On Euler characteristics for large Kronecker quivers

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Abstract

We study Euler characteristics of moduli spaces of stable representations of m-Kronecker quivers for $m >> 0$.

1 Introduction

For each positive integer m, let K^m be the m-Kronecker quiver which consists of two vertices and m arrows from one to the other. For generic non-trivial stability conditions [1] on the category of representations of K^m and moduli spaces of stable representations $M(K^m(a,b))$ of coprime dimension vectors (a,b) [5], we study Euler characteristics $\chi(K^m(a,b))$.

We put some more details in the later section and we go on as follows. Notice that for the Euler form $\langle\cdot,\cdot\rangle$ and a symplectic form $\{\cdot,\cdot\}$, which is an anti-symmetrization of the Euler form, we may take a non-trivial stability condition on the category of representations of K^m such that for representations E, F of K^m and the slope function μ, we have $\mu(E) > \mu(F)$ if and only if $\{E, F\} > 0$.

For objects to study in terms of wall-crossings, stability conditions such that the positivity of the difference of slopes coincides with that of symplectic forms on the Grothendieck group have been commonly called Denef's stability conditions in physics [3]. We employ these special stability conditions and the terminology.

Euler characteristics $\chi(K^m(a,b))$ have been studied extensively. In particular, formulas of Kontsevich-Soibelman and Reineke [7, 10, 11] have been known. In this article, we would like to study quantitative questions for $m >> 0$.

To analyze further, for each coprime a, b and $m > 0$, let us define the bipartite quiver $Q^m(a,b)$ which consist of a source vertices and b terminal vertices with m arrows from each source vertex to each terminal vertex. On representations of $Q^m(a,b)$, we have Denef's stability conditions (see Section 2).

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We denote $M(Q^m(a,b))$ to be the moduli space of stable representations of dimension vectors being one on each vertex of $Q^m(a,b)$ and $\chi(Q^m(a,b))$ to be the corresponding Euler characteristic.

In this paper, we first prove the following.

Theorem 1.1. *For each coprime a,b, and $m >> 0$, we have*

$$\chi(Q^1(a,b)) \sim \frac{a!b!}{m^{a+b-1}} \chi(K^m(a,b)).$$

In terms of physics, we would like to mention that in Theorem 1.1, Euler characteristics in the left-hand and right-hand sides are discussed to be blackhole counting in supergravity [8] and Witten index in superstring theory [2]. In [9], with the framework of Kontsevich's homological mirror symmetry [6], the m-Kronecker quiver K^m has been described in terms of Lagrangian intersection theory.

Key tools to obtain Theorem 1.1 are the recently obtained formula Theorem 2.1 on $\chi(K^m(a,b))$ by Manschot-Pioline-Sen [8] (MPS formula for short) and our Lemma 2.3. We realize that by taking m to be a variable, MPS formula provides the polynomial expansion of $\chi(K^m(a,b))$ whose coefficients involve Euler characteristics of bipartite quivers such as $Q^m(a,b)$. Indeed, we are dealing with nothing but the first-order approximation of $\chi(K^m(a,b))$ for $m >> 0$.

By Theorem 1.1, to compute $\chi(Q^1(a,b))$, we can take the advantage of $\chi(K^m(a,b))$. Since the explicit formula of $\chi(K^m(a,a+1))$ has been provided in [12], we can obtain $\chi(Q^1(a,a+1))$ by taking $m \to \infty$ in Corollary 2.5. Let us mention that for the cases of $a=1$ and arbitrary b, we see that Stirling formula explains Theorem 1.1.

When $a+b=1$, $M(K^m(a,b))$ is a point. Taking logarithms in Theorem 1.1, we have the following.

Corollary 1.2. For $m >> 0$, we have

$$\ln(\chi(K^m(a,b))) \sim (a+b-1)\ln(m).$$

In particular, for $a,b >> 0$ such that $\frac{b}{a} \sim r$ and large enough m depending on a,b, we have

$$\frac{\ln(\chi(K^m(a,b)))}{a} \sim (1+r)\ln(m).$$

Let us mention that Douglas has conjectured the following [4, 12]. For coprime $a,b >> 0$ such that $\frac{b}{a} \sim r$ and each m, we have that $\frac{\ln \chi(K^m(a,b))}{a}$ is a continuous function of r. In [12], assuming the continuity, the quantity has been determined through the explicit formula of $\chi(K^m(a,a+1))$ in [12] mentioned above. So, now we notice that some estimates on $\frac{\ln(\frac{\chi(Q^1(a,b))}{a!b!})}{a+b-1}$ in terms of $\frac{b}{a} \sim r$ for $a >> 0$ would give further understanding of $\frac{\ln \chi(K^m(a,b))}{a}$.

2 Proofs

Let us expand and introduce notions. For each a, let $\overline{a}$ denote a partition of a such that for non-negative integers a_l of $l \geq 1$, we have $\sum_l la_l = a$. We put $S_{\overline{a}} = \sum a_l$ for our convenience. When $a_1 = a$, we simply write a for $\overline{a}$. For a quiver Q and representations E, F of Q, on the Grothendieck group of the category of representations of Q, let $\langle E, F\rangle_Q$ be the Euler form and $\{E, F\}_Q$ be the symplectic form $\langle F, E\rangle_Q - \langle E, F\rangle_Q$. For a dimension vector d, we call a partition $d^1, \ldots, d^s$ of d such that $\sum_{p=1}^{s} d^p = d$ and $\{\sum_{p=1}^{b} d^p, d\}_Q > 0$ for each $b = 1, \ldots, s-1$ to be admissible.

For each $m > 0$ and partitions $\overline{a}, \overline{b}$ of a and b, we define the bipartite quiver $Q^m(\overline{a}, \overline{b})$ as follows. It consists of $S_{\overline{a}}$ source vertices such that for each l, we have a_l vertices v; for our convenience, we say a_l is the label of v and we put $w(v) = l$. It consists of $S_{\overline{b}}$ terminal vertices with labels and $w(\cdot)$ being defined by the same manner. We put $mw(v)w(v')$ arrows from each source vertex v to each terminal vertex v'.

Let us explain Denef's stability conditions in use. For the m-Kronecker quiver K^m, the source vertex $(1, 0)$, and the terminal vertex $(0, 1)$, the slope function μ satisfies $\mu(1, 0) > \mu(0, 1)$. For $Q^m(\overline{a}, \overline{b})$ and vertices v and v' with the labels being a_l and $b_{l'}$, central charges $\frac{Z(v)}{w(v)}$ and $\frac{Z(v')}{w(v')}$ coincide with those of the vertices $(1, 0)$ and $(0, 1)$.

We write $(\overline{a}, \overline{b})$ for the dimension vector which has one on each vertex of the quiver $Q^m(\overline{a}, \overline{b})$. We let $M(Q^m(\overline{a}, \overline{b}))$ to be the moduli space of stable representations of the dimension vector $(\overline{a}, \overline{b})$ of $Q^m(\overline{a}, \overline{b})$. For coprime a, b and moduli spaces of stable representations of quivers such that $M(K^m(a, b))$ and $M(Q^m(\overline{a}, \overline{b}))$, we denote $P(K^m(a, b), y)$ and $P(Q^m(\overline{a}, \overline{b}), y)$ to be Poincare polynomials. For the m-Kronecker quiver K^m, we have the following MPS formula [8, Appendix D].

Theorem 2.1. *(MPS formula) For each coprime a, b and $m > 0$, we have*

$$P(K^m(a,b),y) = y^{-\langle (a,b),(a,b)\rangle_{K^m}} \sum_{\overline{a},\overline{b}} y^{\langle (\overline{a},\overline{b}),(\overline{a},\overline{b})\rangle_{Q^m(\overline{a},\overline{b})}} P(Q^m(\overline{a},\overline{b}),y) \cdot$$

$$\Pi_l \frac{1}{\overline{a}_l!} \left(\frac{y - y^{-1}}{l(y^l - y^{-l})} (-1)^{l-1} \right)^{\overline{a}_l} \cdot$$

$$\Pi_l \frac{1}{\overline{b}_l!} \left(\frac{y - y^{-1}}{l(y^l - y^{-l})} (-1)^{l-1} \right)^{\overline{b}_l} .$$

We shall not repeat their proof of MPS formula, but we would like to mention a key point of the proof as follows. To compute $P(Q^m(\overline{a}, \overline{b}), y)$ with Reineke's formula [10, Corollary 6.8], we start with a partition (α^p, β^p) of $(\overline{a}, \overline{b})$ for $p = 1, \ldots, s$ of some s. For each p and l, we put α^p_l to denote the number of non-zero entries of α^p and of vertices of labels being a_l; we put β^p_l of β^p by the same manner. Observe that, through direct computation on symplectic forms, the partition (α^p, β^p) is admissible if and only if the partition $(\sum_l l\alpha^p_l, \sum_l l\beta^p_l)$ of

(a, b) for $p = 1, \ldots, s$ is admissible. Now, as shown in [8, Appendix D], we can proceed by explicitly computing involved terms for admissible partitions.

For Euler characteristics, we put the following for our convenience.

Corollary 2.2. We have

$$\chi(K^m(a,b)) = \sum_{\overline{a},\overline{b}} \chi(Q^m(\overline{a},\overline{b})) \cdot \Pi_l \frac{1}{\overline{a}_l!} \frac{(-1)^{\overline{a}_l(l-1)}}{l^{2\overline{a}_l}} \cdot \Pi_l \frac{1}{\overline{b}_l!} \frac{(-1)^{\overline{b}_l(l-1)}}{l^{2\overline{b}_l}}.$$

Notice that $M(Q^1(\overline{a},\overline{b}))$ is a non-trivial smooth projective variety, since we have stable representations including ones with invertible maps on every arrows. Now, we have the following.

Lemma 2.3. *We have*

$$\chi(Q^m(\overline{a},\overline{b})) = m^{S_{\overline{a}}+S_{\overline{b}}-1}\chi(Q^1(\overline{a},\overline{b})).$$

Proof. Let us consider the Poincare polynomial $P(Q^m(\overline{a},\overline{b}), y)$ with Reineke's formula [10, Corollary 6.8]. For the dimension vector $(\overline{a},\overline{b})$, we take an admissible partition $d^1, \ldots, d^s$ and the term $(-1)^{s-1}y^{2\sum_{k\le l}\sum_{v\to v'} d^l_v d^k_{v'}}$.

We notice that $\{\cdot,\cdot\}_{Q^m(\overline{a},\overline{b})} = m\{\cdot,\cdot\}_{Q^1(\overline{a},\overline{b})}$. The set of admissible partitions is invariant under choices of m. For each admissible partition, the power of y above is the m times of that for $P(Q^1(\overline{a},\overline{b}), y)$.

We have that $P(Q^1(\overline{a},\overline{b}), y)$ is a non-zero polynomial. Ignoring an overall factor of a power of y and writing y^2 as q for simplicity, for some non-trivial and non-negative integers α_i and β_i, we have

$$P(Q^1(\overline{a},\overline{b}), q) = (q-1)^{1-S_{\overline{a}}-S_{\overline{b}}}\left(\sum_{i\ge 0}\alpha_i(q-1)^{S_{\overline{a}}+S_{\overline{b}}-1}q^{\beta_i}\right).$$

For admissible partitions, the second factor is the sum of terms above. So we have

$$P(Q^m(\overline{a},\overline{b}), q) = (q-1)^{1-S_{\overline{a}}-S_{\overline{b}}}\left(\sum_{i\ge 0}\alpha_i(q^m-1)^{S_{\overline{a}}+S_{\overline{b}}-1}q^{m\beta_i}\right),$$

and the assertion follows. □

We put a proof of Theorem 1.1.

Proof. By Lemma 2.3, we see that the term $\chi(Q^m(a,b))$ carries the highest power of m among $\chi(Q^m(\overline{a},\overline{b}))$ in Corollary 2.2. □

We put a proof of Corollary 1.2.

Proof. We have

$$\ln\left(\frac{\chi(Q^m(a,b))}{a!b!}\right) = \ln\left(\frac{m^{a+b-1}\chi(Q^1(a,b))}{a!b!}\right)$$
$$= (a+b-1)\ln(m) + \ln\left(\frac{\chi(Q^1(a,b))}{a!b!}\right).$$

So for $a+b \neq 1$ and large enough m so that

$$\left|\frac{\ln(\frac{\chi(Q^1(a,b))}{a!b!})}{(a+b-1)\ln(m)}\right| << 1,$$

the assertion follows. □

Let us compute $\chi(Q^1(a,a+1))$ as in the introduction. From [12], we recall the following.

Theorem 2.4. *([12, Theorem 6.6])*

$$\chi(K^m(a,a+1)) = \frac{m}{(a+1)((m-1)a+m)}\binom{(m-1)^2a+(m-1)m}{a}.$$

So, by Theorem 1.1, we have the following.

Corollary 2.5.

$$\chi(Q^1(a,a+1)) = \lim_{m\to\infty}\frac{\chi(K^m(a,a+1))a!(a+1)!}{m^{2a}} = (a+1)!(a+1)^{-2+a}.$$

Remark 2.6. *With the formula of $\chi(K^m(2,2a+1))$ in [10], Manschot told the author that he has proved the following formula.*

$$\chi(Q^1(2,2a+1)) = \frac{(2a+1)!}{a!^2}.$$

This sequence and the one in Corollary 2.5 coincide with A002457 and A066319 at oeis.org.

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