MALCEV DIALGEBRAS

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Abstract. We apply Kolesnikov's algorithm to obtain a variety of nonassociative algebras defined by right anticommutativity and a "noncommutative" version of the Malcev identity. We use computational linear algebra to verify that these identities are equivalent to the identities of degree ≤ 4 satisfied by the dicommutator in every alternative dialgebra. We extend these computations to show that any special identity for Malcev dialgebras must have degree at least 7. Finally, we introduce a trilinear operation which makes any Malcev dialgebra into a Leibniz triple system.

1. Introduction

In this paper we introduce the appropriate generalization of Malcev algebras to the setting of dialgebras. These new structures, which we call Malcev dialgebras, are related to Malcev algebras in the same way that Leibniz algebras are related to Lie algebras; they are related to alternative dialgebras in the same way that Malcev algebras are related to alternative algebras. To obtain the defining identities for Malcev dialgebras, we apply Kolesnikov's algorithm to anticommutativity and the Malcev identity; we obtain right anticommutativity and a "noncommutative" version of the Malcev identity. We use computer algebra to verify that these identities are equivalent to the identities of degree ≤ 4 satisfied by the dicommutator in every alternative dialgebra, and that the resulting identities imply every identity of degree ≤ 6 satisfied by the dicommutator in every alternative dialgebra. We then generalize the construction of Loos, which defines the structure of a Lie triple system on a Malcev algebra, to the setting of dialgebras: we introduce a trilinear operation on a Malcev dialgebra which makes the underlying vector space into a Leibniz triple system in the sense of Bremner and Sánchez-Ortega [2].

2. Preliminaries

Dialgebras were introduced by Loday [8] (see also [9]) to provide a natural setting for Leibniz algebras, a "noncommutative" version of Lie algebras.

Definition 1. (Cuvier [4], Loday [7]) A **Leibniz algebra** is a vector space L together with a bilinear map $L \times L \to L$, denoted $(a, b) \mapsto \langle a, b\rangle$, satisfying the **Leibniz identity**, which says that right multiplications are derivations:

$$\langle\langle a, b\rangle, c\rangle \equiv \langle\langle a, c\rangle, b\rangle + \langle a, \langle b, c\rangle\rangle. \tag{1}$$

If $\langle a, a\rangle \equiv 0$ then the Leibniz identity is the Jacobi identity and L is a Lie algebra.

An associative algebra becomes a Lie algebra if the associative product is replaced by the Lie bracket. The notion of dialgebra gives, by a similar procedure, a Leibniz algebra: one replaces the associative products ab and ba by two distinct operations, so that the resulting bracket is not necessarily skew-symmetric.

Definition 2. (Loday [8]) A **dialgebra** is a vector space D with two bilinear operations $\dashv\colon D \times D \to D$ and $\vdash\colon D \times D \to D$, called the **left** and **right** products.

Definition 3. (Kolesnikov [5]) A **0-dialgebra** is a dialgebra satisfying the **left** and **right bar identities**:

$$(a \dashv b) \vdash c \equiv (a \vdash b) \vdash c, \qquad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c).$$

Definition 4. (Loday [8]) An **associative dialgebra** is a 0-dialgebra satisfying **left, right** and **inner associativity**:

$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \quad (a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \quad (a \vdash b) \dashv c \equiv a \vdash (b \dashv c).$$

Definition 5. In any dialgebra, the **dicommutator** is the bilinear operation

$$\langle a, b \rangle = a \dashv b - b \vdash a.$$

An associative dialgebra gives rise to a Leibniz algebra by considering the same underlying vector space with the product defined to be the dicommutator. The goal of the present paper is to study the same construction for alternative dialgebras.

Definition 6. (Liu [6]) An **alternative dialgebra** is a 0-dialgebra satisfying:

$$(a,b,c)_\dashv + (c,b,a)_\vdash \equiv 0, \quad (a,b,c)_\dashv - (b,c,a)_\vdash \equiv 0, \quad (a,b,c)_\times + (a,c,b)_\vdash \equiv 0,$$

where the **left**, **right** and **inner associators** are defined by

$$(a,b,c)_\dashv = (a \dashv b) \dashv c - a \dashv (b \dashv c), \qquad (a,b,c)_\vdash = (a \vdash b) \vdash c - a \vdash (b \vdash c),$$
$$(a,b,c)_\times = (a \vdash b) \dashv c - a \vdash (b \dashv c).$$

Kolesnikov's algorithm. Kolesnikov [5] (see also Pozhidaev [13]) introduced a general framework for converting the defining identities of a variety of algebras into the defining identities of the corresponding variety of dialgebras. Part 1 converts a multilinear polynomial identity of degree d for a bilinear operation into d multilinear identities of degree d for two new bilinear operations. Part 2 introduces the analogues of the bar identities for each new operation.

Part 1: We consider a bilinear operation, not necessarily associative, denoted by the symbol $\{-,-\}$. Given a multilinear polynomial identity of degree d in this operation, we show how to apply the algorithm to one monomial, and from this the application to the complete identity follows by linearity. Let $\overline{a_1 a_2 \dots a_d}$ be a multilinear monomial of degree d, where the bar denotes some placement of operation symbols. We introduce two new operations, denoted by the same symbol but distinguished by subscripts: $\{-,-\}_1$, $\{-,-\}_2$. For each $i \in \{1, 2, \dots, d\}$ we convert the monomial $\overline{a_1 a_2 \dots a_d}$ in the original operation into a new monomial of the same degree d in the two new operations, according to the following rule, which is based on the position of a_i, called the central argument of the monomial. For each occurrence of the original operation $\{-,-\}$ in the monomial, either a_i occurs within one of the two arguments or not, and we have the following cases:

- If a_i occurs within the j-th argument then we convert the original operation $\{-,-\}$ to the j-th new operation $\{-,-\}_j$.
- If a_i does not occur within either of the two arguments, then either
  - a_i occurs to the left of the original operation, in which case we convert $\{-,-\}$ to the first new operation $\{-,-\}_1$, or
  - a_i occurs to the right of the original operation, in which case we convert $\{-,-\}$ to the second new operation $\{-,-\}_2$.

Part 2: We also include the following two identities, analogous to the left and right bar identities of Definition 3. These identities say that the two new operations are interchangeable in the i-th argument of the j-th new operation when $i \neq j$:

$$\{\{a,b\}_1,c\}_2 \equiv \{\{a,b\}_2,c\}_2, \qquad \{a,\{b,c\}_1\}_1 \equiv \{a,\{b,c\}_2\}_1.$$

Example 7. The definition of associative dialgebra can be obtained by applying Kolesnikov's algorithm to associativity, $\{\{a,b\},c\} \equiv \{a,\{b,c\}\}$. Part 1 produces three new identities of degree 3 by making a, b, c in turn the central argument:

$$\{\{a,b\}_1,c\}_1 \equiv \{a,\{b,c\}_1\}_1, \quad \{\{a,b\}_2,c\}_1 \equiv \{a,\{b,c\}_1\}_2,$$
$$\{\{a,b\}_2,c\}_2 \equiv \{a,\{b,c\}_2\}_2.$$

Combining these identities with the two identities from Part 2, and reverting to the standard notation $a \dashv b = \{a,b\}_1$, $a \vdash b = \{a,b\}_2$, we obtain Definition 4.

Example 8. The definition of alternative dialgebra can be obtained by applying Kolesnikov's algorithm to right and left alternativity, $(a,a,b) \equiv 0$ and $(b,a,a) \equiv 0$, where $(x,y,z) = (xy)z - x(yz)$ is the associator. If we assume characteristic not 2, then these two identities are equivalent to their multilinear forms; we expand the associators and use the operation symbol $\{-,-\}$:

$$\{\{a,b\},c\} - \{a,\{b,c\}\} + \{\{b,a\},c\} - \{b,\{a,c\}\} \equiv 0,$$
$$\{\{a,b\},c\} - \{a,\{b,c\}\} + \{\{a,c\},b\} - \{a,\{c,b\}\} \equiv 0.$$

Part 1 produces six identities relating the two new operations $\{-,-\}_1$ and $\{-,-\}_2$:

$$\{\{a,b\}_1,c\}_1 - \{a,\{b,c\}_1\}_1 + \{\{b,a\}_2,c\}_1 - \{b,\{a,c\}_1\}_2 \equiv 0, \tag{2}$$
$$\{\{a,b\}_2,c\}_1 - \{a,\{b,c\}_1\}_2 + \{\{b,a\}_1,c\}_1 - \{b,\{a,c\}_1\}_1 \equiv 0, \tag{3}$$
$$\{\{a,b\}_2,c\}_2 - \{a,\{b,c\}_2\}_2 + \{\{b,a\}_2,c\}_2 - \{b,\{a,c\}_2\}_2 \equiv 0, \tag{4}$$
$$\{\{a,b\}_1,c\}_1 - \{a,\{b,c\}_1\}_1 + \{\{a,c\}_1,b\}_1 - \{a,\{c,b\}_1\}_1 \equiv 0, \tag{5}$$
$$\{\{a,b\}_2,c\}_1 - \{a,\{b,c\}_1\}_2 + \{\{a,c\}_2,b\}_2 - \{a,\{c,b\}_2\}_2 \equiv 0, \tag{6}$$
$$\{\{a,b\}_2,c\}_2 - \{a,\{b,c\}_2\}_2 + \{\{a,c\}_2,b\}_1 - \{a,\{c,b\}_1\}_2 \equiv 0. \tag{7}$$

Writing $a \dashv b = \{a,b\}_1$ and $a \vdash b = \{a,b\}_2$ and using the dialgebra associators of Definition 6, we rewrite identities (2) to (7) as follows:

$$(a,b,c)_\dashv + (b,a,c)_\times \equiv 0, \qquad (a,b,c)_\times + (b,a,c)_\dashv \equiv 0,$$
$$(a,b,c)_\vdash + (b,a,c)_\vdash \equiv 0, \qquad (a,b,c)_\dashv + (a,c,b)_\dashv \equiv 0,$$
$$(a,b,c)_\times + (a,c,b)_\vdash \equiv 0, \qquad (a,b,c)_\vdash + (a,c,b)_\times \equiv 0.$$

These identities show how the transpositions (ab) and (bc) affect the dialgebra associators. It is now clear that (2) and (3) are equivalent, as are (6) and (7). Furthermore, (4) can be derived from (2), (5) and (6):

$$(a,b,c)_\vdash \overset{(6)}{\equiv} -(a,c,b)_\times \overset{(2)}{\equiv} (c,a,b)_\dashv \overset{(5)}{\equiv} -(c,b,a)_\dashv \overset{(2)}{\equiv} (b,c,a)_\times \overset{(6)}{\equiv} -(b,a,c)_\vdash.$$

Thus the final result is the variety of dialgebras satisfying these identities:

$$(a,b,c)_\dashv + (b,a,c)_\times \equiv 0, \; (a,b,c)_\dashv + (a,c,b)_\dashv \equiv 0, \; (a,b,c)_\times + (a,c,b)_\vdash \equiv 0, \tag{8}$$

which are equivalent to the identities of Definition 6.

Remark 9. The algorithm of Kolesnikov is closely related to general constructions in the theory of operads discussed by Chapoton [3] and Vallette [16].

3. The definition of Malcev dialgebra

In this section we recall the defining identities for Malcev algebras, and then apply Kolesnikov's algorithm to obtain the defining identities for the corresponding variety of dialgebras, which we call Malcev dialgebras.

Definition 10. (Malcev [12]) A **Malcev algebra** is a vector space with a bilinear operation ab satisfying **anticommutativity** and the **Malcev identity**:

$$a^2 \equiv 0, \qquad J(a,b,ac) \equiv J(a,b,c)a,$$

where $J(a,b,c) = (ab)c + (bc)a + (ca)b$ is the Jacobian.

Lemma 11. (Sagle [14]) *If the characteristic is not 2, then an algebra is Malcev if and only if it satisfies the following multilinear identities:*

$$ab + ba \equiv 0, \qquad (ac)(bd) \equiv ((ab)c)d + ((bc)d)a + ((cd)a)b + ((da)b)c.$$

To apply Kolesnikov's algorithm, we write these identities as follows:

$$\begin{aligned}
&\{a,b\} + \{b,a\} \equiv 0,\\
&\{\{a,c\},\{b,d\}\} - \{\{\{a,b\},c\},d\} - \{\{\{b,c\},d\},a\} - \{\{\{c,d\},a\},b\}\\
&\qquad - \{\{\{d,a\},b\},c\} \equiv 0.
\end{aligned}$$

Part 1 gives six identities relating the operations $\{-,-\}_1$ and $\{-,-\}_2$:

$$\{a,b\}_1 + \{b,a\}_2 \equiv 0, \qquad \{a,b\}_2 + \{b,a\}_1 \equiv 0, \tag{9}$$

$$\begin{aligned}
&\{\{a,c\}_1,\{b,d\}_1\}_1 - \{\{\{a,b\}_1,c\}_1,d\}_1 - \{\{\{b,c\}_2,d\}_2,a\}_2\\
&\qquad -\{\{\{c,d\}_2,a\}_2,b\}_1 - \{\{\{d,a\}_2,b\}_1,c\}_1 \equiv 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\{\{a,c\}_2,\{b,d\}_1\}_2 - \{\{\{a,b\}_2,c\}_1,d\}_1 - \{\{\{b,c\}_1,d\}_1,a\}_1\\
&\qquad -\{\{\{c,d\}_2,a\}_2,b\}_2 - \{\{\{d,a\}_2,b\}_2,c\}_1 \equiv 0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\{\{a,c\}_2,\{b,d\}_1\}_1 - \{\{\{a,b\}_2,c\}_2,d\}_1 - \{\{\{b,c\}_2,d\}_1,a\}_1\\
&\qquad -\{\{\{c,d\}_1,a\}_1,b\}_1 - \{\{\{d,a\}_2,b\}_2,c\}_2 \equiv 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\{\{a,c\}_2,\{b,d\}_2\}_2 - \{\{\{a,b\}_2,c\}_2,d\}_2 - \{\{\{b,c\}_2,d\}_2,a\}_1\\
&\qquad -\{\{\{c,d\}_2,a\}_1,b\}_1 - \{\{\{d,a\}_1,b\}_1,c\}_1 \equiv 0.
\end{aligned} \tag{13}$$

The two identities (9) are equivalent; both say $\{a,b\}_2 \equiv -\{b,a\}_1$, so we can eliminate the second operation. Applying this to identities (10)–(13), we obtain:

$$\begin{aligned}
&\{\{a,c\}_1,\{b,d\}_1\}_1 - \{\{\{a,b\}_1,c\}_1,d\}_1 + \{a,\{d,\{c,b\}_1\}_1\}_1\\
&\qquad -\{\{a,\{d,c\}_1\}_1,b\}_1 + \{\{\{a,d\}_1,b\}_1,c\}_1 \equiv 0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\{\{b,d\}_1,\{c,a\}_1\}_1 + \{\{\{b,a\}_1,c\}_1,d\}_1 - \{\{\{b,c\}_1,d\}_1,a\}_1\\
&\qquad +\{b,\{a,\{d,c\}_1\}_1\}_1 - \{\{b,\{a,d\}_1\}_1,c\}_1 \equiv 0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\{\{c,a\}_1,\{b,d\}_1\}_1 + \{\{c,\{b,a\}_1\}_1,d\}_1 - \{\{\{c,b\}_1,d\}_1,a\}_1\\
&\qquad +\{\{\{c,d\}_1,a\}_1,b\}_1 - \{c,\{b,\{a,d\}_1\}_1\}_1 \equiv 0,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\{\{d,b\}_1,\{c,a\}_1\}_1 - \{d,\{c,\{b,a\}_1\}_1\}_1 + \{\{d,\{c,b\}_1\}_1,a\}_1\\
&\qquad -\{\{\{d,c\}_1,a\}_1,b\}_1 + \{\{\{d,a\}_1,b\}_1,c\}_1 \equiv 0.
\end{aligned} \tag{17}$$

Since we now have only one operation, we revert to a simpler notation, and write $\{a,b\}_1$ simply as ab. Identities (14)–(17) take the following form:

$$(ac)(bd) - ((ab)c)d + a(d(cb)) - (a(dc))b + ((ad)b)c \equiv 0, \tag{18}$$

$$(bd)(ca) + ((ba)c)d - ((bc)d)a + b(a(dc)) - (b(ad))c \equiv 0, \tag{19}$$

$$-(ca)(bd) - (c(ba))d + ((cb)d)a - ((cd)a)b + c(b(ad)) \equiv 0, \tag{20}$$

$$-(db)(ca) + d(c(ba)) - (d(cb))a + ((dc)a)b - ((da)b)c \equiv 0. \tag{21}$$

Part 2 gives two identities; rewriting them in terms of the first operation gives

$$\{a, \{b, c\}_1\}_1 \equiv -\{a, \{c, b\}_1\}_1, \qquad -\{c, \{a, b\}_1\}_1 \equiv \{c, \{b, a\}_1\}_1,$$

which are both equivalent to right anticommutativity $a(bc) \equiv -a(cb)$. We note that (21) is a permutation of (20). Furthermore, rearranging the terms in (19) and (20), and applying right anticommutativity, gives (18). Thus we require only one identity in degree 4.

Definition 12. Over a field of characteristic not 2, a **(right) Malcev dialgebra** is a vector space with a bilinear operation ab satisfying **right anticommutativity** and the **di-Malcev identity**:

$$a(bc) + a(cb) \equiv 0, \qquad ((ab)c)d - ((ad)b)c - (a(cd))b - (ac)(bd) - a((bc)d) \equiv 0.$$

4. The dicommutator in an alternative dialgebra

Malcev [12] showed that an alternative algebra becomes a Malcev algebra by considering the same underlying vector space with the new operation $ab - ba$. In this section we extend this result to the setting of dialgebras: we use computer algebra to show that any subspace of an alternative dialgebra closed under the dicommutator is a Malcev dialgebra, and conversely that the dicommutator identities of degrees ≤ 4 are equivalent to right anticommutativity and the di-Malcev identity.

We write FA_n for the multilinear subspace of degree n in the free nonassociative algebra on n generators. The number of association types (distinct placements of parentheses) in degree n is the Catalan number,

$$K_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Since there are $n!$ permutations of n indeterminates, we have $\dim FA_n = n! K_n$.

Lemma 13. *Let X be a set. For any $a_1, \ldots, a_n \in X$, let $w = a_1 \ldots a_n$ be a monomial in the free 0-dialgebra on X, with some placement of parentheses and choice of operations. If $x \dashv y$ or $y \vdash x$ is a submonomial, then y does not depend on the choice of operations: we may regard y as a monomial in the free algebra.*

Proof. Induction on the degree n using the bar identities of Definition 3. □

We write FD_n for the multilinear subspace of degree n in the free 0-dialgebra on n generators.

Lemma 14. *The number of 0-dialgebra association types in degree n is*

$$Z_n = \binom{2n-2}{n-1}.$$

Proof. Suppose that we have enumerated the 0-dialgebra association types up to degree $n-1$. By Lemma 13, any 0-dialgebra association type in degree n is either

$x \dashv y$ or $y \vdash x$ where x is a 0-dialgebra association type in degree $n-i$ and y is an algebra association type in degree i, for some $i < n$. Therefore

$$Z_1 = 1, \qquad Z_n = 2 \sum_{i=1}^{n-1} Z_{n-i} K_i \ \ (n \geq 2).$$

The unique solution to this recurrence relation is $Z_n = nK_n$. □

Lemma 15. *We have* $\dim FD_n = n! Z_n$.

Proposition 16. *Over a field of characteristic not 2 or 3, every multilinear polynomial identity in degree 3 satisfied by the dicommutator in every alternative dialgebra is a consequence of right anticommutativity.*

Proof. There are two algebra association types, $(ab)c$ and $a(bc)$, and 12 basis monomials for FA_3, which we list in lexicographical order:

$$(ab)c,\ (ac)b,\ (ba)c,\ (bc)a,\ (ca)b,\ (cb)a,\ a(bc),\ a(cb),\ b(ac),\ b(ca),\ c(ab),\ c(ba).$$

Lemmas 13–15 imply that we need only 6 dialgebra association types:

$$(a \dashv b) \dashv c, \qquad (a \vdash b) \dashv c, \qquad ab \vdash c, \qquad a \dashv bc, \qquad a \vdash (b \dashv c), \qquad a \vdash (b \vdash c).$$

We therefore have 36 basis monomials for FD_3 in lexicographical order:

$$\begin{array}{llllll}
(a{\dashv}b){\dashv}c, & (a{\dashv}c){\dashv}b, & (b{\dashv}a){\dashv}c, & (b{\dashv}c){\dashv}a, & (c{\dashv}a){\dashv}b, & (c{\dashv}b){\dashv}a, \\
(a{\vdash}b){\dashv}c, & (a{\vdash}c){\dashv}b, & (b{\vdash}a){\dashv}c, & (b{\vdash}c){\dashv}a, & (c{\vdash}a){\dashv}b, & (c{\vdash}b){\dashv}a, \\
ab{\vdash}c, & ac{\vdash}b, & ba{\vdash}c, & bc{\vdash}a, & ca{\vdash}b, & cb{\vdash}a, \\
a{\dashv}bc, & a{\dashv}cb, & b{\dashv}ac, & b{\dashv}ca, & c{\dashv}ab, & c{\dashv}ba, \\
a{\vdash}(b{\dashv}c), & a{\vdash}(c{\dashv}b), & b{\vdash}(a{\dashv}c), & b{\vdash}(c{\dashv}a), & c{\vdash}(a{\dashv}b), & c{\vdash}(b{\dashv}a), \\
a{\vdash}(b{\vdash}c), & a{\vdash}(c{\vdash}b), & b{\vdash}(a{\vdash}c), & b{\vdash}(c{\vdash}a), & c{\vdash}(a{\vdash}b), & c{\vdash}(b{\vdash}a).
\end{array}$$

We rewrite the identities of equation (8) using our basis of FD_3 and obtain

$$\left\{ \begin{array}{c}
(a \dashv b) \dashv c + (b \vdash a) \dashv c - a \dashv bc - b \vdash (a \dashv c) \equiv 0, \\
(a \dashv b) \dashv c + (a \dashv c) \dashv b - a \dashv bc - a \dashv cb \equiv 0, \\
(a \vdash b) \dashv c + ac \vdash b - a \vdash (b \dashv c) - a \vdash (c \vdash b) \equiv 0.
\end{array} \right. \tag{22}$$

Each admits six permutations, giving 18 identities which span the subspace of FD_3 consisting of the multilinear identities for alternative dialgebras. This subspace is the row space of an 18×36 matrix A; the rows correspond to identities and the columns to basis monomials. (See Figure 1, with $+$, $-$, $\cdot$ for 1, -1, 0.)

The linear expansion map $E_3 \colon FA_3 \to FD_3$ is defined on basis monomials by iteration of the dicommutator:

$$E_3 \colon (ab)c = \langle\langle a, b\rangle, c\rangle \longmapsto (a \dashv b) \dashv c - (b \vdash a) \dashv c - c \vdash (a \dashv b) + c \vdash (b \vdash a),$$

$$E_3 \colon a(bc) = \langle a, \langle b, c\rangle\rangle \longmapsto\ - bc \vdash a + cb \vdash a + a \dashv bc - a \dashv cb.$$

It suffices to calculate E_3 on one monomial of each association type; the other expansions are obtained by permutation. We represent these expansions as a 12×36 matrix E in which the (i, j) entry contains the coefficient of the j-th basis monomial of FD_3 in the expansion of the i-th basis monomial of FA_3. (See Figure 1.)

We construct a 30×48 matrix M; columns 1–36 correspond to the basis monomials of FD_3, and columns 37–48 to the basis monomials of FA_3. The upper left

$$
A = \left[\begin{array}{l}
+\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot+\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot+\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot++\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot--\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot-\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot- \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot-\cdot
\end{array} \right]
$$

$$
E = \left[\begin{array}{l}
+\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot+ \\
\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot \\
\cdot\cdot+\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot+\cdot \\
\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot+\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot+\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot+\cdot\cdot\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot\cdot\cdot+\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot++-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot--+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot\cdot+\cdot\cdot\cdot+-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot\cdot-\cdot\cdot\cdot-+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-\cdot+\cdot\cdot\cdot\cdot\cdot\cdot\cdot+-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot+\cdot-\cdot\cdot\cdot\cdot\cdot\cdot\cdot-+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot
\end{array} \right]
$$

FIGURE 1. The matrices A and E from the proof of Proposition 16

18×36 block is A; the lower left 12×36 block is E; the upper right 18×12 block is the zero matrix O; the lower right 12×12 block is the identity matrix I:

$$
M = \left[\begin{array}{c|c} A & O \\ E & I \end{array} \right]. \tag{23}
$$

Any row of the row canonical form $\mathrm{RCF}(M)$ which has its leading 1 to the right of the vertical line represents a polynomial identity in FA_3 satisfied by the dicommutator in every alternative dialgebra. These rows represent dependence relations among the basis monomials of FA_3 resulting from dependence relations, implied by the alternative dialgebra identities, among their expansions in FD_3. A calculation with the computer algebra system Maple shows that there are three such rows, which represent the three permutations of right anticommutativity:

$$
a(bc) + a(cb), \qquad b(ac) + b(ca), \qquad c(ab) + c(ba).
$$

The restriction on the characteristic is required by the fact that 6 is the least common multiple of the denominators of the nonzero entries of RCF(M). □

Theorem 17. *Over a field of characteristic $\neq 2, 3$, every multilinear polynomial identity in degree 4 satisfied by the dicommutator in every alternative dialgebra is a consequence of right anticommutativity and the di-Malcev identity.*

Proof. There are 20 dialgebra association types in degree 4:

$$
\begin{array}{llll}
((a \dashv b) \dashv c) \dashv d, & ((a \vdash b) \dashv c) \dashv d, & (ab \vdash c) \dashv d, & (a \dashv bc) \dashv d, \\
(a \vdash (b \dashv c)) \dashv d, & (a \vdash (b \vdash c)) \dashv d, & (ab)c \vdash d, & a(bc) \vdash d, \\
(a \dashv b) \dashv cd, & (a \vdash b) \dashv cd, & ab \vdash (c \dashv d), & ab \vdash (c \vdash d), \\
a \dashv (bc)d, & a \dashv b(cd), & a \vdash ((b \dashv c) \dashv d), & a \vdash ((b \vdash c) \dashv d), \\
a \vdash (bc \vdash d), & a \vdash (b \dashv cd), & a \vdash (b \vdash (c \dashv d)), & a \vdash (b \vdash (c \vdash d)).
\end{array}
$$

Each identity $P(a, b, c)$ for alternative dialgebras has 10 consequences in degree 4:

$$
\begin{array}{lllll}
P(a \vdash d, b, c), & P(a \dashv d, b, c), & P(a, b \vdash d, c), & P(a, b \dashv d, c), & P(a, b, c \vdash d), \\
P(a, b, c \dashv d), & P(a, b, c) \dashv d, & P(a, b, c) \vdash d, & d \dashv P(a, b, c), & d \vdash P(a, b, c).
\end{array}
$$

We obtain the 30 identities in Table 1; each admits 24 permutations, giving 720 elements of FD_4 which span the subspace of multilinear identities satisfied by every alternative dialgebra. This subspace is the row space of a 720×480 matrix A.

A right anticommutative operation has four association types in degree 4:

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d),$$

since $a(b(cd)) = -a((cd)b)$. Types 2, 3 and 4 have these skew-symmetries:

$$(a(cb))d = -(a(bc))d, \quad (ab)(dc) = -(ab)(cd), \quad a((cb)d) = -a((bc)d). \tag{24}$$

Each skew-symmetry halves the number of multilinear monomials, giving the 60 monomials of Table 2 which form an ordered basis of FRA_4, the multilinear subspace of degree 4 in the free right anticommutative algebra on four generators.

The expansion map $E_4\colon FRA_4 \to FD_4$ is defined on basis monomials by iteration of the dicommutator. The result of applying E_4 to the first basis monomial in each association type is displayed in Table 3. The action of E_4 on the other basis monomials can be obtained by permutation. We represent these expansions as a 60×480 matrix E in which the (i, j) entry contains the coefficient of the j-th basis monomial of FD_4 in the expansion of the i-th basis monomial of FRA_4.

Let O denote the 720×60 zero matrix and let I denote the 60×60 identity matrix. We combine the matrices A, E, O, I into a 780×540 matrix M as in equation (23). Any row of RCF(M) which has its leading 1 to the right of column 480 represents a polynomial identity in FRA_4 satisfied by the dicommutator in every alternative dialgebra. There are 20 such rows; the first represents the di-Malcev identity. Further computations show that all of these identities are linear combinations of permutations of the di-Malcev identity: we create a 24×60 matrix in which row i contains the coefficient vector of the identity obtained by applying permutation i to the di-Malcev identity and straightening the terms using right anticommutativity, and find that this matrix has rank 20. We did these computations using rational arithmetic with the Maple package `LinearAlgebra`. □

$$
\begin{array}{l}
((a \vdash d) \dashv b) \dashv c + (b \vdash (a \vdash d)) \dashv c - (a \vdash d) \dashv bc - b \vdash ((a \vdash d) \dashv c),\\
((a \dashv d) \dashv b) \dashv c + (b \vdash (a \dashv d)) \dashv c - (a \dashv d) \dashv bc - b \vdash ((a \dashv d) \dashv c),\\
(a \dashv bd) \dashv c + (bd \vdash a) \dashv c - a \dashv (bd)c - bd \vdash (a \dashv c),\\
(a \dashv bd) \dashv c + (bd \vdash a) \dashv c - a \dashv (bd)c - bd \vdash (a \dashv c),\\
(a \dashv b) \dashv cd + (b \vdash a) \dashv cd - a \dashv b(cd) - b \vdash (a \dashv cd),\\
(a \dashv b) \dashv cd + (b \vdash a) \dashv cd - a \dashv b(cd) - b \vdash (a \dashv cd),\\
((a \dashv b) \dashv c) \dashv d + ((b \vdash a) \dashv c) \dashv d - (a \dashv bc) \dashv d - (b \vdash (a \dashv c)) \dashv d,\\
(ab)c \vdash d + (ba)c \vdash d - a(bc) \vdash d - b(ac) \vdash d,\\
d \dashv (ab)c + d \dashv (ba)c - d \dashv a(bc) - d \dashv b(ac),\\
d \vdash ((a \dashv b) \dashv c) + d \vdash ((b \vdash a) \dashv c) - d \vdash (a \dashv bc) - d \vdash (b \vdash (a \dashv c)),\\
((a \vdash d) \dashv b) \dashv c + ((a \vdash d) \dashv c) \dashv b - (a \vdash d) \dashv bc - (a \vdash d) \dashv cb,\\
((a \dashv d) \dashv b) \dashv c + ((a \dashv d) \dashv c) \dashv b - (a \dashv d) \dashv bc - (a \dashv d) \dashv cb,\\
(a \dashv bd) \dashv c + (a \dashv c) \dashv bd - a \dashv (bd)c - a \dashv c(bd),\\
(a \dashv bd) \dashv c + (a \dashv c) \dashv bd - a \dashv (bd)c - a \dashv c(bd),\\
(a \dashv b) \dashv cd + (a \dashv cd) \dashv b - a \dashv b(cd) - a \dashv (cd)b,\\
(a \dashv b) \dashv cd + (a \dashv cd) \dashv b - a \dashv b(cd) - a \dashv (cd)b,\\
((a \dashv b) \dashv c) \dashv d + ((a \dashv c) \dashv b) \dashv d - (a \dashv bc) \dashv d - (a \dashv cb) \dashv d,\\
(ab)c \vdash d + (ac)b \vdash d - a(bc) \vdash d - a(cb) \vdash d,\\
d \dashv (ab)c + d \dashv (ac)b - d \dashv a(bc) - d \dashv a(cb),\\
d \vdash ((a \dashv b) \dashv c) + d \vdash ((a \dashv c) \dashv b) - d \vdash (a \dashv bc) - d \vdash (a \dashv cb),\\
(ad \vdash b) \dashv c + (ad)c \vdash b - ad \vdash (b \dashv c) - ad \vdash (c \vdash b),\\
(ad \vdash b) \dashv c + (ad)c \vdash b - ad \vdash (b \dashv c) - ad \vdash (c \vdash b),\\
(a \vdash (b \vdash d)) \dashv c + ac \vdash (b \vdash d) - a \vdash ((b \vdash d) \dashv c) - a \vdash (c \vdash (b \vdash d)),\\
(a \vdash (b \dashv d)) \dashv c + ac \vdash (b \dashv d) - a \vdash ((b \dashv d) \dashv c) - a \vdash (c \vdash (b \dashv d)),\\
(a \vdash b) \dashv cd + a(cd) \vdash b - a \vdash (b \dashv cd) - a \vdash (cd \vdash b),\\
(a \vdash b) \dashv cd + a(cd) \vdash b - a \vdash (b \dashv cd) - a \vdash (cd \vdash b),\\
((a \vdash b) \dashv c) \dashv d + (ac \vdash b) \dashv d - (a \vdash (b \dashv c)) \dashv d - (a \vdash (c \vdash b)) \dashv d,\\
(ab)c \vdash d + (ac)b \vdash d - a(bc) \vdash d - a(cb) \vdash d,\\
d \dashv (ab)c + d \dashv (ac)b - d \dashv a(bc) - d \dashv a(cb),\\
d \vdash ((a \vdash b) \dashv c) + d \vdash (ac \vdash b) - d \vdash (a \vdash (b \dashv c)) - d \vdash (a \vdash (c \vdash b)).
\end{array}
$$

TABLE 1. Alternative dialgebra identities in degree 4

$((ab)c)d$,	$((ab)d)c$,	$((ac)b)d$,	$((ac)d)b$,	$((ad)b)c$,	$((ad)c)b$,
$((ba)c)d$,	$((ba)d)c$,	$((bc)a)d$,	$((bc)d)a$,	$((bd)a)c$,	$((bd)c)a$,
$((ca)b)d$,	$((ca)d)b$,	$((cb)a)d$,	$((cb)d)a$,	$((cd)a)b$,	$((cd)b)a$,
$((da)b)c$,	$((da)c)b$,	$((db)a)c$,	$((db)c)a$,	$((dc)a)b$,	$((dc)b)a$,
$(a(bc))d$,	$(a(bd))c$,	$(a(cd))b$,	$(b(ac))d$,	$(b(ad))c$,	$(b(cd))a$,
$(c(ab))d$,	$(c(ad))b$,	$(c(bd))a$,	$(d(ab))c$,	$(d(ac))b$,	$(d(bc))a$,
$(ab)(cd)$,	$(ac)(bd)$,	$(ad)(bc)$,	$(ba)(cd)$,	$(bc)(ad)$,	$(bd)(ac)$,
$(ca)(bd)$,	$(cb)(ad)$,	$(cd)(ab)$,	$(da)(bc)$,	$(db)(ac)$,	$(dc)(ab)$,
$a((bc)d)$,	$a((bd)c)$,	$a((cd)b)$,	$b((ac)d)$,	$b((ad)c)$,	$b((cd)a)$,
$c((ab)d)$,	$c((ad)b)$,	$c((bd)a)$,	$d((ab)c)$,	$d((ac)b)$,	$d((bc)a)$.

TABLE 2. Right anticommutative monomials in degree 4

$$
\begin{aligned}
&E_4\colon ((ab)c)d = \langle\langle\langle a,b\rangle,c\rangle,d\rangle \longmapsto\\
&\quad ((a \dashv b) \dashv c) \dashv d - ((b \vdash a) \dashv c) \dashv d - (c \vdash (a \dashv b)) \dashv d + (c \vdash (b \vdash a)) \dashv d\\
&\quad - d \vdash ((a \dashv b) \dashv c) + d \vdash ((b \vdash a) \dashv c) + d \vdash (c \vdash (a \dashv b)) - d \vdash (c \vdash (b \vdash a)),\\
&E_4\colon (a(bc))d = \langle\langle a,\langle b,c\rangle\rangle,d\rangle \longmapsto\\
&\quad - (bc \vdash a) \dashv d + (cb \vdash a) \dashv d + (a \dashv bc) \dashv d - (a \dashv cb) \dashv d\\
&\quad + d \vdash (bc \vdash a) - d \vdash (cb \vdash a) - d \vdash (a \dashv bc) + d \vdash (a \dashv cb),\\
&E_4\colon (ab)(cd) = \langle\langle a,b\rangle,\langle c,d\rangle\rangle \longmapsto\\
&\quad (a \dashv b) \dashv cd - (a \dashv b) \dashv dc - (b \vdash a) \dashv cd + (b \vdash a) \dashv dc\\
&\quad - cd \vdash (a \dashv b) + cd \vdash (b \vdash a) + dc \vdash (a \dashv b) - dc \vdash (b \vdash a),\\
&E_4\colon a((bc)d) = \langle a,\langle\langle b,c\rangle,d\rangle\rangle \longmapsto\\
&\quad - ((bc)d) \vdash a + ((cb)d) \vdash a + (d(bc)) \vdash a - (d(cb)) \vdash a\\
&\quad + a \dashv ((bc)d) - a \dashv ((cb)d) - a \dashv (d(bc)) + a \dashv (d(cb)).
\end{aligned}
$$

TABLE 3. Equations defining the expansion map in degree 4

5. SPECIAL IDENTITIES FOR MALCEV DIALGEBRAS

A special identity (s-identity) for Malcev dialgebras is a polynomial identity which is satisfied by the dicommutator in every alternative dialgebra, but which is not a consequence of right anticommutativity and the di-Malcev identity. In this section we describe a computational search for such identities using the representation theory of the symmetric group; we show that there are no s-identities in degrees 5 or 6, so any s-identity must have degree at least 7.

We write R_n for the number of right anticommutative (RAC) association types in degree n. This equals the number of right commutative association types, for which we refer to Bremner and Peresi [1]; see also Sloane [15], sequence A085748. We write Z_n for the number of 0-dialgebra association types (Lemma 14). Given an RAC association type in degree n, we apply it to the identity permutation of the variables, $a_1 \cdots a_n$. We expand this monomial by interpreting each product as the dicommutator, and obtain a linear combination of 2^{n-1} multilinear monomials of degree n in the free 0-dialgebra.

There are three multilinear identities (22) in the definition of alternative dialgebra. Given a multilinear dialgebra identity $I(a_1, \ldots, a_n)$ of degree n, we obtain $2(n+2)$ consequences of degree $n+1$: we introduce another variable a_{n+1} and consider the $2n$ substitutions obtained by replacing a_i by either $a_i \dashv a_{n+1}$ or $a_i \vdash a_{n+1}$; we also consider the four products

$$
I(\cdots) \dashv a_{n+1}, \qquad I(\cdots) \vdash a_{n+1}, \qquad a_{n+1} \dashv I(\cdots), \qquad a_{n+1} \vdash I(\cdots).
$$

This procedure gives $A_n = 2^{n-6}(n+1)!$ $(n \geq 6)$ multilinear identities in degree n which generate the S_n-module of multilinear identities for alternative dialgebras.

Let λ be a partition of n corresponding to an irreducible representation with dimension $d = d_\lambda$ of the symmetric group S_n. Consider a matrix with $(A_n + R_n)d$ rows and $(Z_n + R_n)d$ columns, regarded as a matrix of size $(A_n + R_n) \times (Z_n + R_n)$ in which each entry is a $d \times d$ block; see Figure 2. In the upper left part, the

representation matrices for alternative dialgebra identities ($A_n d \times Z_n d$)	zero matrix ($A_n d \times R_n d$)
representation matrices for expansions of RAC monomials ($R_n d \times Z_n d$)	identity matrix ($R_n d \times R_n d$)

FIGURE 2. Expansion matrix for dicommutator identities in degree n

$n = 3$											
partition λ	3	21	1^3								
multiplicity	1	1	0								
$n = 4$											
partition λ	4	31	2^2	21^2	1^4						
multiplicity	3	8	5	6	1						
$n = 5$											
partition λ	5	41	32	31^2	2^21	21^3	1^5				
multiplicity	8	31	38	43	35	25	5				
$n = 6$											
partition λ	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
multiplicity	19	94	169	185	94	294	179	90	159	84	15

TABLE 4. Multiplicities of representations for degrees 3, 4, 5, 6

$d \times d$ block in position (i, j) contains the representation matrix of the terms in the i-th alternative dialgebra identity which have 0-dialgebra association type j. In the lower left part, the $d \times d$ block in position (i, j) contains the representation matrix of the terms in the expansion of the i-th RAC association type which have 0-dialgebra association type j. The lower right part contains the identity matrix, representing the RAC association types.

For each partition λ, we compute the row canonical form of this matrix, and identify any rows which have leading 1s to the right of the vertical line. These rows represent linear dependence relations among the expansions of the RAC association types resulting from the alternative dialgebra identities; in other words, these rows represent identities satisfied by the dicommutator in every alternative dialgebra. The number of these rows will be called the multiplicity of dicommutator identities for partition λ; see Table 4 for computational results for degrees $3 \leq n \leq 6$.

We perform a second computation to determine which of the dicommutator identities are consequences of the defining identities for Malcev dialgebras. We work in the free RAC algebra so that we only need to consider the consequences of the di-Malcev identity. If $I(a_1, \ldots, a_n)$ is a multilinear nonassociative algebra identity in degree n, then we have $n+2$ consequences in degree $n+1$, obtained by n substitutions and two multiplications. The di-Malcev identity in degree 4 therefore has 6 consequences in degree 5 and 42 consequences in degree 6; in general we call this number D_n. We also consider the skew-symmetries of the RAC association types; for an example see equation (24). In degrees 3, 4, 5, 6 the number of such skew-symmetries is 1, 3, 10, 28 respectively; in general we call this number W_n.

For each partition λ we construct a matrix of size $(W_n+D_n)d \times R_n d$ consisting of an upper part with W_n rows and R_n columns of $d \times d$ blocks, and a lower part with D_n rows and R_n columns of $d\times d$ blocks. The upper part contains the representation matrices for the skew-symmetries, and the lower part contains the representation matrices for the consequences of the di-Malcev identity. We compute the row canonical form and find that in every case its rank is equal to the multiplicity of the dicommutator identities from Table 4. It follows that every identity of degree less than or equal to 6, satisfied by the dicommutator in every alternative dialgebra, is a consequence of right anticommutativity and the di-Malcev identity.

For further information about the application of the representation theory of the symmetric group to polynomial identities for nonassociative algebras, see Bremner and Peresi [1], especially Section 5.

6. Malcev dialgebras with one or two generators

It is well-known that every Malcev algebra on two generators is a Lie algebra. For dialgebras, the corresponding question is whether every two-generated Malcev dialgebra is a Leibniz algebra. In this section we give a negative answer.

We first consider algebras on one generator. A basis of the free right anticommutative algebra on one generator a consists of the elements a^n for $n \geq 1$ defined by $a^1 = a$ and $a^{n+1} = a^n a$; multiplication is determined by the equations

$$a^n a = a^{n+1}, \qquad a^n a^m = 0 \ (m \geq 2). \tag{25}$$

This structure is isomorphic to the free Leibniz algebra on one generator, since Loday and Pirashvili [10] have shown that the free Leibniz algebra on a set X is linearly isomorphic to the free associative algebra on X. Clearly the structure (25) is also the free Malcev dialgebra on one generator; it follows that every Malcev dialgebra with one generator is a Leibniz algebra.

Since the di-Malcev identity has degree 4, in degrees 1, 2, 3 the free Malcev dialgebra on a set X is linearly isomorphic to the free right anticommutative algebra on X. For two generators a, b the following 10 monomials form a basis of the homogeneous subspace of degree 3 in the free right anticommutative algebra:

$$(aa)a, \quad (aa)b, \quad (ab)a, \quad (ab)b, \quad (ba)a, \quad (ba)b, \quad (bb)a, \quad (bb)b, \quad a(ab), \quad b(ab).$$

By the theorem of Loday and Pirashvili, the homogeneous subspace of degree 3 in the free Leibniz algebra on two generators has dimension 8. It follows that the free Malcev dialgebra with two generators is not a Leibniz algebra.

7. Leibniz triple systems from Malcev dialgebras

Loos [11] introduced the following trilinear operation in any Malcev algebra,

$$[a, b, c] = 2(ab)c - (bc)a - (ca)b,$$

and proved that it satisfies the defining identities for Lie triple systems. In this section we extend this result to the setting of dialgebras: we consider the following trilinear operation on any Malcev dialgebra,

$$\langle a, b, c\rangle = 2(ab)c + a(bc) + (ac)b, \tag{LTP}$$

and prove that it satisfies the defining identities for Leibniz triple systems. Thus any subspace of a Malcev dialgebra which is closed under this operation provides an example of a Leibniz triple system.

Definition 18. (Bremner and Sánchez-Ortega [2]) A **Leibniz triple system** is a vector space with a trilinear operation $\langle -,-,-\rangle$ satisfying these identities:

$$\langle a,\langle b,c,d\rangle,e\rangle \equiv \langle\langle a,b,c\rangle,d,e\rangle - \langle\langle a,c,b\rangle,d,e\rangle - \langle\langle a,d,b\rangle,c,e\rangle + \langle\langle a,d,c\rangle,b,e\rangle,$$
$$\langle a,b,\langle c,d,e\rangle\rangle \equiv \langle\langle a,b,c\rangle,d,e\rangle - \langle\langle a,b,d\rangle,c,e\rangle - \langle\langle a,b,e\rangle,c,d\rangle + \langle\langle a,b,e\rangle,d,c\rangle.$$

It follows that any monomial in the second or third association type is equal to a linear combination of monomials in the first association type. The patterns of signs and permutations on the right sides of these identities correspond to the expansions of the Lie triple products $[[b,c],d]$ and $[[c,d],e]$.

Lemma 19. *In a Malcev dialgebra with product ab, the trilinear operation* (LTP) *does not satisfy any polynomial identity in degree* 3*.*

Proof. We construct an 12×18 matrix in which the upper left 6×12 block contains the right anticommutative identities, the lower left 6×12 block contains the expansions of the trilinear monomials, the upper right 6×6 block contains the zero matrix, and the lower right 6×6 block contains the identity matrix. Columns 1–12 correspond to the 12 multilinear monomials of degree 3 in the free nonassociative algebra, and columns 13–18 correspond to the 6 trilinear monomials of degree 3 in the operation (LTP):

$$\langle a,b,c\rangle, \quad \langle a,c,b\rangle, \quad \langle b,a,c\rangle, \quad \langle b,c,a\rangle, \quad \langle c,a,b\rangle, \quad \langle c,b,a\rangle.$$

There are six permutations of the right anticommutative identity $a(bc) + a(cb)$; the (i,j) entry of the upper left block is the coefficient of the j-th nonassociative monomial in the i-th permutation. The $(6+i,j)$ entry of the lower left block is the coefficient of the j-th nonassociative monomial in the expansion of the i-th trilinear monomial. This matrix and its row canonical form are displayed in Figures 3 and 4; we have omitted the zero rows of the RCF. Since there is no row in the RCF which has its leading 1 in the right part of the matrix, there are no dependence relations among the expansions of the trilinear monomials which hold as a result of the right anticommutative identities. □

Theorem 20. *In a Malcev dialgebra with operation ab, every polynomial identity of degree 5 satisfied by the trilinear operation* (LTP) *is a consequence of the defining identities for Leibniz triple systems (Definition 18).*

Proof. The strategy is the same as in the proof of Lemma 19, but the matrix is much larger and some further computations are required. There are 14 association types for a nonassociative binary operation of degree 5:

$$(((ab)c)d)e, \quad ((a(bc))d)e, \quad ((ab)(cd))e, \quad (a((bc)d))e, \quad (a(b(cd)))e,$$
$$((ab)c)(de), \quad (a(bc))(de), \quad (ab)((cd)e), \quad (ab)(c(de)), \quad a(((bc)d)e),$$
$$a((b(cd))e), \quad a((bc)(de)), \quad a(b((cd)e)), \quad a(b(c(de))).$$

Each type admits 5! permutations of the variables, giving 1680 multilinear monomials which correspond to the columns in the left part of the matrix; we order these monomials first by association type and then lexicographically. We need to generate the consequences of degree 5 of the defining identities for Malcev dialgebras.

$$
\left[\begin{array}{cccccccccccc|cccccc}
. & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . \\
\hline
2 & 1 & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . \\
1 & 2 & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . \\
. & . & 2 & 1 & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . \\
. & . & 1 & 2 & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . \\
. & . & . & . & 2 & 1 & . & . & . & . & 1 & . & . & . & . & . & 1 & . \\
. & . & . & . & 1 & 2 & . & . & . & . & . & 1 & . & . & . & . & . & 1
\end{array}\right]
$$

FIGURE 3. The 12×18 matrix from the proof of Lemma 19

$$
\left[\begin{array}{cccccccccccc|cccccc}
1 & . & . & . & . & . & . & -1 & . & . & . & . & \frac{2}{3} & -\frac{1}{3} & . & . & . & . \\
. & 1 & . & . & . & . & . & 1 & . & . & . & . & -\frac{1}{3} & \frac{2}{3} & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & -1 & . & . & . & . & \frac{2}{3} & -\frac{1}{3} & . & . \\
. & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & -\frac{1}{3} & \frac{2}{3} & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & -1 & . & . & . & . & \frac{2}{3} & -\frac{1}{3} \\
. & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & -\frac{1}{3} & \frac{2}{3} \\
. & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & .
\end{array}\right]
$$

FIGURE 4. The row canonical form of the matrix of Figure 3

A multilinear identity $I(a_1,\dots,a_n)$ of degree n produces $n{+}2$ identities of degree $n{+}1$, using n substitutions and two multiplications:

$$I(a_1 a_{n+1},\dots,a_n),\ \dots,\ I(a_1,\dots,a_n a_{n+1}),\ I(a_1,\dots,a_n)a_{n+1},\ a_{n+1}I(a_1,\dots,a_n).$$

The right anticommutative identity of degree 3 produces 5 identities of degree 4, and each of these produces 6 identities of degree 5, for a total of 30. The di-Malcev identity produces 6 identities of degree 5. Altogether we have 36 identities of degree 5, and each admits 5! permutations, giving 4320 identities. Hence the upper left block of the matrix E in degree 5 has size 4320×1680; its (i,j) entry is the coefficient of the j-th nonassociative monomial in the i-th multilinear identity. There are 3 association types for a trilinear operation in degree 5:

$$\langle\langle a,b,c\rangle,d,e\rangle \qquad \langle a,\langle b,c,d\rangle,e\rangle \qquad \langle a,b,\langle c,d,e\rangle\rangle.$$

Each type admits 5! permutations of the variables, giving 360 ternary monomials, corresponding to the columns in the right part of the matrix. The lower left block has size 360×1680; its (i,j) entry is the coefficient of the j-th nonassociative monomial in the expansion, using equation (LTP), of the i-th ternary monomial. The upper right block is the 4320×360 zero matrix, and the lower right block is the 360×360 identity matrix; see Figure 5.

consequences in degree 5 of the Malcev dialgebra identities of Definition 12	zero matrix
expansion using operation (LTP) of the ternary monomials of degree 5	identity matrix

FIGURE 5. The 4680×2040 matrix from the proof of Theorem 20

We compute the row canonical form of this matrix and find that its rank is 1820. We ignore the first 1580 rows since their leading 1s are in the left part, and retain only the last 240 rows which have leading 1s in the right part. We sort these rows by increasing number of nonzero components. We construct another matrix with a 360×360 upper block and a 120×360 lower block. For each of the identities corresponding to the last 240 rows, we apply all 5! permutations of the variables, store the permuted identities in the lower block, and compute the row canonical form; after each iteration, the lower block is the zero matrix. We record the index numbers of the identities which increase the rank:

identity	1	41	71	111	141	143
rank	60	140	160	160	200	240

Further computations show that identities 1, 41, 71 and 111 are consequences of identities 141 and 143: thus these two identities generate the entire 240-dimensional space. Identities 141 and 143 coincide, up to a permutation of the variables, to the identities of Definition 18. We used the Maple package `LinearAlgebra[Modular]` with $p = 101$ for these computations. □

Corollary 21. *Every subspace of a Malcev dialgebra closed under the trilinear operation* $\langle a, b, c\rangle = 2(ab)c + a(bc) + (ac)b$ *is a Leibniz triple system.*

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References

[1] M. R. Bremner, L. A. Peresi: Special identities for quasi-Jordan algebras. *Communications in Algebra* 39, 7 (2011) 2313–2337.

[2] M. R. Bremner, J. Sánchez-Ortega: Leibniz triple systems. 18 pages. Submitted 24 June 2011. `arXiv:1106.5033v1 [math.RA]`

[3] F. Chapoton: Un endofoncteur de la catégorie des opérades. *Dialgebras and Related Operads*, 105–110. Lectures Notes in Mathematics, 1763. Springer-Verlag, 2001.

[4] C. Cuvier: Algèbres de Leibnitz: définitions, propriétés. *Annales Scientifiques de l'École Normale Supérieure* (4) 27, 1 (1994) 1–45.

[5] P. S. Kolesnikov: Varieties of dialgebras and conformal algebras. *Sibirskiĭ Matematicheskiĭ Zhurnal* 49, 2 (2008) 322–339.
[6] D. Liu: Steinberg-Leibniz algebras and superalgebras. *J. Algebra* 283, 1 (2005) 199–221.
[7] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *L'Enseignement Mathématique* (2) 39, 3-4 (1993) 269–293.
[8] J.-L. Loday: Algèbres ayant deux opérations associatives: les digèbres. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique* 321, 2 (1995) 141–146.
[9] J.-L. Loday: Dialgebras. *Dialgebras and Related Operads*, 7–66. Lectures Notes in Mathematics, 1763. Springer-Verlag, 2001.
[10] J.-L. Loday, T. Pirashvili: Universal enveloping algebras of Leibniz algebras and (co)homology. *Mathematische Annalen* 296, 1 (1993) 139–158.
[11] O. Loos: Über eine Beziehung zwischen Malcev-Algebren und Lietripelsystemen. *Pacific Journal of Mathematics* 18 (1966) 553–562.
[12] A. I. Malcev: Analytic loops. *Matematicheskiĭ Sbornik N. S.* 36/78 (1955) 569–576.
[13] A. P. Pozhidaev: 0-dialgebras with bar-unity, Rota-Baxter and 3-Leibniz algebras. *Contemporary Mathematics* 499 (2009) 245–256. American Mathematical Society.
[14] A. A. Sagle: Malcev algebras. *Transactions of the American Mathematical Society* 101 (1961) 426–458.
[15] N. J. A. Sloane: *The On-Line Encyclopedia of Integer Sequences*. http://oeis.org/
[16] B. Vallette: Manin products, Koszul duality, Loday algebras and Deligne conjecture. *Journal für die Reine und Angewandte Mathematik* 620 (2008) 105–164.

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