SURVEY OF DIRICHLET SERIES OF MULTIPLICATIVE ARITHMETIC FUNCTIONS

RICHARD J. MATHAR

ABSTRACT. The manuscript reviews Dirichlet Series of important multiplicative arithmetic functions. The aim is to represent these as products and ratios of Riemann ζ-functions, or, if that concise format is not found, to provide the leading factors of the infinite product over ζ-functions. If rooted at the Dirichlet series for powers, for sums-of-divisors and for Euler's totient, the inheritance of multiplicativity through Dirichlet convolution or ordinary multiplication of pairs of arithmetic functions generates most of the results.

1. SCOPE

1.1. **Definition.** Multiplicative functions are arithmetic functions $a(n)$—functions defined for integer argument $n \in \mathbb{Z}$—for which evaluation commutes with multiplication for coprime arguments:

Definition 1. *(Multiplicative function a)*

$$a(nm) = a(n)a(m) \quad \forall (n,m) = 1. \tag{1.1}$$

The topic of the manuscript is the computation of the Dirichlet series ζ_D of arithmetic functions of that kind for sufficiently large real part of the argument s:

Definition 2. *(Dirichlet generating function ζ_D)*

$$a(n) \mapsto \zeta_D(s) \equiv \sum_{n\geq 1} \frac{a(n)}{n^s}. \tag{1.2}$$

As an immediate consequence of the definition, the ζ_D of the product of n^k times a multiplicative function $a(n)$ is given by replacing $s \to s-k$ in the ζ_D of $a(n)$:

$$n^k a(n) \mapsto \zeta_D(s-k). \tag{1.3}$$

1.2. **Properties.** For ease of reference further down, we summarize well-known features of arithmetic functions.

A consequence of the definition (1.1) is that the function is already entirely defined if specified for prime powers p^e, because all remaining values follow by the prime power factorization of the arguments:

$$a(p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}) = \prod_p a(p^e). \tag{1.4}$$

The equation that explicitly specifies values of $a(p^e)$ will be called the *master equation* of that sequence in the sequel.

Date: June 22, 2011.
2010 *Mathematics Subject Classification.* Primary 11K65, 11Y70; Secondary 30B50, 11M41.
Key words and phrases. Arithmetic Function, multiplicative, Dirichlet Generating Function.

If a is multiplicative, the Dirichlet series reduces to a product over all primes p:
(1.5)
$$\zeta_D(s) = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \sum_{e\ge 1} \frac{a(p^e)}{p^{es}}\right) = \prod_p \sum_{e\ge 0} \frac{a(p^e)}{p^{es}}.$$

This sum over $e \ge 0$ at some fixed p will be called the *Bell series* of that $a(n)$. The product over the primes p may be a finite product and/or ratio of cyclotomic polynomials of some power of p; then it is rephrased as a finite product of Riemann zeta-functions. In the general case, this expansion will lead to an infinite product, and will be represented in the followup chapters in the format

$$\zeta_D(s) = \prod_p \prod_{i=1}^{\infty} (1 - S_i p^{l_i - u_i s})^{\gamma_i}, \tag{1.6}$$

where the vector of the S_i contains sign factors ± 1, and the l_i, u_i and γ_i are integers. The natural order of the factors is smallest u_i first, and if these are the same largest l_i first. This order stresses which terms put the tightest constraints on the region of convergence in the plane of complex s. With Euler's formula, these constituents are equivalent to Riemann ζ-functions:

$$\prod_p (1 - p^{l_i - u_i s})^{\gamma_i} = \zeta^{-\gamma_i}(u_i s - l_i); \quad \prod_p (1 + p^{l_i - u_i s})^{\gamma_i} = \frac{\zeta^{\gamma_i}(u_i s - l_i)}{\zeta^{\gamma_i}(2u_i s - 2l_i)}. \tag{1.7}$$

(We write ζ_D for a generic Dirichlet series and ζ for the Riemann zeta function.) Truncating the product expansion is a tool of numerical evaluation of the $\zeta_D(s)$.

The difference of this work to the Gould-Shonhiwa table [11] of the transformations $a(n) \mapsto \zeta_D(s)$ is that we will (i) cover Dirichlet series which require this type of infinite Euler products, and (ii) will detail the six-digit A-numbers of individual sequences in Sloane's Online Encylopedia of Integer Sequences (OEIS) [25]; due credit to individual's discovery of many formulae that follow is stored in this database.

1.3. **Dirichlet convolution.** Dirichlet convolution is the construction of a new series by summation over divisor and complementary divisor arguments of two arithmetic functions:

Definition 3. *(Dirichlet Convolution of a and b)*

$$(a \star b)(n) \equiv \sum_{d|n} a(d) b(n/d). \tag{1.8}$$

The master equation of a multiplicative function derived via Dirichlet convolution (1.8) is related to the master equations of the factors [29]:

$$(a \star b)(p^e) = \sum_{l=0}^{e} a(p^l) b(p^{e-l}). \tag{1.9}$$

The ordinary (Hadamard) and the Dirichlet (convolution) product of two multiplicative functions, the ordinary (Hadamard) ratio of two multiplicative functions, and the Dirichlet inverse of a multiplicative function are multiplicative [2], which creates multiplicative function by inheritance from "simpler" multiplicative functions. As a special case, the $j-$th power of a multiplicative function is multiplicative and the associated $a(p^e)$ is the the j-th power of the one for the first power.

The Dirichlet series of the Dirichlet product is the (ordinary) product of the Dirichlet series:

$$a \star b \mapsto \zeta_{D,a}(s)\zeta_{D,b}(s). \tag{1.10}$$

As a corollary, the Dirichlet series of the Dirichlet inverse defined by $(a^{(-1)} \star a)(n) \equiv \delta_{1,n}$ is given by the reciprocal Dirichlet series of a:

$$a^{(-1)} \mapsto 1/\zeta_D(s). \tag{1.11}$$

2. Classifications

2.1. **Completely Multiplicative.**

2.1.1. *Generic Properties.* Completely multiplicative functions are a sub-species of multiplicative functions which obey the equation (1.1) for argument pairs n and m irrespective of common divisors:

Definition 4. *(Completely Multiplicative function a)*

$$a(nm) = a(n)a(m). \tag{2.1}$$

The well-known properties of completely multiplicative functions are that the Dirichlet inverse can be written as a multiplication with the Möbius function,

$$a^{(-1)}(n) = \mu(n)a(n), \tag{2.2}$$

that the master equation allows interchange of exponentiation and evaluation,

$$a(p^e) = a^e(p), \tag{2.3}$$

and that the sum over the exponents e in (1.5) is a geometric series [2],

$$\zeta_D = \prod_p \frac{1}{1 - a(p)p^{-s}}. \tag{2.4}$$

2.1.2. *Powers.* The Dirichlet series of powers is obvious from (1.2):

$$n^k \mapsto \zeta(s-k). \tag{2.5}$$

The most important example is the exponent $k = 0$,

$$1 \mapsto \zeta(s). \tag{2.6}$$

Its Dirichlet inverse is the Möbius function $\mu(n)$ (which is not completely multiplicative, A008683), with Dirichlet generating function

$$\mu(n) \mapsto 1/\zeta(s) \tag{2.7}$$

obtained combining (1.11) and (2.6). The master equation of $\mu(n)$ is

$$a(p^e) = \begin{cases} -1, & e = 1; \\ 0, & e > 1. \end{cases} \tag{2.8}$$

Squaring one obtains the Bell series of $\mu^2(n)$,

$$1 + \sum_{e=1} \frac{1}{p^{es}} = 1 + p^{-s} = \frac{1 - p^{-2s}}{1 - p^{-s}} \tag{2.9}$$

which will be used further down as

$$\mu^2(n) \mapsto \zeta(s)/\zeta(2s). \tag{2.10}$$

Remark 1. *These n^k cover $k = 0$ in A000012, $k = 1$ in A000027 with inverse $n\mu(n)$ in A055615, $k = 2$ in A000290, $k = 3$ in A000578, $k = 4$ and 5 in A000583 and A000584, $k = 6$–9 in A001014–A001017, $k = 10$–12 in A008454–A008456, $k = 13$–25 in A010801–A010813, $k = 26$ in A089081, $k = 27$-30 in A122968–A122971* [25].

2.1.3. *Primes to constants.* If the master equation of a completely multiplicative function is

$$a(p) = c, \tag{2.11}$$

the Dirichlet series is usually expanded into an Euler product

$$\zeta_D(s) = \prod_p \frac{1}{1 - c/p^s} = \prod_{j \ge 1} (1 - p^{-s})^{\gamma_j(c)}. \tag{2.12}$$

for numerical efficienty, such that the Dirichlet generating function becomes an (infinite) product of the form $\prod_{j\ge 1} \zeta^{-\gamma_j(c)}(s)$ [21].

Remark 2. *For $c = 2$–11 the $\gamma_j(c)$ are A001037, A027376, A027377, A001692, A032164, A001693, A027380, A027381, A032165, and A032166 in that order. These exponents appear essentially as $\gamma_{r,j}^{(C)}$ in my expansions of Hardy-Littlewood constants* [19, chapt. 7]. *In numerical practise, Wynn's partial-product algorithm is used to accelerate convergence of the Euler products* [31].

The cases of negative c are mapped via

$$\prod_p \frac{1}{1 - cp^{-s}} = \prod_p \frac{1 + cp^{-s}}{1 - c^2 p^{-2s}} \tag{2.13}$$

to a division of two Hardy-Littlewood constants.

Remark 3. *For a fixed integer s, one may factorize the polynomial of $1/p$ over the reals numerically, to face a Weierstrass product representation*

$$\zeta_D = \prod_p \prod_j (1 + \frac{\beta_j}{p^t}). \tag{2.14}$$

The number of factors in the j-product is equivalent to the order of the polynomial, and β_j are essentially its roots. Interchange of the two products rewrites ζ_D as a finite product of prime zeta-functions of squarefree k-almost primes [18]*:*

$$\zeta_D = \prod_j \left[1 + \beta_j \sum_p \frac{1}{p^t} + \beta_j^2 \sum_{p<q} \frac{1}{(pq)^t} + \cdots \right] = \prod_j \left[1 + \sum_{k\ge 1} (-\beta_j)^k P_k^{(\mu)}(t) \right]. \tag{2.15}$$

Remark 4. *This covers A061142 and A165872 ($c = \pm 2$) and A165824 ($c = 3$) up to A165871 ($c = 50$)* [25].

2.1.4. *Liouville.* The Liouville function $\lambda(n)$ (A008836) is the parity of the number $\Omega(n)$ of prime divisors of n. The master equation is [24]

$$a(p) = -1, \tag{2.16}$$

which evaluates by immediate application of (2.4) to [17]

$$\lambda(n) = (-1)^{\Omega(n)} \mapsto \zeta(2s)/\zeta(s), \tag{2.17}$$

the D-inverse of (2.10). With (1.10) follows that $\lambda \star 1$ is the characteristic function of the squares [24]:

$$\lambda \star 1 = \epsilon_2(n) \mapsto \zeta(2s). \tag{2.18}$$

2.2. **Persistently Multiplicative.**

2.2.1. *Definition.* I call a multiplicative function a *persistently* multiplicative if a product of coprime arguments leads to a coprime product of the function,

Definition 5. *(Persistently multiplicative function a)*

$$a(nm) = a(n)a(m) \quad \text{and} \quad (a(n), a(m)) = 1 \quad \forall (n, m) = 1. \tag{2.19}$$

If a persistently multiplicative function g is the inner function of a compositorial product $a(n) = f(g(n))$, and if f is multiplicative, then a is also multiplicative.

Persistently multiplicative are for example those multiplicative functions where master equations only modify the exponent of the prime power through some function E, $a(p^e) = p^{E(e)}$. The important subclass are the powers. This also includes functions which remove all powers of some fixed prime p_j from n, characterized by

$$\zeta_D = \zeta(s-1)(1 - p_j^{1-s})/(1 - p_j^{-s}). \tag{2.20}$$

Remark 5. *Examples with $p_j = 2$, 3 or 5 are A000265, A038502, and A132739.*

Other persistently multiplicative functions permute the prime bases p in the master equation, for example replace primes by their successors (A003961) or swap with the adjacent prime (A061898).

2.2.2. *Squarefree core.* Persistently multiplicative are functions that reduce n to its squarefree ($t = 2$), cubefree ($t = 3$) etc cores, where $E(e) = e \mod t$ is a modulo function which partitions e into periodically modulated classes. The function which reduces n to the t-free core has the Bell series

$$\sum_{r=0}^{t-1} \sum_{e=r,r+t,r+2t,\dots} \frac{p^r}{p^{es}} = \sum_{r=0}^{t-1} \sum_{j=0}^{\infty} \frac{p^r}{p^{(r+jt)s}} = \frac{1 - p^{t(1-s)}}{(1 - p^{-ts})(1 - p^{1-s})} \tag{2.21}$$

and therefore with (1.7) the generating function

$$\text{core}_t(n) \mapsto \zeta(ts)\zeta(s-1)/\zeta(ts-t). \tag{2.22}$$

Remark 6. *This concerns sequences A007913 (squarefree), A050985 (cubefree) and A053165 (4-free).*

2.2.3. *Largest t-free Divisor.* The largest t-free number dividing n, $\text{rad}_t(n)$, is complementary to the functionality of the previous subsection. The master equation admits exponents limited by t and by the exponent in n:

$$a(p^e) = p^{\min(e,t-1)}. \tag{2.23}$$

The Bell series is

$$\sum_{e=0}^{t-1} \frac{p^e}{p^{es}} + \sum_{e \ge t} \frac{p^{t-1}}{p^{es}} = \frac{1 - p^{-s} - p^{t(1-s)} + p^{t(1-s)-1}}{(1 - p^{-s})(1 - p^{1-s})} \tag{2.24}$$

$$= \frac{(1 - p^{-s}) \sum_{l=0}^{t-2} p^{(1-s)l} + p^{(1-s)(t-1)}}{1 - p^{-s}}.$$

The denominator contributes $\zeta(s)$ to the Dirichlet series. For $t = 2$, the Euler expansion of the numerator starts:

$$\prod_p (1+p^{1-s}-p^{-s}) = \prod_p (1+p^{1-s})(1-p^{-s})(1+p^{1-2s})(1-p^{2-3s}) \times (1+p^{1-3s})(1+p^{3-4s})(1-p^{2-4s})(1+p^{1-4s})(1-p^{4-5s})(1+p^{3-5s})^2 \times (1-p^{2-5s})^2(1+p^{1-5s})\cdots, \quad s>2. \tag{2.25}$$

Remark 7. *The cases $t = 2$–3 are shown in A007947–A007948, the case $t = 4$ in A058035.*

2.2.4. *Even-odd Splitting.* Persistently multiplicative are the functions that assign 1 to all odd arguments and some other values to even arguments. A fundamental example maps all even arguments to some constant c, which creates an arithmetic sequence of period length 2:

$$a(p^e) = \left\{ \begin{array}{ll} c, & p=2; \\ 1, & p>2; \end{array} \right\} \mapsto [1+(c-1)\cdot 2^{-s}]\zeta(s). \tag{2.26}$$

In a variant, multiples of 4 could be assigned to some constant c_1, the other even arguments to another constant c_2:

$$a(n) = \left\{ \begin{array}{ll} 1, & n \,\text{odd} \\ c_1, & n \equiv 0 \bmod 4 \\ c_2, & n \equiv 2 \bmod 4 \end{array} \right\} \mapsto [1+(c_2-1)\cdot 2^{-s}+(c_1-c_2)\cdot 4^{-s}]\zeta(s). \tag{2.27}$$

These periodic functions are additive overlays of L-series [20]. The computational strategy usually involves subtracting the Riemann ζ-function, expansion of the remaining $a(n)-1$ into a discrete Fourier series, and writing each component as a Hilbert zeta-function.

Remark 8. *This applies to A109008 ($c_1 = 4, c_2 = 2$), A010121 ($c_1 = 4, c_2 = 1$), A010123 ($c_1 = 6, c_2 = 2$), A010130 ($c_1 = 10, c_2 = 1$), A010131 ($c_1 = 10, c_2 = 2$), A010137 ($c_1 = 12, c_2 = 5$), A010146 ($c_1 = 14, c_2 = 6$), A112132 ($c_1 = 7, c_2 = 3$), A010127 ($c_1 = 8, c_2 = 3$), A089146 ($c_1 = 4, c_2 = 8$), or A010132 ($c_1 = 10, c_2 = 4$).*

3. Core Classes

3.1. **Characteristic Function of t-th powers.** The characteristic function $\epsilon_t(n)$ of the t-th powers equals 1 if the argument is a t-th power of some positive integer b, 0 otherwise [29]. The Dirichlet generating function (1.2) collects $1/b^{ts}$ summing over all $b \ge 1$:

$$\epsilon_t(n) \mapsto \zeta(ts). \tag{3.1}$$

The application of (1.3) with (2.5) yields

$$n^{k/2}\epsilon_2(n) \mapsto \zeta(2s-k). \tag{3.2}$$

Remark 9. *$\sqrt{n}\epsilon_2(n)$ is A037213.*

The characteristic function of the numbers which are t-free (which cannot be divided by a non-trivial t-th power) shall be denoted $\xi_t(n)$. The master equation puts a cap on the maximum power admitted in each factor:

$$a(p^e) = \left\{ \begin{array}{ll} 1, & e<t; \\ 0, & e \ge t. \end{array} \right. \tag{3.3}$$

The Bell series is

$$\sum_{e=0}^{t-1} 1/p^{es} = \frac{1-p^{-st}}{1-p^{-s}}, \tag{3.4}$$

therefore

$$\xi_t(n) \mapsto \zeta(s)/\zeta(st) \tag{3.5}$$

and [29]

$$\xi_t(n) \star \epsilon_t(n) = 1. \tag{3.6}$$

Remark 10. *The case $t = 2$ comprises $\mu^2(n)$, the characteristic function of square-free integers (A008966) [24], the D-inverse of (2.17). The derived $n\xi_2(n)$ is represented by the absolute values of A055615.*

3.2. **Depleted ζ-functions.** Characteristic functions of numbers which are not multiples of some prime power q^k are multiplicative with

$$a(p^e) = \begin{cases} 1, & \text{if} \quad p \neq q; \\ 1, & \text{if} \quad p = q, \quad e < k; \\ 0, & \text{if} \quad p = q, \quad e \geq k. \end{cases} \tag{3.7}$$

The Bell series is $1/(1-p^{-s})$ for all $p \neq q$ and $\sum_{e=0}^{k-1} 1/p^{es} = (1-p^{-sk})(1-p^{-s})$ for $p = q$. The merger of both is

$$\delta_{q^k \nmid n} \mapsto (1-q^{-sk})\zeta(s). \tag{3.8}$$

Remark 11. *Examples are $q^k = 2^1$ in A000035, 2^2 in A166486, 2^3 in A168181, 3^1 in A011655 (multiplied by n in A091684), 3^2 in A168182, 5^1 in A011558 (multiplied by n in A091703), 7^1 in A109720, 11^1 in A145568, or any principal Dirichlet character modulo some prime.*

3.3. **Greatest Common Divisors.** The greatest common divisor (n, c) with respect to a constant c is periodic $(n+c, c) = (n, c)$ [2, §8.1] and multiplicative. (Periodicity is revealed by the Euclidean algorithm which starting from $n + c$ on one hand or c on the other yields the same quotients and remainders already after the first step of the algorithm.)

Let $c = \prod_p p^{e_c}$ specify the prime exponents of the constant; then the master equation is

$$a(p^e) = p^{\min(e, e_c)}. \tag{3.9}$$

The Bell series is again an exercise in geometric series [12, 0.113][15],

$$\sum_{e=0}^{e_c} \frac{p^e}{p^{es}} + \sum_{e>e_c} \frac{p^{e_c}}{p^{es}} = \frac{1-p^{-s}+p^{e_c}p^{-(e_c+1)s}(1-p)}{(1-p^{-s})(1-p^{1-s})}. \tag{3.10}$$

The product over all primes, the Dirichlet series, is the Riemann ζ-function multiplied by a product of rational polynomials over the primes with non-vanishing e_c:

$$(n, c) \mapsto \zeta(s) \prod_{e_c>0} \frac{1-p^{-s}+p^{e_c}p^{-(e_c+1)s}(1-p)}{1-p^{1-s}} \tag{3.11}$$

$$= \zeta(s) \prod_{e_c>0} \left(1 + (p-1)\sum_{l=0}^{e_c-1} p^{l(1-s)-s}\right).$$

Remark 12. *The reference sequences are A109007–A109015 for c =3–12 in the OEIS* [25]*, with the exception of* $c = 6$ *which is A089128.*

3.4. **Least Common Multiples.** The least common multiple $[n, c]$ of n and a constant c is constructed with the master equation $a(p^e) = p^{\max(e,e_c)}$ but is not multiplicative in general. With $(n, c)[n, c] = nc$ and multiplicativity of (n, c), the divided $[n, c]/c = n/(n, c)$ serves as a multiplicative substitute. The master equation of $[n, c]/c$ is

$$a(p^e) = p^{\max(e,e_c)}/p^{e_c} = p^{\max(e-e_c,0)}. \tag{3.12}$$

The Bell series is

$$\sum_{e=0}^{e_c} \frac{1}{p^{es}} + \sum_{e>e_c} \frac{p^{e-e_c}}{p^{es}} = \frac{1 - p^{1-s} + p^{-s(1+e_c)}(p-1)}{(1-p^{-s})(1-p^{1-s})}. \tag{3.13}$$

The analog of (3.11) becomes

$$[n, c]/c \mapsto \zeta(s-1) \prod_{e_c>0} \left(1 + (1-p) \sum_{l=0}^{e_c-1} p^{-(l+1)s}\right). \tag{3.14}$$

Remark 13. *This refers for* $c = 2$–20 *to A026741, A051176, A060819, A060791, A060789, A106608–A106612, A051724, and A106614–A106621.*

3.5. **Sigma: Sum of Divisors.**

3.5.1. *Base Sequence.* The divisors of some number n

$$n = \prod_p p_i^{e_i} \tag{3.15}$$

are of the form $d = \prod_p p_i^{m_i}$ with $0 \le m_i \le e_i$. The sum of the k-th power of divisors is

$$\sigma_k(n) = (1 + p_1^k + p_1^{2k} + \cdots p_1^{e_1 k})(1 + p_2^k + p_2^{2k} + \cdots p_2^{e_2 k}) \cdots \tag{3.16}$$

which is a product of geometric sums [13, p. 239]:

$$\sigma_k(p^e) = \begin{cases} \frac{p^{k(e+1)}-1}{p^k-1}, & k > 0; \\ e+1, & k = 0. \end{cases} \tag{3.17}$$

Inserted into (1.5) provides the Dirichlet series

$$\zeta_D = \prod_p \left(\sum_{e\ge 0} \frac{p^{k(e+1)} - 1}{p^k - 1} \cdot \frac{1}{p^{es}} \right), \quad k > 0, \tag{3.18}$$

and the geometric series is summarized as [27, (1.3.1)][10, p. 293]

$$\sigma_k(n) \mapsto \zeta(s)\zeta(s-k), \quad k \ge 0. \tag{3.19}$$

In view of (1.8) and (2.5) this shows

$$\sigma_k(n) = n^k \star 1. \tag{3.20}$$

Remark 14. *This covers* $k = 0$, $1\star 1 = \sigma_0(n)$*, in A000005 with D-inverse A007427,* $k = 1$ *in A000203 with D-inverse A046692,* $k = 2$ *in A001157 with D-inverse A053822,* $k = 3$ *in A001158 with D-inverse A053825,* $k = 4$ *in A001159 with D-inverse A053826,* $k = 5$ *in A001160 with D-inverse A178448, and* $k = 6$–24 *in A013954–A013972.*

The sum over the inverse k-th powers deals with negative indices of the σ-function. By inspection of the complementary divisors n/d for each d this is

$$\sigma_{-k}(n) = \sum_{d|n} \frac{1}{d^k} = \frac{\sigma_k(n)}{n^k}. \tag{3.21}$$

Applying the shift-theorem (1.3) demonstrates that (3.19) is also valid in the range $k < 0$.

3.5.2. *Convolutions.* With (3.19) we derive for example $\sigma_1(n) \star 1 \mapsto \zeta^2(s)\zeta(s-1)$ (A007429), $\sigma_0(n) \star \sigma_1(n) \mapsto \zeta^3(s)\zeta(s-1)$ (A007430), $\sigma_2(n) \star 1 \mapsto \zeta^2(s)\zeta(s-2)$ (A007433) or $\sigma_1(n) \star \sigma_1(n) \mapsto \zeta^2(s)\zeta^2(s-1)$ (A034761).

3.6. **Sums of Divisors which are t-th Powers.** The sum over all divisors of n which are perfect $t-$th powers is

$$a(n) = \sum_{d|n} d\epsilon_t(d) = n\epsilon_t(n) \star 1 \mapsto \zeta(s)\zeta(ts-t) \tag{3.22}$$

using the notation of the characteristic function ϵ_t (Section 3.1).

Proof. The Dirichlet generating function in (3.22) is derived (i) either by summing the Bell series and noting that the denominators of the intermediate result are cyclotomic polynomials of p^{-s} which allows to express the Euler product as a finite product of ζ-functions, or (ii) more quickly starting from the generating function (3.1) of $\epsilon_t(n)$, using the shift theorem (1.3) to produce the generating function for $n\epsilon_t(n)$,

$$n\epsilon_t(n) \mapsto \zeta(ts-t) \tag{3.23}$$

and exploiting the convolution with 1 via (1.10) and (2.6). □

Remark 15. *The examples are $t = 2$, the sums of the square divisors (A035316), and $t = 3$, the sum of the cube divisors (A113061).*

The master equation is

$$a(p^e) = \sum_{l=0}^{\lfloor e/t \rfloor} p^{lt} = \frac{p^{t(1+\lfloor e/t \rfloor)} - 1}{p^t - 1} \tag{3.24}$$

which can be made more explicit by writing this down for each remainder of e mod t in the style of (2.21).

The largest t-th power dividing $n = \prod_p p^e$ may be written as $\max_{b^t|n}$. For each prime basis p it selects the maximum exponent e which is a multiple of t. This reduces the sum (3.24) over all multiples to its largest term:

$$a(p^e) = p^{t\lfloor e/t \rfloor} \tag{3.25}$$

Substituting $e = kt + r$ in the Bell series yields

$$1 + \sum_{e \geq 1} \frac{p^{e-r}}{p^{es}} = \sum_{r=0}^{t-1} \sum_{k \geq 0} \frac{p^{kt}}{p^{(kt+r)s}} = \sum_{r=0}^{t-1} p^{-rs} \frac{1}{1 - p^{t(1-s)}} = \frac{1 - p^{-st}}{(1 - p^{-s})(1 - p^{t(1-s)})}. \tag{3.26}$$

The product over all primes is

$$\max_{b^t|n} \mapsto \zeta(s)\zeta(ts-t)/\zeta(st). \tag{3.27}$$

Multiplying this ζ-product by (2.22) shows in conjunction with (1.10) and (3.19) that

$$\max_{b^t|n} \star \operatorname{core}_t(n) = \sigma_1(n). \tag{3.28}$$

Remark 16. *Examples are $t = 2$, the largest square dividing n (A008833), $t = 3$, the largest cube dividing n (A008834), or $t = 4$, the largest 4th power dividing n (A008835).*

One can also split the product in view of (3.23) and (3.5),

$$\max_{b^t|n} = \xi_t(n) \star (n\epsilon_t(n)). \tag{3.29}$$

A similar function is the t-th root of the largest t-th power dividing n,

$$a(p^e) = p^{\lfloor e/t \rfloor}, \tag{3.30}$$

pulling the t-th root out of (3.25). Bell and Dirichlet series are

$$\sum_{e\ge 0} \frac{p^{\lfloor e/t\rfloor}}{p^{es}} = \sum_{r=0}^{t-1}\sum_{k\ge 0} \frac{p^k}{p^{(kt+r)s}} = \frac{1-p^{-st}}{(1-p^{-s})(1-p^{1-st})} \mapsto \zeta(st-1)\zeta(s)/\zeta(st). \tag{3.31}$$

Remark 17. *This theory applies to A000118 ($t = 2$), A053150 ($t = 3$) and A053164 ($t = 4$).*

3.7. **Sum of t-free Divisors.** The sum of the k-th powers of t-free divisors of n is —in the notation of section 3.1—

$$\sum_{d|n} d^k \xi_t(d) = (n^k \xi_t(n)) \star 1 \mapsto \zeta(s)\zeta(s-k)/\zeta(ts-tk), \quad k \ge 0. \tag{3.32}$$

This Dirichlet series follows applying (1.3) to (3.5) and then (2.6) and (1.10).

Remark 18. *The count of the squarefree divisors is A034444 with D-inverse in A158522; the count of the cubefree divisors is A073184. The sum of squarefree divisors (A048250) has the master equation*

$$a(p^e) = p + 1. \tag{3.33}$$

Multiplication by n generates A181797. The sum of the cubefree divisors is A073185.

The count of the t-full divisors has the master equation [26]

$$a(p^e) = \max(1, e - t + 2), \tag{3.34}$$

assuming 1 is included in the set of t-full numbers. Compared to the full count of divisors, this eliminates contributions of the powers p^1, p^2,…,p^{t-1} from the prime factorization of the divisors. The Bell series is

$$\sum_{e=0}^{t-1} \frac{1}{p^{es}} + \sum_{e\ge t} \frac{e-t+2}{p^{es}} = \frac{p^{-st} - p^{-s} + 1}{(1-p^{-s})^2}. \tag{3.35}$$

For $t = 2$, the numerator polynomial is the cyclotomic polynomial $\Phi_6(p^{-s})$, and expansion of numerator and denominator with $1 + p^{-s}$ yields

$$\sum_{d:\xi_2(d)=0} 1 \mapsto \zeta(s)\zeta(2s)\zeta(3s)/\zeta(6s). \tag{3.36}$$

For $t = 3$ and $t = 4$ the infinite Euler products start as

(3.37)
$$\sum_{d:\xi_3(d)=0} 1 \mapsto \zeta(s)\prod_p (1+p^{-3s})(1+p^{-4s})(1+p^{-5s})(1+p^{-6s})(1-p^{-9s})\cdots, \; s>1,$$

and

(3.38)
$$\sum_{d:\xi_4(d)=0} 1 \mapsto \zeta(s)\prod_p (1+p^{-4s})(1+p^{-5s})(1+p^{-6s})(1+p^{-7s})(1+p^{-8s})(1-p^{-11s})\cdots, \; s>1.$$

Remark 19. *$t = 2$ is A005361. $t = 3$ is A190867.*

3.8. **Sigma of powers.** $\sigma_k(n^2)$ is an arithmetic function with master equation obtained by the substitution $e \to 2e$ in (3.17):

$$a(p^e) = \frac{p^{k(2e+1)}-1}{p^k-1}, \quad k>0; \quad a(p^e) = 2e+1, \quad k=0. \tag{3.39}$$

The Bell series is

$$\sum_{e\ge 0} \frac{p^{k(2e+1)}-1}{(p^k-1)p^{es}} = \frac{1+p^{k-s}}{(1-p^{2k-s})(1-p^{-s})}, \quad k\ge 0, \tag{3.40}$$

which induces

$$\sigma_k(n^2) \mapsto \zeta(s)\zeta(s-k)\zeta(s-2k)/\zeta(2s-2k). \tag{3.41}$$

If the right hand side is interpreted as the product of $\zeta(s)\zeta(s-2k)$ and $\zeta(s-k)/\zeta(2s-2k)$, equations (3.5) and (3.19) demonstrate

$$\sigma_k(n^2) = \sigma_{2k}(n) \star (n^k\xi_2(n)). \tag{3.42}$$

An alternative interpretation as a product of $\zeta(s-2k)$ and $\zeta(s)\zeta(s-k)/\zeta(2s-2k)$ shows with (3.32)

$$\sigma_k(n^2) = [\sum_{d|n} d^k\xi_2(d)] \star n^{2k}. \tag{3.43}$$

Remark 20. *The case $\sigma_0(n^2)$ in A048691 is documented by Titchmarsh* [27, (1.2.9)] *with $\sigma_0(n^2) \mapsto \zeta^3(s)/\zeta(2s)$. $\sigma_1(n^2)$ is A065764, and $\sigma_2(n^2)$ is A065827.*

Moving on to higher powers in the argument, subsampled sums of divisors, we first meet $\sigma_0(n^t)$ with Bell series

$$1+\sum_{e\ge 1} \frac{te+1}{p^{es}} = \frac{1+(t-1)p^{-s}}{(1-p^{-s})^2}. \tag{3.44}$$

The denominator contributes a factor $\zeta^2(s)$ to the Dirichlet series, and the numerator is covered by division through the associated term of (2.12).

The master equation of $\sigma_1(n^t)$ replaces e by et in (3.17),

$$a(p^e) = \frac{p^{et+1}-1}{p-1}, \tag{3.45}$$

which generates a Bell series

$$1+\sum_{e\ge 1} \frac{p^{te+1}-1}{(p-1)p^{es}} = \frac{1+p^{1-s}\sum_{l=0}^{t-2} p^l}{(1-p^{-s})(1-p^{t-s})}. \tag{3.46}$$

At $t=3$, the Euler expansion starts

$$\sigma_1(n^3) \mapsto \zeta(s)\zeta(s-3)\prod_p (1+p^{2-s})(1+p^{1-s})(1-p^{3-2s})(1+p^{5-3s}) \times (1+p^{4-3s})(1-p^{7-4s})(1-p^{6-4s})(1-p^{5-4s})(1+p^{9-5s}) \times (1+p^{8-5s})^2(1+p^{7-5s})^2(1+p^{6-5s})\cdots, \quad s>4, \tag{3.47}$$

for example. At $t=4$ it is

$$\sigma_1(n^4) \mapsto \zeta(s)\zeta(s-4)\prod_p (1+p^{3-s})(1+p^{2-s})(1+p^{1-s})(1-p^{5-2s}) \times (1-p^{4-2s})(1-p^{3-2s})(1+p^{8-3s})(1+p^{7-3s})^2(1+p^{6-3s})^2 \times (1+p^{5-3s})^2(1+p^{4-3s})(1-p^{11-4s})(1-p^{10-4s})^2(1-p^{9-4s})^4 \times (1-p^{8-4s})^4(1-p^{7-4s})^4(1-p^{6-4s})^2(1-p^{5-4s})\cdots, \quad s>5. \tag{3.48}$$

Remark 21. *Templates of these sequences are $\sigma_0(n^3)$ is A048785, $\sigma_1(n^3)$ in A175926. $\sigma_0(n^2)\star 1$ is A035116. $\sigma_0(n^3)\star 1$ is A061391.*

3.9. **Sum of Gcd or Lcm.** Following (3.9), the gcd of a divisor d and its complementary divisor n/d contributes with a factor $(p^m,p^{e-m})^t = p^{t\cdot\min(m,e-m)}$ to $\sum_{d|n}(d,n/d)^t$. Summing over m from 0 to e yields the master equation

$$a(p^e) = \begin{cases} p^{et/2}+2\sum_{m=0}^{e/2-1}p^{tm} = [(p^t+1)p^{et/2}-2]/(p^t-1), & e\,\text{even};\\ 2\sum_{m=0}^{(e-1)/2}p^{tm} = 2[p^{t(e+1)/2}-1]/(p^t-1), & e\,\text{odd}.\end{cases} \tag{3.49}$$

The Bell series is

$$\sum_{e=0,2,4,\ldots}\frac{(p^t+1)p^{et/2}-2}{(p^t-1)p^{es}} + \sum_{e=1,3,5,\ldots}2\frac{p^{t(e+1)/2}-1}{(p^t-1)p^{es}} = \frac{1+p^{-s}}{(1-p^{-s})(1-p^{t-2s})}, \tag{3.50}$$

which reveals

$$\sum_{d|n}(d,n/d)^t \mapsto \zeta^2(s)\zeta(2s-t)/\zeta(2s). \tag{3.51}$$

The associated analysis for the lcm starts from (3.12). The prime p and exponent m of the divisor d contribute to $\sum_d[d,n/d]^t$ with a term $[p^m,p^{e-m}]^t = p^{t\cdot\max(m,e-m)}$. The master equation splits again into two cases depending on whether a middle term at $m=e/2$ is present or not:

$$a(p^e) = \begin{cases} p^{et/2}+2\sum_{m=0}^{e/2-1}p^{t(e-m)} = [2p^{t(1+e)}-(p^t+1)p^{et/2}]/(p^t-1), & e\,\text{even};\\ 2\sum_{m=0}^{(e-1)/2}p^{t(e-m)} = 2e^{t(e+1)/2}[p^{t(e+1)/2}-1]/(p^t-1), & e\,\text{odd}.\end{cases} \tag{3.52}$$

The Bell series factorizes in the ζ-basis similar to (3.51):

$$\sum_{d|n}[d,n/d]^t \mapsto \zeta^2(s-t)\zeta(2s-t)/\zeta(2s-2t). \tag{3.53}$$

Remark 22. *$\sum_d(d,n/d)^t$ is A055155 for $t=1$ and A068976 for $t=2$. $\sum_d[d,n/d]$ is A057670.*

3.10. **Sigma powers.**

3.10.1. *Ordinary Products.* The t-th power of (3.17) is

$$\sigma_k^t(p^e) = \frac{(p^{k(e+1)}-1)^t}{(p^k-1)^t}, \quad e \ge 0. \tag{3.54}$$

The binomial expansion of the associated Bell series is

$$\begin{aligned}\sum_{e\ge 0}\frac{\sigma_k^t(p^e)}{p^{es}} &= \frac{1}{(p^k-1)^t}\sum_{e\ge 0}\sum_{t'=0}^{t}(-)^{t-t'}\binom{t}{t'}\frac{p^{t'k(e+1)}}{p^{es}} \\ &= \frac{1}{(p^k-1)^t}\sum_{t'=0}^{t}(-)^{t-t'}\binom{t}{t'}\frac{p^{t'k}}{1-p^{(t'k-s)}}.\end{aligned} \tag{3.55}$$

For the squares of σ, $t=2$,

$$\sum_{e\ge 0}\frac{\sigma_k^2(p^e)}{p^{es}} = \frac{1-p^{2k-2s}}{(1-p^{-s})(1-p^{k-s})^2(1-p^{2k-s})} \tag{3.56}$$

produces the Dirichlet series

$$\sigma_k^2(n) \mapsto \frac{\zeta(s)\zeta^2(s-k)\zeta(s-2k)}{\zeta(2s-2k)}. \tag{3.57}$$

Because this equals (3.41) multiplied by $\zeta(s-k)$, we find with (1.3) and (1.8):

$$\sigma_k^2(n) = \sigma_k(n^2) \star n^k. \tag{3.58}$$

Remark 23. *These considerations cover A035116 with the Dirichlet series* [27, (1.2.10)]

$$\sigma_0^2(n) = \sigma_0(n^2)\star 1 = \sigma_0(n)\star\sigma_0^*(n) \mapsto \zeta^4(s)/\zeta(2s), \tag{3.59}$$

where $\sigma_0^*(n)$ is the number of unitary divisors of n (A034444). They also cover $\sigma_1^2(n)$ in A072861.

For larger t, the denominators of (3.55) contribute $\prod_{t'=0}^{t}\zeta(s-t'k)$ to the Dirichlet series (represented for $k=1$ and various t by A001001 and A038991–A038999), but the numerators do not factor as nicely. The examples are

$$\sum_{e\ge 0}\frac{\sigma_k^3(p^e)}{p^{es}} = \frac{p^{3k-2s}+2p^{2k-s}+2p^{k-s}+1}{\prod_{t'=0}^{3}(1-p^{t'k-s})} \tag{3.60}$$

or

$$\sum_{e\ge 0}\frac{\sigma_k^4(p^e)}{p^{es}} = \frac{p^{6k-3s}+(3p^{2k}+5p^k+3)p^{3k-2s}+(3p^{2k}+5p^k+3)p^{k-s}+1}{\prod_{t'=0}^{4}(1-p^{t'k-s})}. \tag{3.61}$$

The Euler product expansions for these two cases start as

$$\begin{aligned}\sigma_k^3(n) &= \zeta(s)\zeta(s-k)\zeta(s-2k)\zeta(s-3k)\prod_p(1+p^{2k-s})^2(1+p^{k-s})^2 \\ &\times(1-p^{4k-2s})(1-p^{3k-2s})^3(1-p^{2k-2s})(1+p^{6k-3s})^2(1+p^{5k-3s})^6 \\ &\times(1+p^{4k-3s})^6(1+p^{3k-3s})^2(1-p^{8k-4s})^3(1-p^{7k-4s})^{12}(1-p^{6k-4s})^{15} \\ &\times(1-p^{5k-4s})^{12}(1-p^{4k-4s})^3\cdots, \quad s>1+3k,\end{aligned} \tag{3.62}$$

and

$$\sigma_k^4(n) = \zeta(s)\zeta(s-k)\zeta(s-2k)\zeta(s-3k)\zeta(s-4k)\prod_p (1+p^{3k-s})^3 \times (1+p^{2k-s})^5(1+p^{k-s})^3(1-p^{6k-2s})^3(1-p^{5k-2s})^{12}(1-p^{4k-2s})^{14}(1-p^{3k-2s})^{12} \times (1-p^{2k-2s})^3(1+p^{9k-3s})^8(1+p^{8k-3s})^{36}(1+p^{7k-3s})^{72}(1+p^{6k-3s})^{88}(1+p^{5k-3s})^{72} \times (1+p^{4k-3s})^{36}(1+p^{3k-3s})^8\cdots, \quad s>1+4k. \tag{3.63}$$

3.10.2. *Hybrid Products.* Dirichlet series of mixed products are [27, (1.3.3)][29]

$$\sigma_a(n)\sigma_b(n) \mapsto \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}, \tag{3.64}$$

of which (3.57) is a special case. An example of this type is $\sigma_0(n)\sigma_1(n)$ in A064840.

3.10.3. *Dirichlet Convolutions.* $\tau_k(n)$ is the number of ways of expressing n as a product of k factors. $\tau_2(n)=\sigma_0(n)$ and iterated convolution with 1 yield the ladder of larger k [27, (1.2.2.)][29]:

$$\tau_k(n) \mapsto \zeta^k(s). \tag{3.65}$$

Remark 24. *τ_2 is A000005, τ_3 is A007425, $\tau_4=\sigma_0(n)\star\sigma_0(n)$ is A007426, τ_5 is A061200, τ_6 is A034695, τ_7–τ_{11} are A111217–A111221, and τ_{12} is A111306.*

Remark 25. *$\sigma_1^2(n)\star 1$ is A065018. $\sigma_0^2(n)\star 1$ is A062367. $\sigma_0^3(n)\star 1$ is A097988.*

3.11. **Powers times Sigma.**

3.11.1. *Ordinary product.* Products with powers have Dirichlet generating functions derived from (1.3) with (3.19) or (3.57):

$$n^t\sigma_k(n) \mapsto \zeta(s-t)\zeta(s-k-t), \quad k\ge 0. \tag{3.66}$$

$$n^t\sigma_k^2(n) \mapsto \frac{\zeta(s-t)\zeta^2(s-k-t)\zeta(s-2k-t)}{\zeta(2s-2k-2t)} \tag{3.67}$$

Remark 26. *This concerns $n\sigma_0(n)$ in A038040, $n^2\sigma_0(n)$ in A034714, and $n\sigma_1(n)$ in A064987.*

3.11.2. *Dirichlet convolutions.* Convolutions with powers have Dirichlet generating functions which are products of (2.5) with (3.19) or (3.57):

$$n^t\star\sigma_k(n) \mapsto \zeta(s)\zeta(s-t)\zeta(s-k), \quad k\ge 0. \tag{3.68}$$

$$n^t\star\sigma_k^2(n) \mapsto \frac{\zeta(s)\zeta(s-t)\zeta^2(s-k)\zeta(s-2k)}{\zeta(2s-2k)}. \tag{3.69}$$

Remark 27. *$n\star\sigma_0(n)$ is A007429. $n\star\sigma_0^2(n)$ is A062369. $n\star\sigma_1(n)$ is A060640. $n^2\star\sigma_0(n)$ is A007433. $n^2\star\sigma_1(n)=n\star\sigma_2(n)$ is A001001. $n^3\star\sigma_1(n)=n\star\sigma_3(n)$ is A027847. Multiplication of (3.19) with $\zeta(s-t)$ shows* [10, p. 285]

$$\sigma_k(n)\star n^t = \sigma_t(n)\star n^k. \tag{3.70}$$

3.12. **Sums of Odd Divisors.** The master equation for the sum of odd divisors of n, $\sigma_k^{(o)}(n) \equiv \sum_{d|n, d\,\mathrm{odd}} d^k$ is

$$a(2^e) = 1; \quad a(p^e) = \begin{cases} \frac{p^{ke+k}-1}{p^k-1}, & k > 0, \quad p > 2. \\ e+1, & k = 0, \quad p > 2. \end{cases} \tag{3.71}$$

The two Bell series for the prime 2 on one hand or any odd prime on the other hand repeat (3.17),

$$\sum_{e\ge 0} \frac{1}{2^{es}} = \frac{1}{1-2^{-s}} \tag{3.72}$$

$$\sum_{e\ge 0} \frac{p^{k(e+1)}-1}{(p^k-1)p^{es}} = \frac{1}{(1-p^{-s})(1-p^{k-s})}, \quad p > 2. \tag{3.73}$$

The Dirichlet series is the interlaced product

$$\sigma_k^{(o)}(n) \mapsto (1-2^{k-s})\zeta(s)\zeta(s-k), \quad k \ge 0. \tag{3.74}$$

Remark 28. *The OEIS examples are $k = 0$ in A001227, $k = 1$ in A000593, and $k = 2$–5 in A050999–A051002.*

3.13. **Jordan Functions.** Dirichlet convolution of n^k and $\mu(n)$ defines Jordan functions J_k. The generating functions are an immediate consequence of (1.3) and (2.7):

$$n^k \star \mu(n) = J_k(n) \mapsto \zeta(s-k)/\zeta(s). \tag{3.75}$$

Remark 29. *OEIS representatives are A007434 ($k = 2$) with D-inverse A046970, A059376 ($k = 3$) with D-inverse A063453, A059377–A059378 ($k = 4$–5), and A069091–A069095 ($k = 6$–10).*

Via (1.9), the master equation for J_k is

$$a(p^e) = p^{k(e-1)}(p^k-1), \quad e > 0. \tag{3.76}$$

3.13.1. *Products.* A000056 is $nJ_2(n)$. A115224 is $n^2J_3(n)$. The convolution products $n^k \star J_k(n) \mapsto \zeta^2(s-k)/\zeta(s)$ generalize Pillai's function [14].

3.13.2. *Dedekind ψ.* The Dedekind ψ-function is the ratio

$$\psi(n) = J_2(n)/J_1(n) \mapsto \zeta(s)\zeta(s-1)/\zeta(2s), \tag{3.77}$$

which can be phrased as

$$\psi(n) = n \star \xi_t(n) \tag{3.78}$$

with the aid of (1.10), (2.5) and (3.5).

Remark 30. *The Möbius transform $\mu(n) \star \psi(n)$ drops the factor $\zeta(s)$ in (3.77) and is found in A063659. The Dirichlet series of $n\psi(n)$ (A000082) and $n^2\psi(n)$ (A033196) follow from (1.3). $\mu(n) \star n\psi(n)$ is A140697.*

Remark 31. *The $J_k(n)/J_1(n)$ for $k = 2$–11 are A001615, A160889, A160891, A160893, A160895, A160897, A160908, A160953, A160957, and A160960.*

The master equation for $J_k(n)/J_1(n)$ is a ratio of terms of (3.76):

$$a(p^e) = \frac{p^{k(e-1)}(p^k-1)}{p^{e-1}(p-1)} = \frac{p^{(k-1)(e-1)}(p^k-1)}{p-1}, \quad e>0, \tag{3.79}$$

with Bell series

$$1+\sum_{e\ge 1}\frac{p^{(k-1)(e-1)}(p^k-1)}{(p-1)p^{es}} = \frac{p-1+p^{k-1-s}-p^{-s}}{(p-1)(1-p^{k-1-s})} = \frac{1+(\sum_{l=0}^{k-2}p^l)p^{-s}}{1-p^{k-1-s}}. \tag{3.80}$$

At $k=2$ this reduces to (3.77). If $k>2$, (3.80) is (3.46) multiplied by $1-p^{-s}$ followed by the substitution $s\to s+1$; the Dirichlet series of $J_k(n)/J_1(n)$ are obtained from prime products like (3.47) by deleting $\zeta(s)$ and the substitution $s\to s+1$, to wit

$$1\star[n\frac{J_k(n)}{J_1(n)}] = \sigma_1(n^k), \quad k=1,2,\ldots. \tag{3.81}$$

Multiplicative sequences $J_{2k}(n)/J_k(n)$ are another generalization which—by virtue of (3.76)—have integer entries governed by

$$a(p^e) = p^{k(e-1)}(p^k+1), \quad e>0. \tag{3.82}$$

The Bell series are

$$1+\sum_{e\ge 1}\frac{p^{ke-k}(p^k+1)}{p^{es}} = \frac{1+p^{-s}}{1-p^{k-s}}, \tag{3.83}$$

and their product over all primes generates

$$J_{2k}(n)/J_k(n) \mapsto \zeta(s)\zeta(s-k)/\zeta(2s). \tag{3.84}$$

Mediated by (3.5) and (3.19), factorizations of this product lead to

$$J_{2k}(n)/J_k(n) = n^k\star\mu^2(n); \tag{3.85}$$

$$\epsilon_2(n)\star[J_{2k}(n)/J_k(n)] = \sigma_k(n). \tag{3.86}$$

Remark 32. *Associated OEIS entries are $J_4(n)/J_2(n)$ (A065958), $J_6(n)/J_3(n)$ (A065959), and $J_8(n)/J_4(n)$ (A065960) .*

3.14. **Euler's Totient.**

3.14.1. *Basis function.* The totient $\varphi(n)$ counts numbers $\le n$ and coprime to n, represented by A000010 and its D-inverse A023900. The master equation is

$$\varphi(p^e) = (p-1)p^{e-1}, \quad e>0. \tag{3.87}$$

The Bell series factorizes in the form [4, p. 111]

$$\varphi(n) = \mu(n)\star n \mapsto \frac{\zeta(s-1)}{\zeta(s)} \tag{3.88}$$

Remark 33. *The sum of the $k-$th powers of the divisors coprime to n, $\varphi_k(n)$, is generally not multiplicative for $k>0$. This is easily shown by finding small indices that violate the defining equation (1.1).*

Remark 34. *Equation (3.88) has been generalized to define $\varphi_{k,l}(n)\equiv\mu(n)n^k\star n^l \mapsto \zeta(s-l)/\zeta(s-k)$ [6]. $\varphi_{1,2}(n)$ is A002618, $\varphi_{1,3}(n)$ is A000056, $\varphi_{2,3}(n)$ is A053191.*

The square of (3.88) defines $\varphi^2(n)$ in A127473,

$$a(p^e) = (p-1)^2 p^{2e-2}, \tag{3.89}$$

which leads to the Bell series

$$1 + \sum_{e\ge 1} \frac{(p-1)^2 p^{2e-2}}{p^{es}} = \frac{1-2p^{1-s}+p^{-s}}{1-p^{2-s}}, \tag{3.90}$$

and the infinite Euler product

$$\begin{aligned} \varphi^2(n) \mapsto \zeta(s-2)\prod_p &(1-p^{1-s})^2(1+p^{-s})(1-p^{2-2s})(1+p^{1-2s})^2 \\ &\times (1-p^{3-3s})^2(1+p^{2-3s})^4(1-p^{1-3s})^2(1-p^{4-4s})^3(1+p^{3-4s})^8(1-p^{2-4s})^5 \\ &\times (1+p^{1-4s})^2(1-p^{5-5s})^6(1+p^{4-5s})^{16}(1-p^{3-5s})^{16} \\ &\times (1+p^{2-5s})^8(1-p^{1-5s})^2\cdots, \quad s>3. \end{aligned} \tag{3.91}$$

Cohen defines a multiplicative function φ' with a simple master equation build from the product of (2.8) squared and (3.87) [7]:

$$\varphi'(n) \equiv \mu^2(n)\varphi(n) \mapsto \prod_p (1+p^{1-s}-p^{-s}). \tag{3.92}$$

This has already been met in (2.25), which can be combined into

$$1 \star \varphi'(n) = \mathrm{rad}_2(n). \tag{3.93}$$

Remark 35. *φ' is given by the absolute values of A097945.*

3.14.2. *Basic Convolution.* $\varphi\star\varphi$ is A029935. (2.10) and (3.88) combine as (A007431, A063659)

$$\mu(n)\star\varphi(n) \mapsto \zeta(s-1)/\zeta^2(s); \quad \mu^2(n)\star\varphi(n) \mapsto \zeta(s-1)/\zeta(2s). \tag{3.94}$$

3.14.3. *Ordinary product with powers.* The $\varphi(n^t)$ are obtained from (3.88) by the substitution $e \to et$, so the Bell series is

$$1 + \sum_{e\ge 1} \frac{(p-1)p^{et-1}}{p^{es}} = \frac{p^s - p^{t-1}}{p^s - p^t} = \frac{1-p^{t-s-1}}{1-p^{t-s}} \tag{3.95}$$

and therefore

$$\varphi(n^t) \mapsto \zeta(s-t)/\zeta(s+1-t). \tag{3.96}$$

Applying (1.3) establishes the well-known [7]

$$\varphi(n^t) = n^{t-1}\varphi(n). \tag{3.97}$$

Remark 36. *This describes $n\varphi(n)$ (A002618), twice the sum of the integers coprime to n and not exceeding n* [6], *and $n^2\varphi(n)$ (A053191).*

3.14.4. *Dirichlet product with powers.* $\varphi\star 1 = n$ is obvious from (3.88) [6]. Building $\varphi^2\star 1$ we generate A029939. From (1.9) and (3.89) its master equation ensues,

$$a(p^e) = 1 + \sum_{l=1..e} (p-1)^2 p^{2l-2} = \frac{p^{2e}(p-1)+2}{p+1}. \tag{3.98}$$

By construction, the Dirichlet series is (3.91) multiplied by $\zeta(s)$.

Remark 37. *$n\star\varphi(n)$ is A018804 with D-inverse in A101035. $n^2\star\varphi(n)$ is A069097.*

3.15. **Sigma times Totient.**

3.15.1. *Ordinary products.* The multiplicative $\sigma_0(n)\varphi(n)$ is represented by A062355. The master equation is the product of (3.17) and (3.87),

$$a(p^e) = (e+1)(p-1)p^{e-1}, \quad e > 0. \tag{3.99}$$

The Bell series is

$$1 + \sum_{e\ge 1}(e+1)(p-1)p^{e-1}/p^{es} = \frac{1-2p^{-s}+p^{1-2s}}{(1-p^{1-s})^2}. \tag{3.100}$$

The infinite Euler product is

$$\sigma_0(n)\varphi(n) \mapsto \zeta^2(s-1)\prod_p (1-p^{-s})^2(1+p^{1-2s})(1-p^{-2s})(1+p^{1-3s})^2 \times (1-p^{-3s})^2(1+p^{1-4s})^4(1-p^{-4s})^3(1-p^{2-5s})^2(1+p^{1-5s})^8\cdots, \quad s>2. \tag{3.101}$$

$\sigma_0(n)\varphi^2(n)$ is A126775 with Bell series

$$1 + \sum_{e\ge 1}\frac{(e+1)(p-1)^2p^{2(e-1)}}{p^{es}} = \frac{1-4p^{1-s}+2p^{3-2s}+2p^{-s}-p^{2-2s}}{(1-p^{2-s})^2} \tag{3.102}$$

and Euler product

$$\zeta_D = \zeta^2(s-2)\prod_p (1-p^{1-s})^4(1+p^{-s})^2(1+p^{3-2s})^2(1-p^{2-2s})^7 \times (1+p^{1-2s})^8(1-p^{-2s})(1+p^{4-3s})^8(1-p^{3-3s})^{28}(1+p^{2-3s})^{34} \times (1-p^{1-3s})^{16}(1+p^{-3s})^2(1-p^{6-4s})\cdots, \quad s>3. \tag{3.103}$$

$\sigma_0^2(n)\varphi(n)$ is A110601 with Bell series

$$1 + \sum_{e\ge 1}\frac{(e+1)^2(p-1)p^{e-1}}{p^{es}} = \frac{p^{1-s}+1-p^{2-3s}+3p^{1-2s}-4p^{-s}}{(1-p^{1-s})^3}. \tag{3.104}$$

$$\sigma_0^2(n)\varphi(n) \mapsto \prod_p (1+p^{1-s})(1-p^{-s})(1+p^{1-2s})^7(1-p^{-2s})^6(1-p^{2-3s})^8 \times (1+p^{1-3s})^{28}(1-p^{-3s})^{20}(1+p^{3-4s})^8(1-p^{2-4s})^{53}(1+p^{1-4s})^{112} \times (1-p^{-4s})^{60}\cdots, \quad s>2. \tag{3.105}$$

The master equation of $\sigma_1(n)\varphi(n)$ (A062354) is a product of (3.17) by (3.87),

$$a(p^e) = p^{e-1}(p^{e+1}-1). \tag{3.106}$$

The Bell series is

$$1 + \sum_{e\ge 1}\frac{p^{e-1}(p^{e+1}-1)}{p^{es}} = \frac{1-p^{1-s}-p^{-s}+p^{2-2s}}{(1-p^{2-s})(1-p^{1-s})}, \tag{3.107}$$

and the Euler product

$$\zeta_D(s) = \zeta(s-2)\prod_p (1-p^{-s})(1+p^{2-2s})(1-p^{1-2s})(1+p^{3-3s}) \times (1-p^{1-3s})(1+p^{4-4s})(1+p^{3-4s})(1-p^{1-4s})(1+p^{4-5s}) \times (1+p^{3-5s})(1-p^{2-5s})(1-p^{1-5s})\cdots, \quad s>3. \tag{3.108}$$

3.15.2. *Dirichlet convolutions.* The application of (1.3) and (1.10) to (3.19) and (3.88) yields [10, p. 293]

$$n^t\varphi(n) \star \sigma_t(n) = n^{1+t} \star 1 \tag{3.109}$$

Remark 38. *Examples of these convolutions are $\sigma_0(n)\star\varphi(n)$ in A000203, $\sigma_0^2(n)\star\varphi(n)$ in A060724, $\sigma_1(n)\star\varphi(n)$ in A038040, and $\sigma_2(n)\star\varphi(n)$ in A064987.*

4. Miscellany

4.1. **Ramanujan sums.** For our purposes the following definition suffices [2]:

Definition 6. *(Ramanujan sum $c_k(n)$)*

$$c_k(n) = \sum_{d|n,d|k} \mu(k/d)d. \tag{4.1}$$

The associated Dirichlet series are [27, 13]:

$$c_n(k) \mapsto \frac{\sigma_{1-s}(k)}{\zeta(s)}, \tag{4.2}$$

$$c_k(n) \mapsto \zeta(s)\sum_{d|k}\mu(k/d)d^{1-s}, \tag{4.3}$$

and

$$c_k(n)\tau(n) \mapsto \zeta^2(s)\sum_{\delta|k}\delta^{1-s}\mu(k/\delta)\prod_{p|\delta}(l+1-lp^{-s}) \tag{4.4}$$

where $\delta \equiv \prod p^l$.

Remark 39. *We find $c_n(1) = \mu(n)$, $c_n(2)$ in A086831, $c_n(3)$ in A085097, $c_n(4)$ in A085384, $c_n(5)$ in A085639, and $c_n(6)$ in A085906. $c_1(n) = 1$, but if the role of the argument and index are swapped, the functions are non-multiplicative in general: $c_2(n) = -(-1)^n$ and $c_3(n)$ in A099837, $c_4(n)$ in A176742, and $c_6(n)$ in A100051.*

4.2. **Unitary Arithmetics.**

4.2.1. *Properties.* The unitary convolution

$$(a \oplus b)(n) \equiv \sum_{d|n,(d,n/d)=1} a(d)b(n/d) \tag{4.5}$$

shows parallels to the Dirichlet convolution. Because it preserves the multiplicative property of its factors [7, 23, 28] and because its basic associated Möbius, Sums-of-Divisors and totient functions are multiplicative, inheritance similar to Section 3 ensues. The formula that parallels (1.9) is

$$(a \oplus b)(p^e) = a(1)b(p^e) + a(p^e)b(1), \quad e > 0. \tag{4.6}$$

Cohen defines for example [7]

$$\sigma'(n) = n\mu^2(n) \oplus 1. \tag{4.7}$$

Because the master equation of $n\mu^2(n) \mapsto \zeta(s-1)/\zeta(2s-2)$ is

$$a(p^e) = \begin{cases} p^e, & e \le 1; \\ 0, & e > 1, \end{cases} \tag{4.8}$$

the master equation of σ' is constructed from (4.6) as

$$a(p^e) = \begin{cases} 1, & e = 0; \\ 1+p, & e = 1; \\ 1, & e > 1. \end{cases} \tag{4.9}$$

The Bell series is $(1 + p^{1-s} - p^{1-2s})/(1 - p^{-s})$, which leads to the Dirichlet series

$$\sigma'(n) \mapsto \zeta(s)\prod_p (1 + p^{1-s} - p^{1-2s}) = \zeta(s)\prod_p (1 + p^{1-s})(1 - p^{1-2s}) \tag{4.10}$$
$$\times (1 + p^{2-3s})(1 - p^{3-4s})(1 + p^{4-5s})(1 + p^{3-5s})(1 - p^{5-6s})(1 - p^{4-6s}) \cdots, \quad s > 2.$$

Remark 40. *σ' is A092261.*

4.2.2. *Unitary μ.* The ω-analog of (2.17) is the unitary Möbius function (A076479) [7, 23, 9]

$$\mu^*(n) = (-1)^{\omega(n)}, \tag{4.11}$$

where $\omega(n)$ is the number of distinct prime factors of n. Master equation and Bell series are [8]

$$a(p^e) = -1; \quad 1 + \sum_{e\ge 1} \frac{a(p^e)}{p^{es}} = \frac{1 - 2p^{-s}}{1 - p^{-s}}. \tag{4.12}$$

The Dirichlet series is $\zeta(s)$ divided by (2.12) at $c = 2$, i. e., $\zeta(s)$ multiplied by the associated Feller-Tornier constant [19, Tab. 6]:

$$\mu^*(n) \mapsto \prod_p (1 - p^{-s})(1 - p^{-2s})(1 - p^{-3s})^2(1 - p^{-4s})^3(1 - p^{-5s})^6 \tag{4.13}$$
$$\times (1 - p^{-6s})^9(1 - p^{-7s})^{18}(1 - p^{-8s})^{30} \cdots, \quad s > 1.$$

Remark 41. *The Dirichlet series of Cohen's exponentially odd numbers $\mu_2^*(n)$ is the same at doubled argument $2s$ [7].*

4.2.3. *Unitary Sigma.* The unitary σ-function sums over the divisors d which are coprime to their complementary divisors n/d:

Definition 7. *(Unitary sigma $\sigma^\star$)*

$$\sigma_k^*(n) = n^k \oplus 1 = \sum_{d|n,(d,n/d)=1} d^k. \tag{4.14}$$

Applying (4.6), the master equation for the k-power of the divisors is [30]

$$a(p^e) = 1 + p^{ke}. \tag{4.15}$$

The Bell series is

$$1 + \sum_{e\ge 1} (1 + p^{ek})/p^{es} = \frac{1 - p^{k-2s}}{(1 - p^{-s})(1 - p^{k-s})}, \tag{4.16}$$

which becomes

$$\sigma_k^*(n) \mapsto \zeta(s)\zeta(s-k)/\zeta(2s-k). \tag{4.17}$$

Multiplication with $\zeta(2s - k)$ generates in view of (3.2) and (3.19)

$$(n^{k/2}\epsilon_2(n)) \star \sigma_k^*(n) = \sigma_k(n). \tag{4.18}$$

The sum of the k-th power of the odd unitary divisors $\sigma_k^{*(o)}(n)$ is determined by a master equation which counts only the first or both of the terms in (4.15) depending on p being even or odd:

$$a(p^e) = \begin{cases} 1, & p = 2; \\ 1 + p^{ek}, & p > 2. \end{cases} \tag{4.19}$$

The Bells series is $1/(1-2^{-s})$ for $p=2$ and (4.16) for $p>2$. In summary

$$\sigma_k^{*(o)}(n) \mapsto \frac{\zeta(s)\zeta(s-k)(1-2^{k-s})}{\zeta(2s-k)(1-2^{k-2s})}. \tag{4.20}$$

Remark 42. *$\sigma_0^*(n)$ is A034444. $\sigma_1^*(n)$ is A034448 with D-inverse in A178450. $\sigma_k^*(n)$ with $k = 2$–8 are A034676–A034682. $\sigma_0^{*(o)}(n)$ is A068068.*

4.2.4. *Unitary Phi.* The unitary totient is the unitary convolution of μ^* and n [7]:

Definition 8. *(Unitary Totient)*

$$\varphi^*(n) = \mu^\star(n) \oplus n. \tag{4.21}$$

The master equation is [16]

$$a(p^e) = p^e - 1 \tag{4.22}$$

which sums to

$$1 + \sum_{e\ge 1} \frac{p^e - 1}{p^{es}} = \frac{1 - 2p^{-s} + p^{1-2s}}{(1-p^{-s})(1-p^{1-s})}. \tag{4.23}$$

Comparison of numerator and denominator with (3.100) shows that the Dirichlet series is given by replacing one of the two $\zeta(s-1)$ in (3.101) by $\zeta(s)$; this can be phrased via (3.88) as

$$\varphi^*(n) \star \varphi(n) = \sigma_0(n)\varphi(n). \tag{4.24}$$

Remark 43. *$\varphi^*(n)$ is A047994*

The unitary Jordan functions generalize $\varphi^\star(n)$ akin to (3.75) [22]:

$$J_k^\star(n) = \mu^\star(n) \oplus n^k. \tag{4.25}$$

Via (4.6), its master equation and Bell series are

$$a(p^e) = p^{ek} - 1, \tag{4.26}$$

$$1 + \sum_{e\ge 1} \frac{p^{ek} - 1}{p^{es}} = \frac{1 - 2p^{-s} + p^{k-2s}}{(1-p^{-s})(1-p^{k-s})}. \tag{4.27}$$

The infinite Euler product becomes

$$J_k^\star(n) \mapsto \zeta(s-k)\prod_p (1-p^{-s})(1+p^{k-2s})(1-p^{-2s})(1+p^{k-3s})^2 \tag{4.28}$$
$$\times (1+p^{k-4s})^4(1-p^{-4s})^3(1-p^{2k-5s})^2(1+p^{k-5s})^8(1-p^{-5s})^6\cdots, \quad s > 1+k.$$

(4.22) and (4.26) are related by the substitution $e \to ek$ on the right hand sides, which shows

$$J_k^*(n) = \varphi^*(n^k). \tag{4.29}$$

Unitary analogues of (3.65) might be created as

$$\tau_2^*(n) = \sigma_0^*(n); \quad \tau_{k+1}^*(n) = \tau_k^*(n) \oplus 1. \tag{4.30}$$

The Bell series is bootstrapped from (4.15) with (4.6),

$$1 + \sum_{e\ge 1} \frac{k}{p^{es}} = \frac{1+(k-1)p^{-s}}{1-p^{-s}}. \tag{4.31}$$

The similarity with (3.44) induces

$$1 \star \tau_k^*(n) = \sigma_0(n^k). \tag{4.32}$$

Remark 44. *$J_1^*(n)$ is A047994. $J_2^*(n)$ is A191414. $\tau_3^*(n)$ is A074816.*

4.3. **Higher Order Möbius.** Apostol's higher order $\mu_k(n)$ generalize (2.8) and are defined as $\mu_k(n) = 0$ if any prime power p^{k+1} divides n, and $\mu_k(n) = (-1)^r$ where r is the number of maximum prime powers p^k which divide n [2, 1, 3]. The master equation is

$$a(p^e) = \begin{cases} 1, & 0 \le e < k; \\ -1, & e = k; \\ 0, & e > k. \end{cases} \tag{4.33}$$

The Bell series is

$$\sum_{e=0}^{k-1} \frac{1}{p^{es}} - \frac{1}{p^{ks}} = \frac{1-2p^{-ks}+p^{-(k+1)s}}{1-p^{-s}}, \tag{4.34}$$

with Dirichlet generating function

$$\begin{aligned} \mu_k(n) \mapsto \zeta(s)\prod_p &(1-p^{-ks})^2(1+p^{-(k+1)s})(1-p^{-2ks})(1+p^{-(2k+1)s})^2 \\ &\times (1-p^{-3ks})^2(1+p^{-(3k+1)s})^4(1-p^{-(3k+2)s})^2(1-p^{-4ks})^3(1+p^{-(4k+1)s})^8 \\ &\times (1-p^{-(4k+2)s})^5(1+p^{-(4k+3)s})^2(1-p^{-5ks})^6(1+p^{-(5k+1)s})^{16}\cdots, \quad s>1. \end{aligned} \tag{4.35}$$

Remark 45. *$\mu_2(n)$–$\mu_4(n)$ are A189021–A189023 in the OEIS* [25].

4.4. **Powers Congruential to Zero.** The number of solutions to $x^t \equiv 0 \pmod n$ in the interval $1 \le x \le n$ is a multiplicative function with [5]

$$a(p^e) = p^{e-\lceil e/t\rceil} = p^{\lfloor (t-1)e/t\rfloor}. \tag{4.36}$$

Proof. It is multiplicative because solutions x for n a product of prime powers are all products of solutions to the individual prime powers, and therefore the cardinality of the solutions equals the product of the cardinality of solutions to the individual prime powers. The master equation is derived by noting that the solutions are $x = cp^{\lceil e/t\rceil}$, $c = 1, 2, \ldots$, with a maximum of $x = p^e$. The number of solutions equals the maximum c, which is the maximum solution divided by the minimum solution. □

The Bell series is accumulated by splitting $e = kt + r$ with remainder $0 \le r < t$, and treating $r = 0$ and $r \ne 0$ separately:

$$\sum_{e\ge 0} \frac{p^{\lfloor (t-1)(k+r/t)\rfloor}}{p^{es}} = \sum_{k\ge 0}\frac{p^{(t-1)k}}{p^{kts}} + \sum_{r=1}^{t-1}\sum_{k\ge 0}\frac{p^{(t-1)k+r-1}}{p^{(kt+r)s}} = \frac{1+\sum_{r=1}^{t-1} p^{r-1-rs}}{1-p^{t-1-st}}. \tag{4.37}$$

The case $t = 2$ is dealt with by plugging $t = 2$ into (3.31). The Euler product for the case $t = 3$ is

$$\prod_p \frac{1+p^{-s}+p^{1-2s}}{1-p^{2-3s}} = \zeta(3s-2)\prod_p (1+p^{-s})(1+p^{1-2s})(1-p^{1-3s}) \times (1+p^{1-4s})(1+p^{2-5s})(1-p^{1-5s})(1-p^{2-6s})(1+p^{1-6s})\cdots, \quad s>1. \tag{4.38}$$

Remark 46. $t = 2$–4 *are A000188–A000190.*

The associated smallest positive x whose t-th power is divisible by n have master equations $a(p^e) = p^{\lceil e/t\rceil}$ and Bell series

$$\sum_{e\ge 0}\frac{p^{\lceil e/t\rceil}}{p^{es}} = \sum_{r=0}^{t-1}\sum_{k\ge 0}\frac{p^{k+\lceil r/t\rceil}}{p^{(tk+r)s}} = \frac{1+\sum_{r=1}^{t-1} p^{1-rs}}{1-p^{1-ts}}. \tag{4.39}$$

For $t = 2$, the product over primes is

$$\min_{x>0, x^2\equiv 0 \bmod n} x \mapsto \prod_p \frac{1+p^{1-s}}{1-p^{1-2s}} = \zeta(2s-1)\zeta(s-1)/\zeta(2s-2). \tag{4.40}$$

For $t = 3$, a variation of (4.10) appears:

$$\min_{x>0, x^3\equiv 0 \bmod n} x \mapsto \prod_p \frac{1+p^{1-s}+p^{1-2s}}{1-p^{1-3s}} = \zeta(3s-1)\prod_p (1+p^{1-s}+p^{1-2s}) = \zeta(3s-1)\prod_p (1+p^{1-s})(1+p^{1-2s})(1-p^{2-3s})(1+p^{3-4s})(1-p^{4-5s}) \times (1+p^{3-5s})(1+p^{5-6s})(1-p^{4-6s})\cdots, \quad s>2. \tag{4.41}$$

Remark 47. *These are A019554 and A019555 for* $t = 2$ *and* $t = 3$, *A053166 for* $t = 4$, *A015052 and A015053 for* $t = 5$ *and* $t = 6$.

References

1. Tom M. Apostol, *Möbius function of order k*, Pac. J. Math. **32** (1970), no. 1, 21–27. MR 0253999 (40 #7212)
2. ______, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer, 1976. MR 0434929 (55#7892)
3. Antal Bege, *Generalized Möbius-type functions and special set of k-free numbers*, Acta Univ. Sapientiae Math. **1** (2009), no. 2, 143–150. MR 2521184 (2010f:11148)
4. Richard Bellman, *Analytic number theory*, Mathematics Lecture note series, vol. 57, Benjamin, 1980. MR 0596579 (83c:1001)
5. Henry Bottomley, *Some Smarandache-type multiplicative sequenes*, Smarandache Notions Journal **13** (2002), no. 1–3, 134–135. MR 1933254
6. E. D. Cashwell and C. J. Everett, *The ring of number-theoretic functions*, Pac. J. Math. **9** (1959), no. 4, 975–985. MR 0108510

7. Eckford Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Zeitschr. **74** (1960), 66–80. MR 0112861 (22#3707)
8. ———, *Unitary products of arithmetical functions*, Acta Arith. **7** (1961/1962), 29–38. MR 0130210 (24 #A77)
9. D. E. Daykin, *Generalized Möbius inversion formulae*, Quart. J. Math. **15** (1964), no. 1b, 349–354. MR 0174508 (30 #4709)
10. Leonard Eugene Dickson, *History of the theory of numbers*, Chelsea, New York, 1966. MR 0245499 (39 #6807a)
11. H. W. Gould and Temba Shonhiwa, *A catalogue of interesting Dirichlet series*, Miss. J. Math. Sci **20** (2008), no. 1.
12. I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed., Harri Deutsch, Thun, 1981. MR 0671418 (83i:00012)
13. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3 ed., 1954. MR 0067125 (16,673c)
14. Pentti Haukkanen, *On a gcd-sum function*, Aequat. Math. **76** (2008), no. 1–2, 168–178. MR 2443468 (2009j:11010)
15. Leetsch Charles Hsu and Evelyn L. Tan, *A refinement of de Bruyn's formula for* $\sum a^k k^p$, Fib. Quart. **38** (2000), no. 1, 56–59. MR 1738647 (2000k:11030)
16. Mohan Lal, *Iterates of the unitary totient function*, Math. Comp. **28** (1974), no. 125, 301–302. MR 0355419 (49 #201)
17. R. Sherman Lehman, *On Liouville's function*, Math. Comp. **14** (1960), no. 72, 311–320. MR 0120198 (22 #10955)
18. Richard J. Mathar, *Series of reciprocal powers of* k*-almost primes*, arXiv:0803.0900 [math.NT] (2008).
19. ———, *Hardy–Littlewood constants embedded into inifinite products over all positive integers*, arXiv:0903.2514 [math.NT] (2009).
20. ———, *Table of Dirichlet L-series and prime zeta modulo functions for small moduli*, arXiv:1008.2547 [math.NT] (2010).
21. Pieter Moree, *The formal series Witt transform*, Discr. Math. **295** (2005), no. 1–3, 143–160. MR 2143453 (2006b:05015)
22. K. Nageswara Rao, *On the unitary analogues of certain totients*, Monatsh. Math. **70** (1966), no. 2, 149–154. MR 0200231 (34# 130)
23. József Sándor and Antal Berge, *The Möbius function: generalizations and extensions*, Adv. Stud. Contemp. Math. (Kyungshang) **6** (2003), no. 2, 77–128. MR 1962765 (2004b:11011)
24. Wacław Sierpiński, *Elementary theory of numbers*, Monografie Matematyczne **42** (1964). MR 0175840 (31 #116)
25. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)
26. D. Suryanarayana and R. Sita Rama Chandra Rao, *The number of square-full divisors of an integer*, Proc. Am. Math. Soc. **34** (1972), no. 1, 79–80. MR 0291104 (45 # 198)
27. E. C. Titchmarch and D. R. Heath-Brown, *The theory of the Riemann zeta-function*, 2 ed., Oxford Science Publications, 1986. MR 0882550 (88c:11049)
28. László Tóth, *On a class of arithmetic convolutions involving arbitrary sets of integers*, Mathem. Pannon. **13** (2002), no. 2, 249–263. MR 1932431
29. R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Am. Math. Soc. **33** (1931), 579–662. MR 1501607
30. Charles R. Wall, *The fifth unitary perfect number*, Canad. Math. Bull. **18** (1975), no. 1, 115–122. MR 0376515
31. P. Wynn, *A note on the generalised Euler transformation*, Comp. J. **14** (1971), no. 4, 437–441. MR 0321266 (47 #9799)

URL: `http://www.strw.leidenuniv.nl/~mathar`
E-mail address: `mathar@strw.leidenuniv.nl`

Leiden Observatory, Leiden University, P.O. Box 9513, 2300 RA Leiden, The Netherlands