [20] introduced the concept of 'regular linearized chord diagram', later often referred to as a 'Stoimenow matching'. A Stoimenow matching of size n is a matching on the set $[2n]$ in which no two nested edges have adjacent endpoints. Stoimenow has introduced these matchings as a tool in the study of Vassiliev invariants of knots, and computed several asymptotic bounds on their number. Later, these bounds were improved by Zagier [22], who also showed that the generating function of Stoimenow matchings enumerated by their size admits a simple formula

$$F(x) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-x)^i\right). \tag{1}$$

Recently, Bousquet-Mélou, Claesson, Dukes and Kitaev [3] have found a sequence of bijections, showing that interval orders are equinumerous with several other combinatorial objects, including Stoimenow matchings and ascent sequences. They have also provided an alternative proof for (1), and derived a formula for the refined generating function that counts interval order with respect to their size and magnitude. These results have prompted a renewed interest in the study of interval orders. Several other papers have focused on bijections between interval orders and other objects. For instance, Dukes and Parviainen [9] have described a direct bijection between Fishburn matrices and ascent sequences, Claesson and Linusson [4] gave a direct mapping from Fishburn matrices to Stoimenow matchings, while the papers of Claesson et al. [5] and Dukes et al. [8] extend the bijection between interval orders and Fishburn matrices to more general combinatorial structures.

Another line of research has focused on refined enumeration of interval orders with respect to some natural poset statistics. Dukes, Kitaev, Remmel and Steingrímsson [7] have found an expression for the generating function that enumerates primitive interval orders with respect to their size and magnitude, and deduced a formula that counts interval orders by their size, magnitude and the number of indistinguishable elements. Kitaev and Remmel [17] have obtained, among other results, the formula

$$F(x,y) = 1 + \sum_{n\geq 0} \frac{xy}{(1-xy)^{n+1}} \prod_{i=1}^{n} \left(1-(1-x)^i\right), \tag{2}$$

where $F(x,y)$ is the generating function of interval orders in which x counts the size and y the number of minimal elements of the interval order. They conjectured that $F(x,y)$ can be also expressed in the following form:

$$F(x,y) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-x)^{i-1}(1-xy)\right). \tag{3}$$

This conjecture has been subsequently confirmed by Yan [21] and independently by Levande [18]. Let us remark formula (3) also appears in Zagier's work [22, Theorem 1], but there it is interpreted in terms of Stoimenow matchings, not interval orders.

In Section 2 of this paper, we generalize the above-mentioned results of Bousquet-Mélou et al. [3], Dukes et al. [7], and Kitaev and Remmel [17], by obtaining a closed-form expression for the generation function of primitive interval orders, counted with respect to their magnitude, their size, and their number of minimal and maximal elements. From this expression, it is possible to directly derive the generating function of general interval orders, or of interval orders with bounded size of indistinguishability classes, counted with respect to the same statistics.

However, the main significance of our results is not in counting more statistics than previous papers, but rather in presenting a much simpler method to derive the generating functions. Previous results were largely based on a bijection, constructed by Bousquet-Mélou et al. [3], which maps interval orders to a certain kind of integer sequences, known as 'ascent sequences'. This bijection has a complicated recursive definition, which consequently leads to difficult recurrences for generating functions, which then require a large amount of ingenuity to be solved into closed form expressions, typically by an application of the kernel method. In contrast, the arguments in this paper are based on the direct encoding of interval orders as Fishburn matrices. We exploit a relationship between Fishburn matrices of dimension m and those of dimension $m+1$ to obtain a recurrence for the generating function that can be easily solved by elementary expression manipulation.

To further illustrate the benefit of this approach, in Section 3 we enumerate self-dual interval orders, by a slightly more elaborate application of the same basic technique. The duality map of interval orders translates into 'obvious' involutions on related combinatorial classes — for example, it corresponds to transposition of a Fishburn matrix along the north-east diagonal, or to the reversal of vertex order in a Stoimenow matching. This suggest that poset duality represents a fundamental symmetry of these classes of objects, and it is therefore natural to consider the enumeration of symmetric objects, that is, of objects that are fixed points of this symmetry map.

The problems of counting self-dual interval orders with respect to their size and of counting primitive self-dual Fishburn matrices with respect to their dimension do not seem to have been addressed by previous research, presumably because there is no known way to characterize self-duality in terms of ascent sequences. In view of this, it is remarkable that the expressions we obtain for the generating functions of self-dual objects are almost as simple as those of their non-self-dual counterparts.

Moreover, in Section 4, we introduce *row-Fishburn matrices*, which are upper-triangular matrices with no zero rows, and we prove that there is a close link between the refined enumeration of these matrices and the refined enumeration of self-dual interval orders. This yields a surprising analogue to the correspondence between general interval orders and Fishburn matrices.

2 Enumeration of Interval Orders

Recall that a primitive poset is a poset that does not contain any two indistinguishable elements. Our main concern will be to find an expression for the generating function of primitive interval orders, or equivalently, of 01-Fishburn matrices.

Let us call an element of a poset P *isolated* if it is not comparable to any other element of P. Note that an element is isolated if and only if it is both minimal and maximal. The number of isolated elements of an interval order P is equal to the value of the top-right cell of the corresponding Fishburn matrix. We will call the top-right cell of the matrix *the corner cell*.

Let us also say that an element of the poset P is *internal* if it is neither minimal nor maximal. We will consider these statistics of an interval order P:

- the magnitude of P, denoted by $\mathrm{mag}(P)$,
- the number of isolated elements, denoted by $\mathrm{iso}(P)$,
- the number of non-isolated minimal elements, denoted by $\min(P)$,
- the number of non-isolated maximal elements, denoted by $\max(P)$, and

statistic	poset interpretation	matrix interpretation	variable
mag	magnitude	number of rows	t
iso	num. of isolated elements	value of corner cell	x
min	num. of non-isolated minimal elements	sum of the first row, except the corner cell	y
max	num. of non-isolated maximal elements	sum of the last column, except the corner cell	v
int	num. of internal elements	sum of cells outside first row and last column	w

Table 1: The statistics of interval orders

- the number of internal elements, denoted by $\mathrm{int}(P)$.

In particular, the size of P is equal to $\mathrm{iso}(P) + \min(P) + \max(P) + \mathrm{int}(P)$, and its number of minimal elements is equal to $\min(P) + \mathrm{iso}(P)$. In Table 1, we summarize these statistics, with their interpretation both in terms of posets and in terms of matrices. As a matter of convenience, we restrict ourselves to non-empty interval orders and non-empty matrices in all the arguments. If M is the Fishburn matrix representing an interval order P, we write $\mathrm{iso}(M)$ as a synonym for $\mathrm{iso}(P)$, and similarly for the other statistics.

Let $\mathcal{P}$ be the set of all non-empty primitive interval orders, and let G be the generating function

$$G(t, v, w, x, y) = \sum_{P \in \mathcal{P}} t^{\mathrm{mag}(P)} x^{\mathrm{iso}(P)} y^{\min(P)} v^{\max(P)} w^{\mathrm{int}(P)}.$$

We shall use the following notation: for an integer $n \geq 0$, we let $V_n(a, b)$ denote the polynomial $(a + 1)(b + 1)^n - 1$.

We can now state the main result of this section.

Theorem 2.1. *The generating function $G(t, v, w, x, y)$ satisfies the identity*

$$G(t, v, w, x, y) = \sum_{n \geq 0} t^{n+1} \frac{V_n(x, y)}{1 + tV_n(v, w)} \prod_{i=0}^{n-1} \frac{V_i(v, w)}{1 + tV_i(v, w)}. \tag{4}$$

Remark 2.2. Let $S_n \equiv S_n(t, v, w, x, y)$ denote the n-th summand of the sum on the right-hand side of (4). Clearly, S_n is a multiple of t^{n+1}. Consequently, the sum on the right-hand side of (4) converges as a sum of power series in t. Moreover, since for each $i \geq 0$ the polynomial $V_i(v, w)$ has constant term equal to zero, it follows that S_n has total degree at least n in the variables v and w. Thus, the sum also converges as a sum of power series in v and w. Furthermore, for all k, the coefficient of t^k in S_n is a polynomial in v, w, x and y, and for all k, ℓ, the coefficient of $v^k w^\ell$ is a polynomial in t, x, and y.

The properties stated in the above remark make the identity (4) amenable to many combinatorially meaningful substitutions. Before we state the proof of the theorem, we demonstrate several possible substitutions. Note that some of the formulas we derive have been previously obtained by different methods.

Corollary 2.3. *[7, Theorem 8] Let p_k be the number of primitive interval orders of size k. The generating function of p_k is equal to*

$$\sum_{k\geq 1} p_k x^k = G(1,x,x,x,x) = \sum_{n\geq 0}\prod_{i=0}^{n}\frac{V_i(x,x)}{1+V_i(x,x)} = \sum_{n\geq 0}\prod_{i=0}^{n}\left(1-\frac{1}{(1+x)^{i+1}}\right).$$

Corollary 2.4. *Let m_k be the number of primitive $k\times k$ Fishburn matrices (or equivalently, the number of primitive interval orders of magnitude k). Then*

$$\sum_{m\geq 1} m_k t^k = G(t,1,1,1,1) = \sum_{n\geq 0} t^{n+1}\prod_{i=0}^{n}\frac{2^{i+1}-1}{1+t(2^{i+1}-1)}.$$

Let $\mathcal{G}$ be the set of non-empty interval orders, and let $G^*(t,v,w,x,y)$ be the corresponding generating function

$$G^*(t,v,w,x,y) = \sum_{P\in\mathcal{G}} t^{\mathrm{mag}(P)}x^{\mathrm{iso}(P)}y^{\mathrm{min}(P)}v^{\mathrm{max}(P)}w^{\mathrm{int}(P)}.$$

Each interval order can be uniquely obtained from a primitive interval order by replacing each element with a group of indistinguishable elements. In terms of generating functions, this means that

$$G^*(t,v,w,x,y) = G(t,\frac{v}{1-v},\frac{w}{1-w},\frac{x}{1-x},\frac{y}{1-y}).$$

By substituting into (4), and using, e.g., the identity

$$V_n(\frac{a}{1-a},\frac{b}{1-b}) = -\frac{V_n(-a,-b)}{(1-a)(1-b)^n} = -\frac{V_n(-a,-b)}{1+V_n(-a,-b)}$$

we get the next corollary.

Corollary 2.5. *The generating function $G^*(t,v,w,x,y)$ is equal to*

$$\sum_{n\geq 0} t^{n+1}\frac{1+V_n(-v,-w)}{1+V_n(-x,-y)}\cdot\frac{V_n(-x,-y)}{(t-1)V_n(-v,-w)-1}\cdot\prod_{i=0}^{n-1}\frac{V_i(-v,-w)}{(t-1)V_i(-v,-w)-1}.$$

From Corollary 2.5 we can obtain yet another proof of the formula (3) derived by Levande [18] and Yan [21] (and indirectly also by Zagier [22]) for the generating function of interval orders counted by their size and number of maximal elements.

Corollary 2.6. *[18, 21, 22] Let $g_{k,\ell}$ be the number of interval orders with k elements and having exactly ℓ maximal elements (including isolated ones). We have*

$$\sum_{k,\ell\geq 1} g_{k,\ell}\, r^k s^\ell = G^*(1,rs,r,rs,r) = \sum_{n\geq 0}\prod_{i=0}^{n} -V_i(-rs,-r)$$
$$= \sum_{n\geq 0}\prod_{i=0}^{n}(1-(1-rs)(1-r)^i).$$

Kitaev and Remmel [17] have obtained a different expression for the generating function from the previous corollary. This alternative expression can also be derived from the general formula for G^*.

Lemma 2.7. *[17] With $g_{k,\ell}$ as in Corollary 2.6, we have*

$$\sum_{k,\ell\geq 1} g_{k,\ell}\, r^k s^\ell = \sum_{n\geq 0} \frac{rs}{(1-rs)^{n+1}} \prod_{i=0}^{n-1} \left(1-(1-r)^{i+1}\right). \tag{5}$$

Proof. Since the dual of an interval order is also an interval order, $g_{k,\ell}$ is also equal to the number of interval orders with k elements and ℓ minimal elements. Therefore, $\sum_{k,\ell} g_{k,\ell} r^k s^\ell = G^*(1,r,r,rs,rs)$, which is equal to

$$\begin{aligned}
&\sum_{n\geq 0}\frac{1+V_n(-r,-r)}{1+V_n(-rs,-rs)}\left(-V_n(-rs,-rs)\right)\prod_{i=0}^{n-1}\left(-V_i(-r,-r)\right)\\
=&\sum_{n\geq 0}\frac{1-(1-rs)^{n+1}}{(1-rs)^{n+1}}\left(1+V_n(-r,-r)\right)\prod_{i=0}^{n-1}\left(-V_i(-r,-r)\right)\\
=&\sum_{n\geq 0}\frac{1-(1-rs)^{n+1}}{(1-rs)^{n+1}}\prod_{i=0}^{n-1}\left(-V_i(-r,-r)\right)-\sum_{n\geq 0}\frac{1-(1-rs)^{n+1}}{(1-rs)^{n+1}}\prod_{i=0}^{n}\left(-V_i(-r,-r)\right)\\
=&\sum_{n\geq 0}\left(\frac{1-(1-rs)^{n+1}}{(1-rs)^{n+1}}-\frac{1-(1-rs)^{n}}{(1-rs)^{n}}\right)\prod_{i=0}^{n-1}\left(-V_i(-r,-r)\right),
\end{aligned}$$

which simplifies to yield (5). ☐

Let us now prove Theorem 2.1. Define $G_k(v,w,x,y) = [t^k]G(t,v,w,x,y)$, that is, G_k is the coefficient of t^k in G. We will state the proof in the language of Fishburn matrices rather than in the equivalent language of interval orders. It is thus convenient to view G_k as the generating function of the primitive $k\times k$ Fishburn matrices.

The next lemma provides the main idea in the proof of Theorem 2.1.

Lemma 2.8. *For any $k\geq 1$, we have*

$$G_{k+1}(v,w,x,y) = vG_k(v+w+vw,w,x+y+xy,y) - vG_k(v,w,x,y). \tag{6}$$

Proof. Let $\mathcal{M}_k$ denote the set of primitive $k\times k$ Fishburn matrices. We will describe an operation which from a given matrix $M\in\mathcal{M}_k$ produces a (typically not unique) new matrix $M'\in\mathcal{M}_{k+1}$. The matrix M' is created by adding to M a new rightmost column and a new bottom row, and filling them according to these rules:

- $M'_{k+1,k+1}=1$, and all the other cells in row $k+1$ of M' have value 0.

- For every $j \leq k$, if $M_{j,k} = 0$, then $M'_{j,k} = M'_{j,k+1} = 0$.
- For every $j \leq k$, if $M_{j,k} = 1$ we choose one of the three possibilities to fill $M'_{j,k}$ and $M'_{j,k+1}$: either $M'_{j,k} = 0$ and $M'_{j,k+1} = 1$, or $M'_{j,k} = M'_{j,k+1} = 1$, or $M'_{j,k} = 1$ and $M'_{j,k+1} = 0$.
- Any other cell has the same value in M' as in M.

If M has p 1-cells in column k, then the above operation can produce 3^p matrices M'. All these 3^p matrices are upper-triangular, all of them have at least one 1-cell in each row, all of them have at least one 1-cell in each column different from column k, and all except for exactly one of them have at least one 1-cell in column k. In other words, for a given $M \in \mathcal{M}_k$ with p 1-cells in column k, the above operation produces $3^p - 1$ matrices M' from $\mathcal{M}_{k+1}$ (and one 'bad' matrix not belonging to $\mathcal{M}_{k+1}$). It is not difficult to see that each matrix $M' \in \mathcal{M}_{k+1}$ can be created in this way from exactly one matrix $M \in \mathcal{M}_k$.

More generally, suppose that $M \in \mathcal{M}_k$ is a matrix with $\mathrm{iso}(M) = a$, $\min(M) = b$, $\max(M) = c$ and $\mathrm{int}(M) = d$, that is, M contributes the monomial $x^a y^b v^c w^d$ into $G_k(v,w,x,y)$. Then all the Fishburn matrices produced from M have generating function $v\left((x+y+xy)^a y^b (v+w+vw)^c w^d - x^a y^b v^c w^d\right)$, where the leftmost factor of v counts the 1-cell $(k+1, k+1)$ of M'. Summing this expression over all $M \in \mathcal{M}_k$ gives the recurrence from the statement of the lemma. □

We remark the recursive procedure that generates matrices of $\mathcal{M}_{k+1}$ from matrices of $\mathcal{M}_k$ is not new. In fact, this idea has already been used by Haxell, McDonald, and Thomason [14] to obtain an efficient algorithm for the enumeration of interval orders. It is also very closely related to the approach that Khamis [16] has used to derive a recurrence formula for the number of interval orders of a given size and height.

We now deduce Theorem 2.1 from Lemma 2.8 by simple manipulation of series. Let us introduce the following notation: for any power series $F(v,w,x,y)$, let $\sigma[F(v,w,x,y)]$ denote the power series $F(v+w+vw, w, x+y+xy, y)$, that is, σ represents the substitution of $v+w+vw$ for v and $x+y+xy$ for x. Let $\sigma^{(i)}[F(v,w,x,y)]$ denote the i-fold iteration of σ; in other words, $\sigma^{(0)}[F] = F$ and for $i \geq 1$, $\sigma^{(i)}[F] = \sigma\left[\sigma^{(i-1)}[F]\right]$. It can be easily checked by induction that

$$\sigma^{(i)}[F(v,w,x,y)] = F(V_i(v,w), w, V_i(x,y), y).$$

Writing G_k for $G_k(v,w,x,y)$ and G for $G(t,v,w,x,y)$, we can rewrite (6) as $G_{k+1} = v\sigma[G_k] - vG_k$. Multiplying this by t^{k+1} and summing for all $k \geq 1$, we get $G - tG_1 = tv\sigma[G] - tvG$. Since $G_1 = x$, this simplifies into

$$G = \frac{tx}{1+tv} + \frac{tv}{1+tv}\sigma[G]. \tag{7}$$

Substituting the right-hand side of this expression for the occurrence of G on the right-hand side, we obtain

$$G = \frac{tx}{1+tv} + \frac{tv}{1+tv}\sigma\left[\frac{tx}{1+tv}\right] + \frac{tv}{1+tv}\sigma\left[\frac{tv}{1+tv}\right]\sigma^{(2)}[G]. \tag{8}$$

We may again substitute the right-hand side of (7) into the right-hand side of (8). In general, iterating this substitution m times gives the identity

$$G = \left(\sum_{n=0}^{m} \sigma^{(n)}\left[\frac{tx}{1+tv}\right] \prod_{i=0}^{n-1} \sigma^{(i)}\left[\frac{tv}{1+tv}\right]\right) + \left(\prod_{i=0}^{m} \sigma^{(i)}\left[\frac{tv}{1+tv}\right]\right) \sigma^{(m+1)}[G]. \tag{9}$$

Since the rightmost summand of the right-hand side of (9) is $\mathcal{O}(t^{m+1})$, we can take the limit as m goes to infinity, to obtain

$$\begin{aligned} G &= \sum_{n\geq 0} \sigma^{(n)}\left[\frac{tx}{1+tv}\right] \prod_{i=0}^{n-1} \sigma^{(i)}\left[\frac{tv}{1+tv}\right] \\ &= \sum_{n\geq 0} t^{n+1} \frac{V_n(x,y)}{1+tV_n(v,w)} \prod_{i=0}^{n-1} \frac{V_i(v,w)}{1+tV_i(v,w)}. \end{aligned}$$

This proves Theorem 2.1.

3 Self-Dual Interval Orders

Recall that a $k \times k$ Fishburn matrix M represents a self-dual interval order if and only if M is invariant under transposition along the north-east diagonal, or in other words, if for each i, j the cell (i, j) has the same value as the cell $(k-j+1, k-i+1)$. We will say that the two cells (i, j) and $(k-j+1, k-i+1)$ form a *symmetric pair.*

As in the previous section, we will first concentrate on enumerating the primitive matrices, and the enumeration of general integer matrices is obtained as a consequence. Unless otherwise noted, the generating functions are only counting nonempty objects.

We distinguish three types of cells in a $k \times k$ matrix M: a cell (i, j) is a *diagonal cell* if $i + j = k + 1$, i.e., (i, j) belongs to the north-east diagonal of the matrix. If $i + j < k + 1$ (i.e., (i, j) is above and to the left of the diagonal) then (i, j) is a *North-West cell*, or *NW-cell*, while if $i + j > k + 1$, then (i, j) is an *SE-cell.* The diagonal cells and SE-cells together uniquely determine a self-dual matrix.

Apart of the statistics introduced in Section 2 (see Table 1), we will also consider three new statistics of a matrix M.

- $\mathrm{diag}(M)$ is the sum of all the diagonal cells except for the corner cell.
- $\mathrm{se}(M)$ is the sum of all the SE-cells that do not belong to the last column.
- $\mathrm{nw}(M)$ is the sum of all the NW-cells that do not belong to the first row.

In particular, the sum of all cells in M is equal to $\mathrm{iso}(M)+\min(M)+\max(M)+\mathrm{se}(M)+\mathrm{nw}(M)+\mathrm{diag}(M)$. In a self-dual matrix we of course have $\min(M) = \max(M)$ and $\mathrm{se}(M) = \mathrm{nw}(M)$, so the above sum is equal to $\mathrm{iso}(M)+\mathrm{diag}(M)+2\max(M)+2\mathrm{se}(M)$.

Notice that if $k > 1$, then among all the $k \times k$ primitive self-dual Fishburn matrices, exactly half have the corner cell filled with 1 and half have the corner cell filed with 0. This is because changing the corner cell from 1 to 0 cannot create an empty row or empty column, since the cells $(1,1)$ and (k,k) are always 1-cells and the value of the corner cell also does not affect the symmetry of the matrix. It is simpler to first enumerate the symmetric matrices whose corner cell has the fixed value 1, and then use this result to obtain the full count, rather than to enumerate all the symmetric matrices at once.

Let $\mathcal{S}^+$ be the set of primitive self-dual Fishburn matrices whose corner cell is equal to 1. Define the generating function $S^+(t,v,w,z)$ by

$$S^+(t,v,w,z) = \sum_{M \in \mathcal{S}^+} t^{\mathrm{mag}(M)} v^{\max(M)} w^{\mathrm{se}(M)} z^{\mathrm{diag}(M)}.$$

The next theorem is the key result of this section.

Theorem 3.1. *The generating function $S^+(t,v,w,z)$ is equal to*

$$\sum_{n \geq 0} t^{2n+1} \frac{1 + tV_n(v,w)}{1 + t^2 V_n(v,w)} (1+z)^n (1+v)^n (1+w)^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{V_i(v,w)}{1 + t^2 V_i(v,w)}. \tag{10}$$

The comments in Remark 2.2 apply to the expression (10) as well.

Before we prove Theorem 3.1, we first state some of its consequences. Although Theorem 3.1 provides all the information we need for our enumerations, it is often more convenient to work with the closely related generating function that counts all Fishburn matrices, rather than just those that belong to $\mathcal{S}^+$. Let $\mathcal{S}$ be the set of primitive self-dual Fishburn matrices, and define

$$S(t,v,w,x,z) = \sum_{M \in \mathcal{S}} t^{\mathrm{mag}(M)} v^{\min(M)+\max(M)} w^{\mathrm{se}(M)+\mathrm{nw}(M)} x^{\mathrm{iso}(M)} z^{\mathrm{diag}(M)}.$$

Lemma 3.2. *The generating function S satisfies the identity*

$$S(t,v,w,x,z) = (1+x) S^+(t, v^2, w^2, z) - t. \tag{11}$$

Consequently, $S(t,v,w,x,z)$ is equal to

$$-t + (1+x) \sum_{n \geq 0} t^{2n+1} \frac{1 + tV_n(v^2,w^2)}{1 + t^2 V_n(v^2,w^2)} (1+z)^n (1+v^2)^n (1+w^2)^{\binom{n}{2}} \times \prod_{i=0}^{n-1} \frac{V_i(v^2,w^2)}{1 + t^2 V_i(v^2,w^2)}.$$

Proof. The factor $(1+x)$ on the right-hand side of (11) corresponds to the fact that each primitive self-dual matrix either belongs to $\mathcal{S}^+$ or is obtained from a matrix in $\mathcal{S}^+$ by changing its corner cell from 1 to 0. The subtracted t accounts for the fact that the 1×1 matrix containing 0 is not a Fishburn matrix, even though it can be obtained from a matrix in $\mathcal{S}^+$ by changing the corner cell. The substitutions into S^+ correspond straightforwardly to the fact that $v^{\min(M)+\max(M)} = v^{2\max(M)}$ for any self-dual matrix, and similarly for w. $\square$

Corollary 3.3. *Let s_m be the number of self-dual primitive interval orders on m elements, with $s_0 = 1$. Then*

$$\sum_{m\geq 0} s_m x^m = 1 + S(1,x,x,x) = \sum_{n\geq 0}(1+x)^{n+1}\prod_{i=0}^{n-1}\left((1+x^2)^{i+1}-1\right).$$

Corollary 3.4. *Let r_m be the number of primitive self-dual $m \times m$ Fishburn matrices. Then*

$$\begin{aligned}\sum_{m\geq 1} r_m t^m &= S(t,1,1,1)\\ &= -t + \sum_{n\geq 0} 2^{\binom{2n+2}{2}} t^{2n+1}\frac{1+(2^{n+1}-1)t}{1+(2^{n+1}-1)t^2}\prod_{i=0}^{n-1}\frac{2^{i+1}-1}{1+(2^{i+1}-1)t^2}.\end{aligned}$$

Let $S^*(t,v,w,x,z)$ be the generating function of (not necessarily primitive) self-dual interval orders, with variables having the same meaning as in $S(t,v,w,x,z)$. Clearly, a Fishburn matrix M representing a self-dual interval order may be obtained in a unique way from a matrix M' representing a primitive self-dual interval order, by changing each diagonal 1-cell of M' into an arbitrary non-zero cell, and by changing a symmetric pair of non-diagonal 1-cells of M' into a pair of nonzero cells having the same value. Repeating the reasoning of Lemma 3.2, we get the identity

$$S^*(t,v,w,x,z) = \frac{1}{1-x} S^+\left(t, \frac{v^2}{1-v^2}, \frac{w^2}{1-w^2}, \frac{z}{1-z}\right) - t.$$

It follows that $S^*(t,v,w,x,z)$ is equal to

$$-t + \sum_{n\geq 0}\frac{t^{2n+1}\left(1+(1-t)V_n(-v^2,-w^2)\right)\prod_{i=0}^{n-1}\frac{-V_i(-v^2,-w^2)}{1+(1-t^2)V_i(-v^2,-w^2)}}{(1-x)(1-z)^n(1-v^2)^n(1-w^2)^{\binom{n}{2}}\left(1+(1-t^2)V_n(-v^2,-w^2)\right)}.$$

Corollary 3.5. *Let g_m be the number of self-dual interval orders on m elements, with $g_0 = 1$. Then*

$$\begin{aligned}\sum_{m\geq 0} g_m x^m &= 1 + S^*(1,x,x,x,x)\\ &= \sum_{n\geq 0}\frac{1}{(1-x^2)^{\binom{n+1}{2}}(1-x)^{n+1}}\prod_{i=0}^{n-1}\left(1-(1-x^2)^{i+1}\right)\\ &= \sum_{n\geq 0}\frac{1}{(1-x)^{n+1}}\prod_{i=0}^{n-1}\left(\frac{1}{(1-x^2)^{i+1}}-1\right).\end{aligned}$$

The proof of Theorem 3.1 is based on the same general idea as the proof of Theorem 2.1. Let us define $S_k^+(v,w,z) = [t^k]S^+(t,v,w,z)$. The next lemma is the self-dual analogue of Lemma 2.8.

Lemma 3.6. *For any $k \geq 1$, we have*

$$S^+_{k+2}(v,w,z) = v(1+v)(1+z)S^+_k(v+w+vw,w,z) - vS^+_k(v,w,z). \quad (12)$$

Proof. Let $\mathcal{S}^+_k$ be the set of matrices of $\mathcal{S}^+$ of size $k \times k$. We will show how a given matrix $M \in \mathcal{S}^+_k$ can be extended into a matrix $M' \in \mathcal{S}^+_{k+2}$. Assume, just for the sake of this proof, that matrices in $\mathcal{S}^+_k$ have rows and columns indexed by $1, 2, \dots, k$, while matrices in $\mathcal{S}^+_{k+2}$ have rows and columns indexed by $0, 1, \dots, k+1$. Thus, if a cell (i,j) is a diagonal cell in $M \in \mathcal{S}^+_k$, then (i,j) is also a diagonal cell in $M' \in \mathcal{S}^+_{k+2}$, and similarly for SE-cells and NW-cells.

The matrix M' is created from M by adding a new left-most and right-most row, and a new top-most and bottom-most column, and then filling the new cells by these rules:

- $M'_{k+1,k+1} = 1$, and any other cell in row $k+1$ of M' has value 0.
- $M'_{0,k+1} = 1$. Note that the cell $(0, k+1)$ is the corner cell of M'.
- $M'_{1,k}$ is filled arbitrarily by 0 or 1, and $M'_{1,k+1}$ is filled arbitrarily by 0 or 1 as well. (Recall that $M_{1,k} = 1$ by the definition of $\mathcal{S}^+$.)
- For any $j \in \{2, \dots, k\}$, if $M_{j,k} = 0$, then $M'_{j,k} = M'_{j,k+1} = 0$.
- For any $j \in \{2, \dots, k\}$, if $M_{j,k} = 1$, we choose one of three possibilities to fill $M'_{j,k}$ and $M'_{j,k+1}$: either $M'_{j,k} = 0$ and $M'_{j,k+1} = 1$, or $M'_{j,k} = M'_{j,k+1} = 1$, or $M'_{j,k} = 1$ and $M'_{j,k+1} = 0$.
- Any other SE-cell or diagonal cell of M' has the same value in M' as in M, and the NE-cells of M' are filled in order to form a self-dual matrix.

From a given matrix $M \in \mathcal{S}^+_k$ with $\max(M) = p$, this procedure creates $4 \cdot 3^p$ distinct self-dual matrices of size $(k+2) \times (k+2)$. Of these $4 \cdot 3^p$ matrices, all belong to $\mathcal{S}^+_{k+2}$ except for one matrix, which has column k (and hence also row k) filled with zeros.

In terms of generating functions, if M is a matrix that contributes a monomial $v^p w^q z^r$ into the generating function $S^+_k(v,w,z)$, then the matrices in $\mathcal{S}^+_{k+2}$ created from M have generating function

$$\sum_{\substack{M' \in \mathcal{S}^+_{k+2} \\ M' \text{ obtained from } M}} v^{\max(M')} w^{\mathrm{se}(M')} z^{\mathrm{diag}(M')} =$$

$$= v\big((1+v)(1+z)(v+w+vw)^p w^q z^r - v^p w^q z^r\big). \quad (13)$$

It is easy to see that each matrix $M' \in \mathcal{S}^+_{k+2}$ can be generated by the above rules from a unique matrix $M \in \mathcal{S}^+_k$. By summing the identity (13) over all $M \in \mathcal{S}^+_k$, we obtain the identity (12). □

To prove Theorem 3.1 from Lemma 3.6, we imitate the proof of Theorem 2.1 from Lemma 2.8. Let us write S^+ instead of $S^+(t,v,w,z)$, and S^+_k instead of $S^+_k(v,w,z)$. Multiplying (12) by t^{k+2} and summing for all $k \geq 1$ gives

$$S^+ - tS^+_1 - t^2 S^+_2 = t^2 v(1+v)(1+z)\sigma[S^+] - t^2 v S^+.$$

Since $S^+_1 = 1$ and $S^+_2 = v$, this gives

$$S^+ = \frac{t + t^2 v}{1 + t^2 v} + \frac{t^2 v(1+v)(1+z)}{1+t^2 v}\sigma[S^+]. \tag{14}$$

Repeatedly substituting for S^+ on the right-hand side of (14), and writing V_n instead of $V_n(v,w)$ to save space, we get for each $m \geq 0$ the identity

$$\begin{aligned}
S^+ &= \left(\sum_{n=0}^{m} \sigma^{(n)} \left[\frac{t+t^2 v}{1+t^2 v} \right] \prod_{i=0}^{n-1} \sigma^{(i)} \left[\frac{t^2 v(1+v)(1+z)}{1+t^2 v} \right] \right) \\
&\qquad + \prod_{i=0}^{m} \sigma^{(i)} \left[\frac{t^2 v(1+v)(1+z)}{1+t^2 v} \right] \sigma^{(m+1)}[S^+] \\
&= \sum_{n=0}^{m} t^{2n+1} \frac{1+tV_n}{1+t^2 V_n} \prod_{i=0}^{n-1} \frac{(1+V_i)(1+z)V_i}{1+t^2 V_i} + \mathcal{O}(t^{2m+2}) \\
&= \sum_{n=0}^{m} t^{2n+1} \frac{1+tV_n}{1+t^2 V_n} \prod_{i=0}^{n-1} \frac{(1+v)(1+w)^i(1+z)V_i}{1+t^2 V_i} + \mathcal{O}(t^{2m+2}) \\
&= \sum_{n=0}^{m} t^{2n+1} \frac{1+tV_n}{1+t^2 V_n} (1+v)^n (1+z)^n (1+w)^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{V_i}{1+t^2 V_i} + \mathcal{O}(t^{2m+2}).
\end{aligned}$$

Taking the limit as m goes to infinity proves Theorem 3.1.

4 Self-Dual Orders Counted by Reduced Size

So far, the main statistic of a Fishburn matrix has always been its size, i.e., the sum of its entries. However, for a self-dual Fishburn matrix M, we may consider an alternative notion of size, which we call *the reduced size of M* and denote by $\mathrm{rs}(M)$, and which we define as the sum of all the diagonal and south-east cells of the matrix. Note that the diagonal and south-east cells determine a self-dual matrix uniquely. With the notation of the previous section, $\mathrm{rs}(M)$ is equal to $\mathrm{diag}(M) + \mathrm{se}(M) + \max(M) + \mathrm{iso}(M)$.

It turns out that the notion of reduced size is in some respects a more natural statistic of self-dual interval orders than the notion of size that we have used so far. In particular, we show that self-dual interval orders of a given reduced size admit a matrix representation by a family of matrices that have simpler structure than self-dual Fishburn matrices.

A *row-Fishburn matrix* is an upper-triangular matrix of nonnegative integers with the property that each row has at least one nonzero entry. A row-Fishburn matrix is *primitive* if all its entries are equal to 0 or 1. As usual, the size of a row-Fishburn matrix is taken to be the sum of its cells. The matrix statistics from Table 1 can be applied to row-Fishburn matrices, even though the poset interpretations listed there are not applicable.

Theorem 4.1. *For any integers $n \geq 1$ and $m \geq 1$, the following three quantities are all equal.*

1. *The number of self-dual Fishburn matrices of reduced size n, whose last column has sum m, and whose diagonal cells are all equal to zero.*

2. *The number of self-dual Fishburn matrices of reduced size n, whose last column has sum m, and whose diagonal cells are not all equal to zero.*

3. *The number of row-Fishburn matrices of size n whose last column has sum m.*

Moreover, for integers n, m, p and q with $p+q \geq 1$, the number of self-dual Fishburn matrices M with $\mathrm{rs}(M) = n$, $\max(M) = m$, $\mathrm{diag}(M) = p$ and $\mathrm{iso}(M) = q$ is equal to the number of row-Fishburn matrices N of size n with $\max(N) = m$, $\min(N) = p$ and $\mathrm{iso}(N) = q$. All these relationships remain valid when restricted to primitive matrices.

Proof. We prove the statements by comparing the generating functions of the relevant objects. We will first prove the statement for primitive matrices, the statement for general matrices then follows easily.

Let $\mathcal{R}$ be the set of primitive row-Fishburn matrices, and consider the generating function

$$R(v, w, x, y) = \sum_{N \in \mathcal{R}} v^{\max(N)} w^{\mathrm{int}(N)} x^{\mathrm{iso}(N)} y^{\min(N)}.$$

This generating function satisfies the identity

$$R(v, w, x, y) = \sum_{n \geq 0} ((1 + x)(1 + y)^n - 1) \prod_{i=0}^{n-1} \left((1 + v)(1 + w)^i - 1\right).$$

To see this, note that the n-th summand on the right-hand side of this expression is the generating function of primitive row-Fishburn matrices with $(n+1)$ rows. Indeed, the factor $(1 + x)(1 + y)^n - 1$ counts the number of possibilities to fill the first row of such a matrix, while the factor $(1 + v)(1 + w)^i - 1$ counts the number of possibilities to fill the row $n + 1 - i$.

Recall that $\mathcal{S}$ is the set of primitive self-dual Fishburn matrices, and let $\mathcal{S}_0$ denote the set of primitive self-dual Fishburn matrices whose north-east diagonal only contains zeroes, while $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}_0$ is the set of primitive self-dual Fishburn matrices whose north-east diagonal contains at least one positive cell. Let $S'(v, w, x, z)$ be the generating function

$$S'(v, w, x, z) = \sum_{M \in \mathcal{S}} v^{\max(M)} w^{\mathrm{se}(M)} x^{\mathrm{iso}(M)} z^{\mathrm{diag}(M)},$$

and let $S'_0(v,w)$ and $S'_1(v,w,x,z)$ be the analogous generating functions for the sets $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively, where S'_0 does not depend on x and z because matrices from $\mathcal{S}_0$ have $\mathrm{iso}(M) = \mathrm{diag}(M) = 0$. From Theorem 3.1, and in analogy to Lemma 3.2, we see that

$$\begin{aligned}
S'(v,w,x,z) &= (1+x)S^+(1,v,w,z) - 1\\
&= -1 + (1+x)\sum_{n\geq 0}(1+z)^n(1+v)^n(1+w)^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{V_i(v,w)}{1+V_i(v,w)}\\
&= -1 + (1+x)\sum_{n\geq 0}(1+z)^n(1+v)^n(1+w)^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{(1+v)(1+w)^i-1}{(1+v)(1+w)^i}\\
&= -1+\sum_{n\geq 0}(1+x)(1+z)^n\prod_{i=0}^{n-1}\left((1+v)(1+w)^i-1\right).
\end{aligned}$$

Since $S'_0(v,w) = S'(v,w,0,0)$ and $S'_1(v,w,x,z) = S'(v,w,x,z) - S'_0(v,w)$, we get

$$S'_0(v,w) = -1 + \sum_{n\geq 0}\prod_{i=0}^{n-1}\left((1+v)(1+w)^i-1\right), \text{ and}$$

$$S'_1(v,w,x,z) = \sum_{n\geq 0}\left((1+x)(1+z)^n-1\right)\prod_{i=0}^{n-1}\left((1+v)(1+w)^i-1\right)$$

We see that $S'_0(v,w) = S'_1(v,w,v,w) = R(v,w,v,w)$, and that $S'_1(v,w,x,z) = R(v,w,x,z)$, proving the theorem for primitive matrices. To prove the non-primitive case, it is enough to observe that the generating functions for general matrices may be obtained from the generating functions of primitive matrices by substituting $\alpha/(1-\alpha)$ for each variable $\alpha \in \{v,w,x,y,z\}$. □

Although Theorem 4.1 follows easily from the generating function formulas established before, it might still be worthwhile to provide a bijective argument. Currently, we are not aware of such an argument.

From Theorem 4.1, we may directly deduce the following corollary.

Corollary 4.2. *Let s_m be the number of primitive self-dual interval orders of reduced size m and let r_m be the number of primitive row-Fishburn matrices of size m. Then $s_m = 2r_m$ for each $m \geq 1$, and*

$$\sum_{m\geq 1} r_m x^m = \sum_{n\geq 0}\prod_{i=0}^{n}\left((1+x)^{i+1}-1\right).$$

Let t_m be the number of self-dual interval orders of reduced size m, and let q_m be the number of row-Fishburn matrices of size m. Then $t_m = 2q_m$ for each $m \geq 1$, and

$$\sum_{m\geq 1} q_m x^m = \sum_{n\geq 0}\prod_{i=0}^{n}\left(\frac{1}{(1-x)^{i+1}}-1\right).$$

Let us remark that the numbers $(r_m)_{m\geq 1}$ from Corollary 4.2 correspond to the sequence A179525 in OEIS [15], while $(q_m)_{m\geq 1}$ conjecturally correspond to A158691. To be more precise, A158691 is the sequence of coefficients of the power series $\sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-x)^{2i-1}\right)$, which, according to the notes in the OEIS entry, are conjectured to be equal to q_m for $m \geq 1$.

5 Final Remarks and Open Problems

The formulas of the form we derived in this paper provide an efficient way to explicitly compute the coefficients of the corresponding generating functions. They are also occasionally useful in establishing correspondences between different combinatorial structures, as shown in Theorem 4.1. It is not clear, however, whether one can use such formulas to extract information about the asymptotic growth of the coefficients. Zagier [22] has used formula (1), together with several non-trivial power series identities, to find a very precise asymptotic estimate of the number of interval orders on n elements. We state a weaker version of this estimate as fact.

Fact 5.1 ([22])**.** *If g_n is the number of interval orders of size n, then*

$$g_n = (\alpha + \mathcal{O}(n^{-1}))n!\left(\frac{6}{\pi^2}\right)^n \sqrt{n} \quad \textit{with } \alpha = \frac{12\sqrt{3}}{\pi^{-5/2}}e^{\pi^2/12}.$$

Drmota [6] has pointed out that from this estimate, we may deduce the asymptotic fraction of primitive posets among all interval orders.

Fact 5.2 ([6])**.** *With g_n as above, and with p_n being the number of primitive interval orders of size n, we have*

$$\lim_{n\to\infty}\frac{p_n}{g_n} = e^{-\pi^2/6}.$$

Proof. The generating functions $F(x) = \sum_{m\geq 0} g_m x^m$ and $G(x) = \sum_{n\geq 0} p_n x^n$ are related by $F(x) = G(x/(1-x))$, or equivalently, $G(x) = F(x/(1+x))$. Thus, for every $n \geq 1$, we have

$$\begin{aligned} p_n &= [x^n]F\left(\frac{x}{1+x}\right) = [x^n]\sum_{m=1}^{n} g_m\left(\frac{x}{1+x}\right)^m = \sum_{m=1}^{n} g_m(-1)^{n-m}\binom{n-1}{m-1} \\ &= \sum_{m=1}^{n} m!\left(\frac{6}{\pi^2}\right)^m \sqrt{m}(-1)^{n-m}\frac{(n-1)!}{(m-1)!(n-m)!}\left(\alpha + \mathcal{O}\left(\frac{1}{m}\right)\right) \\ &= \alpha n!\left(\frac{6}{\pi^2}\right)^n\sqrt{n}\sum_{m=1}^{n}\left(\frac{m}{n}\right)^{3/2}(-1)^{n-m}\left(\frac{6}{\pi^2}\right)^{m-n}\frac{1}{(n-m)!}\left(1+\mathcal{O}\left(\frac{1}{m}\right)\right) \\ &= g_n\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^{3/2}\frac{(-\pi^2/6)^k}{k!}\left(1+\mathcal{O}\left(\frac{1}{n-k}\right)\right) = g_n e^{-\pi^2/6}(1+o(1)). \qquad \square \end{aligned}$$

Our main general open problem is to obtain similar asymptotic estimates for the enumeration of self-dual interval orders, counted either by their size or their reduced size. Using the generating function formulas, we can easily enumerate self-dual interval orders of a given size, and using numerical manipulations similar to those described by Zagier [22, Section 3] to accelerate convergence, we can then make conjectures about the coefficient asymptotics.

Conjecture 5.3. *Let s_n be the number of primitive self-dual interval orders of reduced size n, and let t_n be the number of self-dual interval orders of reduced size n. Then*

$$t_n = (\beta + \mathcal{O}(n^{-1}))n! \left(\frac{12}{\pi^2}\right)^n \text{ with } \beta = \frac{12\sqrt{2}}{\pi^2} e^{\pi^2/24},$$
$$\text{and } \lim_{n\to\infty} \frac{s_n}{t_n} = e^{-\pi^2/12}.$$

Conjecture 5.4. *Let r_n be the number of self-dual interval orders of size n, and let q_n be the number of primitive self-dual interval orders of size n. Then*

$$r_n = (\gamma + \mathcal{O}(n^{-1/2}))\sqrt{n}\left(\frac{\delta n}{e}\right)^{n/2} 2^{\sqrt{\delta n}} \text{ with } \gamma \approx 1.361951039\ldots \text{ and } \delta = \frac{6}{\pi^2},$$
$$\text{and } \lim_{n\to\infty} \frac{q_n}{r_n} = \frac{1}{2} e^{-\pi^2/12}.$$

In a similar vein, one can ask whether the multi-variate generating functions can provide information about the distribution of the relevant statistics within the set of interval orders of a given size, or within the set of Fishburn matrices of a given dimension. Here are two examples of the kind of questions that arise.

Problem 5.5. What is the average sum of entries in a primitive $k \times k$ Fishburn matrix?

Problem 5.6. What is the average number of minimal elements in an n-element interval order?

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