A determinant formula for the partition function $p(7k + a)$

Jerome Malenfant
American Physical Society
Ridge, NY

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We derive expressions for the partition function $p(n)$, with n in the form $7k + a$, as $(k + 1)$-dimensional determinants.

I. INTRODUCTION AND MAIN RESULT

In a recent article [1], a formula was derived for the partition function $p(n)$ as the $(n+1)$-dimensional determinant

$$p(n) \ = \begin{vmatrix} 1 & & & & & & & & 1 \\ -1 & 1 & & & & & & & 0 \\ -1 & -1 & 1 & & & & & & 0 \\ 0 & -1 & -1 & 1 & & & & & 0 \\ 0 & 0 & -1 & -1 & 1 & & & & 0 \\ 1 & 0 & 0 & -1 & -1 & 1 & & & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 & & 0 \\ \vdots & & & & & & & \ddots & \vdots \end{vmatrix}_{(n+1)\times(n+1)}, \tag{1}$$

using the generating function

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty}, \quad \text{where} \quad (q)_\infty \equiv \prod_{k=1}^{\infty} (1 - q^k). \tag{2}$$

The coefficients along the diagonals of the above matrix are from the expansion of $(q)_\infty$ in powers of q, (sequence A010815 in OEIS [2]). In Appendix A of that article, $(k + 1)$-dimensional determinant formulas for the partition functions $p(5k + 4)$, $p(7k + 5)$, and $p(25k + 24)$ were found by using some identities due to Ramanujan [3]. The $p(5k + 4)$ and $p(25k + 24)$ expressions were then extended to $p(5k + a_5)$ and $p(25k + a_{25})$ for $a_5 = 0, 1, 2, 3$ and $a_{25} = 4, 9, 14, 19$, respectively.

In this article we will continue this development and generalize the $p(7k + 5)$ result,

$$p(7k+5) \ = \begin{vmatrix} 1 & & & & & & & & 7 \\ -8 & 1 & & & & & & & 21 \\ 20 & -8 & 1 & & & & & & 14 \\ 0 & 20 & -8 & 1 & & & & & 56 \\ -70 & 0 & 20 & -8 & 1 & & & & -35 \\ 64 & -70 & 0 & 20 & -8 & 1 & & & -28 \\ 56 & 64 & -70 & 0 & 20 & -8 & 1 & & -70 \\ 0 & 56 & 64 & -70 & 0 & 20 & -8 & & 35 \\ \vdots & & & & & & & \ddots & \vdots \end{vmatrix}_{(k+1)\times(k+1)}, \tag{3}$$

which was derived from

$$\sum_{k=0}^{\infty} p(7k+5) q^k \ = \ \frac{(q^7)^7_\infty}{(q)^8_\infty} \left[7 \, \frac{(q)^4_\infty}{(q^7)^4_\infty} + 49q \right] \tag{4}$$

using the expansions

$$\begin{aligned} (q)^8_\infty \ &= \ 1 - 8q + 20q^2 - 70q^4 + 64q^5 + \cdots, \\ 7(q^7)^3_\infty (q)^4_\infty + 49q(q^7)^7_\infty \ &= \ 7 + 21q + 14q^2 + 56q^3 - 35q^4 - \cdots, \end{aligned}$$

to $p(7k + a_7)$ for $a_7 = 0, \ldots, 6$. We will follow Ramanujan [3] and make the replacement $q \to q^{1/7}$ in eq. (2). $(q^{1/7})_\infty$ expands as the sum $\sum b_n q^{n/7}$, for which the only nonzero b_n are for $n \equiv 0, 1, 2, \text{or } 5 \mod 7$. Furthermore, the $n \equiv 2$

mod 7 terms sum to $-(q^7)_\infty q^{2/7}$. We can therefore write,

$$\frac{(q^{1/7})_\infty}{(q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3. \tag{5}$$

The J's are power series in q with integer exponents and satisfy the identities [3] [4]

$$J_1J_2J_3 = -1; \tag{6a}$$

$$\frac{J_1^2}{J_3} + \frac{J_2}{J_3^2} = q; \tag{6b}$$

$$J_1^7 + J_2^7 q + J_3^7 q^5 = \frac{(q)_\infty^8}{(q^7)_\infty^8} + 14q\frac{(q)_\infty^4}{(q^7)_\infty^4} + 57q^2; \tag{6c}$$

$$J_1^3J_2 + J_2^3J_3q + J_1J_3^3q^2 = -\frac{(q)_\infty^4}{(q^7)_\infty^4} - 8q; \tag{6d}$$

$$J_1^2J_2^3 + J_1^3J_3^2q + J_2^2J_3^3q^2 = -\frac{(q)_\infty^4}{(q^7)_\infty^4} - 5q. \tag{6e}$$

We have then

$$\sum_{n=0}^{\infty} p(n)q^{n/7} = \frac{1}{(q^7)_\infty}\,\frac{1}{J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3}. \tag{7}$$

As per Ramanujan, we now multiply and divide $(J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3)$ by the product

$$\prod_{n=1}^{6}\left(J_1 + \omega^n q^{1/7}J_2 - \omega^{2n}q^{2/7} + \omega^{5n}q^{5/7}J_3\right)$$

where $\omega \equiv e^{2\pi i/7}$. To simplify the notation and to avoid writing out a lot of subscripts, we define $x \equiv qJ_2^7/J_1^7$ and $a \equiv J_1/J_2^2$. Then

$$J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3 = J_1\left(1 + x^{1/7} - x^{2/7}a - x^{5/7}a^3\right). \tag{8}$$

Eq. (7) then becomes

$$\sum_{n=0}^{\infty} p(n)q^{n/7} = \frac{1}{(q^7)_\infty}\,\frac{J_1^6\prod_{n=1}^6\left(1+\omega^n x^{1/7} - \omega^{2n}x^{2/7}a - \omega^{5n}x^{5/7}a^3\right)}{J_1^7\prod_{n=0}^6\left(1+\omega^n x^{1/7} - \omega^{2n}x^{2/7}a - \omega^{5n}x^{5/7}a^3\right)} \equiv \frac{1}{(q^7)_\infty}\,\frac{N}{D}. \tag{9}$$

The numerator N and the denominator D are expressible as the sums

$$N = J_1^6\sum_{k=0}^{30} c_k x^{k/7}, \quad D = J_1^7\sum_{k=0}^{5} d_k x^k, \tag{10}$$

with coefficients c_k and d_k, respectively. Expanding the product in the denominator in eq. (9) we get

$$D = J_1^7\left[\ 1 + x\left(1 + 7a + 14a^2 - 7a^4\right) - x^2\left(8a^7 - 14a^8\right) - 14x^3a^{11} - 7x^4a^{16} - x^5a^{21}\ \right]. \tag{11a}$$

Converting to (q, J) notation, this is

$$D = J_1^7 + q\left(J_2^7 + 7J_1J_2^5 + 14J_1^2J_2^3 + 7J_1^5J_3\right) - q^2\left(8 - 14J_1^3J_3^2\right) + 14q^3J_2^2J_3^3 + 7q^4J_2J_3^5 + q^5J_3^7, \tag{11b}$$

which, after some manipulation and using the identities (6a-e) above, becomes

$$D = \frac{(q)_\infty^8}{(q^7)_\infty^8}. \tag{11c}$$

And so

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^{n/7} &= \frac{(q^7)_\infty^7}{(q)_\infty^8}\,J_1^6\sum_{k=0}^{30}c_kx^{k/7}\\ &= \frac{(q^7)_\infty^7}{(q)_\infty^8}\left(H_1 + H_2q^{1/7} + H_3q^{2/7} + H_4q^{3/7} + H_5q^{4/7} + H_6q^{5/7} + H_7q^{6/7}\right),\end{aligned} \tag{12}$$

where

$$
\begin{aligned}
H_1 &= \quad J_1^6 \ \ (\ c_0 + c_7 x + c_{14} x^2 + c_{21} x^3 + c_{28} x^4 \ ); \\
H_2 &= J_1^5 J_2 \ (\ c_1 + c_8 x + c_{15} x^2 + c_{22} x^3 + c_{29} x^4 \ ); \\
H_3 &= J_1^4 J_2^2 \ (\ c_2 + c_9 x + c_{16} x^2 + c_{23} x^3 + c_{30} x^4 \ ); \\
H_4 &= J_1^3 J_2^3 \ (\ c_3 + c_{10} x + c_{17} x^2 + c_{24} x^3 \ ); \\
H_5 &= J_1^2 J_2^4 \ (\ c_4 + c_{11} x + c_{18} x^2 + c_{25} x^3 \ ); \\
H_6 &= J_1 J_2^5 \ (\ c_5 + c_{12} x + c_{19} x^2 + c_{26} x^3 \ ); \\
H_7 &= \quad J_2^6 \ \ (\ c_6 + c_{13} x + c_{20} x^2 + c_{27} x^3 \ ).
\end{aligned}
$$

The generating-function equations are then

$$\sum_{k=0}^{\infty} p(7k + a_7) q^k = \frac{(q^7)_\infty^7}{(q)_\infty^8} H_{1+a_7}(q); \quad a_7 = 0, 1, \ldots, 6, \tag{13}$$

and we have

$$p(7k + a_7) \ = \begin{vmatrix} 1 & & & & & & & & Z_0 \\ -8 & 1 & & & & & & & Z_1 \\ 20 & -8 & 1 & & & & & & Z_2 \\ 0 & 20 & -8 & 1 & & & & & Z_3 \\ -70 & 0 & 20 & -8 & 1 & & & & Z_4 \\ 64 & -70 & 0 & 20 & -8 & 1 & & & Z_5 \\ 56 & 64 & -70 & 0 & 20 & -8 & 1 & & Z_6 \\ 0 & 56 & 64 & -70 & 0 & 20 & -8 & & Z_7 \\ \vdots & & & & & & & \ddots & \vdots \end{vmatrix}_{(k+1)\times(k+1)} \tag{14}$$

where the coefficients in the last column are from the expansions

$$(q^7)_\infty^7 H_{1+a_7}(q) = Z_0 + Z_1 q + Z_2 q^2 + \cdots . \tag{15}$$

Eq. (3) gives the first few of these coefficients for $a_7 = 5$.

The c_k coefficients can be found from the recurrence relation

$$c_n + c_{n-1} - a c_{n-2} - a^3 c_{n-5} = \begin{cases} d_{n/7} & \text{if } n \equiv 0 \mod 7, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where the d_k coefficients are given by eq. (11a). The coefficient c_0 equals 1; the rest are listed below:

c_1	-1	c_{11}	$2a^4 + 3a^5 - 6a^6$	c_{21}	a^{12}
c_2	$1 + a$	c_{12}	$3a^5 + 8a^6 - 4a^7$	c_{22}	$a^{12} - 2a^{13}$
c_3	$-1 - 2a$	c_{13}	a^6	c_{23}	$5a^{13}$
c_4	$1 + 3a + a^2$	c_{14}	$a^6 + 6a^8$	c_{24}	a^{14}
c_5	$-1 - 4a - 3a^2 + a^3$	c_{15}	$3a^7 - 3a^8 + a^9$	c_{25}	a^{15}
c_6	$1 + 5a + 6a^2 - a^3$	c_{16}	$6a^8 - a^9$	c_{26}	a^{15}
c_7	$a + 4a^2 - a^3 - 5a^4$	c_{17}	$6a^9 - 3a^{10}$	c_{27}	$-a^{16}$
c_8	$a^2 + 6a^3 + 2a^4$	c_{18}	$a^9 + 2a^{10}$	c_{28}	0
c_9	$-a^3 - 4a^5$	c_{19}	$4a^{10} + 3a^{11}$	c_{29}	0
c_{10}	$a^3 + 2a^4 + 3a^5 + a^6$	c_{20}	$-4a^{11} + a^{12}$	c_{30}	a^{18}

With these expressions, and using the identity (6b), the H functions become

$$
\begin{aligned}
H_1 &= 2J_1^2J_2^8 + 2J_1^3J_2^6 - J_1^4J_2^4 - 13J_1^5J_2^2 + 11J_1^6; \\
H_2 &= 5J_1^2J_2^7 - 9J_1^3J_2^5 + 15J_1^4J_2^3 - 15J_1^5J_2 - 3J_1^7J_3; \\
H_3 &= 11J_1^2J_2^6 - 31J_1^3J_2^4 + 26J_1^4J_2^2 - 5J_1^5 + J_1^8J_3^2; \\
H_4 &= J_1J_2^7 + 8J_1^2J_2^5 - 18J_1^3J_2^3 + 11J_1^4J_2 + 5J_1^6J_3; \qquad (17) \\
H_5 &= 3J_1J_2^6 + 3J_1^2J_2^4 - 12J_1^3J_2^2 + 12J_1^4 - J_1^7J_3^2; \\
H_6 &= 7J_1J_2^5 - 7J_1^2J_2^3 - 14J_1^3J_2 - 7J_1^5J_3; \\
H_7 &= J_2^6 + J_1J_2^4 + 17J_1^2J_2^2 - 10J_1^3 + 2J_1^6J_3^2.
\end{aligned}
$$

H_6 simplifies to the expression inside the brackets in eq. (4) using the additional identities (6c-e).

It remains to determine the expansions for the J functions. From eq. (5) we get

$$
\begin{aligned}
J_1(q) &= \frac{1}{(q^7)_\infty}\left\{-1 + \sum_{k=0}^{\infty} q^{k(42k-1)}\left[\,1 + q^{2k} + q^{14k+1} - q^{30k+5} - q^{42k+10} - q^{44k+11} - q^{56k+18} + q^{72k+30}\,\right]\right\}, \\
J_2(q) &= \frac{-1}{(q^7)_\infty}\sum_{k=0}^{\infty} q^{k(42k+5)}\left[\,1 + q^{14k+2} - q^{18k+3} - q^{32k+8} - q^{42k+13} - q^{56k+22} + q^{60k+25} + q^{74k+37}\,\right], \qquad (18) \\
J_3(q) &= \frac{1}{(q^7)_\infty}\sum_{k=0}^{\infty} q^{k(42k+11)}\left[\,1 - q^{6k+1} + q^{14k+3} - q^{20k+5} - q^{42k+16} + q^{48k+20} - q^{56k+26} + q^{62k+31}\,\right].
\end{aligned}
$$

($J_1(q)$ corresponds to sequence A108483.) The coefficients in eq. (14) for $a_7 = 0, 1, 2, 3, 4$ and 6 are then

$$
\mathbf{Z}^{(0)} = \begin{pmatrix} 1 \\ 7 \\ 35 \\ 12 \\ 12 \\ -7 \\ 36 \\ -167 \\ \vdots \end{pmatrix};\
\mathbf{Z}^{(1)} = \begin{pmatrix} 1 \\ 14 \\ 20 \\ 34 \\ -1 \\ 21 \\ -111 \\ 34 \\ \vdots \end{pmatrix};\
\mathbf{Z}^{(2)} = \begin{pmatrix} 2 \\ 14 \\ 31 \\ 7 \\ 44 \\ -67 \\ 21 \\ -103 \\ \vdots \end{pmatrix};\
\mathbf{Z}^{(3)} = \begin{pmatrix} 3 \\ 18 \\ 21 \\ 39 \\ -28 \\ 31 \\ -80 \\ -73 \\ \vdots \end{pmatrix};\
\mathbf{Z}^{(4)} = \begin{pmatrix} 5 \\ 16 \\ 37 \\ -2 \\ 35 \\ -47 \\ -28 \\ -117 \\ \vdots \end{pmatrix};\
\mathbf{Z}^{(6)} = \begin{pmatrix} 11 \\ 13 \\ 39 \\ 14 \\ 0 \\ -63 \\ -1 \\ -164 \\ \vdots \end{pmatrix}.
$$

II. CONCLUSION

The reduction of the $(n+1)$-dimensional matrix in eq. (1) to a smaller, $(k+1)$-dimensional one for $p(7k+a_7)$, as well as for $p(5k+a_5)$ and $p(25k+a_{25})$, used results from Ramanujan's study of the congruences of the partition function in ref. 3. However, these reductions did not depend upon the existence of a congruence, but rather they used the property that the product

$(q)_\infty(e^{2\pi i/N}q)_\infty(e^{4\pi i/N}q)_\infty \cdots (e^{(N-1)2\pi i/N}q)_\infty$

is a power-series expansion in q^N. Thus, for any N, with $\omega = e^{2\pi i/N}$, we can write,

$$
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q)_\infty} \times \frac{(\omega q)_\infty \cdots (\omega^{N-1}q)_\infty}{(\omega q)_\infty \cdots (\omega^{N-1}q)_\infty} = \left(\sum_{p=0}^{N-1}\sum_{k=0}^{\infty} Z_k^{(p)} q^{kN+p}\right)\left(\sum_{k=0}^{\infty} D_k q^{kN}\right)^{-1}, \qquad (19)
$$

with $D_0 = 1$ and $Z_0^{(a)} = p(a)$. We have then

$$p(kN+a) \;=\; \begin{vmatrix} 1 & & & & p(a) \\ D_1 & 1 & & & Z_1^{(a)} \\ D_2 & D_1 & 1 & & Z_2^{(a)} \\ D_3 & D_2 & D_1 & & Z_3^{(a)} \\ \vdots & & & \ddots & \vdots \end{vmatrix}_{(k+1)\times(k+1)} . \tag{20}$$

This is however only of practical use in calculating partition functions if compact expressions can be found whose expansions give the D and Z coefficients.

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[1] Malenfant, "Finite, Closed-form Expressions for the Partition Function and for Euler, Bernoulli, and Stirling Numbers". http://arxiv.org/abs/1103.1585

[2] The On-Line Encyclopedia of Integer Sequences. http://oeis.org.

[3] Berndt and Ono, "Ramanujan's Unpublished Manuscript on the Partition and Tau Functions with Proofs and Commentary". http://www.math.wisc.edu/∼ono/reprints/044.pdf.

[4] Note that in ref. 3 there is a misprint in the exponent of the last term in eq. (24.4), corresponding to eq. (6c); $57q^3$ should be replaced in (24.4) by $57q^2$.