SET PARTITIONS WITH NO m-NESTING

MARNI MISHNA AND LILY YEN

ABSTRACT. A partition of $\{1, \dots, n\}$ has an m-nesting if there exists $i_1 < i_2 < \cdots < i_m < j_m < j_{m-1} < \cdots < j_1$, where i_l and j_l are in the same block for all $1 \le l \le m$. We use generating trees to construct the class of partitions with no m-nesting and determine functional equations satisfied by the associated generating functions.

We use algebraic kernel method together with a linear operator to describe a coefficient extraction process. This gives rise to enumerative data, and illustrates the increasing complexity of the coefficient formulas as m increases.

1. Introduction

In this work we address the enumeration of set partitions that avoid a particular class of patterns. The patterns considered here, known as m-nestings, arise in a graphical representation of set partitions. Our goal is to determine useful enumerative information about partitions that contain no m-nesting. This work is in the context of recent studies of other combinatorial objects that avoid similar or related patterns, in particular in the study of protein folding [7]. Our strategy parallels a recent generating tree approach used by Bousquet-Mélou to enumerate a class of pattern avoiding permutations [3]. Here, in a less condensed form, upon exhibiting a general functional equation for all m, we apply both an appropriate transformation of variables and a multiplicative factor to produce symmetry in the kernel of the functional equation. This permits us to apply the algebraic kernel method and generate a telescoping sum, which greatly simplifies the expression. The enumerative formulas are obtained by coefficient extraction. The result of the analysis produces a first look into how such numbers are composed in a recurrence. It is our hope that readers new to the generating tree approach for obtaining multivariate functional equations could understand the enumerative power and closed form limitations of this method.

1.1. Notation and definitions.

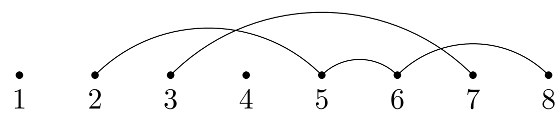
A set partition π of $[n] := \{1, 2, 3, \dots, n\}$, denoted by $\pi \in \Pi_n$, is a collection of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. The number of set partitions of $[n]$ into k blocks is denoted $S(n, k)$, and is known as Stirling number of the second kind. The total number of partitions of $[n]$ is the *Bell* number $B_n = \sum_k S(n, k)$. We represent π by a graph on the vertex set $[n]$ whose edge set consists of arcs connecting elements of each block in numerical order. Such an edge set is called the *standard representation* of the partition π, as seen in [6]. For example, the standard representation of

$$1|2\,5\,6\,8|3\,7|4$$

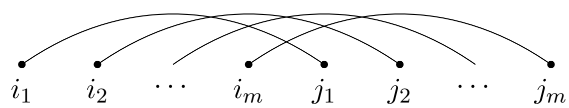
is given by the following graph with edge set $\{(2, 5), (5, 6), (6, 8), (3, 7)\}$:

2000 *Mathematics Subject Classification.* 05A18.

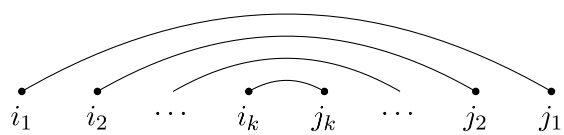
Key words and phrases. set partition, nesting, pattern avoidance, generating tree, algebraic kernel method, coefficient extraction, enumeration.



With this representation, we can define two classes of patterns: crossings and nestings. An *m-crossing* of π is a collection of m edges (i_1, j_1), (i_2, j_2), $\ldots$, (i_m, j_m) such that $i_1 < i_2 < \cdots < i_m < j_1 < j_2 < \cdots < j_m$. Using the standard representation, an m-crossing is drawn as follows:



Similarly, we define an *m-nesting* of π to be a collection of m edges (i_1, j_1), (i_2, j_2), $\ldots$, (i_m, j_m) such that $i_1 < i_2 < \cdots < i_m < j_m < j_{m-1} < \cdots < j_1$. This is drawn:



A partition is m-noncrossing if it contains no m-crossing, and it is said to be m-nonnesting if it contains no m-nesting.

1.2. **Context and plan.** Chen, Deng, Du, Stanley and Yan in [6] and Krattenthaler in [8] gave a non-trivial bijective proof that m-noncrossing partitions of $[n]$ are equinumerous with m-nonnesting partitions of $[n]$, for all values of m and n. A straightforward bijection with Dyck paths illustrates that 2-noncrossing partitions (also called noncrossing partitions) are counted by Catalan numbers. Bousquet-Mélou and Xin in [4] showed that the sequence counting 3-noncrossing partitions is P-recursive, that is, satisfies a linear recurrence relation with polynomial coefficients. Indeed, they determined an explicit recursion, complete with solution and asymptotic analysis. They further conjectured that m-noncrossing partitions are not P-recursive for all $m \geq 4$. Bell numbers are well known not to be P-recursive because of the composed exponentials in the generating function $B(x) = e^{e^x - 1}$ as explained in Example 19 of [2].

Since m-noncrossing partitions of $[n]$ and m-nonnesting partitions of $[n]$ are equinumerous, we study m-nonnesting partitions in this paper and show how to generate the class using generating trees, and how to determine a recursion satisfied by the counting sequence for m-nonnesting partitions.

Our approach is heavily inspired by Bousquet-Mélou's recent work on the enumeration of permutations with no long monotone subsequence in [3]. She combined the ideas of recursive construction for permutations via generating trees and the algebraic kernel method to determine and solve functional equations with multiple catalytic variables.

In Section 2, we employ Bousquet-Mélou's generating tree construction to find functional equations satisfied by the generating functions for set partitions with no m-nesting. The resulting equations, though similar to the equations arising in [3], need a similar multiplicative factor but a different transformation of variables before a comparable analysis using algebraic kernel method techniques is applied. To succeed in obtaining information after applying the algebraic kernel method, a coefficient extraction procedure is required. This is completed in Section 3. Unfortunately for us, unlike her work, we can only find explicit equations parameterised by m, with m catalytic variables for all $m \geq 1$ without obtaining a closed form expression for the number of set partitions of size n avoiding an m-nesting for $m \geq 3$. In the case of $m = 3$, however, it is similar to the functional equation given by Bousquet-Mélou and Xin.

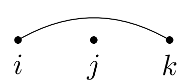
We are able to provide new enumerative data for $m > 4$, and also offer evidence supporting the non-P-recursive conjecture of Bousquet-Mélou and Xin in Section 6.

2. Generating Trees and Functional Equations

The generating tree construction for the class of m-nonnesting partitions is based on a standard generating tree description of partitions, and the constraint is incorporated using a vector labelling system. The generating tree construction has an immediate translation to a functional equation with m-variate series.

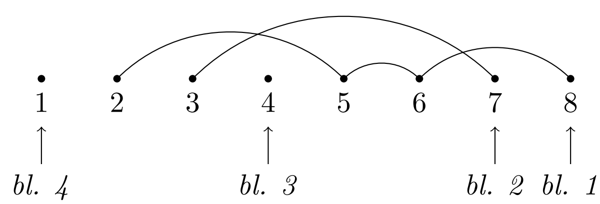
2.1. **A generating tree for set partitions.** Let π be a set partition. Define $\mathrm{ne}(\pi)$ to be the maximal i such that π has an i-nesting, also called the *maximal nesting number* of π, and let $\Pi_n^{(m)}$ be the set of partitions of $[n]$ for $n \geq 0$ (where $n = 0$ means the empty partition) with $\mathrm{ne}(\pi) \leq m$, thus $(m+1)$-nonnesting.

Note that an arc over a fixed point is not a 2-nesting, but a 1-nesting.



We next describe how to generate all set partitions via generating trees in the fashion of [2]. First, order the blocks of a given partition, π, by the maximal element of each block in descending order.

Example 1. The first block of $1|2\,5\,6\,8|3\,7|4$ is $2\,5\,6\,8$; the second block is $3\,7$; the third block is singleton 4; and 1 is the last block. Using the standard representation,



we number the blocks in descending order (from the right to the left) according to the maximal element in each block (that is, the rightmost vertex of each block).

With the order of blocks thus defined, we warm up by generating all set partitions without nesting restriction first. Figure 1 contains the generating tree for all set partitions, in addition to the generating tree for the number of children of each node from the tree of set partitions to indicate how enumeration can be facilitated.

1. Begin with $\emptyset$ as the top node of the tree. It has only one child, so the corresponding node in the tree for the number of children is labelled 1.
2. To produce the $n+1$st level of nodes, take each set partition at the nth level, and either add $n+1$ as a singleton, or join $n+1$ to block j for each $1 \leq j \leq k$ if the set partition has k blocks.

Summarizing the description above in the notation of [2], we recall that the rewriting rule of a generating tree is denoted by:

$$[(s_0), \{(k) \to (e_{1,k})(e_{2,k}) \dots (e_{k,k})\}],$$

where s_0 denotes the degree of the root, and for any node labelled k, that is, with k descendents, the label of each descendent is given by $(e_{j,k})$ for $1 \le j \le k$. Thus, the class of set partitions has a generating tree of labels given by $[(1) : (k) \to (k+1)(k)^{k-1}]$.

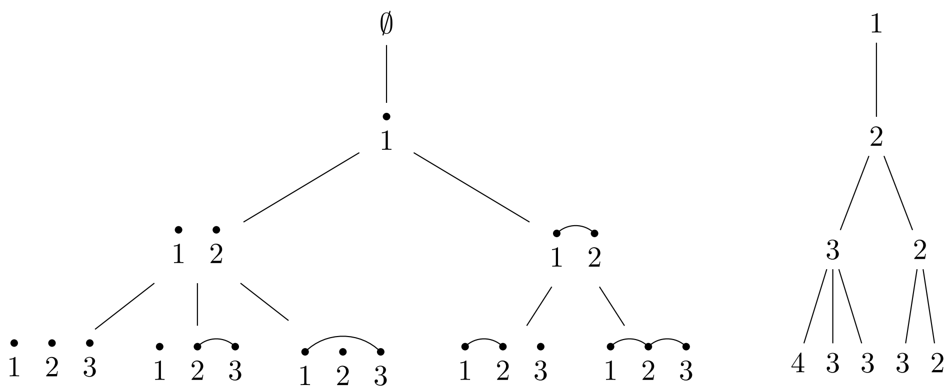


FIGURE 1. *Generating tree for set partitions and its corresponding generating tree of the number of children*

2.2. **A vector label to track nestings.** Note that in Figure 1, the generating tree of set partitions generates all set partitions graded by n, the size of π but it does not keep track of nesting numbers. Also note that in the generating tree for the number of children, the number of children of π is one more than the number of blocks of π for any partition π.

Fix m. In order to keep track of nesting numbers, we need to define the *label* of $\pi \in \Pi^{(m)}$.

Definition 1. Define the label of a partition, $L(\pi) = (a_1(\pi), a_2(\pi), \dots, a_m(\pi))$, or in short, $L(\pi) = (a_1, a_2, \dots, a_m)$ as follows. For $1 \le j \le m$,

$$a_j(\pi) = \begin{cases} 1+ \text{ number of blocks in } \pi, & \text{if } \pi \text{ is } j\text{-nonnesting}, \\ \\ \text{1+ number of blocks ending to the right of the smallest vertex in the rightmost } j\text{-nesting} & \text{otherwise.} \end{cases}$$

By the *rightmost* j-nesting, we mean the minimal element in the j-nesting of a particular partition π that is greater than or equal to all minimal elements in all j-nestings of π.

Example 2. To continue the example, let $\pi = 1|2\,5\,6\,8|3\,7|4$ and suppose $m = 3$. Then $L(1|2\,5\,6\,8|3\,7|4) = (3, 4, 5)$ for the following reasons. The rightmost 1-nesting is the edge with largest vertex endpoint, $(6, 8)$. Hence, $a_1(\pi) = 3$ because blocks 1 and 2 end to the right of vertex 6. The rightmost 2-nesting is the set of edges $\{(5, 6), (3, 7)\}$ hence $a_2(\pi) = 4$ because 3 blocks end to the right of vertex 3. Finally, $a_3(\pi) = 5$ because the diagram has no 3-nesting, and is comprised of 4 blocks. Note that in this convention, the empty set partition has label $(1, 1, \dots, 1)$, since it has no nestings and no blocks.

A set partition in $\Pi^{(m)}$ will have a_m children. This is one plus the number of blocks, if there is no m-nestings (and hence no risk that adding an edge will create an $m+1$-nesting). Otherwise, it indicates one plus the number of blocks to which you can add an edge without creating an $m+1$-nesting. The label of a set partition is sufficient to derive the label of each of its children, and this is described in the next proposition. Also, remark that the label is a non-decreasing sequence, since the rightmost j-nesting either contains the rightmost $j-1$ nestings or is to the left of it.

Proposition 1 (Labels of children)**.** *Let π be in $\Pi_n^{(m)}$, the set of set partitions on $[n]$ avoiding $m+1$-nestings, and suppose the label of π is $L(\pi) = (a_1, a_2, \ldots, a_m)$. Then, the labels of the a_m set partitions of $\Pi_{n+1}^{(m)}$ obtained by recursive construction via the generating tree are*

$$(a_1+1, a_2+1, \ldots, a_m+1) \qquad \textit{(Add } n+1 \textit{ as a singleton to } \pi\textit{)}$$

and

$$\begin{array}{llll}
(\quad 2, & a_2, \quad a_3, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block 1)} \\
(\quad 3, & a_2, \quad a_3, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block 2)} \\
 & & & \vdots \\
(\quad a_1, & a_2, \quad a_3, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1-1\textit{)} \\
(a_1+1, & a_1+1, \quad a_3, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1\textit{)} \\
(a_1+1, & a_1+2, \quad a_3, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1+1\textit{)} \\
 & & & \vdots \\
(a_1+1, & a_2+1, a_2+1, \ldots, & a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_2\textit{)} \\
 & & & \vdots \\
(a_1+1, & a_2+1, a_3+1, \ldots, & a_{m-1}+1, a_{m-1}+1) & \textit{(Add } n+1 \textit{ to block } a_{m-1}\textit{)} \\
 & & & \vdots \\
(a_1+1, & a_2+1, a_3+1, \ldots, & a_{m-1}+1, a_m) & \textit{(Add } n+1 \textit{ to block } a_m-1\textit{)}
\end{array}$$

Proof. By careful inspection. ☐

Example 3. Consider the following partition from $\Pi_8^{(3)}$. The reader can refer to its arc diagram in Example 1 which shows that is is 3-nonnesting, thus also 4-nonnesting. The partition $1|2\,5\,6\,8|3\,7|4$ with label $(3, 4, 5)$ has five children and their respective labels are:

π	$L(\pi)$
$1\|2\,5\,6\,8\|3\,7\|4\|9$	$(4, 5, 6)$
$1\|2\,5\,6\,8\,9\|3\,7\|4$	$(2, 4, 5)$
$1\|2\,5\,6\,8\|3\,7\,9\|4$	$(3, 4, 5)$
$1\|2\,5\,6\,8\|3\,7\|4\,9$	$(4, 4, 5)$
$1\,9\|2\,5\,6\,8\|3\,7\|4$	$(4, 5, 5)$

Notice that in Proposition 1, the first label comes from adding $n+1$ as a singleton to get an element of $\Pi_{n+1}^{(m)}$. The other $a_m - 1$ labels result from adjoining the element $n+1$ to the maximal element of block l for every $1 \le l \le a_m - 1$ blocks without creating an $m+1$-nesting.

Example 4. As we mentioned before, 2-nonnesting set partitions are counted by Catalan numbers. The generating tree construction given in Proposition 1 restricted to this case is given by

$$[(1):(k) \to (2)(3)\dots(k+1)],$$

which is the same construction for Catalan numbers given in [2]. The generating tree for 3-noncrossing partitions is given by

$$[(1,1):(i,j) \to (i+1,j+1)(2,j)(3,j)\cdots(i,j)(i+1,i+1)(i+1,i+2)\dots(i+1,j)].$$

2.3. **A functional equation for the generating function.** The simple structure of the labels of a partition's children in Proposition 1 permits a straightforward translation of the combinatorial construction into a functional equation. Let us define $\tilde{F}(u_1, u_2, \dots, u_m; t)$ to be the ordinary generating function of partitions in $\Pi^{(m)}$ counted by the statistics a_1, a_2, …, a_m and by size,

$$\begin{aligned}\tilde{F}(u_1, u_2, \dots, u_m; t) &:= \sum_{\pi \in \Pi^{(m)}} u_1^{a_1(\pi)} u_2^{a_2(\pi)} \dots u_m^{a_m(\pi)} t^{|\pi|} \\ &= \sum_{a_1, a_2, \dots, a_m} \tilde{F}_{\mathbf{a}}(t) u_1^{a_1} u_2^{a_2} \dots u_m^{a_m},\end{aligned}$$

where $\tilde{F}_{\mathbf{a}}(t)$ is the size generating function for the set partitions of $\Pi^{(m)}$ with the label $\mathbf{a} = (a_1, a_2, \dots, a_m)$. Thus, when $m = 2$,

$$\tilde{F}(\mathbf{u}; t) = u_1u_2 + {u_1}^2{u_2}^2 t + \left({u_1}^3{u_2}^3 + {u_1}^2{u_2}^2\right) t^2 + \left({u_1}^4{u_2}^4 + 2\,{u_1}^3{u_2}^3 + {u_1}^2{u_2}^2 + {u_1}^2{u_2}^3\right) t^3 + \dots .$$

Proposition 1 implies

$$\begin{aligned}\tilde{F}(u_1, u_2, \dots, u_m; t) &= u_1u_2\dots u_m + tu_1u_2\dots u_m \tilde{F}(u_1, u_2, \dots, u_m; t) \\ &\quad + t \sum_{a_1, a_2, \dots, a_m} \tilde{F}_{\mathbf{a}}(t) u_2^{a_2} u_3^{a_3} \dots u_m^{a_m} \sum_{\alpha=2}^{a_1} u_1^{\alpha} \\ &\quad + t \sum_{a_1, a_2, \dots, a_m} \tilde{F}_{\mathbf{a}}(t) \sum_{j=2}^{m} \sum_{\alpha=a_{j-1}+1}^{a_j} u_1^{a_1+1} u_2^{a_2+1} \dots u_{j-1}^{a_{j-1}+1} u_j^{\alpha} u_{j+1}^{a_{j+1}} \dots u_m^{a_m}.\end{aligned}$$

We compactify the sums using finite geometric series sum formula and summarize the above derivation into the following functional equation for the generating function.

Proposition 2.

$$\begin{aligned}(1)\quad \tilde{F}(\mathbf{u}; t) &= u_1u_2\dots u_m + tu_1u_2\dots u_m \tilde{F}(\mathbf{u}; t) \\ &\quad + tu_1 \left(\frac{\tilde{F}(\mathbf{u}; t) - u_1 \tilde{F}(1, u_2, \dots, u_m; t)}{u_1 - 1}\right) \\ &\quad + t \sum_{j=2}^{m} u_1u_2\dots u_j \left(\frac{\tilde{F}(\mathbf{u}; t) - \tilde{F}(u_1, \dots, u_{j-2}, u_{j-1}u_j, 1, u_{j+1}, \dots, u_m; t)}{u_j - 1}\right),\end{aligned}$$

where $\tilde{F}(\mathbf{u}; t) = \tilde{F}(u_1, u_2, \dots, u_m; t)$.

3. Processing the functional equation

To process the functional equation for $\tilde{F}(\mathbf{u};t)$ we transform the variables to get a form more amenable to analysis. We follow [3], and do this in two steps. The first rewrites in a v variable to remove the exponent restriction on the u_i's because $a_1(\pi) \leq a_2(\pi) \leq \cdots \leq a_m(\pi)$; the second transformation to a set of x variables in the next section allows us to analyse the coefficients.

3.1. Removing the exponent restriction.

Define

$$F(\mathbf{v};t) = F(v_1, \ldots, v_m, v_{m+1};t) := \sum_{\pi \in \Pi^{(m)}} v_1^{a_1} v_2^{a_2 - a_1} \ldots v_m^{a_m - a_{m-1}} v_{m+1}^{|\pi| - a_m} t^{|\pi|}.$$

where $(a_1, a_2, \ldots, a_m) = L(\pi)$. Thus, we have eliminated the dependency $a_1 \leq a_2 \leq \cdots \leq a_m$ between the exponents of u_1, u_2, $\ldots$, u_m in $\tilde{F}(\mathbf{u};t)$.

We can write $\tilde{F}$ in terms of F and vice versa:

$$F(v_1, \ldots, v_{m+1};t) = \tilde{F}\left(\frac{v_1}{v_2}, \frac{v_2}{v_3}, \ldots, \frac{v_m}{v_{m+1}}; v_{m+1}t\right),$$

and

$$\tilde{F}(u_1, \ldots, u_m; v_{m+1}t) = F(u_{1,m}v_{m+1}, u_{2,m}v_{m+1}, \ldots, u_m v_{m+1}, v_{m+1};t),$$

where $u_{j,m} = u_j u_{j+1} \ldots u_m$. The function F satisfies a simpler functional equation.

Proposition 3. *The generating function $F(\mathbf{v};t) = F(v_1, v_2, \ldots, v_m, v_{m+1};t)$ of set partitions of $\Pi^{(m)}$ satisfies*

$$(2)\quad F(\mathbf{v};t) = \frac{v_1}{v_{m+1}} + tv_1\Big(F(\mathbf{v};t) + v_{m+1}\sum_{j=1}^{m}\frac{F(\mathbf{v};t)}{v_j - v_{j+1}}\Big) - v_{m+1}tv_1\Big(\frac{v_1}{v_2}\frac{F(v_2, v_2, v_3, \ldots, v_{m+1};t)}{v_1 - v_2} + \sum_{j=2}^{m}\frac{F(v_1, \ldots, v_{j-1}, v_{j+1}, v_{j+1}, v_{j+2}, \ldots, v_m, v_{m+1};t)}{v_j - v_{j+1}}\Big).$$

The series $F(1,1,\ldots,1;t)$ is the generating function for the class of $(m+1)$-nonnesting set partitions.

3.2. A second transformation.

We take the functional equation for $F(\mathbf{v};t)$ in Proposition 3 and rearrange the terms to find the kernel of the functional equation as follows

$$(3)\quad \left(1 - tv_1 - tv_1v_{m+1}\sum_{j=1}^{m}\frac{1}{v_j - v_{j+1}}\right)F(\mathbf{v};t) = \frac{v_1}{v_{m+1}} - v_{m+1}tv_1\Big(\frac{v_1}{v_2}\frac{F(v_2, v_2, v_3, \ldots, v_{m+1};t)}{v_1 - v_2} + \sum_{j=2}^{m}\frac{F(v_1, \ldots, v_{j-1}, v_{j+1}, v_{j+1}, v_{j+2}, \ldots, v_m, v_{m+1};t)}{v_j - v_{j+1}}\Big).$$

The kernel is

$$1 - tv_1 - tv_1 v_{m+1} \sum_{j=1}^{m} \frac{1}{v_j - v_{j+1}}.$$

To exploit invariance properties of the kernel, we introduce the following transformation of the v_j's:

$$\begin{aligned} v_{m+1} &= 1 \\ v_m &= 1 + x_m \\ &\vdots \\ v_2 &= 1 + x_m + \cdots + x_2 \\ v_1 &= 1 + x_m + \cdots + x_2 + x_1. \end{aligned}$$

This transformation enables us to rewrite the kernel as

$$1 - t\left(x_1 + x_2 + \cdots + x_m + 1\right)\left(1 + \sum_{j=1}^{m} \frac{1}{x_j}\right).$$

This new kernel is invariant under $\mathfrak{S}_m$, the symmetric group on $[m]$. To simplify presentation of the functional equation, we use

$$s = x_1 + x_2 + \cdots + x_m + 1, \qquad h = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_m} + 1.$$

Divide both sides of the functional equation by s we just defined, we get

$$\begin{aligned} (4) \quad \big(\frac{1}{s} - th\big)\bar{F}(x_1, x_2, \ldots, x_m; t) = \\ 1 - t\Big(\frac{s}{s - x_1}\frac{\bar{F}(0, x_2, x_3, \ldots, x_m; t)}{x_1} \\ + \sum_{j=2}^{m} \frac{\bar{F}(x_1, \ldots, x_{j-2}, x_{j-1} + x_j, 0, x_{j+1}, \ldots, x_m; t)}{x_j}\Big), \end{aligned}$$

where

$$\bar{F}(x_1, x_2, \ldots, x_m; t) = F(v_1, v_2, \ldots, v_m, v_{m+1}; t),$$

and

$$\bar{F}(0, 0, \ldots, 0; t) = F(1, 1, \ldots, 1; t)$$

is the evaluation that yields the ordinary generating function in t for set partitions avoiding $m+1$-nestings.

3.3. **A multiplicative factor and a telescoping sum.** We introduce a multiplicative factor, $M(\mathbf{x}) := x_1 x_2^2 x_3^3 \ldots x_m^m$, to be applied to Equation (4). Let the new kernel, $K(\mathbf{x}; t)$ be defined by $\frac{1}{s} - ht$. Since the kernel $K(\mathbf{x}; t)$ is invariant under $\mathfrak{S}_m$, when we take the signed orbit sum of the functional Equation (4) under $\mathfrak{S}_m$, namely, $\sum_{\sigma \in \mathfrak{S}_m} \epsilon(\sigma)\sigma(\text{functional equation})$ the left hand side has the kernel as a factor outside the sum; namely,

$$LHS = K(\mathbf{x}; t) \sum_{\sigma \in \mathfrak{S}_m} \epsilon(\sigma)\sigma(M(\mathbf{x})\bar{F}(x_1, \ldots, x_m; t)).$$

On the right hand side of Equation (4), before taking the orbit sum, the effect of multiplying by $M(\mathbf{x})$ is

$$
\begin{aligned}
(5)\qquad M(\mathbf{x}) - t\Big(&x_2^2x_3^3\dots x_m^m\bar{F}(0,x_2,x_3,\dots,x_m;t) + \frac{M(\mathbf{x})\bar{F}(0,x_2,\dots,x_m;t)}{x_2+x_3+\dots+x_m} \\
&+ x_1x_2x_3^3\dots x_m^m\bar{F}(x_1+x_2,0,x_3,\dots,x_m;t) \\
&+ x_1x_2^2x_3^2x_4^4\dots x_m^m\bar{F}(x_1,x_2+x_3,0,x_4,\dots,x_m;t) \\
&+ \dots \\
&+ x_1x_2^2x_3^3\dots x_{m-1}^{m-1}x_m^{m-1}\bar{F}(x_1,\dots,x_{m-2},x_{m-1}+x_m,0;t)\Big)
\end{aligned}
$$

Note that the coefficient of $\bar{F}(0,x_2,x_3,\dots,x_m;t)$ is split because

$$
\frac{s}{s-x_1} = \frac{s-x_1+x_1}{s-x_1} = 1 + \frac{x_1}{s-x_1}
$$

which is easier to manipulate when the orbit sum is taken. Because each of the last $m-1$ terms of the RHS of Equation (5) is invariant under $\sigma_j = (j, j+1)$ for some $j \in [m-1]$, (that is, the generators for $\mathfrak{S}_m$), by forming the signed sum over $\mathfrak{S}_m$ we reduce these $m-1$ terms to zero, leaving only the first three terms:

$$
\begin{aligned}
(6)\quad K(\mathbf{x};t)&\sum_{\sigma\in\mathfrak{S}_m}\epsilon(\sigma)\sigma(M(\mathbf{x})\bar{F}(x_1,\dots,x_m;t)) \\
&= \sum_{\sigma\in\mathfrak{S}_m}\epsilon(\sigma)\sigma(M(x)) - t\sum_{\sigma\in\mathfrak{S}_m}\epsilon(\sigma)\sigma(x_2^2x_3^3\dots x_m^m\bar{F}(0,x_2,\dots,x_m;t)) \\
&\qquad - t\sum_{\sigma\in\mathfrak{S}_m}\epsilon(\sigma)\sigma\Big(\frac{M(\mathbf{x})\bar{F}(0,x_2,\dots,x_m;t)}{x_2+x_3+\dots+1}\Big).
\end{aligned}
$$

3.4. **The constant term extraction operator $\mathcal{CT}$.** Our goal is to obtain the series $\bar{F}(0,0,\dots,0;t)$. Remark, any term in $\bar{F}(x_1,\dots,x_m;t)$ containing non-zero exponents of x_i's for $i \in [m]$ disappears when x_i is set to 0. The exponents of each $(x_i + x_{i+1} + \dots + x_m + 1)$ are all non-negative, implying that to get a constant term, each factor in parentheses must go to the constant, leaving only the variable t, keeping track of the size of the partition. For the sake of brevity in presentation, we define a linear operator for constant term extraction, namely $[x_1^0x_2^0\dots x_m^0]\bar{F}$.

Definition 2. Let $\mathcal{CT}$ be the constant term extraction operator defined on Laurent series by the following action on monomials:

$$
\mathcal{CT}(x_1^{e_1}x_2^{e_2}\dots x_m^{e_m}t^k) = \begin{cases} 0, & \text{if } e_i \neq 0 \text{ for some } i \in [m], \\ t^k & \text{otherwise.} \end{cases}
$$

Before applying our constant term extraction operator $\mathcal{CT}$, to the orbit sum, Equation (6), we first divide Equation (6) by $M(\mathbf{x})K(\mathbf{x};t)$:

$$\begin{aligned}(7)\quad \sum_{\sigma\in\mathfrak{S}_m} \frac{\epsilon(\sigma)\sigma(M(\mathbf{x})\bar{F}(x_1,\dots,x_m;t))}{M(\mathbf{x})}\\ = \frac{s}{1-ths}\Big(\sum_{\sigma\in\mathfrak{S}_m}\frac{\epsilon(\sigma)\sigma(M(\mathbf{x}))}{M(\mathbf{x})}\\ - t\sum_{\sigma\in\mathfrak{S}_m}\frac{\epsilon(\sigma)\sigma(x_2^2x_3^3\dots x_m^m\bar{F}(0,x_2,\dots,x_m;t))}{M(\mathbf{x})}\\ - t\sum_{\sigma\in\mathfrak{S}_m}\frac{\epsilon(\sigma)\sigma(\frac{M(\mathbf{x})\bar{F}(0,x_2,\dots,x_m;t)}{x_2+\dots+x_m+1})}{M(\mathbf{x})}\Big).\end{aligned}$$

On the LHS of Equation (7), after $\mathcal{CT}$ is applied, only the term corresponding to $\sigma = id$ remains, yielding

$$\mathcal{CT}(\bar{F}(x_1,\dots,x_m;t)) = \sum_{\pi\in\Pi^{(m)}} t^{|\pi|}$$

because the other terms all contain a nonzero exponent for some x_i where $i\in[m]$. When we extract the coefficient of t^n from $\mathcal{CT}(\bar{F}(\mathbf{x};t))$, we get precisely the number of set partitions of size n without an $(m+1)$-nesting.

The task is now clear: We need to extract the coefficient of $x_1^0x_2^0\dots x_m^0$ from the RHS of Equation (7). Fortunately, since $\mathcal{CT}$ is a linear operator, we can examine the RHS of Equation (7) term by term, namely, by considering the three surviving orbit sums one at a time. We illustrate this process with an example using nonnesting set partitions.

4. 2-nonnesting set partitions

The generating function derivation in this section is for pedagogical purposes, to illustrate how to manipulate Equation (7). Indeed, there are easier ways to determine the generating function for the Catalan numbers.

To enumerate 2-nonnesting set partitions we set $m = 1$ in the above equations. In this case, $\bar{F}(0;t) = \sum_{\pi\in\Pi^{(1)}} t^{|\pi|}$ which we rewrite as $\bar{F}(0;t) = \sum F_n t^n$. Since $m = 1$, the associated symmetric group is $\mathfrak{S}_1$ which only contains the identity permutation; thus the functional equation is:

$$(8)\qquad \bar{F}(x_1;t) = \frac{x_1+1}{1-t\left(\frac{1}{x_1}+1\right)(x_1+1)} - \frac{t(x_1+1)}{1-t\left(\frac{1}{x_1}+1\right)(x_1+1)}\left(\frac{1}{x_1}+1\right)\bar{F}(0;t).$$

Though this is an easy case, writing out the action of $\mathcal{CT}$ shows us how terms are collected and coefficients computed in a rather slow way. First expand Equation (8) as power series to get

$$(9)\quad \bar{F}(x_1;t) = \sum_{n=0}^{\infty}(x_1+1)^{n+1}\left(\frac{1}{x_1}+1\right)^n t^n - \sum_{n=0}^{\infty}((x_1+1)t)^{n+1}\left(\frac{1}{x_1}+1\right)^{n+1} \times \sum_{n=0}^{\infty} F_n t^n$$

Now apply $\mathcal{CT}$ to get

$$\sum_{n=0}^{\infty} F_n t^n = \mathcal{CT}\bar{F}(x_1;t) = \sum_n \sum_{j=0}^{n} \binom{n+1}{j+1}\binom{n}{j} t^n - \sum_n \sum_{j=0}^{n} \binom{n+1}{j+1}^2 t^{n+1} \times \sum_{n=0}^{\infty} F_n t^n$$
$$= \sum_n t^n \left(\sum_{j=0}^{n} \binom{n+1}{j+1}\binom{n}{j} - \sum_{n^*=0}^{n-1} \Big(\sum_{j=0}^{n^*+1} \binom{n^*+1}{j}^2 \Big) F_{n-n^*-1} \right).$$

We deduce a recurrence for F_n after simplifying the binomial summations:

$$F_n = \frac{1}{2}\binom{2n+2}{n+1} - \sum_{j=0}^{n-1} \binom{2j+2}{j+1} F_{n-1-j}.$$

When all F_k's are collected to the left, we get

$$\sum_{k=0}^{n} \binom{2k}{k} F_{n-k} = \frac{1}{2}\binom{2n+2}{n+1}.$$

Upon noticing that the left hand side is a convolution product, we define

$$f(x) = \sum_0^{\infty} F_k x^k, \qquad \text{and} \qquad g(x) = \sum_0^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

to obtain

$$f(x)g(x) = \frac{1}{2x}(g(x)-1), \qquad \text{or} \qquad f(x) = \frac{1}{2x}(1-\sqrt{1-4x}),$$

the famous Catalan series as expected.

5. 3-NONNESTING SET PARTITIONS

The first non-trivial case is 3-nonnesting set partitions to study how the orbit sum produces sums of products of multinomial coefficients. Notice that the previous example with $m = 2$, no explicit formula for F_n was used; instead, it the F_n's was defined in terms of all previous F_j for all $j \leq n$. However, the convolution product allowed a successful isolation of the generating function $f(x)$. For this reason, it is our opinion that the study of the structure of convolution-like product for $m = 3$ and beyond may shed light in the nature of the generating series. As in the previous example, through the investigating of the action of $\mathcal{CT}$ on the RHS of Equation (7), we show how the conditions of summation indices turn out to reduce to a simple equation, thus restricting the degree of freedom. This exercise, though tedious when carried to the next case, $m = 3$, lends evidence to the conjecture by Bousquet-Mélou and Xin in [4] that the generating function of the 4-nonnesting case is not D-finite. Furthermore, we get enumerative formulas as functions of the label, and an understanding of the structure of the generating functions.

5.1. **First term of RHS of Equation** (7)**.** The first extraction is resolved by a simple combinatorial argument on the total way to combine the exponents to get a constant:

$$
\begin{aligned}
&\mathcal{CT}\Big(\frac{x_1+x_2+1}{1-ths}\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\frac{\sigma(x_1x_2^2)}{x_1x_2^2}\Big)\\
=&\mathcal{CT}\Bigg(\Big(\sum_{n=0}^{\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+1\right)^n(x_1+x_2+1)^{n+1}\,t^n\Big)(1-\frac{x_1}{x_2})\Bigg)\\
=&\sum_{n=0}^{\infty}\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n}}\binom{n}{l_1,l_2,l_3}\binom{n+1}{l_1,l_2,l_3+1}t^n\\
&\qquad-\sum_{n=0}^{\infty}\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n}}\binom{n}{l_1,l_2,l_3}\binom{n+1}{l_1-1,l_2+1,l_3+1}t^n\\
=&\sum_{n=0}^{\infty}t^n\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n}}\binom{n}{l_1,l_2,l_3}\left(1-\frac{l_1}{l_2+1}\right).
\end{aligned}
$$

5.2. **Second term of RHS of Equation** (7)**.** The remaining terms are expressed in terms of

$$
\bar{F}(0,x_2;t)=\sum_{\pi\in\Pi^{(2)}}(x_2+1)^{a_2}t^{|\pi|}=\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}F_n(k)(1+x_2)^k t^n,
$$

where $F_n(k)$ is the number of partitions of $[n]$ in $\Pi^{(2)}$ such that $a_2(\pi)=k$. Remark, we have the relation $F_{n+1}=\sum_{k=1}^{n+1}F_n(k)k$ by the comment that each partition π has $a_2(\pi)$ children. Under the action of $\mathfrak{S}_2$, the orbit sum has two terms, one from the identity and one from interchanging x_1 and x_2:

$$
\begin{aligned}
(10)\quad &\frac{(x_1+x_2+1)t}{1-ths}\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\frac{\sigma\left(x_2^2\bar{F}(0,x_2;t)\right)}{x_1x_2^2}\\
&=\ \Big(\sum_{n=0}^{\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+1\right)^n((x_1+x_2+1)t)^{n+1}\Big)\ \times\ \Big(\frac{1}{x_1}\bar{F}(0,x_2;t)\ -\ \frac{x_1}{x_2^2}\bar{F}(0,x_1;t)\Big).
\end{aligned}
$$

We use the linearity of the operator, and consider this expression in two steps. First, We expand this and apply $\mathcal{CT}$ to the first term, using the definition of $\bar{F}$, to get

$$
\begin{aligned}
(11)\quad & \mathcal{CT}\Big(\sum_{n=0}^{\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+1\right)^n ((x_1+x_2+1)t)^{n+1}\,\frac{1}{x_1}\bar{F}(0,x_2;t)\Big) \\
&= \mathcal{CT}\Big(\sum_{n=0}^{\infty}\sum_{\substack{0\le l_1,l_2,l_3,\\ l_1+l_2+l_3=n}}\sum_{\substack{0\le j_2,j_3\\ j_2+j_3=n-l_1}}\binom{n}{l_1,l_2,l_3}\binom{n+1}{l_1+1,j_2,j_3}x_2^{j_2-l_2}t^{n+1} \\
&\qquad\qquad \times\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}F_n(k)\sum_{i=0}^{k}\binom{k}{i}x_2^i t^n\Big) \\
&= \sum_{n=0}^{\infty}t^n\sum_{n^*\le n-1}\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n-n^*-1}}\sum_{\substack{0\le j_2,j_3\\ j_2+j_3=n-n^*-1-l_1}}\binom{n-n^*-1}{l_1,l_2,l_3}\binom{n-n^*}{l_1+1,j_2,j_3} \\
&\qquad\qquad \times\sum_{k\le n^*+1}F_{n^*}(k)\binom{k}{l_2-j_2}.
\end{aligned}
$$

Remark, to extract the constant coefficient with respect to x_2, we impose $j_2-l_2=-i$ on the inner most summation.

A similar expression is obtained from the second term:

$$
\begin{aligned}
(12)\quad & \mathcal{CT}\Big(\sum_{n=0}^{\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+1\right)^n ((x_1+x_2+1)t)^{n+1}\,\frac{x_1}{x_2}\bar{F}(0,x_1;t)\Big) \\
&= \mathcal{CT}\Big(\sum_{n=0}^{\infty}\sum_{\substack{0\le l_1,l_2,l_3,\\ l_1+l_2+l_3=n}}\sum_{\substack{0\le j_1,j_3\le n,\\ j_1+j_3=n-1-l_2}}\binom{n}{l_1,l_2,l_3}\binom{n+1}{j_1,l_2+2,j_3}x_1^{j_1-l_1+1}t^{n+1} \\
&\qquad\qquad \times\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}F_n(k)\sum_{i=0}^{k}\binom{k}{i}x_1^i t^n\Big) \\
&= \sum_{n=0}^{\infty}t^n\sum_{n^*\le n-1}\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n-n^*-1}}\sum_{\substack{0\le j_1,j_3\\ j_1+j_3=n-n^*-2-l_2}}\binom{n-n^*-1}{l_1,l_2,l_3}\binom{n-n^*}{j_1,l_2+2,j_3} \\
&\qquad\qquad \times\sum_{k\le n^*+1}F_{n^*}(k)\binom{k}{l_1-j_1-1}.
\end{aligned}
$$

where similar conditions as above also apply to surviving terms, namely: $-i=j_1-l_1+1$.

5.3. **Third term of RHS of Equation** (7)**.** Finally, the action of $\mathfrak{S}_2$ on the third term of RHS of Equation (7) yields two terms as in the previous case, and the analysis is almost identical. The third term is:

$$\frac{(x_1+x_2+1)t}{1-ths}\sum_{\sigma\in\mathfrak{S}_2}\frac{\epsilon(\sigma)}{x_1x_2^2}\sigma\left(\frac{x_1x_2^2}{x_2+1}\bar{F}(0,x_2;t)\right)$$
$$=\left(\sum_{n=0}^{\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+1\right)^n((x_1+x_2+1)t)^{n+1}\right)\times\left(\frac{\bar{F}(0,x_2;t)}{x_2+1}-\frac{x_1}{x_2}\frac{\bar{F}(0,x_1;t)}{x_1+1}\right)$$

We take a closer look at the second part:

$$\frac{\bar{F}(0,x_2;t)}{x_2+1}-\frac{x_1}{x_2}\frac{\bar{F}(0,x_1;t)}{x_1+1}=\sum_n\sum_{k=1}^{n+1}F_n(k)t^n\left((x_2+1)^{k-1}-\frac{x_1}{x_2}(x_1+1)^{k-1}\right).$$

Applying $\mathcal{CT}$ to the entire expression yields

$$(13)\quad \sum_n t^n\sum_{n^*\le n-1}\sum_{k\le n^*+1}F_{n^*}(k)\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n-n^*-1}}\binom{n-n^*-1}{l_1,l_2,l_3}$$
$$\left(\sum_{\substack{0\le j_2,j_3\\ j_2+j_3=n-n^*-l_1}}\binom{n-n^*}{l_1,j_2,j_3}\binom{k-1}{l_2-j_2}-\sum_{\substack{0\le j_1,j_3\\ j_1+j_3=n-n^*-l_2-1}}\binom{n-n^*}{j_1,l_2+1,j_3}\binom{k-1}{l_1-j_1-1}\right).$$

5.4. **A complete expression for F_n.** We can put the three components together into one expression for the coefficient of F_n in terms of $F_{n^*}(k)$, a function of the label where $n^*<n$:

$$(14)\quad F_n=\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n}}\binom{n}{l_1,l_2,l_3}\left(1-\frac{l_1}{l_2+1}\right)$$
$$-\sum_{n^*=0}^{n-1}\sum_{k\le n^*+1}F_{n^*}(k)\sum_{\substack{0\le l_1,l_2,l_3\\ l_1+l_2+l_3=n-n^*-1}}\binom{n-n^*-1}{l_1,l_2,l_3}$$
$$\Bigg(\sum_{\substack{0\le j_2,j_3\\ j_2+j_3=n-n^*-1-l_1}}\binom{n-n^*}{l_1+1,j_2,j_3}\binom{k}{l_2-j_2}-\sum_{\substack{0\le j_1,j_3\\ j_1+j_3=n-l_2-2}}\binom{n-n^*}{j_1,l_2+2,j_3}\binom{k}{l_1-j_1-1}$$
$$+\sum_{\substack{0\le j_2,j_3\\ j_2+j_3=n-n^*-l_1}}\binom{n-n^*}{l_1,j_2,j_3}\binom{k-1}{l_2-j_2}-\sum_{\substack{0\le j_1,j_3\\ j_1+j_3=n-n^*-l_2-1}}\binom{n-n^*}{j_1,l_2+1,j_3}\binom{k-1}{l_1-j_1-1}\Bigg).$$

Note how F_n is expressed as a convolution-like sum involving all previous F_{n^*} for $n^*<n$. In this form, the authors are unable to obtain a recurrence for the F_n's.

6. Complexity of $m\ge 3$

These examples give us a strong flavour of the general formula. The first term is always the constant term of a rational function, and hence is always D-finite. There are some sources for added complexity when m is greater than 2. First, the number of terms in the orbit sum grows like $m!$, although one can expect them to be of a similar form, as was the case in the $m=2$ case. The number of parameters that play a role in the formulas is perhaps the key difference. In the $m = 1$ case, we eliminate dependence on the parameter, and determine direct recurrences. In the $m = 2$ case, we use the parameter a_2, but we also have the additional property that the sum of this parameter over all 3-nonnesting partitions of size n is the number of 3-nonnesting partitions of size $n+1$. Thus, there is an additional relation.

We avoid the full treatment of the $m = 3$ case, and rather go directly to the typical effect of $\mathcal{CT}$ on the orbit sum to illustrate how this expression is increasingly complex. The functional equation in this case is

$$\begin{aligned}(15)\quad & \frac{x_1+x_2+x_3+1}{1-ths}\sum_{\sigma\in\mathfrak{S}_3}\epsilon(\sigma)\frac{\sigma(x_1x_2^2x_3^3)}{x_1x_2^2x_3^3} \\ &\qquad -\frac{x_1+x_2+x_3+1}{1-ths}t\sum_{\sigma\in\mathfrak{S}_3}\frac{\epsilon(\sigma)}{x_1x_2^2x_3^3}\sigma\left(x_2^2x_3^3\bar{F}(0,x_2,x_3;t)\right) \\ &\qquad\qquad -\frac{x_1+x_2+x_3+1}{1-ths}t\sum_{\sigma\in\mathfrak{S}_3}\frac{\epsilon(\sigma)}{x_1x_2^2x_3^3}\sigma\left(\frac{x_2^2x_3^3\bar{F}(0,x_2,x_3;t)}{x_2+x_3+1}\right).\end{aligned}$$

Applying $\mathcal{CT}$ to the first term yields a sum of six multinomial summations that simplifies to an expression of the form

$$\sum_{k=0}^{\infty}\sum_{\substack{0\le l_1,l_2,l_3,l_4\le k,\\ l_1+l_2+l_3+l_4=k}}\binom{k}{l_1,l_2,l_3,l_4}R(l_1,l_2,l_3,l_4),$$

where R is a simple rational function, and, as we noted earlier this expression is D-finite, since it is a coefficient extraction of a rational function.

The second and third terms involve

$$\bar{F}(0,x_2,x_3;t)=\sum_{\pi\in\Pi^{(3)}}(x_2+x_3+1)^{a_2(\pi)}(x_3+1)^{a_3(\pi)-a_2(\pi)}t^{|\pi|}.$$

The result is that when $\mathcal{L}$ is applied to the second and third terms, we get the nested summations involving complex expressions of $a_2(\pi)$ and $a_3(\pi)$. The following is a typical sample expression for the coefficient of t^n:

$$\begin{aligned}(16)\quad & \sum_{k\le n-1}\sum_{\substack{0\le l_1,l_2,l_3,l_4\le k,\\ l_1+l_2+l_3+l_4=k}}\binom{k}{l_1,l_2,l_3,l_4}\sum_{\substack{0\le j_2,j_3,j_4\le k\\ l_1+j_2+j_3+j_4=k}}\binom{k+1}{l_1+1,j_2,j_3,j_4} \\ &\qquad\left(\sum_{\substack{\pi\in\Pi^{(3)}\\ |\pi|=n-k-1}}\sum_{p}\binom{a_2(\pi)}{l_2-j_2,p,a_2(\pi)-p-l_2+j_2}\times\binom{a_3(\pi)-a_2(\pi)}{l_3-p-j_3}\right).\end{aligned}$$

		n														
m	OEIS #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	A000108	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440	9694845
2	A108304	1	2	5	15	52	202	859	3930	19095	97566	520257	2877834	16434105	96505490	580864901
3	A108305	1	2	5	15	52	203	877	4139	21119	115495	671969	4132936	26723063	180775027	1274056792
4	A192126	1	2	5	15	52	203	877	4140	21147	115974	678530	4212654	27627153	190624976	1378972826
5	A192127	1	2	5	15	52	203	877	4140	21147	115975	678570	4213596	27644383	190897649	1382919174
6	A192128	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597	27644437	190899321	1382958475

TABLE 1. Numbers of set partitions of n avoiding an $m+1$-nesting. The OEIS numbers refer to entries in the Online Encyclopedia of Integer Sequences [1]

7. Computing series expansions

All three Equations (1), (3), and (4) are used to generate initial terms in the series. To improve convergence, we slightly modify the x equation:

$$\begin{aligned}\bar{F}(x_1, x_2, \ldots, x_m; t) &= s + sth\bar{F}(x_1, x_2, \ldots, x_m; t)\\ &\quad -st\Big(\frac{s}{s-x_1}\frac{\bar{F}(0, x_2, x_3, \ldots x_m); t)}{x_1}\\ &\qquad + \sum_{j=2}^{m} \frac{\bar{F}(x_1, \ldots, x_{j-2}, x_{j-1}+x_j, 0, x_{j+1}, \ldots, x_m; t)}{x_j}\Big),\end{aligned} \tag{17}$$

Notice that in Equation (17), if one has a series expansion of $\bar{F}(\mathbf{x}; t)$ correct up to t^k, then substituting this series into RHS of Equation (17) yields the series expansion of $\bar{F}$ correct to t^{k+1} because the RHS of Equation (17) contains a term free of t, otherwise, the degree of t is increased by 1. We have thus iterated Equation (17) to get enumerative data for up to $m = 9$.

For 3-nonnesting set partitions, an average laptop running Maple15 can produce 70 terms in a reasonable time (less than 24 hours). For $m = 4$, only 38 terms; $m = 5$, 27 terms; $m = 6$, 20 terms; $m = 7$, 16 terms, $m = 8$, 12 terms; and finally $m = 9$, 12 terms. The limitation seems memory space due to the growing complication in the functional equation when m gets larger.

8. Conclusion

Without passing through vacillating lattice walks or tableaux, the generating tree approach permits a direct translation to a functional equation involving an arbitrary number of catalytic variables satisfied by set partitions avoiding $m+1$-nestings for any m. Constant term coefficient extraction analysis gives us insight into why the number of 3-nonnesting set partitions should be more easily controlled than those of higher non-nesting set partitions. The authors are aware of the techniques developed for constant term extraction and are investigating how such techniques can give insight to the analysis of m-nonnesting numbers of set partitions. Though explicit generating trees are given, formulas thus generated still depend on labels of set partitions. Perhaps further study into the nature of generating trees which give rise to D-finite series, along the lines of the study in [2] will help us understand the differences.

A second way that might yield a proof of non-D-finiteness would be to use our expressions to determine bounds on the order and the coefficient degrees of the minimal differential equation satisfied by the generating function. Though a tantalizingly simple idea, the limitation seems still the lack of data when larger m's give so few values relative to the number one would need to test non-D-finiteness. Nevertheless, this would guide searches and a fruitless search would then be a definitive result.

Finally, our generating tree approach is limited only to the non-enhanced case. For a more general treatment of the subject involving enhanced set partitions and permutations, both enhanced and non-enhanced, we refer the reader to [5] by Burrill, Elizalde, Mishna, and Yen.

9. Acknowledgements

We are grateful to an anonymous referee for many constructive suggestions and to Mireille Bousquet-Mélou for her suggestions, Mogens Lemvig Hansen for his tireless generation of numbers with Maple. The first author is partially supported by an Natural Sciences and Engineering Research Council of Canada Discovery Grant.

References

[1] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*, (2011), published electronically at http://oeis.org.

[2] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps, *Generating functions for generating trees*, Discrete Math. (2002), 29–55.

[3] Mireille Bousquet-Mélou, *Counting permutations with no long monotone subsequence via generating trees*, J. Alg. Combin. 33 (2011), no. 4, 571–608.

[4] Mireille Bousquet-Mélou and Guoce Xin, *On partitions avoiding 3-crossings*, Séminaire Lotharingien de Combinatoire (2006), 1–21.

[5] Sophie Burrill, Sergi Elizalde, Marni Mishna, and Lily Yen, *A generating tree approach to k non-nesting partitions and permutations*, *Preprint*, (2011). Available at `arXiv:1108.5615`.

[6] William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Du, Richard P. Stanley, and Catherine H. Yan, *Crossings and nestings of matchings and partitions*, Trans. Amer. Math. Soc. (2007), 1555–1575.

[7] William Y. C. Chen, Hillary S. W. Han, and Christian M. Reidys, *Random k-noncrossing RNA structures*, Proc. Natl. Acad. Sci. USA 106 (2009), no. 52, 2206122066.

[8] Christian Krattenthaler, *Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes*, Adv. Appl. Math. 37 (2006), 404-431.

Marni Mishna, Dept. Mathematics, Simon Fraser University, Burnaby, BC, Canada

Lily Yen, Dept. Mathematics and Statistics, Capilano University, North Vancouver, and Dept. Mathematics, Simon Fraser University, Burnaby, BC, Canada