# Enumeration of (0, 1)-matrices avoiding some 2 × 2 matrices

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**Abstract**

We enumerate the number of $(0,1)$-matrices avoiding $2\times 2$ submatrices satisfying certain conditions. We also provide corresponding exponential generating functions.

## 1 Introduction

Let $M(k,n)$ be the set of $k \times n$ matrices with entries 0 and 1. It is obvious that the number of elements in the set $M(k,n)$ is $2^{kn}$. It would be interesting to consider the number of elements in $M(k,n)$ with certain conditions. For example, how many matrices of $M(k,n)$ do not have $2 \times 2$ submatrices of the forms $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)$? In this article we will give answers to the previous question and other questions.

Consider $M(2,2)$, the set of all possible $2 \times 2$ submatrices. For two elements $P$ and $Q$ in $M(2,2)$, we denote $P \sim Q$ if $Q$ can be obtained from $P$ by row or column exchanges. It is obvious that $\sim$ is an equivalence relation on $M(2,2)$. With this equivalence relation, $M(2,2)$ is partitioned with seven equivalent classes having the following seven representatives.

$$I = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \Gamma = \begin{pmatrix}1&1\\1&0\end{pmatrix}, \quad C = \begin{pmatrix}1&0\\0&0\end{pmatrix},$$
$$T = \begin{pmatrix}1&1\\0&0\end{pmatrix}, \quad L = \begin{pmatrix}1&0\\1&0\end{pmatrix}, \quad J = \begin{pmatrix}1&1\\1&1\end{pmatrix}, \quad O = \begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

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Here $C$, $T$, and $L$ mean "corner", "top", and "left", respectively. Let $\mathcal{S}$ be the set of these representatives, i.e.,

$$\mathcal{S} := \{I,\, \Gamma,\, C,\, T,\, L,\, J,\, O\}.$$

Given an element $S$ in $\mathcal{S}$, a matrix $A$ is an element of the set $M(S)$ if and only if for *every* permutation $\pi_1$ of the rows and $\pi_2$ of the columns, the resulting matrix does not have the submatrix $S$. Equivalently, $A \in M(S)$ means that $A$ has no submatrices in the equivalent class $[S]$. For a subset $\alpha$ of $\mathcal{S}$, $M(\alpha)$ is defined by the set $\cap_{S\in\alpha} M(S)$. Note that the definition of $M(S)$ (also $M(\alpha)$) is different from that in [13, 17]. If $A$ belongs to $M(\alpha)$, then we say that $A$ *avoids* $\alpha$. We let $\phi(k, n; \alpha)$ be the number of $k \times n$ $(0, 1)$-matrices in $M(\alpha)$.

Our goal is to express $\phi(k, n; \alpha)$ in terms of $k$ and $n$ explicitly for each subset $\alpha$ of the set $\mathcal{S}$. For $|\alpha| = 1$, We can easily notice that $\phi(k, n; \Gamma) = \phi(k, n; C)$ and $\phi(k, n; J) = \phi(k, n; O)$ by swapping 0 and 1. We also notice that $\phi(k, n; T) = \phi(n, k; L)$ by transposing the matrices. The number $\phi(k, n; I)$ is well known (see [2, 7, 8]) and $(0, 1)$-matrices avoiding type $I$ are called $(0, 1)$-lonesum matrices (we will define and discuss this in 2.2). In fact, lonesum matrices are the primary motivation of this article and its corresponding work. The study of $M(J)$ (equivalently $M(O)$) appeared in [9, 13, 17], but finding a closed form of $\phi(k, n; J) = \phi(k, n; O)$ is still open. The notion of "$\Gamma$-free matrix" was introduced by Spinrad [16]. He dealt with a totally balanced matrix which has a permutation of the rows and coluums that are $\Gamma$-free. We remark that the set of totally balanced matrices is different from $M(\Gamma)$.

In this paper we calculate $\phi(k, n; \alpha)$, where $\alpha$'s are $\{\Gamma\}$ (equivalently $\{C\}$) and $\{T\}$ (equivalently $\{L\}$). We also enumerate $M(\alpha)$ where $\alpha$'s are $\{\Gamma, C\}$, $\{T, L\}$, and $\{J, O\}$. For the other subsets of $\mathcal{S}$, we discuss them briefly in the last section. Note that some of our result (subsection 3.5) is an independent derivation of some of the results in [11, section 3] by Kitaev et al.; for other relevant papers see [10, 12].

# 2 Preliminaries

## 2.1 Definitions and Notations

A matrix $P$ is called $(0, 1)$-matrix if all the entries of $P$ are 0 or 1. From now on we will consider $(0, 1)$-matrices only, so we will omit "$(0, 1)$" if it causes no confusion. Let $M(k, n)$ be the set of $k \times n$-matrices. Clearly, if $k, n \geq 1$, $M(k, n)$ has $2^{kn}$ elements. For convention we assume that $M(0, 0) = \{\emptyset\}$ and $M(k, 0) = M(0, n) = \emptyset$ for positive integers $k$ and $n$.

Given a matrix $P$, a submatrix of $P$ is formed by selecting certain rows and columns from $P$. For example, if $P = \left(\begin{smallmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{smallmatrix}\right)$, then $P(2, 3; 2, 4) = \left(\begin{smallmatrix} f & h \\ j & l \end{smallmatrix}\right)$.

Given two matrices $P$ and $Q$, we say $P$ contains $Q$, whenever $Q$ is equal to a submatrix of $P$. Otherwise say $P$ avoids $Q$. For example, $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ contains $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ but avoids $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For a matrix $P$ and a set $\alpha$ of matrices, we say that $P$ avoids the type set $\alpha$ if $P$ avoids all the matrices in $\alpha$. If it causes no confusion we will simply say that $P$ avoids $\alpha$.

Given a set $\alpha$ of matrices, let $\phi(k,n;\alpha)$ be the number of $k \times n$ matrices avoiding $\alpha$. From the definition of $M(k,n)$, for any set $\alpha$, we have $\phi(0,0;\alpha) = 1$ and $\phi(k,0;\alpha) = \phi(0,n;\alpha) = 0$ for positive integers $k$ and $n$. Let $\Phi(x,y;\alpha)$ be the bivariate exponential generating function for $\phi(k,n;\alpha)$, i.e.,

$$\Phi(x,y;\alpha) := \sum_{n\geq 0}\sum_{k\geq 0} \phi(k,n;\alpha)\,\frac{x^k}{k!}\frac{y^n}{n!} = 1 + \sum_{n\geq 1}\sum_{k\geq 1} \phi(k,n;\alpha)\,\frac{x^k}{k!}\frac{y^n}{n!}\,.$$

Let $\Phi(z;\alpha)$ be the exponential generating function for $\phi(n,n;\alpha)$, i.e.,

$$\Phi(z;\alpha) := \sum_{n\geq 0} \phi(n,n;\alpha)\,\frac{z^n}{n!}\,.$$

Given $f, g \in \mathbb{C}[[x,y]]$, we denote $f \overset{2}{=} g$ if the coefficients of $x^k y^n$ in $f$ and $g$ are the same, for each $k, n \geq 2$.

## 2.2 $I$-avoiding matrices (Lonesum matrices)

This is related to the lonesum matrices. A lonesum matrix is a $(0,1)$-matrix determined uniquely by its column-sum and row-sum vectors. For example, $\left(\begin{smallmatrix}1&0&1\\0&0&1\\1&0&1\end{smallmatrix}\right)$ is a lonesum matrix since it is a unique matrix determined by the column-sum vector $(2,0,3)$ and the row-sum vector $(2,1,2)^{\mathrm{t}}$. However $\left(\begin{smallmatrix}1&0&1\\0&0&1\\1&0&0\end{smallmatrix}\right)$ is not, since $\left(\begin{smallmatrix}1&0&1\\1&0&0\\0&0&1\end{smallmatrix}\right)$ has the same column-sum vector $(2,0,2)$ and row-sum vector $(2,1,1)^{\mathrm{t}}$.

**Theorem 2.1** (Brewbaker [2])**.** *A matrix is a lonesum matrix if and only if it avoids $I$.*

Theorem 2.1 implies that $\phi(k,n;I)$ is equal to the number of $k \times n$ lonesum matrices.

**Definition 2.2.** *Bernoulli number $B_n$ is defined as following:*

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x\,e^x}{e^x - 1}.$$

Note that $B_n$ can be written explicitly as

$$B_n = \sum_{m=0}^{n} (-1)^{m+n}\,\frac{m!\,S(n,m)}{m+1},$$

where $S(n,m)$ is the Stirling number of the second kind. The poly-Bernoulli number, introduced first by Kaneko [7], is defined as

$$\sum_{n\geq 0} B_n^{(k)}\frac{x^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}},$$

where the polylogarithm $\mathrm{Li}_k(x)$ is the function $\mathrm{Li}_k(x) := \sum_{m\geq 1} \frac{x^m}{m^k}$. Bernoulli numbers are nothing but poly-Bernoulli numbers with $k = 1$. Sanchez-Peregrino [15] proved that $B_n^{(-k)}$ has the following simple expression:

$$B_n^{(-k)} = \sum_{m=0}^{\min(k,n)} (m!)^2\, S(n+1, m+1)\, S(k+1, m+1)\,.$$

Brewbaker [2] and Kim et. al. [8] proved that the number of $k \times n$ lonesum matrices is the poly-Bernoulli number $B_n^{(-k)}$, which yields the following result.

**Proposition 2.3** (Brewbaker [2]; Kim, Krotov, Lee [8])**.** *The number of $k \times n$ matrices avoiding $I$ is equal to $B_n^{(-k)}$, i.e.,*

$$\phi(k, n; I) = \sum_{m=0}^{\min(k,n)} (m!)^2\, S(n+1, m+1)\, S(k+1, m+1)\,. \tag{1}$$

The generating function $\Phi(x, y; I)$, given by Kaneko [7], is

$$\Phi(x, y; I) = e^{x+y} \sum_{m\geq 0} [(e^x - 1)(e^y - 1)]^m = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \tag{2}$$

Also, $\Phi(z; I)$ can be easily obtained as follows:

$$\begin{aligned}\Phi(z; I) = \sum_{n\geq 0} \phi(n, n; I)\, \frac{z^n}{n!} &= \sum_{n\geq 0}\sum_{m\geq 0} (-1)^{m+n} m! S(n, m)(m+1)^n\, \frac{z^n}{n!} \\ &= \sum_{m\geq 0} (-1)^m m! \sum_{n\geq 0} S(n, m)\, \frac{(-(m+1)z)^n}{n!} \\ &= \sum_{m\geq 0} (1 - e^{-(m+1)z})^m\,. \end{aligned} \tag{3}$$

# 3 Main Results

## 3.1 Γ-avoiding matrices (or $C$)

By row exchange and column exchange we can change the original matrix into a block matrix as in Figure 1. Here [**0**] (resp. [**1**]) stands for a 0-block (resp.1-block) and [**0***] stands for a 0-block or an empty block. Diagonal blocks are [**1**]'s except for the last block [**0***], and the off-diagonal blocks are [**0**]'s.

**Theorem 3.1.** *The number of $k \times n$ matrices avoiding $\Gamma$ is given by*

$$\phi(k, n; \Gamma) = \sum_{m=0}^{\min(k,n)} m!\, S(n+1, m+1)\, S(k+1, m+1). \tag{4}$$

| | | | |
|---|---|---|---|
| [1] | [0] | [0] | [0*] |
| [0] | [1] | [0] | [0*] |
| [0] | [0] | [1] | [0*] |
| [0*] | [0*] | [0*] | [0*] |

Figure 1: A matrix avoiding $\Gamma$ can be changed into a block diagonal matrix.

*Proof.* Let $\mu = \{C_1, C_2, \ldots, C_{m+1}\}$ be a set partition of $[n+1]$ into $m+1$ blocks. Here the block $C_l$'s are ordered by the largest element of each block. Thus $n+1$ is contained in $C_{m+1}$. Likewise, let $\nu = \{D_1, D_2, \ldots, D_{m+1}\}$ be a set partition of $[k+1]$ into $m+1$ blocks. Choose $\sigma \in S_{m+1}$ with $\sigma(m+1) = m+1$, where $S_{m+1}$ is the set of all permutations of length $m+1$. Given $(\mu, \nu, \sigma)$ we define a $k \times n$ matrix $M = (a_{i,j})$ by

$$a_{i,j} := \begin{cases} 1, & (i,j) \in C_l \times D_{\sigma(l)} \text{ for some } l \in [m] \\ 0, & \text{otherwise} \end{cases}.$$

It is obvious that the matrix $M$ avoids the type $\Gamma$.

Conversely, let $M$ be a $k \times n$ matrix avoiding type $\Gamma$. Set $(k+1) \times (n+1)$ matrix $\widetilde{M}$ by augmenting zeros to the last row and column of $M$. By row exchange and column exchange we can change $\widetilde{M}$ into a block diagonal matrix $B$, where each diagonal is 1-block except for the last diagonal. By tracing the position of columns (resp. rows) in $\widetilde{M}$, $B$ gives a set partition of $[n+1]$ (resp. $[k+1]$). Let $\{C_1, C_2, \ldots, C_{m+1}\}$ (resp. $\{D_1, D_2, \ldots, D_{m+1}\}$) be the set partition of $[n+1]$ (resp. $[k+1]$). Note that the block $C_i$'s and $D_i$'s are ordered by the largest element of each block. Let $\sigma$ be a permutation on $[m]$ defined by $\sigma(i) = j$ if $C_i$ and $D_j$ form a 1-block in $B$.

The number of set partitions $\pi$ of $[n+1]$ is $S(n+1, m+1)$, and the number of set partitions $\pi'$ of $[k+1]$ is $S(n+1, k+1)$. The cardinality of the set of $\sigma$'s is the cardinality of $S_m$, i.e., $m!$. Since the number of blocks $m+1$ runs through 1 to $\min(k,n)+1$, the sum of $S(k+1, m+1)\, S(n+1, m+1)\, m!$ gives the required formula. □

*Example* 1. Let $\mu = 4/135/26/7$ be a set partition of $[7]$ and $\nu = 25/6/378/149$ of $[9]$ into 4 blocks. Let $\sigma = 3124$ be a permutation in $S_4$ such that $\sigma(4) = 4$. From $(\mu, \nu, \sigma)$ we can construct the $6 \times 8$ matrix $M$ which avoids type $\Gamma$ as in Figure 2.

To find the generating function for $\phi(k, n; \Gamma)$ the following formula (see [5]) is helpful.

$$\sum_{n \geq 0} S(n+1, m+1) \frac{x^n}{n!} = e^x \frac{(e^x - 1)^m}{m!}. \tag{5}$$

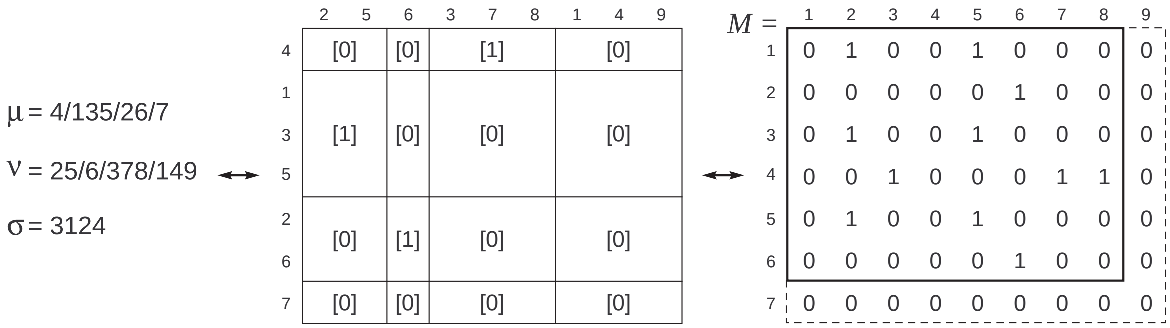


Figure 2: A matrix avoiding Γ corresponds to two set partitions with a permutation.

From Theorem 3.1 and (5), we can express $\Phi(x,y;\Gamma)$ as follows:

$$\begin{aligned}\Phi(x,y;\Gamma) &= \sum_{n,k\geq 0}\sum_{m\geq 0} m!\,S(n+1,m+1)\,S(k+1,m+1)\,\frac{x^k}{k!}\frac{y^n}{n!}\\ &= \sum_{m\geq 0}\frac{1}{m!}\sum_{k\geq 0} m!\,S(k+1,m+1)\frac{x^k}{k!}\sum_{n\geq 0} m!\,S(n+1,m+1)\,\frac{y^n}{n!}\\ &= \sum_{m\geq 0}\frac{1}{m!}\,e^x(e^x-1)^m\;e^y(e^y-1)^m\\ &= \exp[(e^x-1)(e^y-1)+x+y]\,. \qquad (6)\end{aligned}$$

*Remark* 1. It seems to be difficult to find a simple expression of $\Phi(z;\Gamma)$. The sequence $\phi(n,n;\Gamma)$ is not listed in the OEIS [14]. The first few terms of $\phi(n,n;\Gamma)$ $(0\leq n\leq 9)$ are as follows:

$$1, 2, 12, 128, 2100, 48032, 1444212, 54763088, 2540607060, 140893490432, \ldots$$

## 3.2 $\{\Gamma, C\}$-avoiding matrices

Given the equivalence relation $\sim$ on $M(2,2)$, which is defined in Section 1, if we define the new equivalent relation $P \sim' Q$ by $P\sim Q$ or $P \sim \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) - Q$, then $[\Gamma]\cup[C]$ becomes a single equivalent class. Clearly $\phi(k,n;\{\Gamma,C\})$ is the number of $k\times n$ (0,1)-matrix which does not have a submatrix in $[\Gamma]\cup[C]$. From now on we simply write $\phi(k,n;\Gamma,C)$, instead of $\phi(k,n;\{\Gamma,C\})$. The reduced form of a matrix $M$ avoiding $\{\Gamma,C\}$ is very simple as in Figure 3. In this case if the first row and the first column of $M$ are determined then the rest of the entries of $M$ are determined uniquely. Hence the number $\phi(k,n;\Gamma,C)$ of such matrices is

$$\phi(k,n;\Gamma,C) = 2^{k+n-1} \qquad (k,n\geq 1), \qquad (7)$$

and its exponential generating function is

$$\Phi(x,y;\Gamma,C) = 1 + \frac{1}{2}\,(e^{2x}-1)(e^{2y}-1)\,. \qquad (8)$$

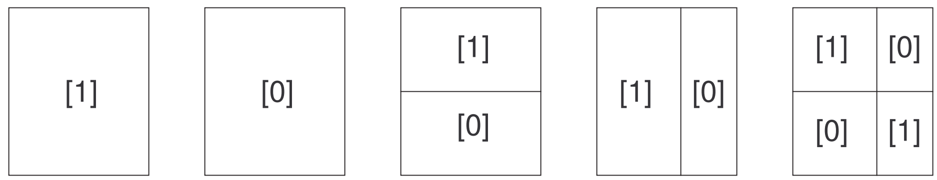


Figure 3: Possible reduced forms of matrices avoiding $\{\Gamma, C\}$

Clearly, $\phi(n, n; \Gamma, C) = 2^{2n-1}$ for $n \geq 1$. Thus its exponential generating function is

$$\Phi(z; \Gamma, C) = \frac{1}{2}\left(e^{4z} + 1\right). \tag{9}$$

## 3.3 $T$-avoiding matrices (or $L$)

Given a $(0,1)$-matrix, 1-column (resp. 0-column) is a column in which all entries consist of 1's (resp. 0's). We denote a 1-column (resp. 0-column) by $\mathbf{1}$ (resp. $\mathbf{0}$). A mixed column is a column which is neither $\mathbf{0}$ nor $\mathbf{1}$. For $k = 0$, we have $\phi(0, n; T) = \delta_{0,n}$. In case $k \geq 1$, i.e., there being at least one row, we can enumerate as follows:

- case 1: there are no mixed columns. Then each column should be $\mathbf{0}$ or $\mathbf{1}$. The number of such $k \times n$ matrices is $2^n$.

- case 2: there is one mixed column. In this case each column should be $\mathbf{0}$ or $\mathbf{1}$ except for one mixed column. The number of $k \times n$ matrices of this case is $2^{n-1}\, n\, (2^k - 2)$.

- case 3: there are two mixed columns. As in case 2, each column should be $\mathbf{0}$ or $\mathbf{1}$ except for two mixed columns, say, $v_1$ and $v_2$. The number of $k \times n$ matrices of this case is the sum of the following three subcases:
  - $v_1 + v_2 = \mathbf{1}$: $2^{n-2}\binom{n}{2}\, 2!\, S(k, 2)$
  - $v_1 + v_2$ has an entry $0$: $2^{n-2}\binom{n}{2}\, 3!\, S(k, 3)$
  - $v_1 + v_2$ has an entry $2$: $2^{n-2}\binom{n}{2}\, 3!\, S(k, 3)$

- case 4: there are $m$ ($m \geq 3$) mixed columns $v_1, \ldots, v_m$. The number of $k \times n$ matrices of this case is the sum of the following four subcases:
  - $v_1 + \cdots + v_m = \mathbf{1}$: $2^{n-m}\binom{n}{m}\, m!\, S(k, m)$
  - $v_1 + \cdots + v_m = (m-1)\mathbf{1}$: $2^{n-m}\binom{n}{m}\, m!\, S(k, m)$
  - $v_1 + \cdots + v_m$ has an entry $0$: $2^{n-m}\binom{n}{m}\, (m+1)!\, S(k, m+1)$
  - $v_1 + \cdots + v_m$ has an entry $m$: $2^{n-m}\binom{n}{m}\, (m+1)!\, S(k, m+1)$

Adding up all numbers in the previous cases yields the following theorem.

**Theorem 3.2.** *For $k$, $n \geq 1$ the number of $k \times n$ matrices avoiding $T$ is given by*

$$\phi(k, n; T) = 2 \sum_{l \geq 1} \binom{n}{l-1} l^k + (n^2 - n - 4)\, 2^{n-2} - n(n+3)\, 2^{n+k-3}. \qquad (10)$$

*Proof.*

$$\begin{aligned}
\phi(k,n;T) &= 2^n + 2^{n-1}\binom{n}{1}(2^k - 2) + 2^{n-2}\binom{n}{2}\,(\,2!\,S(k,2) + 3!\,2\,S(k,3)\,) \\
&\quad + \sum_{m=3}^{n} 2^{n-m+1}\binom{n}{m}\,(\,m!\,S(k,m) + (m+1)!\,S(k,m+1)\,) \\
&= 2\sum_{m=0}^{n} 2^{n-m}\binom{n}{m} m! S(k+1, m+1) + \; (n^2 - n - 4)\, 2^{n-2} - n(n+3)\, 2^{n+k-3} \\
&= 2\sum_{l \geq 1}\binom{n}{l-1} l^k + (n^2 - n - 4)\, 2^{n-2} - n(n+3)\, 2^{n+k-3}.
\end{aligned}$$

□

Note that in the proof of Theorem 3.2 we use the identity

$$\sum_{m \geq 0}\binom{n}{m} m!\, S(k,m)\, 2^{n-m} = \sum_{l \geq 0}\binom{n}{l} l^k\,,$$

where both sides count the number of functions $f$ from $[k]$ to $[n]$ such that each element of $[n] \setminus f([k])$ has two colors.

The generating function $\Phi(x, y; T)$ is given by

$$\begin{aligned}
\Phi(x,y;T) &= 1 + \sum_{n \geq 1}\sum_{k \geq 1} 2 \sum_{l \geq 1}\binom{n}{l-1} l^k \frac{x^k}{k!}\frac{y^n}{n!} \\
&\quad + \sum_{n \geq 1}\sum_{k \geq 1}(n^2 - n - 4)\, 2^{n-2}\frac{x^k}{k!}\frac{y^n}{n!} - \sum_{n \geq 1}\sum_{k \geq 1} n(n+3)\, 2^{n+k-3}\frac{x^k}{k!}\frac{y^n}{n!} \\
&= 1 + \left(2e^x(e^{y(e^x+1)} - 1) - 2e^{2y} + 2\right) \\
&\quad + (e^x - 1)\left((y^2 - 1)e^{2y} + 1\right) - \frac{1}{2}y(y+2)e^{2y}(e^{2x} - 1) \\
&= 2e^{y(e^x+1)+x} - \frac{y^2 + 2y}{2}e^{2x+2y} + (y^2 - 1)e^{x+2y} - e^x - \frac{y^2 - 2y + 2}{2}e^{2y} + 2\,. \qquad (11)
\end{aligned}$$

Note that if we use the symbol "$\overset{2}{=}$" introduced in Section 2.1, then

$$\Phi(x,y;T) \overset{2}{=} 2e^{y(e^x+1)+x} - \frac{y^2 + 2y}{2}e^{2x+2y} + (y^2 - 1)e^{x+2y}\,.$$

For the $n \times n$ matrices we have

$$\phi(n,n;T) = 2\sum_{l\geq 1} \binom{n}{l-1} l^n + (n^2-n-4)\,2^{n-2} - n(n+3)\,2^{2n-3}.$$

Thus the generating function $\Phi(n,n;T)$ is given by

$$\begin{aligned}
\sum_{n\geq 0} \phi(n,n;T)\,\frac{z^n}{n!} &= 2\sum_{n\geq 0}\sum_{l\geq 1}\binom{n+1}{l} l^{n+1}\,\frac{z^n}{(n+1)!} \\
&\quad + \sum_{n\geq 0} \frac{n^2-n-4}{4}\,\frac{(2z)^n}{n!} - \frac{n(n+3)}{8}\,\frac{(4z)^n}{n!} \\
&= \frac{2}{z}\sum_{l\geq 1}\frac{(lz)^l}{l!}\sum_{n\geq l-1}\frac{(lz)^{n-l+1}}{(n-l+1)!} + (z^2-1)e^{2z} - 2z(z+1)e^{4z} \\
&= \frac{2}{z}\sum_{l\geq 1}\frac{l^l}{l!}(ze^z)^l + (z^2-1)e^{2z} - 2z(z+1)e^{4z} \\
&= \frac{2}{z}\,(ze^z\,W'(-ze^z)) + (z^2-1)e^{2z} - 2z(z+1)e^{4z} \\
&= \frac{-2\,W(-ze^z)}{z+z\,W(-ze^z)} + (z^2-1)e^{2z} - 2z(z+1)e^{4z}, \qquad (12)
\end{aligned}$$

where

$$W(x) := \sum_{n\geq 1}(-n)^{n-1}\frac{x^n}{n!}$$

is the Lambert $W$-function which is the inverse function of $f(W) = We^W$. See [3] for extensive study about the Lambert $W$-function.

*Remark* 2. The sequence $\phi(n,n;T)$ is not listed in the OEIS [14]. The first few terms of $\phi(n,n;T)$ $(0 \leq n \leq 9)$ are as follows:

$$1, 2, 14, 200, 3536, 67472, 1423168, 34048352, 927156224, 28490354432, \ldots$$

### 3.4 $\{T, L\}$-avoiding matrices

Given the equivalence relation $\sim$ on $M(2,2)$, which is defined in Section 1, if we define the new equivalent relation $P \sim'' Q$ by $P \sim Q$ or $P \sim Q^{\mathrm{t}}$, then $[T]\cup[L]$ becomes a single equivalent class. Clearly $\phi(k,n;\{T,L\})$ is the number of $k \times n$ $(0,1)$-matrix which does not have any submatrix in $[T]\cup[L]$. By the symmetry of $\{T, L\}$, we have

$$\phi(k,n;T,L) = \phi(n,k;T,L).$$

So it is enough to consider the case $k \geq n$. For $k \leq 2$ or $n \leq 1$, we have

$$\begin{gathered}
\phi(0,n;T,L) = \delta_{0,n}, \quad \phi(1,n;T,L) = 2^n, \\
\phi(k,0;T,L) = \delta_{k,0}, \quad \phi(k,1;T,L) = 2^k, \\
\phi(2,2;T,L) = 12.
\end{gathered}$$

Given a $(0,1)$-vector $v$ with a length of at least 3, $v$ is called 1-dominant (resp. 0-dominant) if all entries of $v$ are 1's (resp. 0's) except one entry.

**Theorem 3.3.** *For $k \geq 3$ and $n \geq 2$, the number of $k \times n$ matrices avoiding $\{T, L\}$ is equal to twice the number of rook positions in the $k \times n$ chessboard. In other words,*

$$\phi(k,n;T,L) = 2 \sum_{m=0}^{\min(k,n)} \binom{k}{m}\binom{n}{m} m!\,. \tag{13}$$

*Proof.* Suppose $M$ is a $k \times n$ matrix avoiding $\{T, L\}$. It is easy to show each of the following steps:

(i) If $M$ has a mixed column $v$, then $v$ should be either 0-dominant or 1-dominant.

(ii) Assume that $v$ is 0-dominant. This implies that other mixed columns(if any) in $M$ should be 0-dominant.

(iii) Any non-mixed column in $M$ should be a 0-column.

(iv) The location of 1's in $M$ corresponds to a rook position in the $k \times n$ chessboard.

If we assume $v$ is 1-dominant in (ii) then the locations of 0's again corresponds to a rook position. The summand of RHS in (13) is the number of rook positions in the $k \times n$ chessboard with $m$ rooks. This completes the proof. □

The generating function $\Phi(x, y; T, L)$ is given by

$$\begin{aligned}\Phi(x,y;T,L) &= 2e^{xy+x+y} - \frac{(xy)^2}{2} \\ &\quad - 2xy + 3 - 2e^x - 2e^y + x(e^y - 2y - 1)(e^y - 1) + y(e^x - 2x - 1)(e^x - 1). \end{aligned} \tag{14}$$

Here the crucial part of the equation (14) can be obtained as follows:

$$\begin{aligned}\sum_{k,n\geq 0}\left(\sum_{m\geq 0}\binom{k}{m}\binom{n}{m}m!\right)\frac{x^k}{k!}\frac{y^n}{n!} &= \sum_{m\geq 0} m!\left(\sum_{k\geq 0}\binom{k}{m}\frac{x^k}{k!}\right)\left(\sum_{n\geq 0}\binom{n}{m}\frac{y^n}{n!}\right) \\ &= \sum_{m\geq 0} m!\left(\frac{x^m}{m!}e^x\right)\left(\frac{y^m}{m!}e^y\right) = \exp(xy + x + y).\end{aligned}$$

Note that if we use the symbol "$\overset{2}{=}$" introduced in Section 2.1, then

$$\Phi(x,y;T,L) \overset{2}{=} 2e^{xy+x+y} - \frac{(xy)^2}{2}.$$

For the $n \times n$ matrices we have

$$\phi(0,0;T,L) = 1, \quad \phi(1,1;T,L) = 2, \quad \phi(2,2;T,L) = 12, \quad \text{and}$$
$$\phi(n,n;T,L) = 2\sum_{m=0}^{n}\binom{n}{m}^2 m!\,. \quad (n \geq 3)$$

Thus the generating function $\Phi(z;T,L)$ is given by

$$\Phi(z;T,L) = \frac{2\,e^{\frac{z}{1-z}}}{1-z} - 1 - 2z - z^2. \tag{15}$$

Note that we use the equation

$$\sum_{n\geq 0}\left(\sum_{m=0}^{n}\binom{n}{m}^2 m!\right)\frac{z^n}{n!} = \frac{e^{\frac{z}{1-z}}}{1-z},$$

which appears in [4, pp. 597–598].

### 3.5 $\{J,O\}$-avoiding matrices

Recall the equivalence relation $\sim'$ defined in subsection 3.2. With this relation, $\{J,O\}$ becomes a single equivalent class. Due to the symmetry of $\{J,O\}$ it is obvious that

$$\phi(k,n;J,O) = \phi(n,k;J,O).$$

The $k$-color bipartite Ramsey number $br(G;k)$ of a bipartite graph $G$ is the minimum integer $n$ such that in any $k$-coloring of the edges of $K_{n,n}$ there is a monochromatic subgraph isomorphic to $G$. Beineke and Schwenk [1] had shown that $br(K_{2,2};2) = 5$. From this we can see that

$$\phi(k,n;J,O) = 0 \quad (k,n \geq 5)\,.$$

For $k=1$ and 2, we have

$$\phi(1,n;J,O) = 2^n, \qquad \phi(2,n;J,O) = (n^2+3n+4)2^{n-2}.$$

Note that the sequence $(n^2+3n+4)\,2^{n-2}$ appears in [14, A007466] and its exponential generating function is $(1+x)^2e^{2x}$.

For $k \geq 3$, we have

$$\begin{aligned}
\phi(3,n;J,O) = \phi(4,n;J,O) &= 0 \qquad \text{for } n \geq 7,\\
\phi(5,n;J,O) = \phi(6,n;J,O) &= 0 \qquad \text{for } n \geq 5,\\
\phi(k,n;J,O) &= 0 \qquad \text{for } k \geq 7 \text{ and } n \geq 3.
\end{aligned}$$

For exceptional cases, due to the symmetry of $\{J,O\}$, it is enough to consider the followings:

$$\phi(3,3;J,O) = 156, \quad \phi(3,4;J,O) = 408, \quad \phi(4,4;J,O) = 840,$$
$$\phi(3,5;J,O) = \phi(3,6;J,O) = \phi(4,5,J,O) = \phi(4,6;J,O) = 720\,.$$

The sequence $\phi(k,n;J,O)$ is listed in Table 1. Note that Kitaev et al. have already calculated $\phi(k,n;J,O)$ in [11, Proposition 5], but the numbers of $\phi(3,3;J,O)$ and $\phi(4,4;J,O)$ are different with ours.

| $k \diagdown n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | $\cdots$ |
| 2 | 4 | 14 | 44 | 128 | 352 | 928 | 2368 | $\cdots$ |
| 3 | 8 | 44 | 156 | 408 | 720 | 720 | 0 | 0 |
| 4 | 16 | 128 | 408 | 840 | 720 | 720 | 0 | 0 |
| 5 | 32 | 352 | 720 | 720 | 0 | 0 | 0 | 0 |
| 6 | 64 | 928 | 720 | 720 | 0 | 0 | 0 | 0 |
| 7 | 128 | 2368 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1: The sequence $\phi(k, n; J, O)$

The generating function $\Phi(x, y; J, O)$ is given by

$$\begin{aligned}\Phi(x,y;J,O) \;=\; & 1 + x\,e^{2y} + y\,e^{2x} + x^2(1+y)^2e^{2y} + y^2(1+x)^2e^{2x} \\ & - \left(x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + 2xy + 2x^2y + 2xy^2 + 14\frac{x^2y^2}{2!2!}\right) + 156\frac{x^3y^3}{3!3!} + 840\frac{x^4y^4}{4!4!} \\ & + 720\left(\frac{x^3y^5}{3!5!} + \frac{x^5y^3}{5!3!} + \frac{x^3y^6}{3!6!} + \frac{x^6y^3}{6!3!} + \frac{x^4y^5}{4!5!} + \frac{x^5y^4}{5!4!} + \frac{x^4y^6}{4!6!} + \frac{x^6y^4}{6!4!}\right). \end{aligned} \tag{16}$$

In particular, the generating function $\Phi(z; J, O)$ is given by

$$\Phi(z; J, O) = 1 + 2z + 7z^2 + 26z^3 + 35z^4\,. \tag{17}$$

# 4 Concluding remarks

Table 2 summarizes our results (except the $I$-avoiding case). Due to the amount of difficulty, we are not able to enumerate the number $\phi(k, n; J)$, hence $\phi(k, n; O)$. Note that $\phi(k, n; J)$ is equal to the following:

(a) The number of labeled $(k, n)$-bipartite graphs with girth of at least 6, i.e., the number of $C_4$-free labeled $(k, n)$-bipartite graphs, where $C_4$ is a cycle of length 4.

(b) The cardinality of the set $\{(B_1, B_2, \ldots, B_k) \,:\, B_i \subseteq [n]\ \forall\, i,\ \ |B_i \cap B_j| \le 1\ \forall\, i \ne j\}$.

For the other subsets $\alpha$ of $\mathcal{S}$ which is not listed in Table 2, we have calculated $\phi(k, n; \alpha)$ in [6]. Note that if the size of $\alpha$ increases then enumeration of $M(\alpha)$ becomes easier.

For further research, we suggest the following problems.

1. In addition to $(0, 1)$-matrices, one can consider $(0, 1, \ldots, r)$-matrices with $r \ge 2$.

2. Consideration of the results of adding the line sum condition to each individual case given in the first column of Table 2.

| $\alpha$ | $\phi(k,n;\alpha)$ | $\Phi(x,y;\alpha)$ | $\Phi(z;\alpha)$ |
|---|---|---|---|
| $I$ | (1) | (2) | (3) |
| $\Gamma$ (or $C$) | (4) | (6) | complicated |
| $\{\Gamma, C\}$ | (7) | (8) | (9) |
| $T$ (or $L$) | (10) | (11) | (12) |
| $\{T, L\}$ | (13) | (14) | (15) |
| $J$ (or $O$) | unknown | unknown | unknown |
| $\{J, O\}$ | Table 1 | (16) | (17) |

Table 2: Formulas and generating functions avoiding $\alpha$.

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