On the cogrowth of Thompson's group F*

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Abstract

We investigate the cogrowth and distribution of geodesics in R. Thompson's group F.

1 Introduction

In this article we study the cogrowth and distribution of geodesics in Richard Thompson's group F, in an attempt to decide experimentally whether or not F is amenable.

The *cogrowth* of a finitely generated group G is defined as follows. Suppose $S = \{a_1, \dots, a_k\}$ generates G^1, and consider the Cayley graph $\mathcal{G}$ of (G, S). Let r_n be the number of paths in this graph of length n starting and ending at the identity element — let us call such paths *returns*. Since we can concatenate any two such paths to get another we have

$$r_n r_k \leq r_{n+k} \tag{1}$$

and then by Fekete's lemma (see, for example, [23])

$$\rho = \limsup_{n\to\infty} r_n^{1/n} \tag{2}$$

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[1]Formally, we consider G as the epimorphic image from the free monoid generated by $S \cup S^{-1}$, rather than $S \cup S^{-1}$ as being a subset of G

exists. This constant is called the *cogrowth* for (G, S). Since we consider generators and their inverses to label distinct edges in $\mathcal{G}$, then $\rho \leq 2k$.

The connection between this growth rate and amenability was established by Grigorchuk and independently by Cohen:

Theorem 1 ([12, 7]). *Let G, S and ρ be as above. G is amenable if and only if $\rho = 2k$.*

Let p_n be the number of returns of length n on $\mathcal{G}$ which do not contain immediate reversals. Again concatenation shows that p_n is supermultiplicative so Fekete's lemma gives

$$\alpha = \limsup_{n \to \infty} p_n^{1/n} \tag{3}$$

exists. In this case since there are $2k(2k-1)^{n-1}$ freely reduced words of length n in the $2k$ generators and their inverses, we have $\alpha \leq 2k - 1$.

The previous theorem can then be restated as:

Theorem 2 ([12, 7]). *Let G, S and α be as above. G is amenable if and only if $\alpha = 2k - 1$.*

Note that lim sups are required since, for example, if G has a presentation where all relators have even length, the number of returns of odd length (with or without immediate reversals) is 0.

In this article we compute bounds on the cogrowth rates of a number of 2-generator groups: Thompson's group F, the free and free abelian groups on 2 generators, Baumslag-Solitar groups, and various wreath products. Each of these examples, apart from F, is known to be either amenable or non-amenable. We compare the data obtained for F against these examples, to see whether F behaves more like an amenable or a non-amenable group.

The question of the amenability of Thompson's group F has captivated many researchers for some time, initially since F has exponential growth but no nonabelian free subgroups, making it a prime candidate for a counterexample to von Neumann's conjecture that a group is non-amenable if and only if it contains a nonabelian free subgroup. In 1980 Ol'shanskii constructed a finitely generated non-amenable group with no nonabelian free subgroups [14], and in 1982 Adyan gave further examples [1]. In 2002 Ol'shanskii and Sapir constructed finitely presented examples [15]. In spite of these results the amenability or non-amenability of F remains an intensely studied problem.

In the second half of the article we extend our techniques to study the distribution of geodesic words in Thompson's group.

This work is in the same spirit as previous papers by Burillo, Cleary and Wiest [5], and Arzhantseva, Guba, Lustig, and Préaux [2], who also applied computational techniques to consider the amenability of F. We refer the reader to these papers for more background on Thompson's group and the problem of deciding its amenability computationally.

The article is organised as follows. In Section 2 we compute rigorous lower bounds on the cogrowth by computing the dominant eigenvalue of the adjacency matrix of truncated Cayley graphs. We then extrapolate these bounds to estimate the cogrowth and compare and contrast those extrapolations for F and other groups. In Section 3 we use a weighted random sampling of random words in the generators to estimate the exponential growth rate of trivial words in several different groups. As a byproduct we estimate the distribution of geodesic lengths as a function of word-length.

2 Bounding returns and cogrowth

2.1 Bounding the number of returns

Consider the Cayley graph $\mathcal{G}$ of some group G with finite generating set — for the discussion at hand, let us assume that G is generated by two nontrivial elements a, b.

As noted above, an upper bound for the cogrowth ρ is 4. We can compute lower bounds for the number of returns, and thus the cogrowth, as follows.

Consider the following sequence of finite connected subgraphs, $\mathcal{G}_N$ of N vertices that contain the identity.

Set $\mathcal{G}_1$ to be the identity vertex. Record the list of edges incident to $\mathcal{G}_1$. Define $\mathcal{G}_2, \mathcal{G}_3, \ldots$ by appending edges from this list, one at a time. Once the list is exhausted (so $\mathcal{G}_N = B(1)$), repeat the process. It follows that for each $\mathcal{G}_N$ there is an R so that $B(R) \subseteq \mathcal{G}_N \subseteq B(R+1)$.

We can then define $r_{N,n}$ be the number of returns of length n in $\mathcal{G}_N$. Since $\mathcal{G}_N \subset \mathcal{G}_{N+1}$, the sequence $\{r_{N,n}\}$ is supermultiplicative, so $\rho_N = \limsup_{n\to\infty} r_{N,n}^{1/n}$ exists by Fekete's lemma. Further we must have $r_n \geq r_{N,n}$ and so $\rho \geq \rho_N$. Hence we can bound ρ by computing ρ_N.

Using the Perron-Frobenius theorem (in one of its many guises — Proposition V.7 from [11] for example) the growth rate ρ_N of such paths on $\mathcal{G}_N$ is given by the dominant eigenvalue of the corresponding adjacency matrix, provided it is irreducible. We construct $\mathcal{G}_N$ so that it is connected and so the corresponding adjacency matrix is be irreducible.

In some cases we can also demonstrate that the adjacency matrix is aperiodic, which implies that the dominant eigenvalue is simple and dominates all other eigenvalues. This also implies that the corresponding generating function

$\sum p_{N,n} z^n$ has a simple pole at the reciprocal of that eigenvalue. Unfortunately many of the matrices we study are not aperiodic, but they do have period 2.

Perhaps the easiest way to prove that the matrix is aperiodic is to show the existence of two circuits of relatively prime length (see chapter V.5 in [11] for example). In order to show that a matrix has period 2 it suffices (providing the matrix is finite) to show that if there is a path of length k between any two nodes, then there is a path of length $k + 2\ell$ between those two nodes for any ℓ.

It follows that the adjacency matrix for $\mathcal{G}_N$ is aperiodic whenever the group G has a presentation with an odd length relator (since aa^{-1} and the odd length relator form circuits of relatively prime lengths) and if all relators have even length, the matrix has period 2 (since a path of length k can be made into a path of length $k + 2\ell$ by inserting $(aa^{-1})^\ell$).

In particular we have that subgraphs of Baumslag-Solitar groups $BS(p, q)$ with $p + q$ odd are aperiodic, while subgraphs of $BS(p, q)$ with $p + q$ even, Thompson's group F, $\mathbb{Z}^2$, $\mathbb{Z} \wr \mathbb{Z}$ and F_2 (the free group on 2 generators), all with the usual generating sets, have period 2.

Since all of above groups except $\mathbb{Z}^2$ grow exponentially, and $B(R) \subseteq \mathcal{G}_N \subseteq B(R + 1)$, then the radius of $\mathcal{G}_N$ is $O(\log N)$. In the case of $\mathbb{Z}^2$ the radius of $\mathcal{G}_N$ is $O(\sqrt{N})$

We used this method to compute ρ_N for a selection of groups. However, we found significantly better bounds by considering only freely reduced words, *i.e.* paths that did not contain immediate reversals, essentially since there is less to count.

2.2 Bounding the cogrowth

Let p_n be the number of returns of length n on $\mathcal{G}$ which do not contain immediate reversals. We similarly define $p_{N,n}$ to be similar paths on the subgraph $\mathcal{G}_N$. Again we define the exponential growth of these quantities by

$$\alpha = \limsup_{n\to\infty} p_n^{1/n} \qquad\qquad \alpha_N = \limsup_{n\to\infty} p_{N,n}^{1/n}$$

and $\alpha \geq \alpha_N$.

In this case, we cannot now simply concatenate two freely reduced paths to obtain another freely reduced path since it may create an immediate reversal. Thus we do not have similar supermultiplicative relations. We can, however, relate r_n to p_n and ρ to α using the following result of Kouksov [13] which we have specialised to the case of 2 generator groups.

Lemma 3. *(from [13]) Let $R(z) = \sum r_n z^n$ and $C(z) = \sum p_n z^n$ be the generating functions of returns and freely reduced returns respectively. Then*

$$C(z) = \frac{1-z^2}{1+3z^2} R\left(\frac{z}{1+3z^2}\right) \qquad \text{and equivalently}$$
$$R(z) = \frac{-1+2\sqrt{1-12z^2}}{1-16z^2} C\left(\frac{1-\sqrt{1-12z^2}}{6z}\right).$$

A careful generating function argument gives the second equation (and the first is simply its inverse). Consider any freely reduced returning path; it can be mapped to an infinite set of returning paths by replacing each edge s by any returning path in the free group on 2 generators that does start with s^{-1}. At the level of generating functions, this is exactly the substitution

$$z \mapsto \frac{1-\sqrt{1-12z^2}}{6z}.$$

A very general result for generating functions then links the dominant singularity of $R(z)$ to the value of ρ:

Theorem 4. *([11], page 240.) If $f(z)$ is analytic at 0 and ρ is the modulus of a singularity nearest to the origin, then the coefficient $f_n = [z^n]f(z)$ satisfies:*

$$\limsup_{n\to\infty} |f_n|^{-1/n} = \rho$$

Combining these two results (and using the positivity of r_n, p_n) we obtain

Corollary 5. *The constants ρ and α are related by*

$$\rho = \frac{\alpha^2+3}{\alpha}$$

Further if β is a lower bound for α, then $\beta' = \dfrac{\beta^2+3}{\beta}$ is an lower bound for ρ.

We are unable to prove a similar exact relationship between ρ_N and α_N, but we do have the following bound:

Lemma 6. *For a fixed value of N we have $\rho_N \leq \dfrac{\alpha_N^2+3}{\alpha_N}$.*

Proof. Consider the generating functions of returns and freely reduced returns on $\mathcal{G}_n$.

$$R_N(z) = \sum_{n\geq 0} r_{N,n} z^n \qquad\qquad C_N(z) = \sum_{n\geq 0} p_{N,n} z^n$$

It suffices to show that

$$C_N(z) \geq \frac{1-z^2}{1+3z^2} R_N\left(\frac{z}{1+3z^2}\right) \qquad \text{or equivalently}$$

$$R_N(z) \leq \frac{-1+2\sqrt{1-12z^2}}{1-16z^2} C_N\left(\frac{1-\sqrt{1-12z^2}}{6z}\right)$$

within the respective radii of convergence. Let ω be any freely reduced returning path of length k in $\mathcal{G}_N$ — this path contributes z^k to the generating function $C_N(z)$. The substitution $z \mapsto \dfrac{1-\sqrt{1-12z^2}}{6z}$ maps ω to an infinite set of non-reduced returning paths by replacing each edge with freely reduced words from F_2. Some of the resulting words will lie entirely within $\mathcal{G}_N$ and so be enumerated by the generating function $R_N(z)$. However an infinite number of these words will not be contained in $\mathcal{G}_N$. These words are enumerated by

$$\frac{-1+2\sqrt{1-12z^2}}{1-16z^2} C_N\left(\frac{1-\sqrt{1-12z^2}}{6z}\right)$$

but not by $R_N(z)$. Thus the inequality follows. $\square$

To compute α_N we relate it to the dominant eigenvalue of an adjacency matrix. Unfortunately there is no simple way to reuse the adjacency matrix of $\mathcal{G}_N$, in order to enumerate paths without immediate reversal. Instead we construct a new graph $\mathcal{H}_N$ which encodes freely reduced paths in $\mathcal{G}$ as follows: $\mathcal{H}_N$ has N vertices labeled by pairs $(1,-)$ or (g,s) where $g \in G$ and $s \in S$. The vertex $(1,-)$ corresponds to being at the identity vertex of $\mathcal{G}$, and (g,s) to being at the group element $g \in \mathcal{G}$ after having just read a letter s. The edges of $\mathcal{H}_N$ are

$$E(\mathcal{H}_N) = \{((g,s),(h,t)) \in (V(\mathcal{H}_N))^2 \mid h = gt \text{ and } st \neq 1\} \qquad (4)$$

So a path $(1,-),(g_1,s_1),(g_2,s_2),\ldots(g_k,s_k)$ corresponds to a path in $\mathcal{G}$ starting at 1 with $g_1 = s_1, g_2 = s_1s_2, \ldots g_k = s_1s_2\ldots s_k$ a freely reduced word.

We construct $\mathcal{H}_N$ using a breadth-first search similar to the construction of $\mathcal{G}_N$, starting with $\mathcal{H}_1 = (1,-)$ and appending vertices one at a time so that $\{g \in G \mid (g,s) \in \mathcal{H}_N\}$ lies between two balls of a given radius. It follows that $\mathcal{H}_N$ is necessarily connected, and the corresponding adjacency matrices are irreducible.

We then compute the growth rate of paths (and so freely reduced returns) on $\mathcal{H}_N$ by computing the dominant eigenvalue of the corresponding adjacency matrix.

2.3 Exact lower bounds

Since each node of $\mathcal{G}_N$ has outdegree at most 4 and those of the $\mathcal{H}_N$ excluding $(1, -)$ have outdegree at most 3 (vertices on the boundary may have smaller degree) the corresponding adjacency matrices are sparse. We found that the power method and Rayleigh quotients (see [20] for example) converged very quickly to the dominant eigenvalue and so the growth rate.

We constructed $\mathcal{G}_N$ and $\mathcal{H}_N$ for many different values of N ranging between 10^2 and 10^7. Our calculations on Thompson's group F as well as the Baumslag-Solitar groups $BS(2,2), BS(2,3)$ and $BS(3,5)$, yielded the following result.

Theorem 7. *The following are exact lower bounds on the cogrowth, α, of the indicated groups.*

$$BS(2,2) \geq 2.5904 \quad BS(2,3) \geq 2.42579 \quad BS(3,5) \geq 2.06357$$

Thompson's group ≥ 2.17329

This implies that the growth rate of all trivial words, ρ, in these groups are bounded as indicated.

$$BS(2,2) \geq 3.78522 \quad BS(2,3) \geq 3.66250 \quad BS(3,5) \geq 3.51736$$

Thompson's group ≥ 3.55368

Note that all of these bounds were computed using information from $\mathcal{H}_N$ and Corollary 5. We observed that the bounds obtained from $\mathcal{G}_N$ were worse — typically differing in the second or third significant digit. We also note that the above result for $BS(2,3)$ is improves on a result in [9] (the preprint was withdrawn by the authors since it contained an error).

These computations were done on a desktop computer using about 4Gb of memory. It should be noted that while our techniques require both exponential time and memory, it was memory that was the constraining factor. We did implement some simple space-saving methods. Perhaps the most effective of these was to store elements as geodesic words in the generators rather than as their more standard normal forms (eg tree-pairs for Thompson's group or words in the normal form implied by Britton's lemma for the Baumslag-Solitar groups). These geodesic words could then be stored as bit-strings rather than ASCII strings. We believe that by running these computations on a computer with more memory we could improve the bounds, but the returns are certainly diminishing.

2.4 Extrapolation and comparison

The results of the previous section can be extended by considering the sequence of lower bounds α_N and using simple numerical methods to extrapolate them to $N \to \infty$. This required only minimal changes to our computations; after computing the adjacency matrix of $\mathcal{H}_N$ for the maximal value of N, we computed the dominant eigenvalue of submatrices. The corresponding estimate of the eigenvector was then used as an initial vector for estimating the eigenvalue of the next submatrix. This meant that we could compute a sequence of lower bounds in not much more time than it took to compute our best bounds.

In Figure 1 we have plotted α_N against $^1/_{\log N}$ for three non-amenable Baumslag-Solitar groups and F. We found that this gave approximately linear

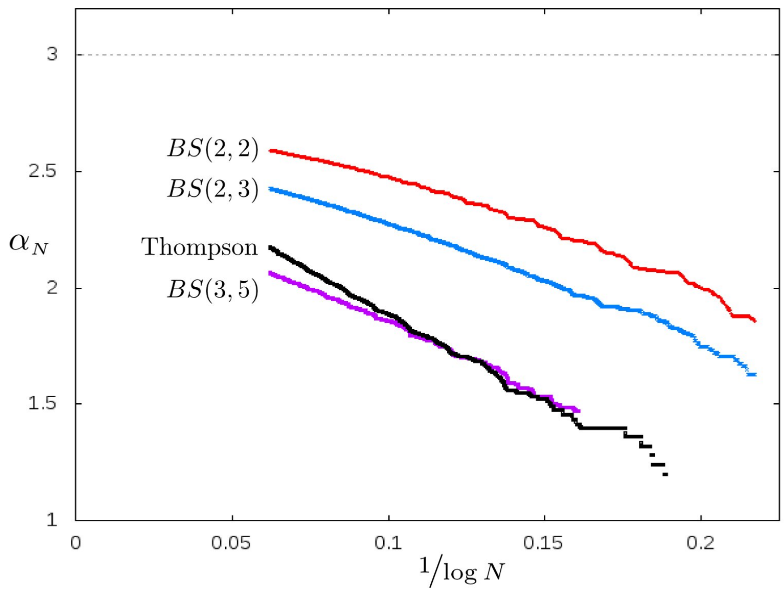

Figure 1: A plot of cogrowth lower bounds α_N against $^1/_{\log N}$. We see that the groups that known to be non-amenable are converging to numbers strictly below 3. The Thompson's group sequence has a clear upward inflection (as $N \to \infty$ or $^1/_{\log N} \to 0$) and so it is difficult to estimate whether the limit is 3 or less than 3.

plots and so this suggests that

$$\alpha_N \approx \alpha_\infty - ^\lambda/_{\log N}.$$

Since α_N is a monotonically increasing sequence and is bounded above by 3, we have that $\alpha_N \to \alpha_\infty$ exists. Unfortunately we cannot prove that $\alpha_\infty = \alpha$, but certainly $\alpha_\infty \leq \alpha$.

Note that the curves terminate at $N = 10^7$ but start at different N-values. This is because the graphs $\mathcal{G}_N$ do not contain freely-reduced loops for small values of N. The smallest value of N for which $\mathcal{G}_N$ contains a freely reduced loop depends on the length of the relations of the group and on the details of the breadth-first search used to construct the graph.

One can observe that the Baumslag-Solitar groups all seem to behave similarly and that the sequences of bounds are clearly converging to constants strictly less than 3. This is completely consistent with the non-amenability of these groups. Thompson's group behaves quite differently — in particular we see that the curve has some upward inflection (as $x \to 0$) and it makes it very unclear as to whether or not α_∞ converges to 3 or below 3.

For the sake of comparison we decided to repeat the above analysis for a set of amenable groups and so we computed similar sequences of lower bounds for $BS(1,2), BS(1,3), \mathbb{Z}^2$ and $\mathbb{Z} \wr \mathbb{Z}$. These results are plotted in Figure 2.

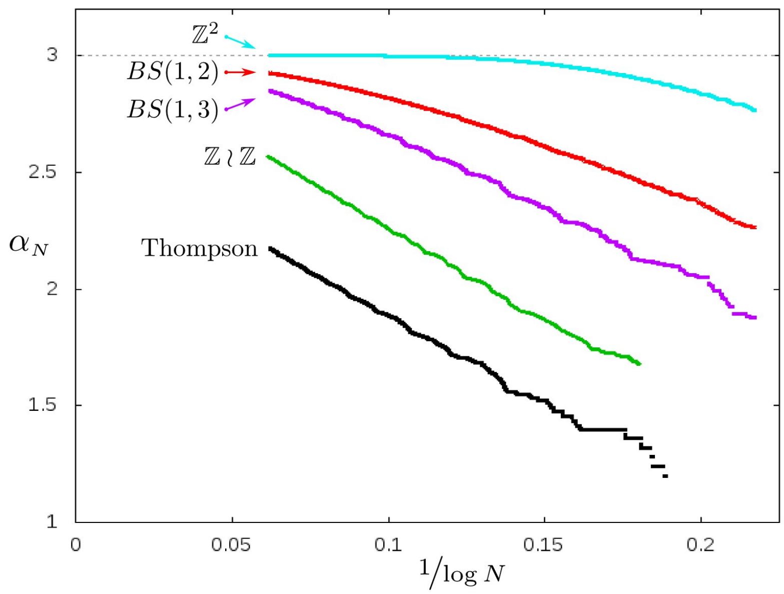

Figure 2: A plot of cogrowth lower bounds α_N against $1/\log N$. We see that the groups that known to be amenable are clearly converging to 3. Again we see that Thompson's group behaves quite differently.

Note that no sequence gives a perfectly straight line and so to estimate α_∞ we fitted the data to the form

$$\alpha_n = \alpha_\infty + \lambda/(\log N)^\delta.$$

We varied the number of data points by removing small-N points and we also varied the value of δ. For any fixed number of points we varied δ to find a value that minimised the R^2 statistic. This gives an "optimal" value of α_∞ and λ.

For some groups, we found that these optimal values were quite sensitive to changes in δ, while other groups were quite robust. To include some measure of this systematic error we moved δ through a range of values so that the R^2 statistic was allowed to move to 5% below its optimal value. These results are summarised in Tables 1 and 2.

The results for all the groups *except* Thompson's group are as one might expect — the amenable groups all give estimates of α_∞ close to 3, and the non-amenable groups give $\alpha_\infty < 3$. Hence it would appear as though this technique is a reasonable test to differentiate amenable and non-amenable groups. Unfortunately it is not sufficiently sensitive to determine the amenability of Thompson's group. In particular we find that the results are too sensitive to variations in δ and to removal of low-N data points. A possible reason for this atypical behaviour is the presence of nested wreath products which converge very slowly to their asymptotic behaviour.

Because of this, we turn to numerical methods based on random sampling and approximate enumeration.

Group	Number of points	Optimal R^2 Value	δ range	α_∞ estimate
$BS(2,2)$	4501	0.998	1.74 ± 0.05	2.682 ± 0.007
	2500	0.998	1.85 ± 0.13	2.672 ± 0.009
$BS(2,3)$	4501	0.999	1.36 ± 0.04	2.597 ± 0.012
	2500	0.999	1.57 ± 0.07	2.562 ± 0.009
$BS(3,5)$	4101	0.998	1.33 ± 0.05	2.29 ± 0.01
	2000	0.998	1.65 ± 0.19	2.24 ± 0.03
F	3947	0.998	0.83 ± 0.07	2.79 ± 0.08
	2000	0.998	0.93 ± 0.16	2.69 ± 0.12
	1700	0.998	0.65 ± 0.21	2.95 ± 0.38

Table 1: Results of fitting eigenvalue data for non-amenable groups and Thompson's group. The Baumslag-Solitar groups all give good results, but Thompson's group does not. There is some upward drift in the estimate of α_∞ as one cuts out small N data, but at the same time the error in the estimates blows up.

Group	Number of points	Optimal R^2 Value	δ range	α_∞ estimate
$BS(1,2)$	4501	0.99975	1.7316 ± 0.0225	3.0158 ± 0.0031642
	2500	0.99981	1.9472 ± 0.0552	2.9975 ± 0.0031542
$BS(1,3)$	4501	0.99894	1.354 ± 0.046	3.0722 ± 0.016473
	2500	0.99855	1.54 ± 0.151	3.0261 ± 0.026664
$\mathbb{Z}^2$	4501	0.99613	5.1624 ± 0.1134	3.002 ± 0.000364
	2500	0.99932	10.996 ± 0.154	$3 \pm 1.7248 \times 10^{-6}$
$\mathbb{Z} \wr \mathbb{Z}$	3947	0.99925	0.8592 ± 0.0436	3.1807 ± 0.050903
	2000	0.99915	1.0237 ± 0.1251	3.0476 ± 0.082052
F	3947	0.99796	0.83 ± 0.072	2.7866 ± 0.083778
	2000	0.99869	0.9344 ± 0.1548	2.6917 ± 0.12318
	1700	0.99848	0.6464 ± 0.2016	2.9532 ± 0.38051

Table 2: Results of fitting eigenvalue data for amenable groups and Thompson's group. All the amenable groups give good results quite close to 3, though $\mathbb{Z} \wr \mathbb{Z}$ is not as good as the others. Also note that since balls in $\mathbb{Z}^2$ grow quadratically with radius rather than exponentially, better results can be obtained by fitting against $1/N^\delta$ rather than $1/(\log N)^\delta$.

2.5 An aside — cogrowth series

As a byproduct of our computations we obtained the first few terms of the cogrowth series for all of these groups. It is well know that the number of trivial words in $\mathbb{Z}^2$ is given by $\binom{2n}{n}^2$ (see A002894 [22]); the corresponding generating function is not algebraic and is expressible as a complete elliptic integral of the first kind. The number of trivial words in F_2 is just the number of returning paths in a quadtree and its generating function is $3(1 + 2\sqrt{1 - 12z^2})^{-1}$ (see A035610 [22]).

Unfortunately we have been unable to find (using tools such as GFUN [21]) any useful explicit or implicit expressions for the cogrowth series (or the generating functions) for any of the other groups we have examined. For completeness we include our data in Table 3.

3 Distribution of geodesic lengths

In this section we broaden our study from the growth rate of trivial words to the distribution of geodesic lengths of all words by sampling random words. In

n	F	BS(1,2)	BS(1,3)	BS(2,2)	BS(2,3)	BS(3,5)	$\mathbb{Z} \wr \mathbb{Z}$
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	10	0	0	0	0	0
6	0	0	12	12	0	0	0
7	0	20	0	0	14	0	0
8	0	64	40	40	0	0	16
9	0	96	0	0	28	0	0
10	20	338	264	224	60	20	72
11	0	736	0	0	84	0	0
12	64	2052	1604	1236	240	64	272
13	0	5208	0	0	564	0	0
14	336	13336	9748	7252	1090	280	1504
15	0	36330	0	0	2760	0	0
16	1160	92636	61720	41192	6492	1048	8576
17	0	248816	0	0	13496	0	0
18	5896	665196	412072	247272	33728	4660	46080
19	0	1771756	0	0	75768	0	0
20	24652	4776094	2750960	1491136	174760	17964	257160
21	0	12848924	0	0	411234	0	0
22	117628	34765448	18725784	9119452	958364	77508	1475592

Table 3: The first few terms of the cogrowth series $C(z)$ for various groups, *i.e.* the number of freely reduced words equivalent to the identity. The first few terms of the returns series $R(z)$ can be obtained from the above using Lemma 3.

previous work of Burillo *et al* [5], random words in Thompson's group F were sampled using simple sampling; words were grown by appending generators one-by-one uniformly at random. Those authors observed only very trivial words and so then sampled uniformly at random from a subset of those words, namely the set of words with balanced numbers of each generator and their inverses. Again, very few trivial words were observed. Indeed if Thompson's group is non-amenable, the probability of observing a trivial word using simple sampling will decay exponentially quickly.

We will proceed along a similar line but using a more powerful random sampling method based on flat-histogram ideas used in the FlatPERM algorithm

[17, 18]. Each sample word is grown in a similar manner to simple sampling — append one generator at a time chosen uniformly at random. The weight of a word of n symbols is simply 1, so that the total weight of all possible words at any given length is just 4^n. As the word grows we keep track of its geodesic length. We now deviate from simple sampling by "pruning" and "enriching" the words.

Consider a word of length n, geodesic length ℓ and weight W. If we have "too many" samples of such words, then with probability $1/2$ prune the current sample or otherwise continue to grow the current sample but with weight $2W$. Similarly if we have "too few" samples of the current length and geodesic length, then enrich by making 2 copies of the current word and then growing a sample from both each with weight $W/2$. Of course, one is free to play around with the precise meaning of "too few" or "too many". We refer the reader to [17, 18] for more details on the implementation of this algorithm. The mean weight (multiplied by 4^n) of all samples of length n and geodesic length ℓ, $c_{n,\ell}$, is then an estimate of the number of such words.

In order to run the above algorithm we need to be able to compute the geodesic length of the element generated by a given random word. Computing geodesic lengths from a normal form is, in general, a very difficult problem and remains stubbornly unsolved for many interesting groups, such as $BS(2,3)$. Because of this we restrict our studies to Thompson's group and a number of different wreath products.

- Thompson's group — a method for computing the geodesic length of an element from its tree-pair representation was first given by Fordham [4], though we found it easier to implement the method of Belk and Brown [3].
- Wreath products — we use the results of [6] to find the geodesic lengths in $\mathbb{Z} \wr \mathbb{Z}$, $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ and $\mathbb{Z} \wr F_2$.

We note that the geodesic problem for Baumslag-Solitar groups has recently been solved in the cases $BS(1,n)$ [10] and $BS(n,kn)$ [8], but we have not implemented these approaches.

3.1 Distributions

We used the random sampling algorithm described above to estimate the distribution of geodesic lengths in Thompson's group F, as well as $\mathbb{Z}\wr\mathbb{Z}$, $\mathbb{Z}\wr F_2$ and $\mathbb{Z}\wr(\mathbb{Z}\wr\mathbb{Z})$. Each run took approximately 1 day on a modest desktop computer. To visualise the results, we started by normalising the data by dividing by the total number of words (*i.e.* 4^n or 6^n). The resulting peak-heights still decay with length, and we found that multiplying by $\sqrt{n}$ compensated for this. The normalised distributions are plotted in Figures 3, 4 and 5.

In each case we see similar behaviour. At short word lengths (*i.e.* small n) the distribution of geodesic lengths is quite wide, but settles to what appears to be a bell-shaped distribution at moderate lengths. This suggests that the geodesic length has an approximately Gaussian distribution about the mean length and that the tails of the distribution are exponentially suppressed. This also explains why the normalising factor of $\sqrt{n}$ works well.

If this is indeed the case, then we expect that trivial words, having geodesic length zero, will be exponentially fewer than 4^n — implying that Thompson's group is non-amenable. Unfortunately things cannot be so simple, because the same reasoning would imply that $\mathbb{Z} \wr \mathbb{Z}$ is non-amenable.

One obvious difference between the graphs is the movement of the peak of the distribution, that is the rate of growth of the mean geodesic length. It is clear that the mean geodesic length of $\mathbb{Z} \wr F_2$ grows linearly, and so the group has a nontrivial rate of escape — exactly as one would expect of a non-amenable group. Similarly we see that the mean geodesic lengths of the other wreath products grow sublinearly, so their rates of escape are zero. When we examine the movement of the peak of Thompson's group's distribution, things are less clear; the mean geodesic length appears to be very nearly linear.

Estimating the mean geodesic length for Thompson's group was substantially easier. We constructed 2^{12} random words of length 2^{16}. As each word was constructed generator-by-generator, the geodesic length was computed and added to our statistics. So while there is correlation between the geodesic lengths at different word lengths *within* a given sample, there is no correlation *between* samples. This took approximately 3 days on a modest desktop computer. Our data is plotted in Figure 6.

We assume that the mean geodesic length grows as n^ν. Linear regression on a log-log plot estimates $\nu \approx 0.98$. Further, if we fit a moving "window", we find that the local estimates of ν increase as the positioning of the window increases. This strongly suggests that the mean geodesic length grows linearly.

To test linearity further, we generated a small number words of length $2^{20} = 1048576$. It took approximately 1 hour to generate each word and compute the corresponding geodesic length, so this was too slow to generate meaningful statistics. In each case we observed that the ratio ℓ/n appeared to converge to approximately 0.28. Of course, this does not preclude more exotic sublinear behaviour such as $n^\nu(\log n)^\theta$. Such logarithmic corrections are extremely difficult to detect or rule out.

We now estimate the rate of escape by assuming linear growth with a

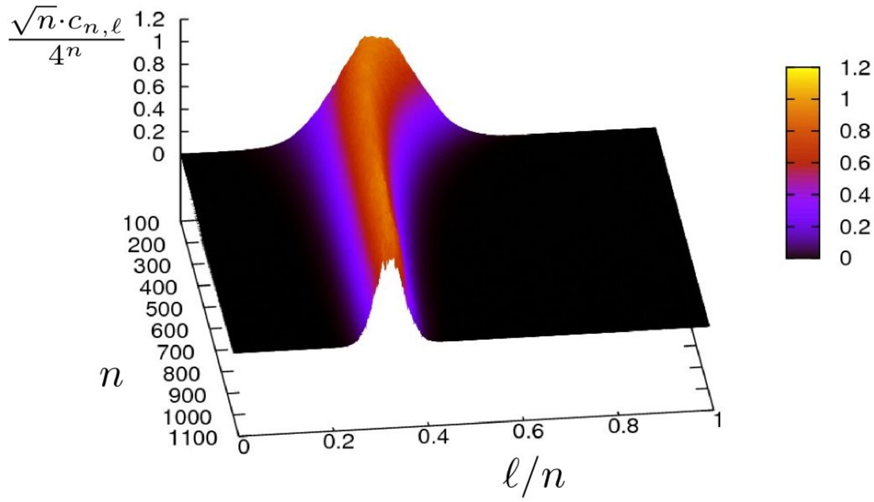

Figure 3: A plot of the normalised distribution of the number of words $c_{n,\ell}$ of length n and geodesic length ℓ in Thompson's group F. Notice that the peak position is quite stable, indicating that the mean geodesic length grows roughly linearly with word length.

polynomial subdominant correction term

$$\langle \ell \rangle_n = An + bn^{\delta}. \tag{5}$$

Our estimates were quite sensitive to changes in δ:

δ	0	$1/4$	$1/3$	$1/2$	$2/3$	$3/4$
A	0.281	0.279	0.279	0.276	0.272	0.267
b	176	17	8.0	1.8	0.47	0.25

(6)

Hence we conclude that the rate of escape is approximately 0.27 with an error of ± 0.01.

We would like to conclude that this positive rate of escape implies that Thompson's group is non-amenable, however there are examples of amenable groups with nontrivial rate of escape. The group $\mathbb{Z}^3 \wr \mathbb{Z}_2$ is amenable but has positive rate of escape [19]. Unfortunately, computing geodesics in this group is equivalent to solving the traveling salesman problem on $\mathbb{Z}^3$ [16] and so beyond these techniques.

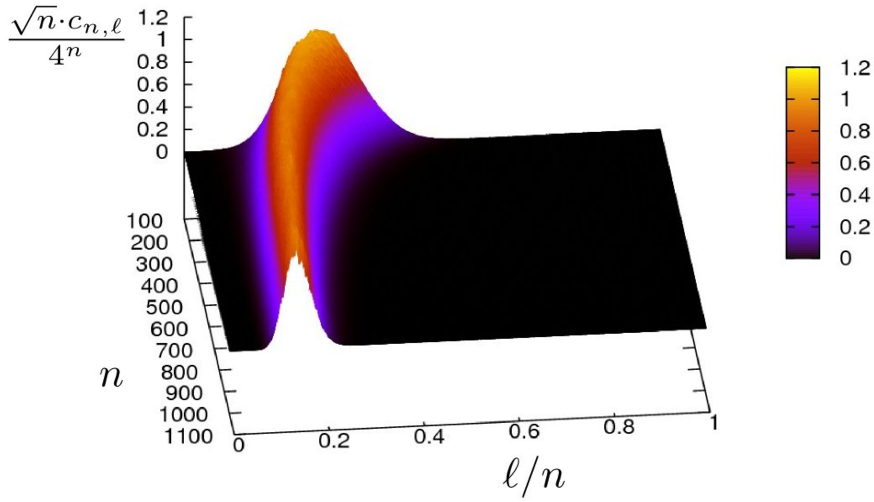

Figure 4: A plot of the normalised distribution of the number of words $c_{n,\ell}$ of length n and geodesic length ℓ in $\mathbb{Z} \wr \mathbb{Z}$. Observe that the peak position is clearly moving towards the left of the plot suggesting that the mean geodesic length grows sublinearly.

4 Conclusions

We have computed exact lower bounds on the cogrowth of several groups including Thompson's group F. In particular, the cogrowth (α) of Thompson's group must be greater than 2.17329. By extrapolating the sequences of lower bounds we see that the bounds for the amenable groups clearly converge to 3, while those of the non-amenable groups converge to numbers strictly less than 3. Thompson's group appears to behave quite differently from the other groups we examined. Our extrapolations do not give clear results, though perhaps they point towards non-amenability.

To further probe this group we used flat histogram methods to estimate the distribution of geodesic lengths in random words. The data suggests that geodesic lengths have an approximately Gaussian distribution about their mean length. Similar Gaussian distributions were observed for other groups, both amenable and non-amenable.

The mean geodesic length of the amenable groups studied grow sublinearly,

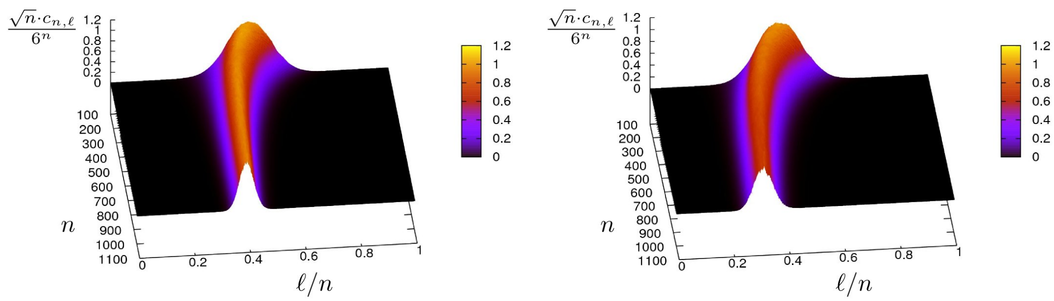

Figure 5: Plot of the normalised distribution of the number of words $c_{n,\ell}$ of length n and geodesic length ℓ in $\mathbb{Z} \wr F_2$ (left) and $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ (right). Observe that the peak is quite stable in the left-hand plot indicating the mean geodesic length is linear, while the right-hand plot the peak shows clear a left drift indicating that the geodesics grow sublinearly.

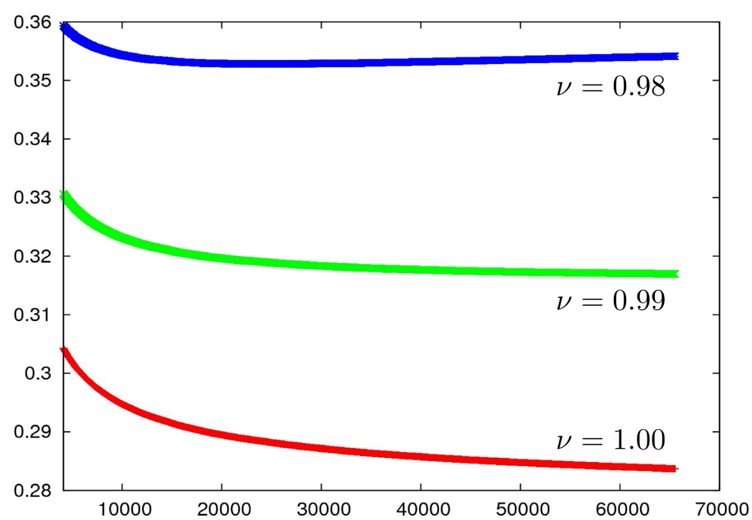

Figure 6: Plot of the mean geodesic length divided by n^ν; for $\nu = 0.98, 0.99$ and 1. This data strongly suggests that Thompson's group has a nontrivial rate of escape. Note that the statistical error was smaller than the symbols used.

while those of $\mathbb{Z} \wr F_2$ and Thompson's group are observed to grow linearly. Using simple sampling we estimate that the mean geodesic length of Thompson's group does indeed grow linearly and that the rate of escape is 0.27 ± 0.01.

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References

[1] S. I. Adyan. Random walks on free periodic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(6):1139–1149, 1343, 1982.

[2] G.N. Arzhantseva, V.S. Guba, M. Lustig, J. Préaux. Testing Cayley graph densities. (Tester les densités de graphes de Cayley.) (French) *Ann. Math. Blaise Pascal* 15(2):233–286, 2008.

[3] J.M. Belk and K.S. Brown. Forest diagrams for elements of Thompson's group F. *International Journal of Algebra and Computation*, 15(6):815–850, 2005.

[4] S. Blake Fordham. Minimal length elements of Thompson's group F. *Geometriae Dedicata*, 99(1):179–220, 2003.

[5] José Burillo, Sean Cleary, and Bert Wiest. Computational explorations in Thompson's group F. In *Geometric group theory*, Trends Math., pages 21–35. Birkhäuser, Basel, 2007.

[6] S. Cleary and J. Taback. Metric properties of the lamplighter group as an automata group. *Geometric Methods In Group Theory: AMS Special Session Geometric Group Theory, October 5-6, 2002, Northeastern University, Boston, Massachusetts: Special Session At The First Joint Mee*, 372:207, 2005.

[7] Joel M. Cohen. Cogrowth and amenability of discrete groups. *J. Funct. Anal.*, 48(3):301–309, 1982.

[8] V. Diekert and J. Laun. On computing geodesics in Baumslag-Solitar groups. *Internat. J. Algebra Comput.*, 21(1–2):119–145, 2011.

[9] K. Dykema and D. Redelmeier. Lower bounds for the spectral radii of adjacency operators on Baumslag-Solitar groups. *Arxiv preprint arXiv:1006.0556*, 2010.

[10] M. Elder. A linear-time algorithm to compute geodesics in solvable Baumslag-Solitar groups. *Illinois J. Math.*, 54(1):109–128, 2010.

[11] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge Univ Pr, 2009.

[12] R. I. Grigorchuk. Symmetrical random walks on discrete groups. In *Multicomponent random systems*, volume 6 of *Adv. Probab. Related Topics*, pages 285–325. Dekker, New York, 1980.

[13] D. Kouksov. On rationality of the cogrowth series. *Proceedings of the American Mathematical Society*, 126(10):2845–2847, 1998.

[14] Alexander Yu. Ol'shanskii. On the question of the existence of an invariant mean on a group. *Uspekhi Mat. Nauk*, 35(4(214)):199–200, 1980.

[15] Alexander Yu. Ol'shanskii and Mark V. Sapir. Non-amenable finitely presented torsion-by-cyclic groups. *Publ. Math. Inst. Hautes Études Sci.*, No. 96:43–169 (2003), 2002.

[16] Walter Parry. Growth series of some wreath products. *Trans. Amer. Math. Soc.*, 331(2):751–759, 1992.

[17] T. Prellberg and J. Krawczyk. Flat histogram version of the pruned and enriched Rosenbluth method. *Physical review letters*, 92(12):120602, 2004.

[18] T. Prellberg, J. Krawczyk, and A. Rechnitzer. Polymer Simulations with a Flat Histogram Stochastic Growth Algorithm. In *Computer simulation studies in condensed-matter physics XVI: proceedings of the seventeenth workshop, Athens, GA, USA, February 16-20, 2004*, page 122. Springer Verlag, 2006.

[19] David Revelle. Rate of escape of random walks on wreath products and related groups. *Ann. Probab.*, 31(4):1917–1934, 2003.

[20] Y. Saad. *Iterative methods for sparse linear systems.* Society for Industrial Mathematics, 2003.

[21] Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[22] N. J. A. Sloane (ed). The On-Line Encyclopedia of Integer Sequences. published electronically at http://oeis.org/.

[23] J.H. van Lint and R.M. Wilson. *A course in combinatorics.* Cambridge Univ Pr, 2001.