REPETITION IN REDUCED DECOMPOSITIONS

BRIDGET EILEEN TENNER

Abstract. Given a permutation w, we show that the number of repeated letters in a reduced decomposition of w is always less than or equal to the number of 321- and 3412-patterns appearing in w. Moreover, we prove bijectively that the two quantities are equal if and only if w avoids the ten patterns 4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, and 45231.

Keywords: permutation, reduced decomposition, pattern

1. Introduction

Permutations can be described in a variety of ways, including as a product of simple reflections and in one-line notation. These two were studied extensively by the author in [10], and a means for translating properties of one presentation into properties of the other was given. The first of these presentations is most relevant to the generalized setting of Coxeter groups and the Bruhat order. There is a rich literature studying various properties of reduced decompositions, including [1] and [7]. The second of these presentations, one-line notation, is primarily useful when discussing the notion of permutation patterns. This topic originated in work of Rodica Simion and Frank Schmidt [5], and has become a popular subfield of combinatorics.

Given any permutation w, one can calculate its length, and one can also calculate the number of distinct simple reflections that appear in any reduced decomposition of w. The difference between these two quantities, denoted $\mathsf{rep}(w)$ in this paper, would thus count the number of repeated letters in any reduced decomposition of w. These statistics are readily computed from the presentation of a permutation as a product of simple reflections.

When written in one-line notation, one often looks at the patterns in (or not in) a permutation. In particular, one can count the number of distinct 321- and 3412-patterns in a permutation w, and this total will be denoted $[\mathsf{321;3412}](w)$ here.

It was shown in previous work by the author that $\mathsf{rep}(w) = 0$ if and only if $[\mathsf{321;3412}](w) = 0$ [9]. Additionally, Daniel Daly shows that $\mathsf{rep}(w) = 1$ if and only if $[\mathsf{321;3412}](w) = 1$ [3]. Other than these results, not much has been known about the quantity or type of repetition that might occur within a reduced decomposition of a given permutation.

The ideal conclusion based on the results of [9] and [3], that $\mathsf{rep}(w)$ and $[\mathsf{321;3412}](w)$ would always be equal, is not actual the case, as can be seen with $\mathsf{rep}(4321) = 3$ and $[\mathsf{321;3412}](4321) = 4$. However, the main result of this paper (Theorem 3.2) is that $\mathsf{rep}(w)$ is always less than or equal to $[\mathsf{321;3412}](w)$, and the two quantities are equal exactly when w

2010 *Mathematics Subject Classification.* 05A05; 05A19; 05E15.
Research partially supported by a DePaul University Faculty Summer Research Grant.

avoids each of the patterns

$$\{4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\}.$$

Moreover, in Corollary 5.3, we give a crude lower bound on the difference $[321;3412](w) - \mathsf{rep}(w)$ when w contains some of the patterns listed above.

In Section 2 of the paper, we introduce the necessary objects and terminology for this work. Section 3 suggests the relevance of the ten patterns listed above and states the main theorem, while the proof of this theorem is spread over Sections 4 and 5.

2. Definitions

This section summarizes the primary objects studied in this work. More background on this material can be found in [1] and [4].

Let $\mathfrak{S}_n$ be the symmetric group on n elements. The group $\mathfrak{S}_n$ is generated by the simple reflections (also called adjacent transpositions) $\{s_1, \ldots, s_{n-1}\}$, where s_i is the permutation interchanging i and $i+1$, and fixing all other elements. These permutations satisfy the Coxeter relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i, \\ s_i s_j &= s_j s_i && \text{if } |i-j| > 1, \text{ and} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned}$$

As is customary, a map acts to the right, meaning that $s_i w$ interchanges the positions of the values i and $i+1$ in the one-line notation of w, and $w s_i$ interchanges the values in positions i and $i+1$ in the one-line notation of w.

A permutation $w \in \mathfrak{S}_n$ can also be written in one-line notation as $w = w(1)w(2)\cdots w(n)$.

Example 2.1. The permutation $3241 \in \mathfrak{S}_4$ maps 1 to 3, 2 to itself, 3 to 4, and 4 to 1.

We have now described two substantially different presentations for permutations: products of simple reflections and one-line notation. A means of translating between these two, and of inferring properties of one from properties of the other, was given in [10].

Definition 2.2. If $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ where $\ell(w)$ is minimal, then $s_{i_1} \cdots s_{i_{\ell(w)}}$ is a *reduced decomposition* (or *reduced word*) of w. This $\ell(w)$ is the *length* of w.

The set of reduced decompositions of a permutation has been studied from several viewpoints, including connections to Young tableaux as described in [7]. In this paper, we will study repetition among the letters in a reduced decomposition of a permutation. To that end, we make the following definition.

Definition 2.3. Given a permutation w, the *support* of w is the set $\mathsf{supp}(w)$ of distinct letters appearing in a reduced decomposition of w.

It is important to clarify why this definition is sound.

Lemma 2.4. *The set* $\mathsf{supp}(w)$ *is well defined.*

Proof. We must prove that the set of letters in a reduced decomposition of a permutation is independent of the particular reduced decomposition chosen as a representative. Any reduced decomposition of w can be obtained from any other by a series of Coxeter relations [2]. These do not change the underlying set of distinct letters in the reduced decomposition, so the set $\mathsf{supp}(w)$ is well defined. That is, given any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$,

$$\mathsf{supp}(w) = \{s_{i_1}, \ldots, s_{i_\ell}\}.$$

□

Example 2.5. Let $w = 32154 \in \mathfrak{S}_5$. One reduced decomposition for w is $s_2 s_1 s_2 s_4$, so $\mathsf{supp}(w) = \{s_1, s_2, s_4\}$. Note that $s_2 s_1 s_4 s_2$ and $s_1 s_2 s_1 s_4$ are also reduced decompositions for w, and they each yield the same set $\mathsf{supp}(w)$.

The following statistics will be crucial in our proof of the main theorem.

Definition 2.6. Fix $w \in \mathfrak{S}_n$ and $k \in \{1, \ldots, n-1\}$. Let

$$M_k(w) = \max\{w(1), \ldots, w(k)\}$$

and

$$m_k(w) = \min\{w(k+1), \ldots, w(n)\}.$$

Lemma 2.7. *For any $w \in \mathfrak{S}_n$, the values of $M_k(w)$ satisfy*

$$M_1(w) \le M_2(w) \le M_3(w) \le \cdots \le M_{n-1}(w),$$

and the values of $m_k(w)$ satisfy

$$m_1(w) \le m_2(w) \le m_3(w) \le \cdots \le m_{n-1}(w).$$

We have strict inequality $M_k(w) < M_{k+1}(w)$ exactly when $w(k+1) > M_k(w)$, and $m_k(w) < m_{k+1}(w)$ exactly when $w(k+1) < m_{k+1}(w)$.

Proof. These inequalities follow immediately from the definitions of $M_k(w)$ and $m_k(w)$. □

The next lemma is a consequence of the definition of the support of a permutation.

Lemma 2.8. *Fix a permutation $w \in \mathfrak{S}_n$. The following statements are equivalent:*

- $s_k \in \mathsf{supp}(w)$,
- $\{w(1), \ldots, w(k)\} \neq \{1, \ldots, k\}$,
- $\{w(k+1), \ldots, w(n)\} \neq \{k+1, \ldots, n\}$,
- $M_k(w) > k$,
- $m_k(w) < k+1$, *and*
- $M_k(w) > m_k(w)$.

Proof. Suppose that $s_k \in \mathsf{supp}(w)$. This means that s_k appears at least once in each reduced decomposition of w, which means that there is some inversion $w(i) > w(j)$ in w, where $i \le k < j$. Thus the set $\{w(1), \ldots, w(k)\}$ cannot equal $\{1, \ldots, k\}$, and, equivalently, the set $\{w(k+1), \ldots, w(n)\}$ cannot equal $\{k+1, \ldots, n\}$. Also equivalently, the set $\{w(1), \ldots, w(k)\}$ contains an element larger than k, and, equivalently, the set $\{w(k+1), \ldots, w(n)\}$ contains an element less than $k+1$.

If, on the other hand, $s_k \notin \mathsf{supp}(w)$, then there is no inversion such as described in the previous paragraph. Therefore $w(1) \cdots w(k)$ is a permutation of $\{1, \ldots, k\}$ and $w(k+1) \cdots w(n)$ is a permutation of $\{k+1, \ldots, n\}$. Thus $M_k(w) = k$ and $m_k(w) = k+1$. □

In this paper, we will study the relationship between two statistics of a permutation. The first of these is related to the support of a permutation.

Definition 2.9. Given a permutation w, let $\mathsf{rep}(w)$ be the quantity

$$\mathsf{rep}(w) = \ell(w) - |\mathsf{supp}(w)|. \tag{1}$$

This quantity is so named because it counts the number of simple reflections in a reduced decomposition of w, when reading from one end to the other, which repeat previously seen letters. The fact that this latter description is well defined may not be immediately obvious, given that a permutation may have more than one reduced decomposition. However, this does not affect $\mathsf{supp}(w)$, as shown by Lemma 2.4, and so $\mathsf{rep}(w)$ is well defined, by equation (1).

Example 2.10. Let $w = 35412$, where $\ell(w) = 7$ and $\mathsf{supp}(w) = \{s_1, s_2, s_3, s_4\}$. Thus $\mathsf{rep}(w) = 7 - 4 = 3$. Relatedly, one reduced decomposition for w is $s_2s_1s_3s_2s_4s_3s_2$, and reading from left to right we encounter the repeated simple reflections which are marked in

$$s_2s_1s_3\boxed{s_2}s_4\boxed{s_3}\boxed{s_2}.$$

There are three such letters, so $\mathsf{rep}(w) = 3$.

The other statistic we will consider relates to permutation patterns.

Definition 2.11. Let $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ for $k \leq n$. The permutation w *contains the pattern* p if there exist $i_1 < \cdots < i_k$ such that $w(i_1)\cdots w(i_k)$ is in the same relative order as $p(1)\cdots p(k)$, in which case $w(i_1)\cdots w(i_k)$ is an *occurrence* of p in w. For notational clarity, we will sometimes denote this pattern by $\{w(i_1), \ldots, w(i_k)\}$. If $N = \max\{w(i_1), \ldots, w(i_k)\}$, then this $w(i_1)\cdots w(i_k)$ is an *N-occurrence* of p. If w does not contain p, then w *avoids* p, or is *p-avoiding.*

The set of all occurrences of a pattern p in a permutation w can be partitioned by the largest letter appearing in the occurrence:

$$\{\text{occurrences of } p \text{ in } w\} = \bigsqcup_N \{N\text{-occurrences of } p \text{ in } w\}.$$

Example 2.12. Continuing Example 2.10, there are two occurrences of 3412 in w: 3512 and 3412. The first of these is a 5-occurrence, and the second is a 4-occurrence. The permutation w is 123-avoiding because it has no increasing subsequence of length 3.

There is much interest in enumeration related to permutation patterns. The portion of this scholarship relevant to the current work is the enumeration of occurrences of a pattern p appearing in a permutation w.

Definition 2.13. Given a permutation w and a pattern p, let $[\mathsf{p}]_N(w)$ denote the number of N-occurrences of p in w. Let

$$[\mathsf{p}](w) = \sum_N [\mathsf{p}]_N(w)$$

be the total number of occurrences of p in w.

Example 2.14. Continuing Example 2.10, we have $[\mathsf{321}]_5(w) = 2$ and $[\mathsf{321}]_i(w) = 0$ for all $i \neq 5$. Also, $[\mathsf{3412}]_4(w) = [\mathsf{3412}]_5(w) = 1$, and $[\mathsf{3412}]_i(w) = 0$ otherwise.

For reasons that will be suggested by Theorem 2.17, we are most concerned with the patterns 321 and 3412, and we will count the number of distinct occurrences of these patterns.

Definition 2.15. Given a permutation w, and a positive integer N, let

$$[321;3412]_N(w) = [321]_N(w) + [3412]_N(w).$$

Let $[321;3412](w)$ be the quantity

$$\begin{aligned} [321;3412](w) &= [321](w) + [3412](w) \\ &= \sum_N [321;3412]_N(w). \end{aligned} \tag{2}$$

Example 2.16. Continuing Example 2.10, let us calculate $[321;3412](w)$. The distinct occurrences of 321 in w are $\{541, 542\}$, and the distinct occurrences of 3412 in w are $\{3512, 3412\}$. Thus

$$\begin{aligned} [321;3412]_4(w) &= 0 + 1 = 1, \\ [321;3412]_5(w) &= 2 + 1 = 3, \text{ and} \\ [321;3412](w) &= 2 + 2 = 1 + 3 = 4. \end{aligned}$$

Using the notation defined above, the following results were shown previously, the first by the author and the second by Daniel Daly.

Theorem 2.17 ([9] and [3])**.** *For any permutation w,*

(a) $\mathsf{rep}(w) = 0$ *if and only if* $[321;3412](w) = 0$*, and*
(b) $\mathsf{rep}(w) = 1$ *if and only if* $[321;3412](w) = 1$*.*

Theorem 2.17 gives a clear indication that the statistic rep is related to whether a permutation contains the patterns 321 or 3412. This arises from the previously mentioned work by the author in [10], relating patterns (and hence the one-line presentation of a permutation) with the presentation of a permutation as a product of simple reflections.

The statistics rep and [321;3412] are not always equal, as shown by Examples 2.10 and 2.16:

$$\mathsf{rep}(35412) < [321;3412](35412).$$

In this paper, we will show that $\mathsf{rep}(w)$ never exceeds $[321;3412](w)$, and we will characterize equality of the two quantities by pattern avoidance.

3. The main theorem

Rather surprisingly, the potential equality of the statistics rep and [321;3412] mentioned at the end of the last section depends solely on the avoidance of ten patterns, the set of which we will denote Φ.

Definition 3.1. Let

$$\Phi = \{4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\} \subset (\mathfrak{S}_4 \cup \mathfrak{S}_5).$$

To suggest the relevance of the set Φ, let us compare $\mathsf{rep}(\phi)$ and $[321;3412](\phi)$ for all $\phi \in \Phi$, writing $\mathsf{rep}(\phi)$ as the difference $\ell(\phi) - |\mathsf{supp}(\phi)|$, and $[321;3412](\phi)$ as the sum $[321](\phi) +$

$[3412](\phi)$ in equation (2).

$\phi \in \Phi$	$\mathsf{rep}(\phi)$	$[321;3412](\phi)$
4321	$6-3=3$	$4+0=4$
34512	$6-4=2$	$0+3=3$
45123	$6-4=2$	$0+3=3$
35412	$7-4=3$	$2+2=4$
43512	$7-4=3$	$2+2=4$
45132	$7-4=3$	$2+2=4$
45213	$7-4=3$	$2+2=4$
53412	$8-4=4$	$4+1=5$
45312	$8-4=4$	$4+1=5$
45231	$8-4=4$	$4+1=5$

Observe that for each $\phi \in \Phi$, we have $\mathsf{rep}(\phi) < [321;3412](\phi)$.

We are now able to state the main theorem of the paper.

Theorem 3.2. *If a permutation w avoids every pattern in the set Φ, then*

$$\mathsf{rep}(w) = [321;3412](w).$$

Otherwise,

$$\mathsf{rep}(w) < [321;3412](w).$$

This characterization of equality between rep and $[321;3412]$ if and only if the set Φ is avoided is recorded in entry P0022 of the Database of Permutation Pattern Avoidance [8], and is enumerated by A191721 in [6].

Observe that Theorem 3.2 recovers the result in Theorem 2.17, since a permutation w in which $[321;3412](w) \in \{0,1\}$ necessarily avoids every pattern in Φ. Note also that 0 and 1 are the only values for which $\mathsf{rep}(w)$ and $[321;3412](w)$ are necessarily equal, because there are permutations $\phi \in \Phi$ with $\mathsf{rep}(\phi) = 2$ but $[321;3412](\phi) = 3$.

Suppose $w \in \mathfrak{S}_N$. Theorem 3.2 is proved by induction on N and involves an assignment of at least one N-occurrence of 321 or 3412 to each previously used letter involved in positioning N in the one-line notation of w, after first positioning all other letters relative to each other. We must be wary of overcounting these N-occurrences of 321 and 3412. The details of the proof are covered in Sections 4 and 5.

4. Preliminaries for proving the main theorem

Definition 4.1. Consider $w \in \mathfrak{S}_N$. Define $\overline{w} \in \mathfrak{S}_{N-1}$ by

$$\overline{w}(i) = \begin{cases} w(i) & \text{if } i < w^{-1}(N), \text{ and} \\ w(i+1) & \text{if } i > w^{-1}(N). \end{cases}$$

The one-line notation of $\overline{w}$ is obtained from the one-line notation of w by deleting the letter N and sliding all subsequent letters one space to the left. Moreover, if we think of $\overline{w}$ as a permutation in $\mathfrak{S}_N$ that fixes N, then

$$w = \overline{w} s_{N-1} s_{N-2} \cdots s_{w^{-1}(N)}, \tag{3}$$

and

$$\ell(w) = \ell(\overline{w}) + N - w^{-1}(N).$$

Example 4.2. If $w = 35412$, then $\overline{w} = 3412$. If we consider $\overline{w}$ to be the element $34125 \in \mathfrak{S}_5$, then

$$w = \overline{w}s_4s_3s_2.$$

One reduced decomposition of $\overline{w}$ is $s_2s_1s_3s_2$, and so $s_2s_1s_3s_2s_4s_3s_2$ is a reduced decomposition of w.

Throughout the rest of this section, let w be a permutation in $\mathfrak{S}_N$, and $\overline{w} \in \mathfrak{S}_{N-1}$ be as defined above.

The following set will be crucial in the proof of Theorem 3.2, describing the letters in a reduced word of w, but not of $\overline{w}$, which count as repeated letters for w.

Definition 4.3. Let $\mathsf{new\text{-}rep}(w) = \{k : s_k \in \mathsf{supp}(\overline{w}) \text{ and } w^{-1}(N) \leq k\}$.

Lemma 4.4.

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{supp}(\overline{w}) \cap \{s_{N-1}s_{N-2}\cdots s_{w^{-1}(N)}\}| \\ &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)|. \end{aligned}$$

Proof. This follows from equation (3). □

Recall the functions M_k and m_k from Definition 2.6.

Lemma 4.5. *If $M_k(\overline{w}) > m_k(\overline{w})$ and $w^{-1}(N) \leq k$, then $k \in \mathsf{new\text{-}rep}(w)$.*

Proof. This is immediate from the definition of $\mathsf{new\text{-}rep}(w)$. □

To show that $\mathsf{rep}(w)$ is a lower bound for $[321;3412](w)$, we would like to assign, to each element of $\mathsf{new\text{-}rep}(w)$, at least one N-occurrence in w of one of the patterns $\{321, 3412\}$. This assignment should be done carefully to avoid overcounting. Additionally, to characterize when $\mathsf{rep}(w)$ and $[321;3412](w)$ are equal, we would like to understand when each N-occurrence in w of the patterns $\{321, 3412\}$ corresponds to some element of $\mathsf{new\text{-}rep}(w)$.

For the remainder of this section, set $M_k = M_k(\overline{w})$ and $m_k = m_k(\overline{w})$ for all k.

Definition 4.6. Consider $k \in \mathsf{new\text{-}rep}(w)$. Define $\mathfrak{p}_k(w)$ as follows.

I. If $w^{-1}(N) < w^{-1}(M_k)$, then $\mathfrak{p}_k(w) = \{N, M_k, m_k\}$, which is a 321-pattern in w.
II. If $w^{-1}(N) > w^{-1}(M_k)$ and $\overline{w}(k) > m_k$, then $\mathfrak{p}_k(w) = \{N, \overline{w}(k), m_k\}$, which is a 321-pattern in w.
III. Otherwise, set $\mathfrak{p}_k(w) = \{M_k, N, \overline{w}(k), m_k\}$, which is a 3412-pattern in w.

This $\mathfrak{p}_k(w)$ is undefined if $k \notin \mathsf{new\text{-}rep}(w)$.

Note that $\mathfrak{p}_k(w)$ is always an N-occurrence of either 321 or of 3412 because $k \in \mathsf{new\text{-}rep}(w)$ and thus $M_k > m_k$ by Lemma 2.8. However, it is not clear when $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ coincide for $k \neq k'$, nor which N-occurrences of 321 or of 3412 have the form $\mathfrak{p}_k(w)$ for some k.

4.1. **Issues of overcounting.** Consider whether the patterns $\mathfrak{p}_k(w)$ might overcount N-occurrences of 321 or 3412 in w.

Proposition 4.1.1. *There are no distinct $k, k' \in \mathsf{new\text{-}rep}(w)$ for which $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ are the same N-occurrence of* 3412 *in w.*

Proof. If this were the case, then $(M_k, \overline{w}(k), m_k) = (M_{k'}, \overline{w}(k'), m_{k'})$. But then $\overline{w}(k) = \overline{w}(k')$, implying that $k = k'$. □

Therefore, if there is any overcounting of N-occurrences of 321 or 3412 among the $\{\mathfrak{p}_k(w)\}$, it must be that $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ are the same N-occurrence of 321.

Proposition 4.1.2. *If there exist distinct $k, k' \in \mathsf{new\text{-}rep}(w)$ with $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$, then w has an N-occurrence of* 4321.

Proof. Suppose that there exist $k, k' \in \mathsf{new\text{-}rep}(w)$, with $k < k'$, such that $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$. Proposition 4.1.1 implies that these coincident patterns must be N-occurrences of 321 in w.

These coincident $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ cannot both be of type II as in Definition 4.6, because that would mean that $\overline{w}(k) = \overline{w}(k')$, and so $k = k'$.

Now suppose that one is of type I, and the other of type II. Thus $\overline{w}(k) = M_{k'}$ and $m_k = m_{k'}$. Then $\{N, \overline{w}(k) = M_{k'}, \overline{w}(k), m_k = m_{k'}\}$ forms an N-occurrence of 4321 in w. Note also in this case that we must have $M_k = \overline{w}(k) = M_{k'}$, since otherwise M_k would lie to the left of $\overline{w}(k) = M_{k'}$, and be greater than $\overline{w}(k)$ by definition, which would contradict the maximality of $M_{k'}$.

It remains to consider the case when both patterns are of type I, and so $(M_k, m_k) = (M_{k'}, m_{k'}) = (M, m)$. We can assume that $M \notin \{\overline{w}(k), \overline{w}(k')\}$ because that case was already addressed. Then the one-line notation of $\overline{w}$, and hence of w, looks like

$$\cdots\ M\ \cdots\ \overline{w}(k)\ \cdots\ \overline{w}(k')\ \cdots\ m\ \cdots .$$

Consider where N lies in relation to the values $\{M, \overline{w}(k), \overline{w}(k'), m\}$. Because both patterns have type I, we must have that $w^{-1}(N) < w^{-1}(M)$, and so N lies to the left of M. The definitions of M and m require that $M > \overline{w}(k), \overline{w}(k')$, and $m < \overline{w}(k')$. Thus the letters $\{N, M, \overline{w}(k'), m\}$ form an N-occurrence of 4321. □

Corollary 4.1.3. *If w has no N-occurrence of* 4321*, then*

$$|\mathsf{new\text{-}rep}(w)| \le [\mathsf{321;3412}]_N(w).$$

Proof. This follows from Propositions 4.1.1 and 4.1.2, because there do not exist distinct $k, k' \in \mathsf{new\text{-}rep}(w)$ with $\mathfrak{p}_k(w)$ equalling $\mathfrak{p}_{k'}(w)$. □

Therefore, by Corollary 4.1.3, the procedure for assigning to each $k \in \mathsf{new\text{-}rep}(w)$ an N-occurrence of either 321 or 3412 is injective if w has no N-occurrence of 4321. We must now consider what happens to this assignment when w does have such a pattern.

Proposition 4.1.4. *Suppose that w has an N-occurrence of* 4321*. If there exist distinct $k, k' \in \mathsf{new\text{-}rep}(w)$ and $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$, then there are two other N-occurrences $\mathfrak{p}^+_{k'}(w)$ and $\mathfrak{p}^-_{k'}(w)$ of* 321 *in w, which are not equal to $\mathfrak{p}_j(w)$ for any j.*

Proof. Suppose that there are such $k < k'$. Then we know from Proposition 4.1.2 that $(M_k, m_k) = (M_{k'}, m_{k'}) = (M, m)$, and $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w) = \{N, M, m\}$. Also, we know that the one-line notation of w looks like

$$\cdots\ N\ \cdots\ M\ \cdots\ \overline{w}(k)\ \cdots\ \overline{w}(k')\ \cdots\ m\ \cdots,$$

where M and $\overline{w}(k)$ could possibly be equal. Because $M = M_{k'}$, we must have $M > \overline{w}(k')$. Also, because $m = m_k$, we must have $m < \overline{w}(k')$. Thus

$$\mathfrak{p}^+_{k'}(w) = \{N > M > \overline{w}(k')\} \text{ and } \mathfrak{p}^-_{k'}(w) = \{N > \overline{w}(k') > m\}$$

are both N-occurrences of 321 in w.

Note that $\mathfrak{p}^+_{k'}(w)$ is not equal to $\mathfrak{p}_j(w)$ for any j, because $\overline{w}(k')$ is not equal to m_j for any j: there exists a letter (for example, m) to the right of $\overline{w}(k')$ which is less than $\overline{w}(k')$. Similarly, $\mathfrak{p}^-_{k'}(w) \neq \mathfrak{p}_j(w)$ for any j, because $\overline{w}(k')$ cannot equal M_j. □

It is also helpful to note that for all k', the same reasoning as in Proposition 4.1.4 implies that $\{\mathfrak{p}^+_{k'}(w), \mathfrak{p}^-_{k'}(w)\} \cap \{\mathfrak{p}^+_j(w), \mathfrak{p}^-_j(w) : j \neq k'\} = \emptyset$.

Corollary 4.1.5. *If w has an N-occurrence of* 4321*, then*

$$|\textsf{new-rep}(w)| < [321;3412]_N(w).$$

Proof. Partition the set $\textsf{new-rep}(w)$ into sets $S_1, S_2, \ldots, S_t$ so that $(M_k, m_k) = (M(i), m(i))$ for each $k \in S_i$. Suppose $S_i = \{k_{i_1} < k_{i_2} < \cdots < k_{i_{|S_i|}}\}$, and define

$$\mathfrak{p}_{S_i}(w) = \left\{\mathfrak{p}_{k_{i_1}}(w), \mathfrak{p}^+_{k_{i_2}}(w), \mathfrak{p}^-_{k_{i_2}}(w), \ldots, \mathfrak{p}^+_{k_{i_{|S_i|}}}(w), \mathfrak{p}^-_{k_{i_{|S_i|}}}(w)\right\}.$$

Note that if $|S_i| = 1$, then $|\mathfrak{p}_{S_i}(w)| = 1$. Also, if $|S_i| > 1$, then $|\mathfrak{p}_{S_i}(w)| = 2|S_i| - 1 > |S_i|$. Moreover, the elements of $\mathfrak{p}_{S_i}(w)$ are all N-occurrences of either 321 or 3412 in w. Finally, by design of the partition $\textsf{new-rep}(w) = S_1 \sqcup S_2 \sqcup \cdots$, the sets $\{\mathfrak{p}_{S_i}(w)\}$ are disjoint.

If w has an N-occurrence of 4321, then there exists some S_i containing at least two elements. Thus $|\textsf{new-rep}(w)| = |S_1| + |S_2| + \cdots + |S_t| < |\mathfrak{p}_{S_1}(w)| + |\mathfrak{p}_{S_2}(w)| + \cdots + |\mathfrak{p}_{S_t}(w)| \leq [321;3412]_N(w)$. □

Using the notation from the proof of Corollary 4.1.5, we can also rewrite its result to say that the map

$$\xi_n : j \mapsto \begin{cases} \mathfrak{p}_j(w) & \text{if } j \text{ is the minimal element in } S_i, \text{ and} \\ \mathfrak{p}^+_j(w) & \text{otherwise.} \end{cases} \tag{4}$$

is an injection.

4.2. **Issues of undercounting.** We have now addressed the issue of whether the set $\{\mathfrak{p}_k(w) : k \in \textsf{new-rep}(w)\}$ might overcount some N-occurrences of 321 (never of 3412). We must now consider when this set might undercount these N-occurrences. As we have seen in Proposition 4.1.4, undercounting is certainly a possibility. What we will show now is that if w avoids the ten patterns in the set Φ, then there is no undercounting, and thus the inequality of Corollary 4.1.3 is actually an equality.

To examine potential undercounting, we must decide if and when an N-occurrence of 321 or of 3412 might not equal $\mathfrak{p}_k(w)$ for some k.

Proposition 4.2.1. *If any N-occurrence $\{N > a > b\}$ of* 321 *in w is such that $b \notin \{m_k : k \in \textsf{new-rep}(w)\}$, then w has an N-occurrence of* 4321*.*

Proof. Suppose there is an N-occurrence of 321 in w where $b \neq m_k$ for any k. Then to the right of b in the one-line notation w, there exists $c < b$, preventing b from equalling any such m_k. Thus $\{N > a > b > c\}$ is an N-occurrence of 4321. Now, suppose there is no such c, and set $k = \overline{w}^{-1}(b) - 1$. Then $b = m_k$, and, since $a > b$ appears to the left of b, we must have $M_k \geq a > b = m_k$. Therefore, by Lemma 4.5, $k \in \textsf{new-rep}(w)$. $\square$

Proposition 4.2.2. *Suppose w is* 4321*-avoiding. If any N-occurrence $\{N > a > b\}$ of* 321 *in w is such that there exists no $k \in \textsf{new-rep}(w)$ with $(a, b) \in \{(M_k, m_k), (\overline{w}_k, m_k)\}$, then w has an N-occurrence of at least one of the patterns $\{45312, 53412\}$.*

Proof. By Proposition 4.2.1, we know that $b = m_k$ for at least one value of $k \in \textsf{new-rep}(w)$. Suppose that $a \notin \{M_k, \overline{w}(k)\}$.

Suppose $a > M_k$. Then, by maximality of M_k, this a must appear to the right of both M_k and $\overline{w}(k)$ in the one-line notation of w. But then, setting $k' = \overline{w}^{-1}(a)$, we must have $m_{k'} = m_k = b$, and so $(a, b) = (\overline{w}(k'), m_{k'})$. By definition, $k' > k$, and so $m_{k'} = m_k < k+1 < k'+1$, where the first inequality is because $k \in \textsf{new-rep}(w)$. Therefore $k' \in \textsf{new-rep}(w)$ as well.

Now suppose that $a < M_k$. If $w^{-1}(N) > w^{-1}(M_k)$, then the one-line notation of w looks like

$$\cdots \; M_k \; \cdots \; N \; \underbrace{\cdots\cdots}_{<a} \; a \; \underbrace{\cdots\cdots}_{>a \text{ or } <m_k} \; b = m_k \; \cdots,$$

because w is 4321-avoiding. If all values appearing between a and b in w are larger than a, then we can set $k' = \overline{w}^{-1}(a)$, and we have $(M_{k'}, m_{k'}) = (M_k, m_k)$, and again $k' \in \textsf{new-rep}(w)$. Thus suppose that there is some value c in this portion of w with $c < m_k$. Then $\{M_k, N, a, c, b\}$ is an N-occurrence of 45312 in w.

Finally, suppose that $w^{-1}(N) < w^{-1}(M_k)$, where k is minimal with this property. So the one-line notation of w looks like

$$\cdots \; N \; \cdots \; a \; \cdots \; M_k \; \cdots \; b = m_k \; \cdots,$$

again because w is 4321-avoiding. There must exist some $c < m_k$ between M_k and m_k in the one-line notation for w preventing us from choosing a different value $k' < k$ so that $a \in \{M_{k'}, \overline{w}(k')\}$ and $m_{k'} = m_k$. Such a k' would be in $\textsf{new-rep}(w)$ because $a > b$ while $\{a, b\} \cap \{\overline{w}(1), \ldots, \overline{w}(k')\} = \{a\}$, so $\max\{\overline{w}(1), \ldots, \overline{w}(k')\} > k'$. Then $\{N, a, M_k, c, m_k = b\}$ would form a 53412-pattern in w. $\square$

Propositions 4.2.1 and 4.2.2 now imply the following result.

Corollary 4.2.3. *If w has no N-occurrences of the patterns $\{4321, 45312, 53412\}$, then every N-occurrence of* 321 *in w is equal to $\mathfrak{p}_k(w)$ for some k.*

Proposition 4.2.4. *If any N-occurrence $\{a, N, b, c\}$ of* 3412 *in w is such that $c \notin \{m_k : k \in \textsf{new-rep}(w)\}$, then w has an N-occurrence of at least one of the patterns $\{45231, 45132\}$.*

Proof. Suppose there is such an N-occurrence of 3412 in w. This means that to the right of c in the one-line notation of w, there exists a $d < c$, preventing c from equalling any such m_k. Thus $\{a, N, b, c, d\}$ is an N-occurrence of either 45231 or of 45132, depending on whether $b > d$ or $b < d$. Now suppose that there is no such d, and set $k = \overline{w}^{-1}(c) - 1$. Then $c = m_k$, and, since $a > c$ appears to the left of c, we must have $M_k > m_k$. Therefore, by Lemma 4.5, $k \in \textsf{new-rep}(w)$. $\square$

Proposition 4.2.5. *Suppose w is 45231- and 45132-avoiding. If any N-occurrence $\{a, N, b, c\}$ of 3412 in w is such that there exists no $k \in \textsf{new-rep}(w)$ with $(a, c) = (M_k, m_k)$, then w has an N-occurrence of at least one of the patterns $\{43512, 34512, 35412\}$.*

Proof. By Proposition 4.2.4, we know that $c = m_k$ for some $k \in \textsf{new-rep}(w)$. Choose the minimal such k; that is, choose k so that $\overline{w}(k) < c$ (and thus, necessarily, $\overline{w}(j) \geq c$ for all $j > k+1$). There are now three places M_k might appear relative to the letters $\{a, N, \overline{w}(k), c\}$, which themselves form an N-occurrence of 3412 in w:

$$\underbrace{\cdots\cdots}_{M_k?}\ a\ \underbrace{\cdots\cdots}_{M_k?}\ N\ \underbrace{\cdots\cdots}_{M_k?}\ \overline{w}(k)\ \cdots\ c = m_k\ \cdots.$$

By definition, $M_k \geq a$. Thus, if $M_k \neq a$, then these three possibilities create N-occurrences of 43512, 34512, or 35412 in w, respectively. $\square$

Proposition 4.2.6. *Suppose w avoids the patterns*

$$\{45231, 45132, 43512, 34512, 35412\}.$$

If any N-occurrence $\{a, N, b, c\}$ of 3412 in w is such that there exists no $k \in \textsf{new-rep}(w)$ with $(a, b, c) = (M_k, \overline{w}(k), m_k)$, then w has an N-occurrence of at least one of the patterns $\{45123, 45213\}$.

Proof. By Propositions 4.2.4 and 4.2.5, we know that $(a, c) = (M_k, m_k)$ for some $k \in \textsf{new-rep}(w)$. If $b \neq \overline{w}(k)$, then $\overline{w}(k)$ either lies between N and b, or between b and $c = m_k$. In fact, $\overline{w}(k)$ must lie to the right of b, because $b < c = m_k = \min\{\overline{w}(k+1), \ldots, \overline{w}(n)\}$. We also know that $\overline{w}(k) < M_k = a$. Therefore, since w is 45132-avoiding, the set $\{a, N, b, \overline{w}(k), c\}$ forms an N-occurrence of either 45123 or 45213. $\square$

Propositions 4.2.4, 4.2.5, and 4.2.6 now imply the following result.

Corollary 4.2.7. *If w has no N-occurrences of the patterns*

$$\{45231, 45132, 43512, 34512, 35412, 45123, 45213\},$$

then every N-occurrence of 3412 in w is equal to $\mathfrak{p}_k(w)$ for some $k \in \textsf{new-rep}(w)$.

This addresses the concern about undercounting the N-occurrences of 321 and 3412 in w.

Corollary 4.2.8. *If w has no N-occurrence of any of the patterns in the set*

$$\{45231, 45132, 43512, 34512, 35412, 45123, 45213\},$$

then

$$|\textsf{new-rep}(w)| \geq [\textsf{321;3412}]_N(w).$$

Proof. This follows from Corollaries 4.2.3 and 4.2.7. $\square$

4.3. **Conclusions.** We now combine the previous two subsections to draw the following conclusion.

Corollary 4.3.1. *If w has no N-occurrence of any of the patterns in the set Φ, then*

$$|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w).$$

In other words, if w has no N-occurrence of any of the patterns in the set Φ, then the map ξ_N of equation (4) *is a bijection.*

Proof. Combine the inequalities in Corollaries 4.1.3 and 4.2.8. □

It is natural now to wonder about the implications of containing an N-occurrence of a pattern in Φ. In fact, for each w containing an N-occurrence of some $\phi \in \Phi$, there is an N-occurrence $\mathfrak{p}_\phi(w)$ of either 321 or 3412 which is not equal to $\mathfrak{p}_k(w)$ or to $\mathfrak{p}_k^+(w)$ (as defined in Proposition 4.1.4) for any k, as is shown in the following table. In this table, the N-occurrence $\mathfrak{p}_\phi(w)$ will be written as a substring of ϕ, and will refer to those respective letters of the N-occurrence of ϕ in w.

$\phi \in \Phi$	$\mathfrak{p}_\phi(w)$
4321	421
34512	3512
45123	4513
35412	3512
43512	3512
45132	4513
45213	4523
53412	532
45312	532
45231	4523

Note that for $4321 \in \Phi$, the subpattern 432 is also not equal to any $\mathfrak{p}_k(w)$. However, it could equal some $\mathfrak{p}_k^+(w)$, so to avoid this possibility we set $\mathfrak{p}_{4321}(w) = 421$.

Proposition 4.3.2. *Let $w \in \mathfrak{S}_N$ be a permutation containing an N-occurrence of some pattern $\phi \in \Phi$. Then $\mathfrak{p}_\phi(w)$ is not equal to $\mathfrak{p}_k(w)$ for any $k \in \mathsf{new\text{-}rep}(w)$, nor to any $\mathfrak{p}_k^+(w)$, as defined in Proposition 4.1.4. That is, the injection ξ_N of equation* (4) *is not surjective.*

Proof. This follows from the definitions of the patterns $\mathfrak{p}_\phi(w)$, $\mathfrak{p}_k(w)$, and $\mathfrak{p}_k^+(w)$. □

This proposition has the following corollary.

Corollary 4.3.3. *If w has an N-occurrence of at least one of the patterns in the set Φ, then*

$$|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w).$$

5. Proof of the main theorem

Proof of Theorem 3.2. We prove this by induction on the number of letters in a permutation.

The result is easy to verify for small cases, so assume that the theorem holds for all permutations in $\mathfrak{S}_n$ for all $n < N$, and consider $w \in \mathfrak{S}_N$. Define $\overline{w} \in \mathfrak{S}_{N-1}$ as in Section 4.

Since $N-1 < N$, we know that $\mathsf{rep}(\overline{w})$ is equal to $[\mathsf{321;3412}](\overline{w})$ if $\overline{w}$ avoids the patterns in the set Φ, and that $\mathsf{rep}(\overline{w})$ is less than $[\mathsf{321;3412}](\overline{w})$ if $\overline{w}$ contains at least one pattern in Φ.

Suppose first that $\overline{w}$ avoids the patterns in Φ. If w has no N-occurrences of any of the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w)$. Thus

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &= [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w). \end{aligned}$$

On the other hand, if w does have an N-occurrence of at least one the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w)$, and so

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &< [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w). \end{aligned}$$

Now assume that $\overline{w}$ does not avoid the patterns in Φ. If w has no N-occurrences of any of the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w)$. Thus

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &< [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w). \end{aligned}$$

On the other hand, if w does have an N-occurrence of at least one the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w)$, and so

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &< [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w). \end{aligned}$$

This completes the proof. $\square$

Definition 5.1. Consider a permutation $w \in \mathfrak{S}_N$. Let $\overline{w}_{(0)} = w$, and for $i \in \{1, \ldots, N-1\}$, let $\overline{w}_{(i+1)} = \overline{\overline{w}_{(i)}}$.

Corollary 5.2. *If a permutation $w \in \mathfrak{S}_N$ avoids every pattern in the set Φ, then the maps $\{\xi_n : n \leq N\}$ define a bijection from the set $\{\mathsf{new\text{-}rep}(\overline{w}_{(i)}) : i \in \{0, \ldots, N-1\}\}$ to the set of all* 321- *and* 3412*-patterns in w.*

Additionally, the proof of Theorem 3.2 can be adapted to show the following.

Corollary 5.3. *For any permutation w,*

$$[\mathsf{321;3412}](w) - \mathsf{rep}(w) \geq |\{r : w \textit{ has an } r\textit{-occurrence of a pattern in } \Phi\}|.$$

Acknowledgements

Special thanks are due to Kári Ragnarsson for writing the Java code to compute examples and to confirm conjectures through $\mathfrak{S}_{10}$.

References

[1] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics 231, Springer, New York, 2005.

[2] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres IV–VI*, Masson, Paris, 1981.

[3] D. Daly, Reduced decompositions with one repetition and permutation pattern avoidance, preprint.

[4] I. G. Macdonald, *Notes on Schubert Polynomials*, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, Montreal, 1991.

[5] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985), 383-406.
[6] N. J. A. Sloane, Online encyclopedia of integer sequences, published electronically at `http://oeis.org`.
[7] R. P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, *European J. Combin.* **5** (1984), 359–372.
[8] B. E. Tenner, Database of permutation pattern avoidance, published electronically at `http://math.depaul.edu/bridget/patterns.html`.
[9] B. E. Tenner, Pattern avoidance and the Bruhat order, *J. Combin. Theory Ser. A* **114** (2007), 888–905.
[10] B. E. Tenner, Reduced decompositions and permutation patterns, *J. Algebraic Combin.* **24** (2006), 263–284.

Department of Mathematical Sciences, DePaul University, Chicago, IL 60614
E-mail address: `bridget@math.depaul.edu`

REPETITION IN REDUCED DECOMPOSITIONS

BRIDGET EILEEN TENNER

ABSTRACT. Given a permutation w, we show that the number of repeated letters in a reduced decomposition of w is always less than or equal to the number of 321- and 3412-patterns appearing in w. Moreover, we prove bijectively that the two quantities are equal if and only if w avoids the ten patterns 4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, and 45231.

Keywords: permutation, reduced decomposition, pattern

1. Introduction

Permutations can be described in a variety of ways, including as a product of simple reflections and in one-line notation. These two were studied extensively by the author in [13], and a means for translating properties of one presentation into properties of the other was given. The first of these presentations is most relevant to the generalized setting of Coxeter groups and the Bruhat order. There is a rich literature studying various properties of reduced decompositions, including [2] and [10]. The second of these presentations, one-line notation, is primarily useful when discussing the notion of permutation patterns. This topic originated in work of Rodica Simion and Frank Schmidt [8], and has become a popular subfield of combinatorics.

Given any permutation w, one can calculate its length, and one can also calculate the number of distinct simple reflections that appear in any reduced decomposition of w. The difference between these two quantities, denoted $\mathsf{rep}(w)$ in this paper, would thus count the number of repeated letters in any reduced decomposition of w. These statistics are readily computed from the presentation of a permutation as a product of simple reflections.

When written in one-line notation, one often looks at the patterns in (or not in) a permutation. In particular, one can count the number of distinct 321- and 3412-patterns in a permutation w, and this total will be denoted $[\mathsf{321;3412}](w)$ here.

It was shown in previous work by the author that $\mathsf{rep}(w) = 0$ if and only if $[\mathsf{321;3412}](w) = 0$ [12]. Additionally, Daniel Daly shows that $\mathsf{rep}(w) = 1$ if and only if $[\mathsf{321;3412}](w) = 1$ [4]. Other than these results, not much has been known about the quantity or type of repetition that might occur within a reduced decomposition of a given permutation.

The ideal conclusion based on the results of [12] and [4], that $\mathsf{rep}(w)$ and $[\mathsf{321;3412}](w)$ would always be equal, is not actually the case, as can be seen with $\mathsf{rep}(4321) = 3$ and $[\mathsf{321;3412}](4321) = 4$. However, the main result of this paper (Theorem 3.2) is that $\mathsf{rep}(w)$ is always less than or equal to $[\mathsf{321;3412}](w)$, and the two quantities are equal exactly when w

2010 *Mathematics Subject Classification.* 05A05; 05A19; 05E15.
Research partially supported by a DePaul University Faculty Summer Research Grant.

avoids each of the patterns

$$\{4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\}.$$

Moreover, in Corollary 5.3, we give a crude lower bound on the difference $[\mathsf{321;3412}](w) - \mathsf{rep}(w)$ when w contains some of the patterns listed above.

In Section 2 of the paper, we introduce the necessary objects and terminology for this work. Section 3 suggests the relevance of the ten patterns listed above and states the main theorem, while the proof of this theorem is spread over Sections 4 and 5.

2. Definitions

This section summarizes the primary objects studied in this work. More background on this material can be found in [2] and [5].

Let $\mathfrak{S}_n$ be the symmetric group on n elements. The group $\mathfrak{S}_n$ is generated by the simple reflections (also called adjacent transpositions) $\{s_1, \ldots, s_{n-1}\}$, where s_i is the permutation interchanging i and $i+1$, and fixing all other elements. These permutations satisfy the Coxeter relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i, \\ s_i s_j &= s_j s_i && \text{if } |i-j| > 1, \text{ and} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned}$$

We adopt the custom that $s_i w$ interchanges the positions of the values i and $i+1$ in the one-line notation of w, and ws_i interchanges the values in positions i and $i+1$ in the one-line notation of w.

A permutation $w \in \mathfrak{S}_n$ can also be written in one-line notation as $w = w(1)w(2)\cdots w(n)$.

Example 2.1. The permutation $3241 \in \mathfrak{S}_4$ maps 1 to 3, 2 to itself, 3 to 4, and 4 to 1.

We have now described two substantially different presentations for permutations: products of simple reflections and one-line notation. A means of translating between these two, and of inferring properties of one from properties of the other, was given in [13].

Definition 2.2. If $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ where $\ell(w)$ is minimal, then $s_{i_1} \cdots s_{i_{\ell(w)}}$ is a *reduced decomposition* of w. This $\ell(w)$ is the *length* of w.

The set of reduced decompositions of a permutation has been studied from several viewpoints, including connections to Young tableaux as described in [10]. In this paper, we will study repetition among the letters in a reduced decomposition of a permutation. To that end, we make the following definition.

Definition 2.3. Given a permutation w, the *support* of w is the set $\mathsf{supp}(w)$ of distinct letters appearing in a reduced decomposition of w.

It is important to clarify why this definition is sound.

Lemma 2.4. *The set* $\mathsf{supp}(w)$ *is well defined.*

Proof. We must prove that the set of letters in a reduced decomposition of a permutation is independent of the particular reduced decomposition chosen as a representative. Any reduced decomposition of w can be obtained from any other by a series of Coxeter relations ([7] and

[14], independently). These do not change the underlying set of distinct letters in the reduced decomposition, so the set $\mathsf{supp}(w)$ is well defined. That is, given any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$,

$$\mathsf{supp}(w) = \{s_{i_1}, \ldots, s_{i_\ell}\}.$$

□

Example 2.5. Let $w = 32154 \in \mathfrak{S}_5$. One reduced decomposition for w is $s_2s_1s_2s_4$, so $\mathsf{supp}(w) = \{s_1, s_2, s_4\}$. Note that $s_2s_1s_4s_2$ and $s_1s_2s_1s_4$ are also reduced decompositions for w, and they each yield the same set $\mathsf{supp}(w)$.

The following statistics will be crucial in our proof of the main theorem.

Definition 2.6. Fix $w \in \mathfrak{S}_n$ and $k \in \{1, \ldots, n-1\}$. Let

$$M_k(w) = \max\{w(1), \ldots, w(k)\}$$

and

$$m_k(w) = \min\{w(k+1), \ldots, w(n)\}.$$

Lemma 2.7. *For any $w \in \mathfrak{S}_n$, the values of $M_k(w)$ satisfy*

$$M_1(w) \le M_2(w) \le M_3(w) \le \cdots \le M_{n-1}(w),$$

and the values of $m_k(w)$ satisfy

$$m_1(w) \le m_2(w) \le m_3(w) \le \cdots \le m_{n-1}(w).$$

We have strict inequality $M_k(w) < M_{k+1}(w)$ exactly when $w(k+1) > M_k(w)$, and $m_k(w) < m_{k+1}(w)$ exactly when $w(k+1) < m_{k+1}(w)$.

Proof. These inequalities follow immediately from the definitions of $M_k(w)$ and $m_k(w)$. □

The next lemma is a consequence of the definition of the support of a permutation.

Lemma 2.8. *Fix a permutation $w \in \mathfrak{S}_n$. The following statements are equivalent:*

- $s_k \in \mathsf{supp}(w)$,
- $\{w(1), \ldots, w(k)\} \neq \{1, \ldots, k\}$,
- $\{w(k+1), \ldots, w(n)\} \neq \{k+1, \ldots, n\}$,
- $M_k(w) > k$,
- $m_k(w) < k+1$, *and*
- $M_k(w) > m_k(w)$.

Proof. Suppose that $s_k \in \mathsf{supp}(w)$. This means that s_k appears at least once in each reduced decomposition of w, which means that there is some inversion $w(i) > w(j)$ in w, where $i \le k < j$. Thus the set $\{w(1), \ldots, w(k)\}$ cannot equal $\{1, \ldots, k\}$, and, equivalently, the set $\{w(k+1), \ldots, w(n)\}$ cannot equal $\{k+1, \ldots, n\}$. Also equivalently, the set $\{w(1), \ldots, w(k)\}$ contains an element larger than k, and, equivalently, the set $\{w(k+1), \ldots, w(n)\}$ contains an element less than $k+1$.

If, on the other hand, $s_k \notin \mathsf{supp}(w)$, then there is no inversion such as described in the previous paragraph. Therefore $w(1) \cdots w(k)$ is a permutation of $\{1, \ldots, k\}$ and $w(k+1) \cdots w(n)$ is a permutation of $\{k+1, \ldots, n\}$. Thus $M_k(w) = k$ and $m_k(w) = k+1$. □

In this paper, we will study the relationship between two statistics of a permutation. The first of these is related to the support of a permutation.

Definition 2.9. Given a permutation w, let $\mathsf{rep}(w)$ be the quantity

$$\mathsf{rep}(w) = \ell(w) - |\mathsf{supp}(w)|. \tag{1}$$

This quantity is so named because it counts the number of simple reflections in a reduced decomposition of w, when reading from one end to the other, which repeat previously seen letters. The fact that this latter description is well defined may not be immediately obvious, given that a permutation may have more than one reduced decomposition. However, this does not affect $\mathsf{supp}(w)$, as shown by Lemma 2.4, and so $\mathsf{rep}(w)$ is well defined, by equation (1).

Example 2.10. Let $w = 35412$, where $\ell(w) = 7$ and $\mathsf{supp}(w) = \{s_1, s_2, s_3, s_4\}$. Thus $\mathsf{rep}(w) = 7 - 4 = 3$. Relatedly, one reduced decomposition for w is $s_2s_1s_3s_2s_4s_3s_2$, and reading from left to right we encounter the repeated simple reflections which are marked in

$$s_2s_1s_3\boxed{s_2}s_4\boxed{s_3}\boxed{s_2}.$$

There are three such letters, so $\mathsf{rep}(w) = 3$.

The other statistic we will consider relates to permutation patterns.

Definition 2.11. Let $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ for $k \leq n$. The permutation w *contains the pattern* p if there exist $i_1 < \cdots < i_k$ such that $w(i_1)\cdots w(i_k)$ is in the same relative order as $p(1)\cdots p(k)$, in which case $w(i_1)\cdots w(i_k)$ is an *occurrence* of p in w. For notational clarity, we will sometimes denote this pattern by $\{w(i_1), \ldots, w(i_k)\}$. If $N = \max\{w(i_1), \ldots, w(i_k)\}$, then this $w(i_1)\cdots w(i_k)$ is an *N-occurrence* of p. If w does not contain p, then w *avoids* p, or is *p-avoiding*.

The set of all occurrences of a pattern p in a permutation w can be partitioned by the largest letter appearing in the occurrence:

$$\{\text{occurrences of } p \text{ in } w\} = \bigsqcup_N \{N\text{-occurrences of } p \text{ in } w\}.$$

Example 2.12. Continuing Example 2.10, there are two occurrences of 3412 in w: 3512 and 3412. The first of these is a 5-occurrence, and the second is a 4-occurrence. The permutation w is 123-avoiding because it has no increasing subsequence of length 3.

There is much interest in enumeration related to permutation patterns (see, for example, [3, 6, 8]). The portion of this scholarship relevant to the current work is the enumeration of occurrences of a pattern p appearing in a permutation w.

Definition 2.13. Given a permutation w and a pattern p, let $[\mathsf{p}]_N(w)$ denote the number of N-occurrences of p in w. Let

$$[\mathsf{p}](w) = \sum_N [\mathsf{p}]_N(w)$$

be the total number of occurrences of p in w.

Example 2.14. Continuing Example 2.10, we have $[\mathsf{321}]_5(w) = 2$ and $[\mathsf{321}]_i(w) = 0$ for all $i \neq 5$. Also, $[\mathsf{3412}]_4(w) = [\mathsf{3412}]_5(w) = 1$, and $[\mathsf{3412}]_i(w) = 0$ otherwise.

For reasons that will be suggested by Theorem 2.17, we are most concerned with the patterns 321 and 3412, and we will count the number of distinct occurrences of these patterns.

Definition 2.15. Given a permutation w, and a positive integer N, let

$$[321;3412]_N(w) = [321]_N(w) + [3412]_N(w).$$

Let $[321;3412](w)$ be the quantity

$$\begin{aligned} [321;3412](w) &= [321](w) + [3412](w) \\ &= \sum_N [321;3412]_N(w). \end{aligned} \tag{2}$$

Example 2.16. Continuing Example 2.10, let us calculate $[321;3412](w)$. The distinct occurrences of 321 in w are $\{541, 542\}$, and the distinct occurrences of 3412 in w are $\{3512, 3412\}$. Thus

$$\begin{aligned} [321;3412]_4(w) &= 0 + 1 = 1, \\ [321;3412]_5(w) &= 2 + 1 = 3, \text{ and} \\ [321;3412](w) &= 2 + 2 = 1 + 3 = 4. \end{aligned}$$

Using the notation defined above, the following results were shown previously, the first by the author and the second by Daniel Daly.

Theorem 2.17 ([12] and [4])**.** *For any permutation w,*

(a) $\mathsf{rep}(w) = 0$ *if and only if* $[321;3412](w) = 0$*, and*
(b) $\mathsf{rep}(w) = 1$ *if and only if* $[321;3412](w) = 1$.

Theorem 2.17 gives a clear indication that the statistic rep is related to whether a permutation contains the patterns 321 or 3412. This arises from the previously mentioned work by the author in [13], relating patterns (and hence the one-line presentation of a permutation) with the presentation of a permutation as a product of simple reflections.

The statistics rep and [321;3412] are not always equal, as shown by Examples 2.10 and 2.16:

$$\mathsf{rep}(35412) < [321;3412](35412).$$

In this paper, we will show that $\mathsf{rep}(w)$ never exceeds $[321;3412](w)$, and we will characterize equality of the two quantities by pattern avoidance.

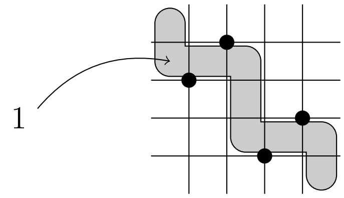
3. The main theorem

Rather surprisingly, the potential equality of the statistics rep and [321;3412] mentioned at the end of the last section depends solely on the avoidance of ten patterns, the set of which we will denote Φ.

Definition 3.1. Let

$$\Phi = \{4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\} \subset (\mathfrak{S}_4 \cup \mathfrak{S}_5).$$

Note that the subset $\{34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\} \subset \Phi$ can be expressed as the single marked mesh pattern



where the marking of this region is 1, as indicated. The reader is referred to [15] for more information about these objects.

To suggest the relevance of the set Φ, let us compare $\mathsf{rep}(\phi)$ and $[321;3412](\phi)$ for all $\phi \in \Phi$, writing $\mathsf{rep}(\phi)$ as the difference $\ell(\phi) - |\mathsf{supp}(\phi)|$, and $[321;3412](\phi)$ as the sum $[321](\phi) + [3412](\phi)$ in equation (2).

$\phi \in \Phi$	$\mathsf{rep}(\phi)$	$[321;3412](\phi)$
4321	$6-3=3$	$4+0=4$
34512	$6-4=2$	$0+3=3$
45123	$6-4=2$	$0+3=3$
35412	$7-4=3$	$2+2=4$
43512	$7-4=3$	$2+2=4$
45132	$7-4=3$	$2+2=4$
45213	$7-4=3$	$2+2=4$
53412	$8-4=4$	$4+1=5$
45312	$8-4=4$	$4+1=5$
45231	$8-4=4$	$4+1=5$

Observe that for each $\phi \in \Phi$, we have $\mathsf{rep}(\phi) < [321;3412](\phi)$.

We are now able to state the main theorem of the paper.

Theorem 3.2. *If a permutation w avoids every pattern in the set Φ, then*

$$\mathsf{rep}(w) = [321;3412](w).$$

Otherwise,

$$\mathsf{rep}(w) < [321;3412](w).$$

This characterization of equality between rep and $[321;3412]$ if and only if the set Φ is avoided is recorded in entry P0022 of the Database of Permutation Pattern Avoidance [11], and is enumerated by A191721 in [9].

Observe that Theorem 3.2 recovers the result in Theorem 2.17, since a permutation w in which $[321;3412](w) \in \{0, 1\}$ necessarily avoids every pattern in Φ. Note also that 0 and 1 are the only values for which $\mathsf{rep}(w)$ and $[321;3412](w)$ are always equal, because there are permutations $\phi \in \Phi$ with $\mathsf{rep}(\phi) = 2$ but $[321;3412](\phi) = 3$.

Suppose $w \in \mathfrak{S}_N$. Theorem 3.2 is proved by induction on N and involves an assignment of at least one N-occurrence of 321 or 3412 to each previously used letter involved in positioning N in the one-line notation of w, after first positioning all other letters relative to each other. We must be wary of overcounting these N-occurrences of 321 and 3412. The details of the proof are covered in Sections 4 and 5.

4. Preliminaries for proving the main theorem

4.1. Notation and elementary results to be used in the proof.

Definition 4.1.1. Consider $w \in \mathfrak{S}_N$. Define $\overline{w} \in \mathfrak{S}_{N-1}$ by

$$\overline{w}(i) = \begin{cases} w(i) & \text{if } i < w^{-1}(N), \text{ and} \\ w(i+1) & \text{if } i > w^{-1}(N). \end{cases}$$

The one-line notation of $\overline{w}$ is obtained from the one-line notation of w by deleting the letter N and sliding all subsequent letters one space to the left. Moreover, if we think of $\overline{w}$ as a permutation in $\mathfrak{S}_N$ that fixes N, then

$$w = \overline{w} s_{N-1} s_{N-2} \cdots s_{w^{-1}(N)}, \tag{3}$$

and

$$\ell(w) = \ell(\overline{w}) + N - w^{-1}(N).$$

Example 4.1.2. If $w = 35412$, then $\overline{w} = 3412$. If we consider $\overline{w}$ to be the element $34125 \in \mathfrak{S}_5$, then

$$w = \overline{w} s_4 s_3 s_2.$$

One reduced decomposition of $\overline{w}$ is $s_2 s_1 s_3 s_2$, and so $s_2 s_1 s_3 s_2 s_4 s_3 s_2$ is a reduced decomposition of w.

Throughout the rest of this section, let w be a permutation in $\mathfrak{S}_N$, and $\overline{w} \in \mathfrak{S}_{N-1}$ be as defined above.

The following set will be crucial in the proof of Theorem 3.2, describing the letters in a reduced word of w, but not of $\overline{w}$, which count as repeated letters for w.

Definition 4.1.3. Let $\mathsf{new\text{-}rep}(w) = \{k : s_k \in \mathsf{supp}(\overline{w}) \text{ and } w^{-1}(N) \le k\}$.

Lemma 4.1.4.

$$\begin{aligned} \mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{supp}(\overline{w}) \cap \{s_{N-1} s_{N-2} \cdots s_{w^{-1}(N)}\}| \\ &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)|. \end{aligned}$$

Proof. This follows from equation (3). □

Recall the functions M_k and m_k from Definition 2.6.

Lemma 4.1.5. *$M_k(\overline{w}) > m_k(\overline{w})$ and $w^{-1}(N) \le k$ if and only if $k \in \mathsf{new\text{-}rep}(w)$.*

Proof. The forward direction of the statement is immediate from the definition of $\mathsf{new\text{-}rep}(w)$. The converse of this follows from Lemma 2.8 and Definition 4.1.3. □

To show that $\mathsf{rep}(w)$ is a lower bound for $[321;3412](w)$, we would like to assign, to each element of $\mathsf{new\text{-}rep}(w)$, at least one N-occurrence in w of one of the patterns $\{321, 3412\}$. This assignment should be done carefully to avoid overcounting. Additionally, to characterize when $\mathsf{rep}(w)$ and $[321;3412](w)$ are equal, we would like to understand when each N-occurrence in w of the patterns $\{321, 3412\}$ corresponds to some element of $\mathsf{new\text{-}rep}(w)$.

For the remainder of this section, set $M_k = M_k(\overline{w})$ and $m_k = m_k(\overline{w})$ for all k.

Definition 4.1.6. Consider $k \in \mathsf{new\text{-}rep}(w)$. Define $\mathfrak{p}_k(w)$ as follows.

I. If $w^{-1}(N) < w^{-1}(M_k)$, then $\mathfrak{p}_k(w) = \{N, M_k, m_k\}$, which is a 321-pattern in w.
II. If $w^{-1}(N) > w^{-1}(M_k)$ and $\overline{w}(k) > m_k$, then $\mathfrak{p}_k(w) = \{N, \overline{w}(k), m_k\}$, which is a 321-pattern in w.
III. Otherwise, set $\mathfrak{p}_k(w) = \{M_k, N, \overline{w}(k), m_k\}$, which is a 3412-pattern in w.

This $\mathfrak{p}_k(w)$ is undefined if $k \notin \textsf{new-rep}(w)$.

Note that $\mathfrak{p}_k(w)$ is always an N-occurrence of either 321 or of 3412 because $k \in \textsf{new-rep}(w)$ and thus $M_k > m_k$ by Lemma 2.8. However, it is not clear when $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ coincide for $k \neq k'$, nor which N-occurrences of 321 or of 3412 have the form $\mathfrak{p}_k(w)$ for some k.

4.2. **Issues of overcounting.** Consider whether the patterns $\mathfrak{p}_k(w)$ might overcount N-occurrences of 321 or 3412 in w.

Proposition 4.2.1. *There are no distinct $k, k' \in \textsf{new-rep}(w)$ for which $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ are the same N-occurrence of* 3412 *in w.*

Proof. If this were the case, then $(M_k, \overline{w}(k), m_k) = (M_{k'}, \overline{w}(k'), m_{k'})$. But then $\overline{w}(k) = \overline{w}(k')$, implying that $k = k'$. □

Therefore, if there is any overcounting of N-occurrences of 321 or 3412 among the $\{\mathfrak{p}_k(w)\}$, it must be that $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ are the same N-occurrence of 321.

Proposition 4.2.2. *If there exist distinct $k, k' \in \textsf{new-rep}(w)$ with $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$, then w has an N-occurrence of* 4321.

Proof. Suppose that there exist $k, k' \in \textsf{new-rep}(w)$, with $k < k'$, such that $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$. Proposition 4.2.1 implies that these coincident patterns must be N-occurrences of 321 in w.

These coincident $\mathfrak{p}_k(w)$ and $\mathfrak{p}_{k'}(w)$ cannot both be of type II as in Definition 4.1.6, because that would mean that $\overline{w}(k) = \overline{w}(k')$, and so $k = k'$.

Now suppose that the patterns have different types. Thus $\overline{w}(k) = M_{k'}$ and $m_k = m_{k'}$. Then $\{N, \overline{w}(k) = M_{k'}, \overline{w}(k'), m_k = m_{k'}\}$ forms an N-occurrence of 4321 in w. Note also in this case that we must have $M_k = \overline{w}(k) = M_{k'}$, since otherwise M_k would lie to the left of $\overline{w}(k) = M_{k'}$, and be greater than $\overline{w}(k)$ by definition, which would contradict the maximality of $M_{k'}$.

It remains to consider the case when both patterns are of type I, and so $(M_k, m_k) = (M_{k'}, m_{k'}) = (M, m)$. We can assume that $M \notin \{\overline{w}(k), \overline{w}(k')\}$ because that case was already addressed. Then the one-line notation of $\overline{w}$, and hence of w, looks like

$$\cdots\ M\ \cdots\ \overline{w}(k)\ \cdots\ \overline{w}(k')\ \cdots\ m\ \cdots.$$

Consider where N lies in relation to the values $\{M, \overline{w}(k), \overline{w}(k'), m\}$. Because both patterns have type I, we must have that $w^{-1}(N) < w^{-1}(M)$, and so N lies to the left of M. The definitions of M and m require that $M > \overline{w}(k), \overline{w}(k')$, and $m < \overline{w}(k')$. Thus the letters $\{N, M, \overline{w}(k'), m\}$ form an N-occurrence of 4321. □

Corollary 4.2.3. *If w has no N-occurrence of* 4321*, then*

$$|\textsf{new-rep}(w)| \le [321;3412]_N(w).$$

Proof. This follows from Propositions 4.2.1 and 4.2.2, because there do not exist distinct $k, k' \in \textsf{new-rep}(w)$ with $\mathfrak{p}_k(w)$ equalling $\mathfrak{p}_{k'}(w)$. □

Therefore, by Corollary 4.2.3, the procedure for assigning to each $k \in \textsf{new-rep}(w)$ an N-occurrence of either 321 or 3412 is injective if w has no N-occurrence of 4321. We must now consider what happens to this assignment when w does have such a pattern.

Proposition 4.2.4. *Suppose that w has an N-occurrence of* 4321.

(a) *If there exist distinct $k, k' \in \textsf{new-rep}(w)$ and $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w)$, then there are two other N-occurrences $\mathfrak{p}^+_{k'}(w)$ and $\mathfrak{p}^-_{k'}(w)$ of* 321 *in w, which are not equal to $\mathfrak{p}_j(w)$ for any j. (Let such a k' be called "duplicating.")*

(b) *Let i and j both be duplicating. If $i \neq j$, then $\{\mathfrak{p}^+_i(w), \mathfrak{p}^-_i(w)\} \cap \{\mathfrak{p}^+_j(w), \mathfrak{p}^-_j(w)\} = \emptyset$.*

Proof. First we will prove statement (a). Suppose that there are such $k < k'$. Then we know from Proposition 4.2.2 that $(M_k, m_k) = (M_{k'}, m_{k'}) = (M, m)$, and $\mathfrak{p}_k(w) = \mathfrak{p}_{k'}(w) = \{N, M, m\}$. Also, we know that the one-line notation of w looks like

$$\cdots\ N\ \cdots\ M\ \cdots\ \overline{w}(k)\ \cdots\ \overline{w}(k')\ \cdots\ m\ \cdots,$$

where M and $\overline{w}(k)$ could possibly be equal. Because $M = M_{k'}$, we must have $M > \overline{w}(k')$. Also, because $m = m_k$, we must have $m < \overline{w}(k')$. Thus

$$\mathfrak{p}^+_{k'}(w) = \{N > M > \overline{w}(k')\} \text{ and } \mathfrak{p}^-_{k'}(w) = \{N > \overline{w}(k') > m\} \tag{4}$$

are both N-occurrences of 321 in w.

Note that $\mathfrak{p}^+_{k'}(w)$ is not equal to $\mathfrak{p}_j(w)$ for any j, because $\overline{w}(k')$ is not equal to m_j for any j: there exists a letter (for example, m) to the right of $\overline{w}(k')$ which is less than $\overline{w}(k')$. Similarly, $\mathfrak{p}^-_{k'}(w) \neq \mathfrak{p}_j(w)$ for any j, because $\overline{w}(k')$ cannot equal M_j.

The proof of statement (b) is similar to the previous argument. Suppose that i and j are duplicating, with $i \neq j$. If $\mathfrak{p}^+_i(w) = \mathfrak{p}^+_j(w)$ or $\mathfrak{p}^-_i(w) = \mathfrak{p}^-_j(w)$, as defined in equation (4), then $\overline{w}(i) = \overline{w}(j)$. This would mean that $i = j$, which is a contradiction. Thus it remains to consider the situation $\mathfrak{p}^+_i(w) = \mathfrak{p}^-_j(w)$. Then $M_i = \overline{w}(j)$ and $\overline{w}(i) = m_j$, and $i > j$. Once again, we cannot have $\overline{w}(i) = m_j$, because the letter m_i appears to the right of $\overline{w}(i)$ and is less than $\overline{w}(i)$. This completes the proof. □

Corollary 4.2.5. *If w has an N-occurrence of* 4321*, then*

$$|\textsf{new-rep}(w)| < [\textsf{321;3412}]_N(w).$$

Proof. Partition the set $\textsf{new-rep}(w)$ into sets $S_1, S_2, \ldots, S_t$ so that $(M_k, m_k) = (M(i), m(i))$ for each $k \in S_i$. Suppose $S_i = \{k_{i_1} < k_{i_2} < \cdots < k_{i_{|S_i|}}\}$, and define

$$\mathfrak{p}_{S_i}(w) = \left\{\mathfrak{p}_{k_{i_1}}(w), \mathfrak{p}^+_{k_{i_2}}(w), \mathfrak{p}^-_{k_{i_2}}(w), \ldots, \mathfrak{p}^+_{k_{i_{|S_i|}}}(w), \mathfrak{p}^-_{k_{i_{|S_i|}}}(w)\right\}.$$

Note that if $|S_i| = 1$, then $|\mathfrak{p}_{S_i}(w)| = 1$. Also, if $|S_i| > 1$, then $|\mathfrak{p}_{S_i}(w)| = 2|S_i| - 1 > |S_i|$. Moreover, the elements of $\mathfrak{p}_{S_i}(w)$ are all N-occurrences of either 321 or 3412 in w. Finally, by Proposition 4.2.4(b), the sets $\{\mathfrak{p}_{S_i}(w)\}$ are disjoint.

If w has an N-occurrence of 4321, then there exists some S_i containing at least two elements. Thus $|\textsf{new-rep}(w)| = |S_1| + |S_2| + \cdots + |S_t| < |\mathfrak{p}_{S_1}(w)| + |\mathfrak{p}_{S_2}(w)| + \cdots + |\mathfrak{p}_{S_t}(w)| \leq [\textsf{321;3412}]_N(w)$. □

Using the notation from the proof of Corollary 4.2.5, we can also rewrite its result to say that the map

$$\xi_n : j \mapsto \begin{cases} \mathfrak{p}_j(w) & \text{if } j \text{ is the minimal element in } S_i, \text{ and} \\ \mathfrak{p}_j^+(w) & \text{otherwise.} \end{cases} \tag{5}$$

is an injection.

4.3. **Issues of undercounting.** We have now addressed the issue of whether the set $\{\mathfrak{p}_k(w) : k \in \mathsf{new\text{-}rep}(w)\}$ might overcount some N-occurrences of 321 or of 3412 (in fact, we have shown that only 321-patterns may be overcounted). We must now consider when this set might undercount these N-occurrences. As we have seen in Proposition 4.2.4, undercounting is certainly a possibility. What we will show now is that if w avoids the ten patterns in the set Φ, then there is no undercounting, and thus the inequality of Corollary 4.2.3 is actually an equality.

To examine potential undercounting, we must decide if and when an N-occurrence of 321 or of 3412 might not equal $\mathfrak{p}_k(w)$ for some k.

Proposition 4.3.1. *If any N-occurrence $\{N > a > b\}$ of 321 in w is such that $b \not\in \{m_k : k \in \mathsf{new\text{-}rep}(w)\}$, then w has an N-occurrence of 4321.*

Proof. Suppose there is an N-occurrence of 321 in w where $b \neq m_k$ for any k. Then to the right of b in the one-line notation w, there exists $c < b$, preventing b from equalling any such m_k. Thus $\{N > a > b > c\}$ is an N-occurrence of 4321. Now, suppose there is no such c, and set $k = \overline{w}^{-1}(b) - 1$. Then $b = m_k$, and, since $a > b$ appears to the left of b, we must have $M_k \geq a > b = m_k$. Therefore, by Lemma 4.1.5, $k \in \mathsf{new\text{-}rep}(w)$. □

Proposition 4.3.2. *Suppose w is 4321-avoiding. If any N-occurrence $\{N > a > b\}$ of 321 in w is such that there exists no $k \in \mathsf{new\text{-}rep}(w)$ with $(a, b) \in \{(M_k, m_k), (\overline{w}_k, m_k)\}$, then w has an N-occurrence of at least one of the patterns $\{45312, 53412\}$.*

Proof. By Proposition 4.3.1, we know that $b = m_k$ for at least one value of $k \in \mathsf{new\text{-}rep}(w)$. Suppose that $a \not\in \{M_k, \overline{w}(k)\}$.

Suppose $a > M_k$. Then, by maximality of M_k, this a must appear to the right of both M_k and $\overline{w}(k)$ in the one-line notation of w. But then, setting $k' = \overline{w}^{-1}(a)$, we must have $m_{k'} = m_k = b$, and so $(a, b) = (\overline{w}(k'), m_{k'})$. By definition, $k' > k$, and so $m_{k'} = m_k < k + 1 < k' + 1$, where the first inequality is because $k \in \mathsf{new\text{-}rep}(w)$. Therefore $k' \in \mathsf{new\text{-}rep}(w)$ as well, by Lemma 4.1.5.

Now suppose that $a < M_k$. If $w^{-1}(N) > w^{-1}(M_k)$, then the one-line notation of w looks like

$$\cdots\ M_k\ \cdots\ N\ \underbrace{\cdots\cdots}_{<a}\ a\ \underbrace{\cdots\cdots}_{>a \text{ or } <m_k}\ b = m_k\ \cdots,$$

because w is 4321-avoiding. If all values appearing between a and b in w are larger than a, then we can set $k' = \overline{w}^{-1}(a)$, and we have $(M_{k'}, m_{k'}) = (M_k, m_k)$, and again $k' \in \mathsf{new\text{-}rep}(w)$ by Lemma 4.1.5. Thus suppose that there is some value c in this portion of w with $c < m_k$. Then $\{M_k, N, a, c, b\}$ is an N-occurrence of 45312 in w.

Finally, suppose that $w^{-1}(N) < w^{-1}(M_k)$, where k is minimal with this property. So the one-line notation of w looks like

$$\cdots\ N\ \cdots\ a\ \cdots\ M_k\ \cdots\ b = m_k\ \cdots,$$

again because w is 4321-avoiding. If $M_k = \overline{w}(k)$, then the value of k was not chosen to be minimal, a contradiction. Thus the entry $\overline{w}(k)$ must lie strictly between M_k and $b = m_k$. By definition, $\overline{w}(k) < M_k$. Moreover, to avoid the pattern 4321, we must have $\overline{w}(k) < b = m_k$. Thus $\{N, a, M_k, \overline{w}(k), b = m_k\}$ forms a 53412-pattern in w. $\square$

Propositions 4.3.1 and 4.3.2 now imply the following result.

Corollary 4.3.3. *If w has no N-occurrences of the patterns $\{4321, 45312, 53412\}$, then every N-occurrence of 321 in w is equal to $\mathfrak{p}_k(w)$ for some k.*

Proposition 4.3.4. *If any N-occurrence $\{a, N, b, c\}$ of 3412 in w is such that $c \notin \{m_k : k \in \textsf{new-rep}(w)\}$, then w has an N-occurrence of at least one of the patterns $\{45231, 45132\}$.*

Proof. Suppose there is such an N-occurrence of 3412 in w. This means that to the right of c in the one-line notation of w, there exists a $d < c$, preventing c from equalling any such m_k. Thus $\{a, N, b, c, d\}$ is an N-occurrence of either 45231 or of 45132, depending on whether $b > d$ or $b < d$. Now suppose that there is no such d, and set $k = \overline{w}^{-1}(c) - 1$. Then $c = m_k$, and, since $a > c$ appears to the left of c, we must have $M_k > m_k$. Therefore, by Lemma 4.1.5, $k \in \textsf{new-rep}(w)$. $\square$

Proposition 4.3.5. *Suppose w is 45231- and 45132-avoiding. If any N-occurrence $\{a, N, b, c\}$ of 3412 in w is such that there exists no $k \in \textsf{new-rep}(w)$ with $(a, c) = (M_k, m_k)$, then w has an N-occurrence of at least one of the patterns $\{43512, 34512, 35412\}$.*

Proof. By Proposition 4.3.4, we know that $c = m_k$ for some $k \in \textsf{new-rep}(w)$. Choose the minimal such k; that is, choose k so that $\overline{w}(k) < c$ (and thus, necessarily, $\overline{w}(j) \geq c$ for all $j > k+1$). There are now three places M_k might appear relative to the letters $\{a, N, \overline{w}(k), c\}$, which themselves form an N-occurrence of 3412 in w:

$$\underbrace{\cdots\cdots}_{M_k?}\ a\ \underbrace{\cdots\cdots}_{M_k?}\ N\ \underbrace{\cdots\cdots}_{M_k?}\ \overline{w}(k)\ \cdots\ c = m_k\ \cdots.$$

By definition, $M_k \geq a$. Thus, if $M_k \neq a$, then these three possibilities create N-occurrences of 43512, 34512, or 35412 in w, respectively. $\square$

Proposition 4.3.6. *Suppose w avoids the patterns*

$$\{45231, 45132, 43512, 34512, 35412\}.$$

If any N-occurrence $\{a, N, b, c\}$ of 3412 in w is such that there exists no $k \in \textsf{new-rep}(w)$ with $(a, b, c) = (M_k, \overline{w}(k), m_k)$, then w has an N-occurrence of at least one of the patterns $\{45123, 45213\}$.

Proof. By Propositions 4.3.4 and 4.3.5, we know that $(a, c) = (M_k, m_k)$ for some $k \in \textsf{new-rep}(w)$. If $b \neq \overline{w}(k)$, then $\overline{w}(k)$ either lies between N and b, or between b and $c = m_k$. In fact, $\overline{w}(k)$ must lie to the right of b, because $b < c = m_k = \min\{\overline{w}(k+1), \ldots, \overline{w}(n)\}$. We also know that $\overline{w}(k) < M_k = a$. Therefore, since w is 45132-avoiding, the set $\{a, N, b, \overline{w}(k), c\}$ forms an N-occurrence of either 45123 or 45213. $\square$

Propositions 4.3.4, 4.3.5, and 4.3.6 now imply the following result.

Corollary 4.3.7. *If w has no N-occurrences of the patterns*

$$\{45231, 45132, 43512, 34512, 35412, 45123, 45213\},$$

then every N-occurrence of 3412 in w is equal to $\mathfrak{p}_k(w)$ for some $k \in \mathsf{new\text{-}rep}(w)$.

This addresses the concern about undercounting the N-occurrences of 321 and 3412 in w.

Corollary 4.3.8. *If w has no N-occurrence of any of the patterns in the set*

$$\{4321, 45312, 53412, 45231, 45132, 43512, 34512, 35412, 45123, 45213\},$$

then

$$|\mathsf{new\text{-}rep}(w)| \geq [\mathsf{321;3412}]_N(w).$$

Proof. This follows from Corollaries 4.3.3 and 4.3.7. □

4.4. **Conclusions.** We now combine the previous two subsections to draw the following conclusion.

Corollary 4.4.1. *If w has no N-occurrence of any of the patterns in the set Φ, then*

$$|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w).$$

In other words, if w has no N-occurrence of any of the patterns in the set Φ, then the map ξ_N of equation (5) is a bijection.

Proof. Combine the inequalities in Corollaries 4.2.3 and 4.3.8. □

It is natural now to wonder about the implications of containing an N-occurrence of a pattern in Φ. In fact, for each w containing an N-occurrence of some $\phi \in \Phi$, there is an N-occurrence $\mathfrak{p}_\phi(w)$ of either 321 or 3412 which is not equal to $\mathfrak{p}_k(w)$ or to $\mathfrak{p}_k^+(w)$ (as defined in Proposition 4.2.4) for any k, as is shown in the following table. In this table, the N-occurrence $\mathfrak{p}_\phi(w)$ will be written as a substring of ϕ, and will refer to those respective letters of the N-occurrence of ϕ in w.

$\phi \in \Phi$	$\mathfrak{p}_\phi(w)$
4321	421
34512	3512
45123	4513
35412	3512
43512	3512
45132	4513
45213	4523
53412	532
45312	532
45231	4523

Note that for $4321 \in \Phi$, the subpattern 432 is also not equal to any $\mathfrak{p}_k(w)$. However, it could equal some $\mathfrak{p}_k^+(w)$, so to avoid this possibility we set $\mathfrak{p}_{4321}(w) = 421$.

Proposition 4.4.2. *Let $w \in \mathfrak{S}_N$ be a permutation containing an N-occurrence of some pattern $\phi \in \Phi$. Then $\mathfrak{p}_\phi(w)$ is not equal to $\mathfrak{p}_k(w)$ for any $k \in \mathsf{new\text{-}rep}(w)$, nor to any $\mathfrak{p}_k^+(w)$, as defined in Proposition 4.2.4. That is, the injection ξ_N of equation* (5) *is not surjective.*

Proof. This follows from the definitions of the patterns $\mathfrak{p}_\phi(w)$, $\mathfrak{p}_k(w)$, and $\mathfrak{p}_k^+(w)$. □

This proposition has the following corollary.

Corollary 4.4.3. *If w has an N-occurrence of at least one of the patterns in the set Φ, then*

$$|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w).$$

5. Proof of the main theorem

Proof of Theorem 3.2. We prove this by induction on the number of letters in a permutation.

The result is easy to verify for small cases, so assume that the theorem holds for all permutations in $\mathfrak{S}_n$ for all $n < N$, and consider $w \in \mathfrak{S}_N$. Define $\overline{w} \in \mathfrak{S}_{N-1}$ as in Section 4. Since $N-1 < N$, we know that $\mathsf{rep}(\overline{w})$ is equal to $[\mathsf{321;3412}](\overline{w})$ if $\overline{w}$ avoids the patterns in the set Φ, and that $\mathsf{rep}(\overline{w})$ is less than $[\mathsf{321;3412}](\overline{w})$ if $\overline{w}$ contains at least one pattern in Φ.

Suppose first that $\overline{w}$ avoids the patterns in Φ. If w has no N-occurrences of any of the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w)$. Thus

$$\begin{aligned}\mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &= [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w).\end{aligned}$$

On the other hand, if w does have an N-occurrence of at least one the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w)$, and so

$$\begin{aligned}\mathsf{rep}(w) &= \mathsf{rep}(\overline{w}) + |\mathsf{new\text{-}rep}(w)| \\ &< [\mathsf{321;3412}](\overline{w}) + [\mathsf{321;3412}]_N(w) = [\mathsf{321;3412}](w).\end{aligned} \tag{6}$$

Now assume that $\overline{w}$ does not avoid the patterns in Φ. If w has no N-occurrences of any of the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| = [\mathsf{321;3412}]_N(w)$. Thus inequality (6) holds. On the other hand, if w does have an N-occurrence of at least one the patterns in Φ, then $|\mathsf{new\text{-}rep}(w)| < [\mathsf{321;3412}]_N(w)$, and so inequality (6) holds again.

This completes the proof. □

Definition 5.1. Consider a permutation $w \in \mathfrak{S}_N$. Let $\overline{w}_{(0)} = w$, and for $i \in \{1, \ldots, N-1\}$, let $\overline{w}_{(i+1)} = \overline{\overline{w}_{(i)}}$.

Corollary 5.2. *If a permutation $w \in \mathfrak{S}_N$ avoids every pattern in the set Φ, then the maps $\{\xi_n : n \leq N\}$ define a bijection from the set $\{\mathsf{new\text{-}rep}(\overline{w}_{(i)}) : i \in \{0, \ldots, N-1\}\}$ to the set of all* 321- *and* 3412*-patterns in w.*

Additionally, the proof of Theorem 3.2 can be adapted to show the following.

Corollary 5.3. *For any permutation w,*

$$[\mathsf{321;3412}](w) - \mathsf{rep}(w) \geq |\{r : w \textit{ has an } r\textit{-occurrence of a pattern in } \Phi\}|.$$

Using [1] and the MAPLE package [16], Vince Vatter has subsequently found a generating function for the number of permutations in $\mathfrak{S}_N$ avoiding the ten patterns in Φ [17]. This generating function is

$$g(x) = \frac{1 - 4x + x^3}{(1-x)(1-4x-x^2+x^3)}.$$

Acknowledgements

Special thanks are due to Kári Ragnarsson for writing the Java code to compute examples and to confirm conjectures through $\mathfrak{S}_{10}$, as well as to an anonymous referee for many thoughtful comments.

References

[1] M. H. Albert, S. Linton, N. Ruškuc, The insertion encoding of permutations, Electron. J. Combin. 12 (2005), R47.

[2] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Graduate Texts in Mathematics 231, Springer, New York, 2005.

[3] A. Claesson, S. Kitaev, Classification of bijections between 321- and 132-avoiding permutations, Sém. Lothar. Combin. 60 (2008), B60d.

[4] D. Daly, Reduced decompositions with one repetition and permutation pattern avoidance, preprint.

[5] I. G. Macdonald, Notes on Schubert Polynomials, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, Montreal, 1991.

[6] A. Marcus, G. Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture, J. Combin. Theory Ser. A 107 (2004), 153–160.

[7] H. Matsumoto, Générateurs et relations des groupes de Weyl généralisés, C. R. Acad. Sci. Paris 258 (1964), 3419–3422.

[8] R. Simion, F. W. Schmidt, Restricted permutations, European J. Combin. 6 (1985), 383-406.

[9] N. J. A. Sloane, Online encyclopedia of integer sequences, published electronically at http://oeis.org.

[10] R. P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, European J. Combin. 5 (1984), 359–372.

[11] B. E. Tenner, Database of permutation pattern avoidance, published electronically at http://math.depaul.edu/bridget/patterns.html.

[12] B. E. Tenner, Pattern avoidance and the Bruhat order, J. Combin. Theory Ser. A 114 (2007), 888–905.

[13] B. E. Tenner, Reduced decompositions and permutation patterns, J. Algebraic Combin. 24 (2006), 263–284.

[14] J. Tits, Le problème des mots dans les groupes de Coxeter, Symposia Mathematica (INDAM, Rome, 1967/68), Academic Press, London, 1969, vol. 1, pp.175–185.

[15] H. Úlfarsson, A unification of permutation patterns related to Schubert varieties, to appear in Pure Math. Appl.

[16] V. Vatter, the MAPLE package INSENC, published electronically at http://www.math.ufl.edu/˜vatter/publications/ins-enc/.

[17] V. Vatter, private communication.

Department of Mathematical Sciences, DePaul University, Chicago, IL 60614
E-mail address: `bridget@math.depaul.edu`