

# Resummation of small- $x$ double logarithms in QCD: Semi-inclusive electron-positron annihilation

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## Abstract

We have derived the coefficients of the highest three  $1/x$ -enhanced small- $x$  logarithms of all time-like splitting functions and the coefficient functions for the transverse fragmentation function in one-particle inclusive  $e^+e^-$  annihilation at (in principle) all orders in massless perturbative QCD. For the longitudinal fragmentation function we present the respective two highest contributions. These results have been obtained from KLN-related decompositions of the unfactorized fragmentation functions in dimensional regularization and their structure imposed by the mass-factorization theorem. The resummation is found to completely remove the huge small- $x$  spikes present in the fixed-order results, allowing for stable results down to very small values of the momentum fraction and scaling variable  $x$ . Our calculations can be extended to (at least) the corresponding  $\alpha_s^n \ln^{2n-\ell} x$  contributions to the above quantities and their counterparts in deep-inelastic scattering.

# 1 Introduction

One-hadron inclusive electron-positron annihilation,  $e^+e^- \rightarrow \gamma, Z \rightarrow h + X$  where  $h$  denotes the observed hadron (or a sum over all charged hadron species) and  $X$  any inclusive hadronic final state, is an important benchmark process in perturbative QCD which has been measured accurately over a wide range of centre-of-mass (CM) energies  $\sqrt{s}$  [1]. The results provide crucial inputs for fit determinations of the fragmentation distributions (or parton fragmentation functions)  $D_p^h(x, Q^2)$ , see Refs. [2–4], where  $x$  represents the fraction of the momentum of the final-state parton  $p$  transferred to the outgoing hadron  $h$  and  $Q^2$  is a hard scale, for instance the squared four-momentum  $q$  of the timelike virtual photon or  $Z$ -boson in the above semi-inclusive annihilation (SIA) process,  $Q^2 = q^2 = s$ . SIA data have also provided constraints on the strong coupling constant  $\alpha_s$  [5].

The theoretical description of semi-inclusive  $e^+e^-$  annihilation is analogous to that of electron-hadron deep-inelastic scattering (DIS),  $ep \rightarrow e + X$ , via the exchange of a (spacelike) virtual photon or  $Z$ -boson. The SIA differential cross section can be written in terms of transverse ( $T$ ), longitudinal ( $L$ ) and asymmetric ( $A$ ) fragmentation functions (timelike structure functions) [6],

$$\frac{1}{\sigma_0} \frac{d^2\sigma}{dx d\cos\theta} = \frac{3}{8} (1 + \cos^2\theta) F_T^h(x, Q^2) + \frac{3}{4} \sin^2\theta F_L^h(x, Q^2) + \frac{3}{4} \cos\theta F_A^h(x, Q^2). \quad (1.1)$$

Here  $x = 2E_h/\sqrt{s} \leq 1$  and  $\theta$  are the scaled energy of the hadron  $h$  and its angle relative to the electron beam, respectively, in the CM frame; and for photon exchange  $\sigma_0 = n_c 4\pi\alpha^2/3s$  is the total cross section for Bhabha scattering times the number of colours  $n_c$ . Disregarding corrections suppressed by inverse powers of  $Q$ , the fragmentation functions are related to the fragmentation distributions by

$$F_a^h(x, Q^2) = \sum_{p=q, \bar{q}, g} \int_x^1 \frac{dz}{z} c_{a,p}(z, \alpha_s(Q^2)) D_p^h\left(\frac{x}{z}, Q^2\right). \quad (1.2)$$

The coefficient functions  $c_{a,p}$  in Eq. (1.2) are known to order  $\alpha_s^2$  [7–10], see also Ref. [11], i.e., to the next-to-next-to-leading order (NNLO) for  $F_T$  and  $F_A$  and to the next-to-leading order (NLO) for  $F_L$  which vanishes for  $\alpha_s = 0$ . Here and throughout this article we identify, without loss of information, the  $\overline{\text{MS}}$  renormalization and mass factorization scales with the physical hard scale  $Q^2$ .

The scale dependence of the (process-independent) final-state fragmentation distributions is analogous to that of the initial-state parton distributions and given by

$$\frac{d}{d\ln Q^2} D_i^h(x, Q^2) = \sum_{j=q, \bar{q}, g} \int_x^1 \frac{dz}{z} P_{ji}^T(z, \alpha_s(Q^2)) D_j^h\left(\frac{x}{z}, Q^2\right). \quad (1.3)$$

The (timelike) splitting functions  $P_{ji}^T$  can be expanded in powers of  $a_s \equiv \alpha_s(Q^2)/(4\pi)$ ,

$$P_{ji}^T(x, \alpha_s(Q^2)) = a_s P_{ji}^{(0)T}(x) + a_s^2 P_{ji}^{(1)T}(x) + a_s^3 P_{ji}^{(2)T}(x) + \dots \quad (1.4)$$

The leading-order (LO) and NLO contributions  $P^{(0)T}$  and  $P^{(1)T}$  to Eq. (1.4) have been known for a long time [12–16]. A direct calculation of the NNLO corrections  $P^{(2)T}$  has not been performed so far. However, an indirect determination [11, 17], using non-trivial relations to the spacelike DIS case [18] and the supersymmetric limit [13, 15, 19–23] has been completed recently [24] up to a minor caveat, which is not relevant in the present context, concerning the quark-gluon splitting.

Eq. (1.4) and the corresponding fixed-order approximations to the coefficient functions (see below) are adequate except for  $1-x \ll 1$  and  $x \ll 1$ , where higher-order corrections generally include double logarithms which can spoil the perturbative expansions. Here we focus on the small- $x$  case, where the leading contributions to the  $N^{\text{th}}$ LO splitting functions are of the form

$$P_{ji}^{(n)T}(x) = \delta_{jg} a_i^{(n)} \frac{1}{x} \ln^{2n} x + \dots, \quad a_q^{(n)} = \frac{C_F}{C_A} a_g^{(n)} \quad (1.5)$$

where  $\delta_{ij}$  is the Kronecker symbol, and  $C_A$  and  $C_F$  are the standard SU(N) colour factors, with  $C_A = n_c = 3$  and  $C_F = 4/3$  in QCD. The coefficients  $a_i^{(n)}$  and the corresponding subleading contributions lead to corrections which are numerically far larger than the corresponding single-logarithmic enhancement of the analogous spacelike  $N^{\text{th}}$ LO splitting functions governing the DIS case [25–29]; for  $n = 2$  see Figs. 1 of Refs. [17] and [24]. On the other hand, the all-order Mellin-space summation of the leading-logarithmic (LL) contributions (1.5) leads to [30]

$$\frac{C_A}{C_F} P_{gq}^T(N, \alpha_s) = P_{gg}^T(N, \alpha_s) = \frac{1}{4} (N-1) \left\{ \left( 1 + \frac{32 C_A a_s}{(N-1)^2} \right)^{1/2} - 1 \right\} + \text{NLL terms} \quad (1.6)$$

which can be expanded to all orders in  $x$ -space via the standard Mellin transform

$$\mathbf{M} \left[ \frac{1}{x} \ln^k x \right] (N) \equiv \int_0^1 dx x^{N-1} \frac{1}{x} \ln^k x = \frac{(-1)^k k!}{(N-1)^{k+1}}. \quad (1.7)$$

Eq. (1.6) corresponds to a small and oscillating function in  $x$ -space, suggesting that the small- $x$  enhancement of  $P_{gi}^{(1)T}(x, \alpha_s)$  and  $P_{gi}^{(2)T}(x, \alpha_s)$  – which is negative in the former and positive in the latter case, see below – is unphysical and can be removed by extending Eq. (1.6) to the next-to-leading logarithmic (NLL) and next-to-next-to-leading logarithmic (NNLL) small- $x$  accuracy. Even the former extension has not been performed in the  $\overline{\text{MS}}$  scheme so far, as the results of Ref. [31] are not given in this scheme (and consequently do not agree with the NNLO next-to-leading logarithms of Refs. [17, 24]). For a detailed discussion see Ref. [32] where also the LL result for the  $\overline{\text{MS}}$  transverse coefficient function  $c_{T,g}$  corresponding to Eq. (1.6) has been derived.

In this article we employ constraints provided by the structure of the unfactorized fragmentation functions in dimensional regularization [33] and the all-order mass-factorization formula to derive the coefficients of the respective highest three non-vanishing logarithms for all four timelike splitting functions  $P_{ji}^T(x, \alpha_s)$ ,  $i, j = q, g$ , as well as the corresponding coefficients for both coefficient functions for  $F_T$ , to all (in practice sixteen) orders in  $\alpha_s$ . The derivation of the second/third logarithms is made possible by the NLO/NNLO fixed-order results; consequently only the highest two logarithms can be resummed for the longitudinal fragmentation function  $F_L$ .

The remainder of this article is organized as follows: In Section 2 we describe the theoretical framework used to perform the resummation and comment on the calculations which were carried out using the latest version of FORM and TFORM [34, 35]. The resummed splitting functions are written down and discussed in Section 3. The corresponding results for the transverse and longitudinal coefficient functions are presented in Sections 4 and 5, respectively. Our findings are summarized in Section 6, which also provides a brief outlook to future applications and extensions.

## 2 Method and calculation

The main quantities in our resummation are the unfactorized flavour-singlet partonic fragmentation functions in  $D = 4 - 2\varepsilon$  dimensions

$$\widehat{F}_{a,k}(N, a_s, \varepsilon) = \widetilde{C}_{a,i}(N, a_s, \varepsilon) Z_{ik}^T(N, a_s, \varepsilon) \quad (2.1)$$

where the summation over  $i = q, g$  and the  $\overline{\text{MS}}$  removal of  $(4\pi)^\varepsilon$  and  $\gamma_e$  factors [36] are understood. The  $D$ -dimensional coefficient functions  $\widetilde{C}_{a,i}$  include all non-negative powers of  $\varepsilon$  in Eq. (2.1),

$$\widetilde{C}_{a,i}(N, a_s, \varepsilon) = \delta_{aT} \delta_{iq} + \delta_{a\phi} \delta_{ig} + \sum_{\ell=1}^{\infty} a_s^\ell \sum_{k=0}^{\infty} \varepsilon^k c_{a,i}^{(\ell,k)}(N). \quad (2.2)$$

Besides the fragmentation functions  $F_T$  and  $F_L$  of Eq. (1.1) –  $F_A$  is a non-singlet quantity without  $1/x$  terms – we consider SIA with an intermediate scalar  $\phi$  coupling directly only to gluons via an additional term  $\phi G^{\mu\nu} G_{\mu\nu}$  in the Lagrangian, where  $G^{\mu\nu}$  represents the gluon field strength tensor. Such an interaction, suggested as a QCD trick in Ref. [37], does occur in the Standard Model for the Higgs boson in the limit of a very heavy top quark [38]. The NLO and NNLO quark and gluon coefficient functions for the resulting fragmentation function  $F_\phi$  have been presented in Ref. [24].

The final-state transition functions  $Z_{ik}^T$  collecting all negative powers of  $\varepsilon$  are related to the matrix of the splitting functions in Eq. (1.4) by

$$P^T = \frac{dZ^T}{d\ln Q^2} (Z^T)^{-1} = \beta_D(a_s) \frac{dZ^T}{da_s} (Z^T)^{-1} \quad (2.3)$$

with

$$-\gamma \equiv P^T = \begin{pmatrix} P_{qq}^T & P_{gq}^T \\ P_{qg}^T & P_{gg}^T \end{pmatrix} \quad \text{and} \quad \beta_D(a_s) = -\varepsilon a_s - \beta_0 a_s^2 - \beta_1 a_s^3 - \dots \quad (2.4)$$

As we shall see from the next equation, only the leading coefficient of the four-dimensional beta function of QCD,  $\beta_0 = 11/3 C_A - 2/3 n_f$  [39] where  $n_f$  stands for the number of effectively massless quark flavours, enters the resummation of the highest three small- $x$  logarithms.

Eq. (2.3) can be solved for  $Z$  order by order in  $\alpha_s$ . Suppressing all functional dependences, as already done for most quantities in the previous two equations, the first four orders read

$$\begin{aligned} Z^T = & 1 + a_s \frac{1}{\varepsilon} \gamma_0 + a_s^2 \left\{ \frac{1}{2\varepsilon^2} (\gamma_0 - \beta_0) \gamma_0 + \frac{1}{2\varepsilon} \gamma_1 \right\} \\ & + a_s^3 \left\{ \frac{1}{6\varepsilon^3} (\gamma_0 - \beta_0) (\gamma_0 - 2\beta_0) \gamma_0 + \frac{1}{6\varepsilon^2} \left[ (\gamma_0 - 2\beta_0) \gamma_1 + (\gamma_1 - \beta_1) 2\gamma_0 \right] + \frac{1}{3\varepsilon} \gamma_2 \right\} \\ & + a_s^4 \left\{ \frac{1}{24\varepsilon^4} (\gamma_0 - \beta_0) (\gamma_0 - 2\beta_0) (\gamma_0 - 3\beta_0) \gamma_0 \right. \\ & \quad + \frac{1}{24\varepsilon^3} \left[ (\gamma_0 - 2\beta_0) (\gamma_0 - 3\beta_0) \gamma_1 + (\gamma_0 - 3\beta_0) (\gamma_1 - \beta_1) 2\gamma_0 + (\gamma_1 - 2\beta_1) (\gamma_0 - \beta_0) 3\gamma_0 \right] \\ & \quad \left. + \frac{1}{24\varepsilon^2} \left[ (\gamma_0 - 3\beta_0) 2\gamma_2 + (\gamma_1 - 2\beta_1) 3\gamma_1 + (\gamma_2 - \beta_2) 6\gamma_0 \right] + \frac{1}{4\varepsilon} \gamma_3 \right\} + \dots \quad (2.5) \end{aligned}$$

The corresponding higher-order contributions have been generated in FORM to order  $\alpha_s^{16}$ . It is clear from these results that the N<sup>n</sup>LO corrections, i.e., the splitting functions up to  $\gamma_n \equiv \gamma^{(n)}$  defined analogous to Eq. (1.4) together with  $\beta_0, \dots, \beta_n$ , determine the highest  $n+1$  powers of  $1/\varepsilon$  in Eq. (2.5) at all orders in  $\alpha_s$ . Keeping in mind  $\gamma_n \propto 1/(N-1)^{2n+1}$ , one notices that  $\beta_0$  and  $\beta_0^2$  enter at NLL and NNLL small- $x$  accuracy only. Moreover  $\beta_1$  is suppressed by three powers in  $1/(N-1)$  relative to  $\gamma_1$ ; hence this coefficient contributes only at the level of the fourth logarithms, i.e., beyond our present accuracy.

The same considerations apply to the unfactorized structure functions (2.1), which at N<sup>n</sup>LO require the coefficients  $c_{a,i}^{(\ell,k)}$  with  $\ell+k \leq n$  for  $F_T$  and  $F_\phi$ , and  $\ell+k \leq n+1$  for  $F_L$  in Eq. (2.2). For the convenience of the reader we collect the coefficient function results which form the input of the small- $x$  resummation discussed below. The expansions about  $\bar{N} \equiv N-1=0$  for  $F_T$  read

$$\begin{aligned} c_{T,q}^{(1,0)} &= C_F + O(\bar{N}) , & c_{T,q}^{(1,1)} &= O(\bar{N}^0) , \\ c_{T,q}^{(2,0)} &\cong \frac{64}{3} C_F n_f \bar{N}^{-3} + \frac{16}{3} C_F n_f \bar{N}^{-2} - \frac{80}{27} C_F n_f \bar{N}^{-1} \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} c_{T,g}^{(1,0)} &\cong -8C_F \bar{N}^{-2} - 4C_F \bar{N}^{-1} + \left(\frac{27}{2} - 4\zeta_2\right) C_F \bar{N}^0 , \\ c_{T,g}^{(1,1)} &\cong -16C_F \bar{N}^{-3} - 8C_F \bar{N}^{-2} - (12 - 6\zeta_2) C_F \bar{N}^{-1} , \\ c_{T,g}^{(2,0)} &\cong 160C_F C_A \bar{N}^{-4} - \frac{232}{3} C_F C_A \bar{N}^{-3} - \left[\left(\frac{248}{3} + 16\zeta_2\right) C_F C_A + 8C_F^2\right] \bar{N}^{-2} . \end{aligned} \quad (2.7)$$

The corresponding results for  $F_\phi$  are given by

$$\begin{aligned} c_{\phi,q}^{(1,0)} &= -\frac{14}{3} n_f + O(\bar{N}) , & c_{\phi,q}^{(1,1)} &= O(\bar{N}^0) , \\ c_{\phi,q}^{(2,0)} &\cong \frac{64}{3} C_A n_f \bar{N}^{-3} - \frac{16}{3} C_A n_f \bar{N}^{-2} - \frac{296}{27} C_A n_f \bar{N}^{-1} \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} c_{\phi,g}^{(1,0)} &\cong -8C_A \bar{N}^{-2} + \left[\left(\frac{73}{2} - 4\zeta_2\right) C_A - \frac{7}{3} n_f\right] \bar{N}^0 , \\ c_{\phi,g}^{(1,1)} &\cong -16C_A \bar{N}^{-3} + 6\zeta_2 C_A \bar{N}^{-1} , \\ c_{\phi,g}^{(2,0)} &\cong 160C_A^2 \bar{N}^{-4} - \left[\frac{440}{3} C_A^2 + \frac{16}{3} C_A n_f - \frac{32}{3} C_F n_f\right] \bar{N}^{-3} \\ &\quad - \left[\left(\frac{2092}{9} + 16\zeta_2\right) C_A^2 - \frac{260}{9} C_A n_f - \frac{152}{9} C_F n_f\right] \bar{N}^{-2} . \end{aligned} \quad (2.9)$$

The coefficient functions for  $F_L$  are, to the lesser accuracy required in the present context,

$$\begin{aligned} c_{L,q}^{(1,0)} &= 2C_F + O(\bar{N}) , & c_{L,q}^{(1,1)} &= O(\bar{N}^0) , \\ c_{L,q}^{(2,0)} &\cong -\frac{32}{3} C_F n_f \bar{N}^{-2} - 8C_F n_f \bar{N}^{-1} \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} c_{L,g}^{(1,0)} &\cong 4C_F \bar{N}^{-1} - 4C_F \bar{N}^0 , & c_{L,g}^{(1,1)} &\cong 8C_F \bar{N}^{-2} + 8C_F \bar{N}^{-1} , \\ c_{L,g}^{(2,0)} &\cong -64C_F C_A \bar{N}^{-3} + \left[\frac{176}{3} C_F C_A - 16C_F^2\right] \bar{N}^{-2} . \end{aligned} \quad (2.11)$$

Note that our normalizations of  $c_{T,g}$  and  $c_{L,g}$  differ by a factor of  $1/2$  from those in Refs. [9, 10]. Eqs. (2.6) – (2.11), and some contributions with a higher  $\ell + k$  used to further overconstrain the systems of equations discussed below Eq. (2.15), have been obtained from the full  $x$ -space expressions in terms of harmonic polylogarithms (HPLs) as discussed in Ref. [40] and coded in the HARMPOL package for FORM [34] together with Eq. (1.7).

The corresponding expressions for the NLO and NNLO splitting functions can be read off directly from Eqs. (13) and (14) in Ref. [17] and Eqs. (20) – (23) in Ref. [24]. For completeness we finally give the small- $\bar{N}$  expansions of the LO splitting functions which we need to order  $\bar{N}^1$ ,

$$\begin{aligned} P_{\text{qq}}^{(0)T} &= \left(\frac{5}{2} - 4\zeta_2\right) C_F \bar{N} + O(\bar{N}^2), & P_{\text{qg}}^{(0)} &= \frac{4}{3} n_f - \frac{13}{9} n_f \bar{N} + O(\bar{N}^2), \\ P_{\text{gq}}^{(0)T} &= 4C_F \bar{N}^{-1} - 3C_F + \frac{7}{2} C_F \bar{N} + O(\bar{N}^2), \\ P_{\text{gg}}^{(0)T} &= 4C_A \bar{N}^{-1} - \frac{11}{3} C_A - \frac{2}{3} n_f + \left(\frac{67}{9} C_A - 4\zeta_2\right) C_A \bar{N} + O(\bar{N}^2). \end{aligned} \quad (2.12)$$

An easy way to obtain the coefficients of any desired positive power of  $\bar{N}$  is to transform the functions to  $N$ -space harmonic sums [41], multiply by a sufficiently large power of  $\bar{N}^{-1}$ , transform back to  $x$ -space and proceed as above. Routines for the Mellin transform of the HPLs and its inverse are also provided by the HARMPOL package.

Inserting the  $N$ -space small- $x$  expansions (2.6) – (2.12) into Eqs. (2.1) – (2.5), we obtain the highest three (two) logarithms for the  $\alpha_s^n \varepsilon^{-n+\ell}$ ,  $\ell = 0, 1, 2$  ( $\ell = 1, 2$ ), contributions to  $\widehat{F}_{T,k}$  and  $\widehat{F}_{\phi,k}$  ( $\widehat{F}_{L,k}$ ) to all orders in  $\alpha_s$  for which the higher-order extension of Eq. (2.5) has been coded. It turns out that the  $a_s^n$  contributions to  $\widehat{F}_{a,g}$  for  $a = T, \phi$  can be written as

$$\widehat{F}_{a,g}^{(n)}(N, \varepsilon) = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} \frac{1}{N-1-2(n-\ell)\varepsilon} (A_{a,g}^{(\ell,n)} + \varepsilon B_{a,g}^{(\ell,n)} + \varepsilon^2 C_{a,g}^{(\ell,n)} + \dots) \quad (2.13)$$

or

$$\widehat{F}_{a,g}^{(n)}(x, \varepsilon) = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} (A_{a,g}^{(\ell,n)} + \varepsilon B_{a,g}^{(\ell,n)} + \varepsilon^2 C_{a,g}^{(\ell,n)} + \dots) \quad (2.14)$$

up to terms of order  $(N-1)^0$ , i.e., non- $x^{-1}$  contributions. Eqs. (2.13) and (2.14) and the corresponding results for  $\widehat{F}_{T,q}$ ,  $\widehat{F}_{\phi,q}$  and  $\widehat{F}_{L,i}$  given below form the crucial observation of this article.

Focusing for a moment on the leading logarithms, Eq. (2.14) decomposes  $\widehat{F}_{a,g}^{(n)}$ , which includes terms of the form  $x^{-1} \ln^{n+\delta-1} x$  at all orders  $\varepsilon^{-n+\delta}$  with  $\delta = 0, 1, 2, \dots$ , into  $n$  contributions of the form

$$\varepsilon^{-2n+1} x^{-1-k\varepsilon} = \varepsilon^{-2n+1} x^{-1} \left[ 1 - k\varepsilon \ln x + \frac{1}{2} (k\varepsilon)^2 \ln^2 x + \dots \right] \quad (2.15)$$

with  $k = 2, 4, \dots, 2n$ . Since  $\widehat{F}_{a,g}^{(n)}$  only starts at order  $\varepsilon^{-n}$ , the coefficients  $A_{a,g}^{(\ell,n)}$  in Eq. (2.14) have to be such that the coefficients of  $\varepsilon^0, \dots, \varepsilon^{n-2}$  in the square bracket in Eq. (2.15) cancel in the sum of these  $n$  contributions. Together with the three non-vanishing coefficients of  $\varepsilon^{-n+\ell}$ ,  $\ell = 0, 1, 2$ , in  $\widehat{F}_{a,g}^{(n)}$  known from the above NNLO results, we thus have an overconstrained system of  $n+2$  linear equations for the  $n$  coefficients  $A_{a,g}^{(\ell,n)}$  at each order  $n$  of the strong coupling. It is non-trivial that all these systems have solutions, e.g., there would be no solutions if the factor of two in front of  $(n-\ell)$  in Eqs. (2.13) and (2.14) was absent, or if the sign of this term was different.

The situation is completely analogous for the second and third logarithms. The splitting functions and coefficient functions up to NNLO lead to  $n + 1$  equations for the coefficients  $B_{\ell,n}$ , and to  $n$  equations for the coefficients  $C_{\ell,n}$  in Eqs. (2.13) and (2.14). Also the latter system can be overconstrained at all orders except for  $n = 3$  and  $n = 4$ , from which the corresponding contributions to the N<sup>3</sup>LO coefficient functions  $c_{a,i}^{(3)} \equiv c_{a,i}^{(3,0)}$  and splitting functions  $P_{ji}^{(3)T}$  are determined.

The decomposition corresponding to Eq. (2.13) for  $\widehat{F}_{a,q}^{(n)}$ ,  $a = T, \phi$ , which are suppressed by one power of  $(N - 1)^{-1}$  or  $\ln x$  relative to the gluonic quantities, is given by

$$\widehat{F}_{a,q}^{(n)}(N, \varepsilon) = \frac{1}{\varepsilon^{2n-2}} \sum_{\ell=0}^{n-2} \frac{1}{N-1-2(n-\ell)\varepsilon} (A_{a,q}^{(\ell,n)} + \varepsilon B_{a,q}^{(\ell,n)} + \varepsilon^2 C_{a,q}^{(\ell,n)} + \dots) \quad (2.16)$$

for  $n > 1$  (there are no  $x^{-1}$  terms at order  $\alpha_s$  in these cases, see Eqs. (2.6), (2.8) and (2.12) above). The missing equation, due to the lack of an  $\varepsilon^{-2n+1}$  contribution, is compensated by the absence of an  $x^{-1-2\varepsilon}$  term in the decomposition. Consequently also the three coefficients written out in Eq. (2.16) can be determined from the NNLO quantities given above.

We have solved the systems of equations for these coefficients and their gluonic counterparts in Eq. (2.13) at all orders evaluated for  $Z^T$  in Eq. (2.5), i.e., to order  $\alpha_s^{16}$ . Re-inserting the results into these equations then determines the respective highest three logarithms in  $\widehat{F}_{a,k}^{(n \leq 16)}$  for  $a = T, \phi$  and  $k = q, g$  to all orders in  $\varepsilon$ , after which the mass-factorization can be performed to this order in  $\alpha_s$ . It is worthwhile to recall that, since the coefficients of  $\varepsilon^{-n}, \dots, \varepsilon^{-2}$  at order  $\alpha_s^n$  are given in terms of lower-order quantities, this process includes a very large number of automatic checks. Also these steps have been carried out using FORM and, for the more involved last step, TFORM. The resulting splitting functions and coefficient functions are presented in the next two sections.

Analogous to Eqs. (2.13) and (2.16) the unfactorized partonic longitudinal fragmentation functions at all orders  $n \geq 2$  can be decomposed as

$$\widehat{F}_{L,g}^{(n)}(N, \varepsilon) = \frac{1}{\varepsilon^{2n-2}} \sum_{\ell=0}^{n-1} \frac{1}{N-1-2(n-\ell)\varepsilon} (A_{L,g}^{(\ell,n)} + \varepsilon B_{L,g}^{(\ell,n)} + \dots), \quad (2.17)$$

$$\widehat{F}_{L,q}^{(n)}(N, \varepsilon) = \frac{1}{\varepsilon^{2n-3}} \sum_{\ell=0}^{n-2} \frac{1}{N-1-2(n-\ell)\varepsilon} (A_{L,q}^{(\ell,n)} + \varepsilon B_{L,q}^{(\ell,n)} + \dots) \quad (2.18)$$

up to terms of order  $(N - 1)^0$ . Due to the additional factor of  $\varepsilon$  relative to the previous cases, the determination of the third coefficients  $C_{L,i}^{(\ell,n)}$  would require the presently unknown third-order coefficient functions. The determination of the two highest logarithms in  $c_{L,i}^{(n,0)}$  is performed in the manner discussed in the previous paragraph, and provides additional checks of the splitting functions determined from  $F_T$  and  $F_L$ . The resulting coefficient functions are presented in Section 5.

Like their counterparts for the large- $x$  limit in DIS in Ref. [42] (the publication of the corresponding analysis of SIA is in preparation [43]), see also Refs. [44,45], the decompositions (2.13)–(2.18) are inspired by and related (but not identical) to the decomposition into purely real-emission and the various mixed real-virtual contributions. The cancellations of, e.g., the  $\varepsilon^{-2n+1}, \dots, \varepsilon^{-n+1}$  terms between the  $n$  contributions to Eq. (2.13) are thus related to the KLN theorem [46].

### 3 Resummed timelike splitting functions

We are now ready to present our (mostly) new all-order small- $x$  results. With the exception of graphical illustrations, we will continue to work in Mellin- $N$  space. Recall that the connection to  $x$ -space is simple except for the coefficients of  $(N-1)^k$  with  $k \geq 0$  in the expansion of the lowest order quantities about  $N=1$  which are required for the all-order mass factorization. These coefficients are not included in the all-order formulae below.

In this section we present the resummed timelike splitting functions to next-to-next-to-leading logarithmic (NNLL) accuracy,

$$P_{ij}^T(N) = \sum_{n=0}^{\infty} a_s^{n+1} \left( \delta_{ig} P_{ij,LL}^{(n)T}(N) + P_{ij,NLL}^{(n)T}(N) + P_{ij,NNL}^{(n)T}(N) + \dots \right). \quad (3.1)$$

The leading log (LL) and next-to-leading log (NLL) contributions for  $P_{gg}^T$  and  $P_{gq}^T$  have the form

$$P_{gg,LL}^{(n)T}(N) = \frac{C_A}{C_F} P_{gq,LL}^{(n)T}(N) = -\frac{(-8C_A)^{n+1}}{2(N-1)^{2n+1}} A_{gi}^{(n)} \quad (3.2)$$

and

$$P_{gg,NLL}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-1}}{3(N-1)^{2n}} \left[ (11C_A^2 + 2C_A n_f) B_{gg,1}^{(n)} - 2C_F n_f B_{gg,2}^{(n)} \right], \quad (3.3)$$

$$P_{gq,NLL}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-2} C_F}{3(N-1)^{2n}} \left[ C_A^2 B_{gq,1}^{(n)} + 2C_A n_f B_{gq,2}^{(n)} - 2C_F n_f B_{gq,3}^{(n)} \right]. \quad (3.4)$$

The coefficients in Eqs. (3.2) – (3.4) have been determined to order  $\alpha_s^{16}$  ( $n=15$  in Eq. (3.1)), and are given in Table 1 to the tenth order in  $\alpha_s$  – for the next six orders see the text below Eq. (3.13). The the highest two contributions to  $P_{gg}^T$  and  $P_{gq}^T$  can be written as

$$P_{gg,NNL}^{(n)T}(N) = \frac{C_A}{C_F} P_{qq,NNL}^{(n)T}(N) = \frac{(-8C_A)^n n_f}{3(N-1)^{2n}} 2A_{qi}^{(n)} \quad (3.5)$$

and

$$P_{qq,NNL}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-2} n_f}{9(N-1)^{2n-1}} \left[ C_A^2 B_{qq,1}^{(n)} + C_A n_f B_{qq,2}^{(n)} - C_F n_f B_{qq,3}^{(n)} \right], \quad (3.6)$$

$$P_{qq,NNL}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-3} C_F n_f}{9(N-1)^{2n-1}} \left[ C_A^2 B_{qq,1}^{(n)} + C_A n_f B_{qq,2}^{(n)} - C_F n_f B_{qq,3}^{(n)} \right]. \quad (3.7)$$

The coefficients in Eqs. (3.5) – (3.7) are given in Table 2 to the sixteenth order in  $\alpha_s$ , for brevity using a numerical form for  $n \geq 12$ .

The general form and generating function for these series are known at this point (to this author) only for Eq. (3.2) and the non- $C_F$  terms in the square brackets in Eqs. (3.3) and (3.4), i.e., those entries that do not involve factorial denominators.  $A_{gi}^{(n)}$  are the Catalan numbers [47, 48],

$$A_{gi}^{(n)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n}. \quad (3.8)$$



$n$	$A_{gi}^{(n)}$	$B_{gg,1}^{(n)}$	$B_{gg,2}^{(n)}$	$B_{gq,1}^{(n)}$	$B_{gq,2}^{(n)}$	$B_{gq,3}^{(n)}$	$A_{qi}^{(n)}$
0	1	1	–	9	–	–	–
1	1	1	2	9	–	–	–
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

Table 1: The coefficients of the LL and NLL small- $x$  approximations in  $N$ -space (3.2) – (3.4) for the timelike gluon-gluon and gluon-quark splitting functions for the first ten orders in  $\alpha_s$ . Also shown (last column) are the related NLL quark-parton coefficients in Eq. (3.5).

$B_{gg,1}^{(n)}$  and  $B_{gq,2}^{(n)}$  are given by [49]

$$B_{gg,1}^{(n)} = \binom{2n-1}{n}, \quad B_{gq,2}^{(n)} = \binom{2n-1}{n-2} = B_{gg,1}^{(n)} - A_{gi}^{(n)}, \quad (3.9)$$

and the remaining coefficient in Eq. (3.4) is related to these results by

$$B_{gq,1}^{(n)} = 11B_{gg,1}^{(n)} - 2A_{gi}^{(n)}. \quad (3.10)$$

Furthermore it is interesting to note that the last entries in Eqs. (3.3) and (3.4) have a much simpler difference,

$$B_{gg,2}^{(n)} - B_{gq,3}^{(n)} = 2A_{gi}^{(n)}, \quad (3.11)$$

and that the quark-parton coefficients in Eq. (3.5) are related to the above quantities by

$$A_{qi}^{(n)} + B_{gg,2}^{(n)} = 2B_{gq,1}^{(n)}. \quad (3.12)$$

Hence only one more complicated series is contained in  $B_{gg,2}^{(n)}$ ,  $B_{gq,3}^{(n)}$  and  $A_{qi}^{(n)}$ ; and an analytic formula for any of these quantities would lead to closed expressions for all  $\alpha_s^{n+1}/(N-1)^{2n-1}$  contributions to the timelike splitting functions. The coefficients in Eq. (3.5) for  $n = 10, \dots, 15$  are

$$\begin{aligned} A_{qi}^{(10)} &= \frac{12229277}{630}, & A_{qi}^{(11)} &= \frac{136789507}{1980}, & A_{qi}^{(12)} &= \frac{245398487}{990}, \\ A_{qi}^{(13)} &= \frac{16139182231}{18018}, & A_{qi}^{(14)} &= \frac{6986603759}{2145}, & A_{qi}^{(15)} &= \frac{102190158383}{8580}. \end{aligned} \quad (3.13)$$

$n$	$B_{\text{qg}}^{(n)1}$	$B_{\text{qg}}^{(n)2}$	$B_{\text{qg}}^{(n)3}$	$B_{\text{qq}}^{(n)1}$	$B_{\text{qq}}^{(n)2}$	$B_{\text{qq}}^{(n)3}$
2	2	1	2	1	–	–
3	$\frac{89}{3}$	$\frac{17}{3}$	10	26	2	$\frac{8}{3}$
4	$\frac{863}{6}$	$\frac{49}{2}$	$\frac{253}{6}$	$\frac{395}{3}$	$\frac{37}{3}$	$\frac{107}{6}$
5	$\frac{3028}{5}$	$\frac{997}{10}$	$\frac{5153}{30}$	$\frac{16961}{30}$	$\frac{892}{15}$	$\frac{913}{10}$
6	$\frac{219389}{90}$	$\frac{3574}{9}$	$\frac{20701}{30}$	$\frac{207263}{90}$	$\frac{11807}{45}$	$\frac{12617}{30}$
7	$\frac{6069467}{630}$	$\frac{197447}{126}$	$\frac{57770}{21}$	$\frac{412927}{45}$	$\frac{349373}{315}$	$\frac{64299}{35}$
8	$\frac{38066605}{1008}$	$\frac{2216267}{360}$	$\frac{9169289}{840}$	$\frac{36475945}{1008}$	$\frac{11537219}{2520}$	$\frac{6518189}{840}$
9	$\frac{35395649}{240}$	$\frac{8688247}{360}$	$\frac{7255001}{168}$	$\frac{143112541}{1008}$	$\frac{46944767}{2520}$	$\frac{81079091}{2520}$
10	$\frac{1449188057}{2520}$	$\frac{595482761}{6300}$	$\frac{537123949}{3150}$	$\frac{22226523}{40}$	$\frac{157729997}{2100}$	$\frac{59261597}{450}$
11	$\frac{41422167289}{18480}$	$\frac{17097960349}{46200}$	$\frac{93203672711}{138600}$	$\frac{120436395671}{55440}$	$\frac{41718615557}{138600}$	$\frac{8228126859}{15400}$
12	$8.7381596 \cdot 10^6$	$1.4491517 \cdot 10^6$	$2.6500373 \cdot 10^6$	$8.4902823 \cdot 10^6$	$1.2012745 \cdot 10^6$	$2.1542828 \cdot 10^6$
13	$3.4082509 \cdot 10^7$	$5.6761999 \cdot 10^6$	$1.0438829 \cdot 10^7$	$3.3186784 \cdot 10^7$	$4.7804744 \cdot 10^6$	$8.6473779 \cdot 10^6$
14	$1.3302661 \cdot 10^8$	$2.2242725 \cdot 10^7$	$4.1111839 \cdot 10^7$	$1.2976945 \cdot 10^8$	$1.8985567 \cdot 10^7$	$3.4597524 \cdot 10^7$
15	$5.1960779 \cdot 10^8$	$8.7203656 \cdot 10^7$	$1.6190693 \cdot 10^8$	$5.0769752 \cdot 10^8$	$7.5293381 \cdot 10^7$	$1.3808638 \cdot 10^8$

Table 2: The corresponding coefficients in Eqs. (3.5) – (3.7) for the timelike quark-gluon and quark-quark splitting functions to the sixteenth order in the strong coupling constant.

The corresponding expressions for Eqs. (3.2) – (3.4) can be inferred from Eqs. (3.8) – (3.12).

The results (3.8) – (3.10) lead to the closed NLL expressions

$$\begin{aligned}
P_{\text{gg}}^T(N) \Big|_{C_F=0} &= \left\{ (1-4\xi)^{1/2} - 1 \right\} \frac{1}{4} (N-1) \\
&\quad - \left\{ (1-4\xi)^{-1/2} + 1 \right\} a_s \left( \frac{11}{6} C_A + \frac{1}{3} n_f \right) + P_{\text{gg,NNL}}^{(n)T}(N), \quad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
\left[ \frac{C_A}{C_F} P_{\text{gq}}^T(N) \right]_{C_F=0}^{\text{NLL}} &= \left\{ (1-4\xi)^{1/2} - 1 \right\} \frac{1}{24} (N-1)^2 (1+n_f/C_A) \\
&\quad - \left\{ (1-4\xi)^{-1/2} + 1 \right\} \left( \frac{11}{6} C_A + \frac{1}{3} n_f \right) a_s \quad (3.15)
\end{aligned}$$

with

$$\xi = -\frac{8C_A a_s}{(N-1)^2} \quad \text{and} \quad a_s \equiv \frac{\alpha_s}{4\pi}. \quad (3.16)$$

The first line of Eq. (3.14) and the directly related LL part of  $P_{\text{gq}}^T(N)$  agree, of course, with the classical result (1.6) of Refs. [30]. Already at order  $\alpha_s^3$  [17, 24], the NLL second line of Eq. (3.14) is not the same as the result in Ref. [31] which does not refer to the  $\overline{\text{MS}}$  scheme, see Ref. [32].

The expressions for the third logarithms (NNLL for  $P_{\text{gi}}$  and  $\text{N}^3\text{LL}$  for  $P_{\text{qi}}$ ) are far more lengthy. We therefore confine ourselves here to the full analytic expressions at order  $\alpha_s^4$ , and present the higher-order coefficients only in numerical form for the case of QCD,  $C_A = 3$  and  $C_F = 4/3$ . The leading  $N \rightarrow 1$  behaviour of  $P_{\text{gg}}^{(3)T}$  and  $P_{\text{gq}}^{(3)T}$  is given by

$$\begin{aligned} P_{\text{gg}}^{(3)T}(N) = & -\frac{512}{(N-1)^7} 20C_A^4 + \frac{512}{(N-1)^6} \left\{ \frac{110}{3} C_A^4 + \frac{20}{3} C_A^3 n_f - \frac{98}{9} C_A^2 C_F n_f \right\} \\ & + \frac{512}{(N-1)^5} \left\{ \left( -\frac{899}{18} + 16\zeta_2 \right) C_A^4 - 15C_A^3 n_f + \frac{76}{3} C_A^2 C_F n_f - \frac{2}{3} C_A^2 n_f^2 \right. \\ & \left. + \frac{14}{9} C_A C_F n_f^2 - \frac{16}{27} C_F^2 n_f^2 \right\} + \dots \end{aligned} \quad (3.17)$$

with  $\zeta_2 = \pi^2/6$  and

$$\begin{aligned} P_{\text{gq}}^{(3)T}(N) = & -\frac{512}{(N-1)^7} 20C_A^3 C_F + \frac{512}{(N-1)^6} \left\{ \frac{100}{3} C_A^3 C_F + \frac{10}{3} C_A^2 C_F n_f - \frac{38}{9} C_A^2 C_F n_f \right\} \\ & + \frac{512}{(N-1)^5} \left\{ \left( -\frac{110}{3} + \frac{26}{3} \zeta_2 \right) C_A^3 C_F - \left( \frac{55}{12} - \frac{22}{3} \zeta_2 \right) C_A^2 C_F^2 - \frac{793}{108} C_A^2 C_F n_f \right. \\ & \left. + \frac{221}{27} C_A C_F^2 n_f - \frac{1}{9} C_A C_F n_f^2 + \frac{4}{27} C_F^2 n_f^2 \right\} + \dots \end{aligned} \quad (3.18)$$

The corresponding results for  $P_{\text{qg}}^{(3)T}$  and  $P_{\text{qq}}^{(3)T}$  read

$$\begin{aligned} P_{\text{qg}}^{(3)T}(N) = & -\frac{512}{(N-1)^6} \frac{22}{9} C_A^3 n_f + \frac{512}{(N-1)^5} \left\{ \frac{89}{27} C_A^3 n_f + \frac{17}{27} C_A^2 n_f^2 - \frac{10}{9} C_A C_F n_f^2 \right\} \\ & + \frac{512}{(N-1)^4} \left\{ \left( -\frac{187}{72} + \frac{4}{3} \zeta_2 \right) C_A^3 n_f + \left( \frac{1}{9} + \frac{2}{9} \zeta_2 \right) C_A^2 C_F n_f - \frac{23}{18} C_A^2 n_f^2 \right. \\ & \left. + \frac{185}{81} C_A C_F n_f^2 - \frac{1}{27} C_A n_f^3 + \frac{2}{27} C_F n_f^3 \right\} + \dots \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} P_{\text{qq}}^{(3)T}(N) = & -\frac{512}{(N-1)^6} \frac{22}{9} C_A^2 C_F n_f + \frac{512}{(N-1)^5} \left\{ \frac{26}{9} C_A^2 C_F n_f + \frac{2}{9} C_A C_F n_f^2 - \frac{8}{27} C_F^2 n_f^2 \right\} \\ & + \frac{512}{(N-1)^4} \left\{ \left( -\frac{763}{648} + \frac{4}{9} \zeta_2 \right) C_A^2 C_F n_f - \left( \frac{4}{9} - \frac{10}{9} \zeta_2 \right) C_A C_F^2 n_f \right. \\ & \left. - \frac{46}{81} C_A C_F n_f^2 + \frac{50}{81} C_F^2 n_f^2 \right\} + \dots, \end{aligned} \quad (3.20)$$

where the dots indicate terms beyond the present accuracy of the expansion in powers of  $1/(N-1)$ . The respective NNLL and  $\text{N}^3\text{LL}$  higher-order expressions are written as

$$P_{\text{gi,NNL}}^{(n)T}(N) = \frac{(-1)^n}{(N-1)^{2n-1}} \left( 96^n C_{\text{gi},0}^{(n)} + 96^{n-1} C_{\text{gi},1}^{(n)} n_f + 96^{n-2} C_{\text{gi},2}^{(n)} n_f^2 \right) \quad (3.21)$$

and

$$P_{\text{qi}, \text{N}^3\text{L}}^{(n)T}(N) = \frac{(-1)^n}{(N-1)^{2n-2}} \left( 96^{n-1} C_{\text{qi},1}^{(n)} n_f + 96^{n-2} C_{\text{qi},2}^{(n)} n_f^2 + 96^{n-3} C_{\text{qi},3}^{(n)} n_f^3 \right). \quad (3.22)$$

The coefficients for Eq. (3.21) and Eq. (3.22) are given in Table 3 and Table 4, respectively. Here the relative normalization of the coefficients of different orders in  $\alpha_s$  is such that the ratios  $C_{ij,1}^{(n)}/C_{ij,1}^{(n-1)}$  will tend to one for  $n \rightarrow \infty$ , if the NNLL correction have the same convergence properties as the LL and NLL contributions in Eqs. (3.14) and (3.15). The present calculations have not been carried out to an order sufficient to definitely decide whether this is indeed the case.

The fixed-order and resummed timelike splitting functions are illustrated and compared in Figs. 1 – 4 at a standard reference scale,  $Q^2 \simeq M_Z^2$ , for  $n_f = 5$  effectively massless flavours. For the corresponding value  $\alpha_s \simeq 0.12$  of the strong coupling constant, the expansions to order  $\alpha_s^{16}$  are sufficient, and for some of the NNLL results required, for an accuracy of 0.1% or better down to the lowest  $x$ -values shown,  $x = 10^{-4}$ . An extension of the maximal order to cover one more order of magnitude in  $x$  is definitely feasible, but does not appear to be warranted for any foreseeable analyses of experimental data.

It is clear from Figs. 1 and 2 that the available fixed-order approximations to the splitting functions are not reliable at  $x \lesssim 10^{-3}$  for the gluon-parton cases – recall Eq. (1.3) and the form (2.4) of the timelike splitting function matrix, which is transposed relative to the spacelike case of the initial-state parton distributions – and  $x \lesssim 10^{-2}$  for the quark-parton cases. Obviously it is also insufficient to only add the previously known leading-logarithmic resummation [30] from order  $\alpha_s^4$  to the NNLO gluon-quark and gluon-gluon splitting functions in Fig. 1. On the other hand, a near-perfect cancellation of the strong  $x$ -dependences is exhibited by the NNLO+NNLL results for  $xP_{ji}^T$  especially in these cases. The situation is somewhat less clear-cut for the quark-parton splitting functions in Fig. 2 where, as already at order  $\alpha_s^3$  but unlike the gluon-parton cases, the effects of the second and third logarithms have the same sign. Within the present uncertainties all results appear to be consistent with  $xP_{ji}^T \approx 0$  at  $x < 10^{-2}$ .

In Figs. 3 and 4 the known three fixed-order approximations are compared by their resummed counterparts obtained by adding the ‘appropriate’ higher-order resummations to the respective fixed-order results, i.e., forming the LO+LL (for the gluon-parton cases), NLO+NLL and NNLO+NNLL combinations. The differences between the two expansions at  $x < 10^{-2}$  are striking. Some questions remain due to the relatively large NNLO+NNLL corrections in Fig. 3 and the corresponding behaviour at  $x < 10^{-3}$  in Fig. 4. Their answer will require the calculation of the fourth-order ( $\text{N}^3\text{LO}$ ) splitting functions (from which the  $\text{N}^3\text{LL}$  resummations for  $P_{\text{gq}}^T$  and  $P_{\text{gg}}^T$  can be inferred analogously to the present calculations) which, unfortunately, is not expected in the near future. In the meantime the NNLO+NNLL results, and their comparison with the previous NLO+NLL resummed order, should be sufficient for practical data analysis including estimates of the effect of the presently unknown higher orders.

$n$	$C_{gg,0}^{(n)}$	$C_{gg,1}^{(n)}$	$C_{gg,2}^{(n)}$	$C_{gq,0}^{(n)}$	$C_{gq,1}^{(n)}$	$C_{gq,2}^{(n)}$
1	0	3.4074074	–	–0.3148148	–	–
2	0.7923411	5.4814815	2.3703704	0.2174233	2.2469136	–
3	1.1074453	5.6111111	4.4334705	0.3976321	2.4698217	0.9657064
4	1.3336401	5.6790123	5.0809328	0.5129934	2.5064300	1.3924707
5	1.5204469	5.7204475	5.2181070	0.6035361	2.5046339	1.5768328
6	1.6839029	5.7522248	5.1713306	0.6809114	2.4969915	1.6546258
7	1.8313932	5.7823077	5.0603175	0.7498896	2.4920010	1.6831461
8	1.9670281	5.8140519	4.9316765	0.8128873	2.4915980	1.6880472
9	2.0933792	5.8486686	4.8040710	0.8713173	2.4957684	1.6815451
10	2.2121870	5.8864117	4.6847584	0.9260927	2.5039523	1.6697449
11	2.3246982	5.9271265	4.5761405	0.9778468	2.5154984	1.6558008
12	2.4318435	5.9705020	4.4785239	1.0270426	2.5298070	1.6413704
13	2.5343410	6.0161860	4.3913363	1.0740322	2.5463666	1.6273227
14	2.6327593	6.0638358	4.3136775	1.1190916	2.5647543	1.6140995
15	2.7275579	6.1131386	4.2445696	1.1624423	2.5846246	1.6019061

Table 3: The numerical coefficients of the NNLL small- $x$  approximations (3.21) in  $N$ -space for the timelike gluon-gluon and gluon-quark splitting functions in QCD to the sixteenth order in  $\alpha_s$ .

$n$	$C_{qg,1}^{(n)}$	$C_{qg,2}^{(n)}$	$C_{qg,3}^{(n)}$	$C_{qq,1}^{(n)}$	$C_{qq,2}^{(n)}$	$C_{qq,3}^{(n)}$
2	–7.0398681	3.1604938	–	–3.3757439	–	–
3	0.2881972	12.609054	6.3209877	–0.1122633	6.2624600	–
4	3.8811194	19.180041	12.349337	1.4382426	9.0594422	2.2240512
5	6.2663008	23.451903	16.193141	2.4695470	10.673136	3.9805314
6	8.1028556	26.382647	18.550343	3.2699679	11.725549	5.2086911
7	9.6308924	28.524947	20.014717	3.9411132	12.479542	6.0662810
8	10.960497	30.185034	20.944287	4.5288220	13.063470	6.6786404
9	12.150652	31.537321	21.546983	5.0574954	13.545017	7.1280203
10	13.236613	32.685054	21.945288	5.5417090	13.961930	7.4670190
11	14.241194	33.691738	22.213216	5.9909234	14.336239	7.7296206
12	15.180069	34.597838	22.396683	6.4116708	14.681305	7.9382263
13	16.064511	35.429968	22.524875	6.8086793	15.005512	8.1079291
14	16.902933	36.206143	22.616740	7.1854997	15.314277	8.2491157
15	17.701801	36.938872	22.684797	7.5448791	15.611200	8.3690760

Table 4: The corresponding N<sup>3</sup>LL coefficients in (3.22) for the timelike quark-gluon and quark-quark splitting functions in QCD to the sixteenth order in the strong coupling constant.

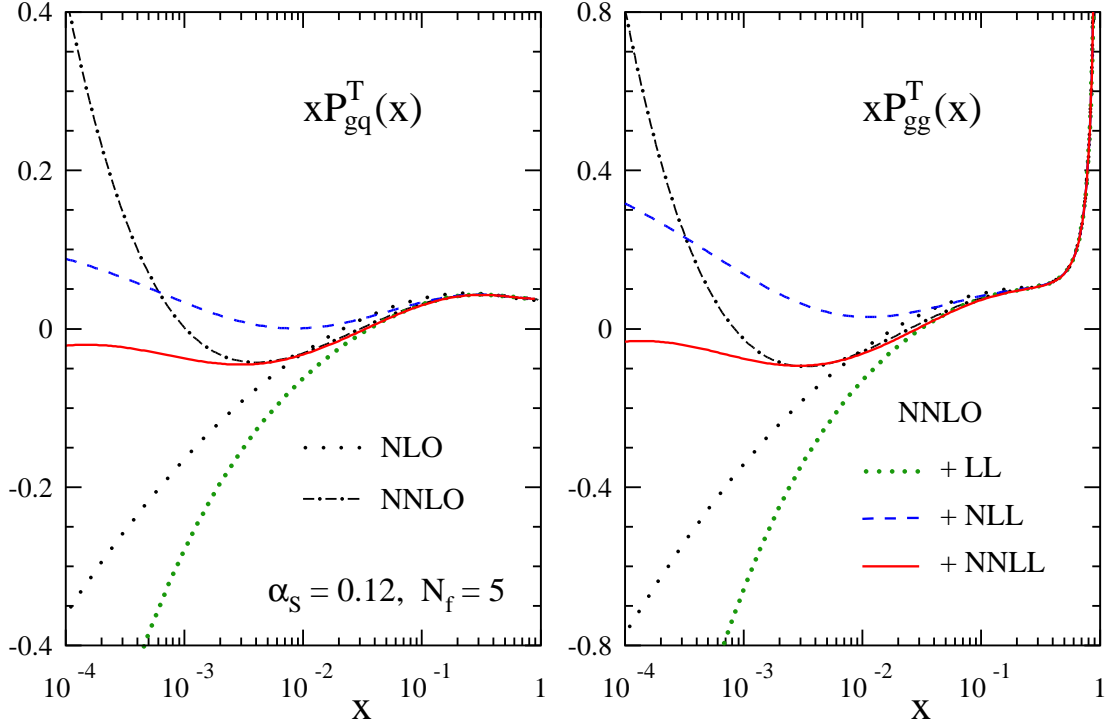


Figure 1: The timelike gluon-quark and gluon-gluon splitting functions at a typical value of the strong coupling constant  $\alpha_s$ , multiplied by  $x$  for display purposes. Shown are the NLO and NNLO approximations, and the consequences of adding the leading ( $\alpha_s^{n-1} \ln^{2n} x$ ), next-to-leading and next-to-next-to-leading small- $x$  logarithms to the latter at all higher orders in  $\alpha_s$ .

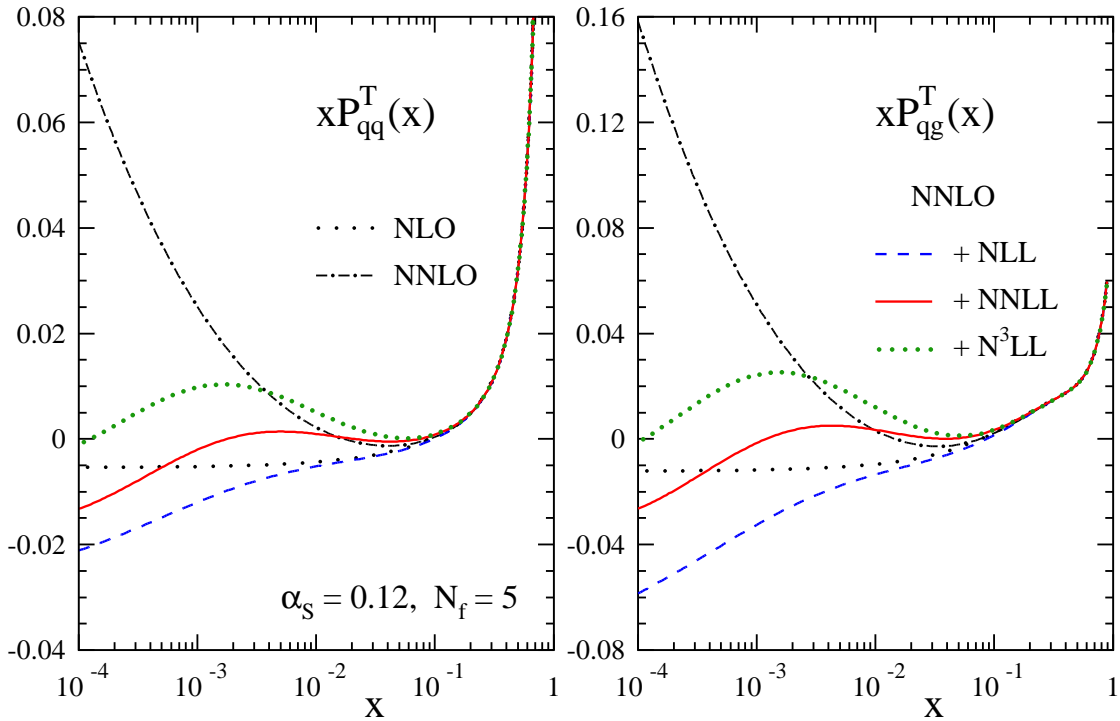


Figure 2: As Fig. 1, but for the timelike quark-quark and quark-gluon splitting functions, where the highest logarithms are of the next-to-leading logarithmic form  $\alpha_s^{n-1} \ln^{2n-1} x$ .

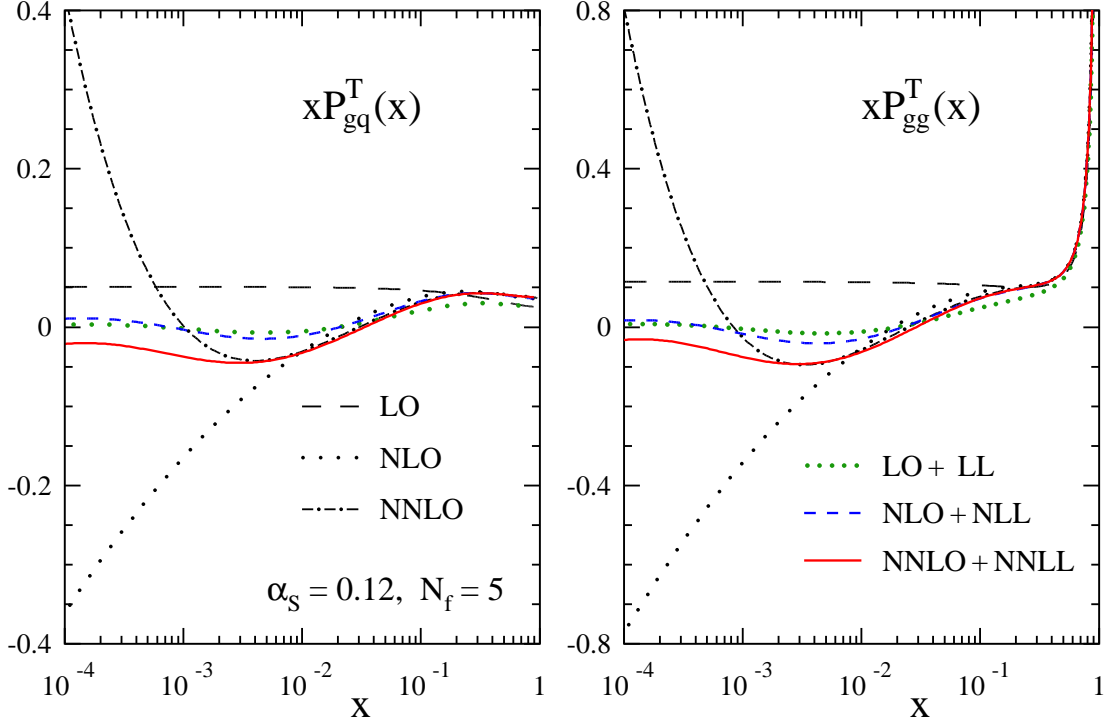


Figure 3: The timelike gluon-quark and gluon-gluon splitting functions at a typical value of  $\alpha_s$ , multiplied by  $x$  for display purposes. The LO, NLO and NNLO fixed-order approximations are compared with the small- $x$  resummed results obtained by respectively adding the LL, NLL and NNLL contributions at all numerically relevant higher orders in  $\alpha_s$ .

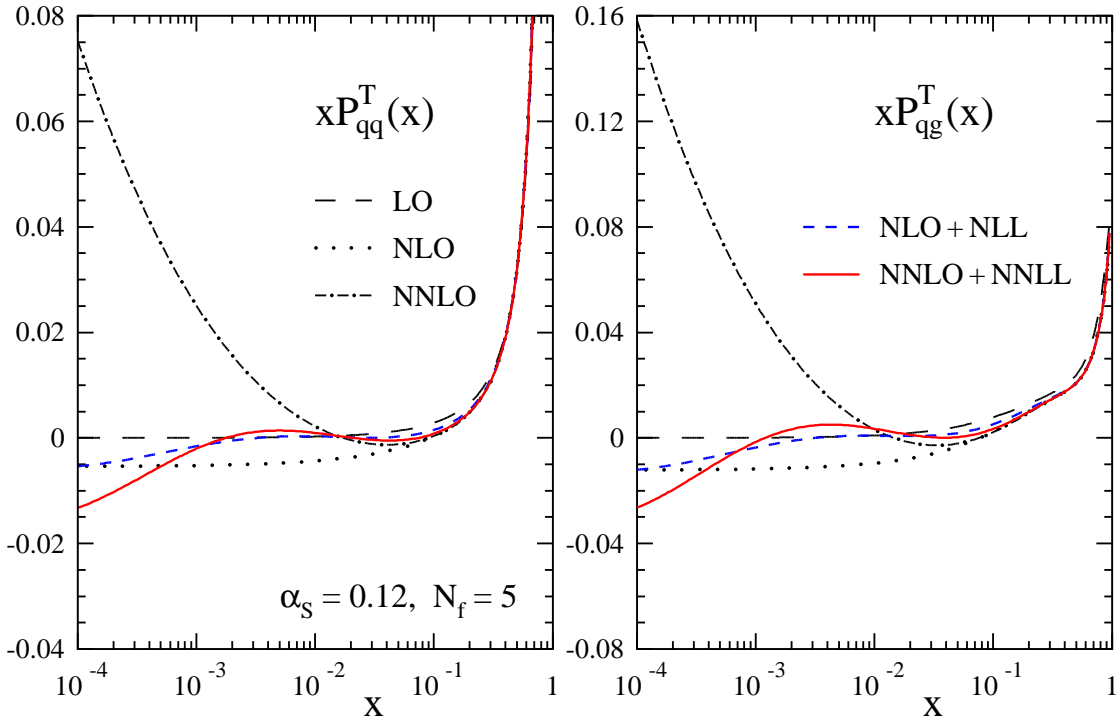


Figure 4: As Fig. 3, but for the timelike quark-quark and quark-gluon splitting functions which do not receive leading logarithmic (LL) corrections. Hence only two resummed curves are shown.

## 4 Resummed coefficient functions for $F_T$

We now turn to the coefficient functions. For brevity, we will not discuss the  $\phi$ -exchange case here (beyond the respective highest logarithms which are directly related to those for  $F_T$ ), as it will be of only theoretical interest in the near future. The corresponding results are included, however, in the FORM file of results distributed with the arXiv version of this article.

The moments of the small- $x$  resummed terms of the transverse coefficient functions are

$$c_{T,i}(N) = \sum_{n=1}^{\infty} a_s^n \left( \delta_{\text{ig}} c_{T_i, \text{LL}}^{(n)}(N) + c_{T_i, \text{NLL}}^{(n)}(N) + c_{T_i, \text{NNL}}^{(n)}(N) + \dots \right). \quad (4.1)$$

The leading and next-to-leading logarithmic contributions for  $c_{T,g}$  can be written as

$$c_{Tg, \text{LL}}^{(n)}(N) = \frac{(-4)^n C_F C_A^{n-1}}{(N-1)^{2n}} A_{T,g}^{(n)} \quad (4.2)$$

and

$$c_{Tg, \text{NL}}^{(n)}(N) = \frac{(-4)^n C_F C_A^{n-3}}{9(N-1)^{2n-1}} \left[ -C_A^2 B_{Tg,1}^{(n)} + C_A n_f B_{Tg,2}^{(n)} + 8 C_F n_f B_{Tg,3}^{(n)} \right]. \quad (4.3)$$

The coefficients in Eqs. (3.2) and (3.4) are given in Table 5 analytically to the twelfth order in  $\alpha_s$  (see the FORM file for the remaining four orders) and numerically for  $n = 13, \dots, 16$ . In this case the general form and the generating function is obvious only for the leading-logarithmic coefficients in Eq. (4.2) with [50]

$$A_{T,g}^{(n)} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (4k+1) \quad (4.4)$$

and

$$c_{T,g}^{\text{LL}}(N) = \frac{C_F}{C_A} \left( c_{\phi,g}^{T, \text{LL}}(N) - 1 \right) = \frac{C_F}{C_A} \left\{ \left( 1 + \frac{32 C_A a_s}{(N-1)^2} \right)^{-1/4} - 1 \right\}. \quad (4.5)$$

Eq. (4.5) agrees with the corresponding result of Ref. [32] for  $c_{T,g}$  up to a factor of two arising from the different normalization of this coefficient function already mentioned below Eq. (2.11).

The relations for the quark coefficient functions corresponding to Eqs. (4.2) and (4.3) can be cast in the form

$$c_{Tq, \text{NLL}}^{(n)}(N) = \frac{C_F}{C_A} c_{\phi,q}^{T, \text{NLL}}(N) = \frac{(-4)^n C_F n_f C_A^{n-2}}{3(N-1)^{2n-1}} 4 A_{T,q}^{(n)} \quad (4.6)$$

and

$$c_{Tq, \text{NNL}}^{(n)}(N) = \frac{(-4)^n C_F n_f C_A^{n-4}}{3(N-1)^{2n-2}} \left[ -C_A^2 B_{Tq,1}^{(n)} + \frac{8}{3} C_A n_f B_{Tq,2}^{(n)} + \frac{8}{3} C_F n_f B_{Tq,3}^{(n)} \right] \quad (4.7)$$

with  $n \geq 2$ . The first sixteen coefficients in Eqs. (4.6) and (4.7) can be found in Table 6. Note the the faster growth of these coefficients with  $n$ , as compared to the corresponding splitting function results in Tables 1 and 2, is largely (but only only) due to the different normalization in Eqs. (4.2) and Eqs. (4.6), which was employed to have mainly integer coefficient in Table 1.



$n$	$A_{T,g}^{(n)}$	$B_{Tg,1}^{(n)}$	$B_{Tg,2}^{(n)}$	$B_{Tg,3}^{(n)}$
1	2	-9	-	-
2	10	87/2	-	-
3	60	779	2	8
4	390	8620	67	115
5	2652	84224	1100	$\frac{6193}{5}$
6	18564	778449	14028	$\frac{59811}{5}$
7	132600	6974466	157500	$\frac{765402}{7}$
8	961350	61261449	1639437	$\frac{13548231}{14}$
9	7049900	530773430	16238552	$\frac{176155814}{21}$
10	52169260	4552643821	155338216	$\frac{1505191630}{21}$
11	388898120	38750254946	1448362604	$\frac{140014436692}{231}$
12	2916735900	327823740972	13242606390	$\frac{391609950056}{77}$
13	$2.1987701 \cdot 10^{10}$	$2.7596825 \cdot 10^{12}$	$1.1922955 \cdot 10^{11}$	$4.2417307 \cdot 10^{10}$
14	$1.6647831 \cdot 10^{11}$	$2.3136533 \cdot 10^{13}$	$1.0602610 \cdot 10^{12}$	$3.5208088 \cdot 10^{11}$
15	$1.2652351 \cdot 10^{12}$	$1.9330232 \cdot 10^{14}$	$9.3330885 \cdot 10^{12}$	$2.9110969 \cdot 10^{12}$
16	$9.6474181 \cdot 10^{12}$	$1.6102477 \cdot 10^{15}$	$8.1461913 \cdot 10^{13}$	$2.3992885 \cdot 10^{13}$

Table 5: The first sixteen  $N$ -space coefficients in Eqs. (4.2) – (4.3) for the LL and NLL small- $x$  approximations to the gluon coefficient function for the transverse fragmentation function.

As for the splitting functions, the next contributions to both transverse coefficient functions are considerably more complex. Since the third-order SIA coefficients functions have not been published so far, we give the third- and fourth-order quantities analytically. The higher orders are presented numerically for  $C_A = 3$  and  $C_F = 4/3$  below. The third-order results are given by

$$c_{T,g}^{(3)}(N) = -\frac{64}{(N-1)^6} 60 C_A^2 C_F + \frac{64}{(N-1)^5} \left\{ \frac{779}{9} C_A^2 C_F - \frac{2}{9} C_A C_F n_f - \frac{64}{9} C_F n_f^2 \right\} \quad (4.8)$$

$$- \frac{64}{(N-1)^4} \left\{ \left( \frac{395}{12} - \frac{68}{3} \zeta_2 \right) C_A^2 C_F - \left( \frac{5}{3} + \frac{4}{3} \zeta_2 \right) C_A C_F^2 + \frac{67}{27} C_A C_F n_f - \frac{47}{9} C_F n_f^2 \right\} + \dots$$

$$c_{T,q}^{(3)}(N) = -\frac{64}{(N-1)^5} \frac{92}{9} C_A C_F n_f + \frac{64}{(N-1)^4} \left\{ \frac{340}{27} C_A C_F n_f - \frac{8}{9} C_F n_f^2 \right\} \quad (4.9)$$

$$+ \frac{64}{(N-1)^3} \left\{ \left( \frac{169}{324} - \frac{2}{9} \zeta_2 \right) C_A C_F n_f - \left( \frac{1}{3} - \frac{8}{3} \zeta_2 \right) C_F^2 n_f - \frac{2}{9} C_F n_f^2 \right\} + \dots$$

$n$	$A_{T,q}^{(n)}$	$B_{Tq,1}^{(n)}$	$B_{Tq,2}^{(n)}$	$B_{Tq,3}^{(n)}$
2	1	-1	-	-
3	$\frac{23}{3}$	$\frac{340}{9}$	1	-
4	$\frac{329}{6}$	$\frac{4790}{9}$	$\frac{133}{12}$	$\frac{26}{3}$
5	$\frac{5884}{15}$	$\frac{50917}{9}$	$\frac{1604}{15}$	$\frac{4211}{30}$
6	$\frac{14166}{5}$	$\frac{2454268}{45}$	$\frac{29869}{30}$	$\frac{16313}{10}$
7	$\frac{144694}{7}$	$\frac{157792304}{315}$	$\frac{319122}{35}$	$\frac{1743793}{105}$
8	$\frac{2130333}{14}$	$\frac{156504164}{35}$	$\frac{34659563}{420}$	$\frac{13254173}{84}$
9	$\frac{71114144}{63}$	$\frac{7404527591}{189}$	$\frac{93174769}{126}$	$\frac{7191782}{5}$
10	$\frac{530983954}{63}$	$\frac{45735067426}{135}$	$\frac{10344355237}{1575}$	$\frac{20099736449}{1575}$
11	$\frac{43854388318}{693}$	$\frac{861656350072}{297}$	$\frac{111399799846}{1925}$	$\frac{1925137106758}{17325}$
12	$\frac{110281846025}{231}$	$\frac{769021780130564}{31185}$	$\frac{10525291437281}{20790}$	$\frac{3306988478369}{3465}$
13	$3.6165904 \cdot 10^9$	$2.0836819 \cdot 10^{11}$	$4.4007722 \cdot 10^9$	$8.1132902 \cdot 10^9$
14	$2.7496227 \cdot 10^{10}$	$1.7521629 \cdot 10^{12}$	$3.8036176 \cdot 10^{10}$	$6.8422246 \cdot 10^{10}$
15	$2.0971243 \cdot 10^{11}$	$1.4675034 \cdot 10^{13}$	$3.2707410 \cdot 10^{11}$	$5.7338592 \cdot 10^{11}$
16	$1.6039639 \cdot 10^{12}$	$1.2249474 \cdot 10^{14}$	$2.7996798 \cdot 10^{12}$	$4.7804829 \cdot 10^{12}$

Table 6: As Table 5, but for the NLL and NNLL quark coefficient function in Eqs. (4.6) – (4.7).

The expansions of the fourth-order transverse coefficient functions about  $N = 1$  read

$$\begin{aligned}
c_{T,g}^{(4)}(N) &= \frac{256}{(N-1)^8} 390 C_A^3 C_F - \frac{256}{(N-1)^7} \left\{ \frac{8620}{9} C_A^3 C_F - \frac{67}{9} C_A^2 C_F n_f - \frac{920}{9} C_A C_F n_f^2 \right\} \\
&+ \frac{256}{(N-1)^6} \left\{ \left( \frac{219007}{216} - 244 \zeta_2 \right) C_A^3 C_F - \left( \frac{15}{4} + 18 \zeta_2 \right) C_A^2 C_F^2 + \frac{451}{36} C_A^2 C_F n_f \right. \\
&\quad \left. - \frac{5308}{27} C_A C_F^2 n_f + \frac{31}{27} C_A C_F n_f^2 + \frac{44}{9} C_F^2 n_f^2 \right\} + \dots
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
c_{T,q}^{(4)}(N) &= \frac{256}{(N-1)^7} \frac{658}{9} C_A C_F n_f - \frac{256}{(N-1)^6} \left\{ \frac{4790}{27} C_A^2 C_F n_f - \frac{266}{27} C_A C_F n_f^2 - \frac{208}{27} C_F^2 n_f^2 \right\} \\
&+ \frac{256}{(N-1)^5} \left\{ \left( \frac{32423}{216} - \frac{80}{9} \zeta_2 \right) C_A^2 C_F n_f + \left( \frac{71}{9} - \frac{262}{9} \zeta_2 \right) C_A C_F^2 n_f - \frac{958}{36} C_A^2 C_F n_f \right. \\
&\quad \left. - \frac{838}{81} C_F^2 n_f^2 + \frac{8}{9} C_F n_f^3 \right\} + \dots
\end{aligned} \tag{4.11}$$

For the coefficients of the third logarithms in Table 7 we use the notation

$$c_{Tg,NNL}^{(n)}(N) = \frac{(-1)^n}{(N-1)^{2n-2}} \left( 96^n C_{Tg,0}^{(n)} - 96^{n-1} C_{Tg,1}^{(n)} n_f + 96^{n-2} C_{Tg,2}^{(n)} n_f^2 \right), \quad (4.12)$$

$$c_{Tq,N^3L}^{(n)}(N) = \frac{(-1)^n}{(N-1)^{2n-3}} \left( 96^{n-1} C_{Tq,1}^{(n)} n_f - 96^{n-1} C_{Tq,2}^{(n)} n_f^2 + 96^{n-2} C_{Tq,3}^{(n)} n_f^3 \right). \quad (4.13)$$

These results are illustrated in Fig. 5 for the same reference point and  $x$ -range as in the previous section. The situation for  $xc_{T,g}$  and  $xc_{T,q}$  is largely analogous to that for the corresponding splitting functions  $xP_{gq}^T$  and  $xP_{qq}^T$  in Figs. 3 and 4. The NLO and NNLO fixed-order approximations (the LO coefficient function  $c_{T,q} = \delta(1-x)$  is obviously not visible in this figure) are unreliable here from even larger  $x$ -values than above. The small- $x$  rise of the NNLO coefficient functions is removed by adding the NLL and NNLL resummations from order  $\alpha_s^3$ , leaving us with functions oscillating about  $xc_{T,k} \approx 0$ . The same behaviour, if with a considerably smaller amplitude, can be established down to extremely small values of  $x$  for the exactly known LL gluon coefficient function (4.5) already determined in Ref. [32]. Also here it would be very interesting to know one more order in  $\alpha_s$  and the N<sup>3</sup>LL resummation of  $xc_{T,g}$ . The latter, however, again requires (at least in the present framework) the calculation of the fourth-order contribution to the splitting function  $P_{gq}^T$ .

It is instructive to briefly address the impact of the (scheme-independent) LL splitting functions (1.6) and (scheme-dependent) LL coefficient functions, given in  $\overline{\text{MS}}$  by Eq. (4.5), on the scale dependence of the fragmentation function  $F_T$  and its ‘gluonic’ counterpart  $F_\phi$ . This is best done by considering the ‘timelike’ physical evolution kernels  $K_{ab}$  in Mellin space,

$$\frac{d}{d \ln Q^2} \begin{pmatrix} F_T \\ F_\phi \end{pmatrix} = \begin{pmatrix} K_{TT} & K_{T\phi} \\ K_{\phi T} & K_{\phi\phi} \end{pmatrix} \begin{pmatrix} F_T \\ F_\phi \end{pmatrix}, \quad (4.14)$$

which are given by the matrix elements of

$$K = CP^T C^{-1} + \beta(a_s) \frac{dC}{da_s} C^{-1} \quad \text{with} \quad C = \begin{pmatrix} c_{T,q} & c_{T,g} \\ c_{\phi,q} & c_{\phi,g} \end{pmatrix}, \quad (4.15)$$

and the splitting function matrix (2.4). In terms of powers of  $(N-1)^{-1}$ , the first term could be different from  $P^T$  already at leading logarithmic ( $\alpha_s^n (N-1)^{-2n+2}$ ) accuracy. However, the relations (1.6) and Eq. (4.5) imply

$$P_{LL}^T = \begin{pmatrix} 0 & \frac{C_F}{C_A} P_{gg,LL}^T \\ 0 & P_{gg,LL}^T \end{pmatrix} \quad \text{and} \quad C_{LL} = \begin{pmatrix} 1 & \frac{C_F}{C_A} c_{LL} \\ 0 & 1 + c_{LL} \end{pmatrix} \quad (4.16)$$

with  $c_{LL}$  given by the curly bracket in Eq. (4.5). Due to Eq. (4.16) all such contributions to the matrix  $K$  cancel, and the factorization-scheme independent physical kernels are correctly given by

$$K_{TT,LL} = K_{\phi T,LL} = 0 \quad , \quad K_{T\phi,LL} = P_{gq,LL}^T \quad , \quad K_{\phi\phi,LL} = P_{gg,LL}^T. \quad (4.17)$$

A study of the physical kernels (4.14) beyond the leading logarithmic accuracy could be interesting, but is beyond the scope of the present article.

$n$	$C_{Tg,0}^{(n)}$	$C_{Tg,1}^{(n)}$	$C_{Tg,2}^{(n)}$	$C_{Tq,1}^{(n)}$	$C_{Tq,2}^{(n)}$	$C_{Tq,3}^{(n)}$
2	-0.0488460	-	-	-0.0411523	-	-
3	-0.0052813	-0.0044582	-	-0.0543741	-0.0020576	-
4	0.0648580	0.2598844	0.3689986	0.4087098	0.0190107	0.0329218
5	0.1603366	0.7502113	1.1226934	1.1073150	0.0595507	0.0993878
6	0.2804175	1.4552223	2.2807323	2.0085092	0.1178470	0.1996540
7	0.4247389	2.3722220	3.8617425	3.1012694	0.1933078	0.3345043
8	0.5931376	3.5017009	5.8830239	4.3806502	0.2857698	0.5048923
9	0.7855611	4.8454857	8.3606018	5.8442114	0.3952535	0.7118052
10	1.0020223	6.4059897	11.309349	7.4907736	0.5218662	0.9562145
11	1.2425739	8.1858750	14.743115	9.3198698	0.6657575	1.2390556
12	1.5072942	10.187889	18.674852	11.331466	8.2709784	1.5612246
13	1.7962784	12.414785	23.116717	13.525807	1.0060672	1.9235757
14	2.1096323	14.869274	28.080167	15.903318	1.2028488	2.3269235
15	2.4474687	17.554012	33.576041	18.464554	1.4176262	2.7720455
16	2.8099054	20.471582	39.614620	21.210152	1.6505807	3.2596851

Table 7: The numerical coefficients of the third small- $x$  contributions (4.12) and (4.13) to the  $N$ -space gluon and quark coefficient functions for the fragmentation function  $F_T$  to order  $\alpha_s^{16}$ .

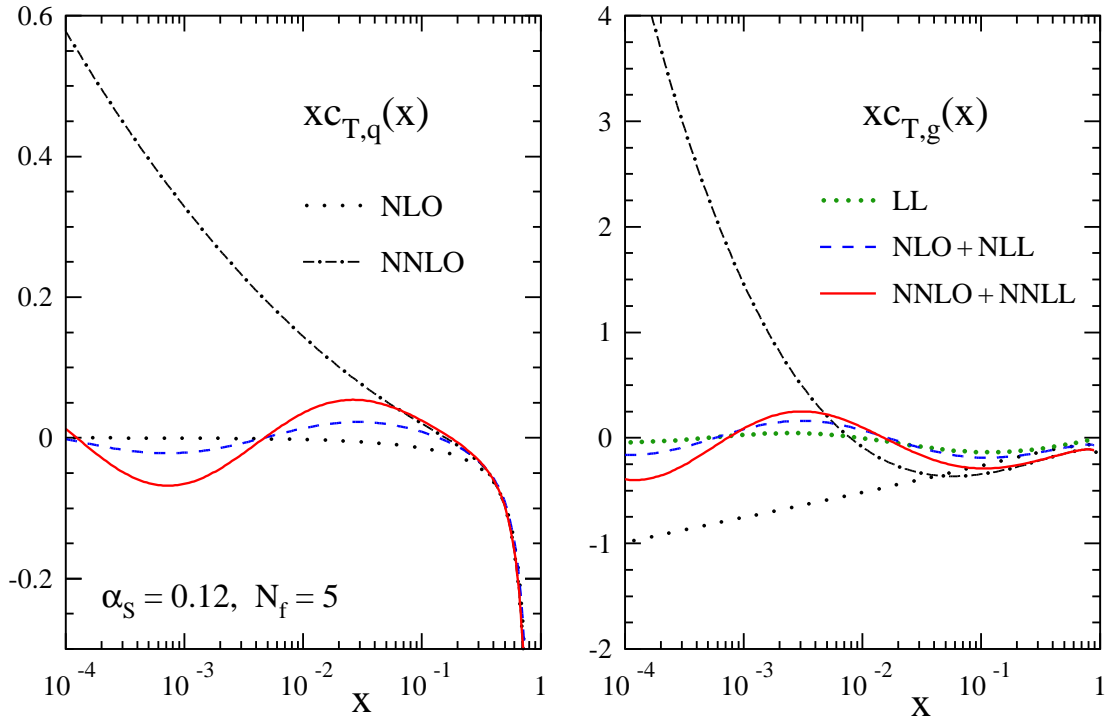


Figure 5: The quark and gluon coefficient functions for  $F_T$  at a typical value of  $\alpha_s$ . Shown are the NLO and NNLO fixed-order approximations, and the matched LL, NLL and NNLL resummed results obtained (beyond LL) by adding the respective small- $x$  terms at all relevant higher orders.

## 5 Resummed coefficient functions for $F_L$

Finally we briefly present the resummed results for the longitudinal fragmentation function  $F_L$ . Since the NNLO (third-order) coefficient functions for this observable are not yet known, only the respective two highest logarithms can be resummed for both the gluon and quark coefficient functions. The corresponding  $N$ -space expressions can be written as

$$c_{L,i}(N) = \sum_{n=1} a_s^n \left( \delta_{\text{ig}} c_{Li,LL}^{(n)}(N) + c_{Li,NLL}^{(n)}(N) + c_{Li,NNL}^{(n)}(N) + \dots \right), \quad (5.1)$$

with the gluon case given by

$$c_{Lg,LL}^{(n)}(N) = - \frac{(-4)^n C_F C_A^{n-1}}{(N-1)^{2n-1}} A_{L,g}^{(n)} \quad (5.2)$$

and

$$c_{Lg,NLL}^{(n)}(N) = \frac{(-4)^n C_F C_A^{n-3}}{9(N-1)^{2n-2}} \left[ C_A^2 B_{Lg,1}^{(n)} - 9 C_A C_F B_{Lg,2}^{(n)} - C_A n_f B_{Lg,3}^{(n)} - C_F n_f B_{Lg,4}^{(n)} \right]. \quad (5.3)$$

As in the transverse case, the quark coefficient functions for  $F_L$  are suppressed by one power of  $\ln x$  or  $(N-1)^{-1}$ , but for  $n > 1$  take the otherwise analogous forms

$$c_{Lq,NLL}^{(n)}(N) = - \frac{(-4)^n C_F n_f C_A^{n-2}}{3(N-1)^{2n-2}} A_{L,q}^{(n)} \quad (5.4)$$

and

$$c_{Lq,NNL}^{(n)}(N) = \frac{(-4)^n C_F n_f C_A^{n-4}}{9(N-1)^{2n-2}} \left[ C_A^2 B_{Lq,1}^{(n)} - C_A C_F B_{Lq,2}^{(n)} - C_A n_f B_{Lq,3}^{(n)} - C_F n_f B_{Lq,4}^{(n)} \right]. \quad (5.5)$$

The coefficients in Eqs. (5.2) – (5.5) are given in Tables 8 and 9, as before giving the thirteenth to sixteenth order in a numerical form for brevity (the exact expressions can be found in the FORM file distributed with this article). In this case the general formula is not even known for the LL coefficients which, like all other ‘unsolved’ series above, involve unpleasantly large prime numbers early in the expansion. For instance, the prime-factor decomposition of  $A_{L,g}^{(7)}$  reads  $4 \cdot 10691$ .

These results are illustrated in Fig. 6 in the same manner as those for  $F_T$  in Fig. 5 above. While neither of the first-order (LO) coefficient functions includes any  $x^{-1} \ln x$  terms in the present case, also here the (now negative) small- $x$  spike of both second-order (NLO) coefficient functions is completely removed by adding the corresponding all-order resummations of the small- $x$  logarithms, leaving small oscillating functions with  $x c_{L,p} \approx 0$  at  $x \lesssim 10^{-2}$ .

One may expect that the small- $x$  resummation of the longitudinal fragmentation function will be the first to be extended to a higher accuracy as, in contrast to the timelike splitting functions and the transverse fragmentation function in the previous sections, ‘only’ a third-order calculation is required for deriving the NNLO + NNLL resummation. Note, however, that already the present results are sufficient for the corresponding resummation of the total fragmentation function, obtained by integrating Eq. (1.1) over  $\theta$ , as the coefficient functions  $c_{L,p}^{(n)}$  are suppressed by one power of  $\ln x$  or  $(N-1)^{-1}$  with respect to their transverse counterparts.

$n$	$A_{L,g}^{(n)}$	$B_{Lg,1}^{(n)}$	$B_{Lg,2}^{(n)}$	$B_{Lg,3}^{(n)}$	$B_{Lg,4}^{(n)}$
1	1	9	–	–	–
2	4	33	1	–	–
3	22	723/2	5	3	30
4	136	3530	30	56	376
5	894	32447	195	722	3754
6	6104	288590	1326	8000	$\frac{172544}{5}$
7	42764	2515565	9282	81722	$\frac{1522436}{5}$
8	305232	21633684	66300	793968	$\frac{91820496}{35}$
9	2209526	184263400	480675	7457476	$\frac{779145058}{35}$
10	16171672	1558144566	3524950	68371776	$\frac{19627939136}{105}$
11	119414516	13101831041	26084630	615603170	$\frac{163580958068}{105}$
12	888212208	109672261452	194449060	5465590416	$\frac{14911681259824}{1155}$
13	$6.6468218 \cdot 10^9$	$9.1464728 \cdot 10^{11}$	$1.4583680 \cdot 10^9$	$4.7987650 \cdot 10^{10}$	$1.0652372 \cdot 10^{11}$
14	$4.9997395 \cdot 10^{11}$	$7.6044089 \cdot 10^{12}$	$1.0993851 \cdot 10^{10}$	$4.1752477 \cdot 10^{11}$	$8.7587508 \cdot 10^{11}$
15	$3.7774611 \cdot 10^{11}$	$6.3057119 \cdot 10^{13}$	$8.3239155 \cdot 10^{10}$	$3.6055542 \cdot 10^{12}$	$7.1815660 \cdot 10^{12}$
16	$2.8649548 \cdot 10^{12}$	$5.2169677 \cdot 10^{14}$	$6.3261758 \cdot 10^{11}$	$3.0939814 \cdot 10^{13}$	$5.8747864 \cdot 10^{13}$

$n$	$A_{L,q}^{(n)}$	$B_{Lq,1}^{(n)}$	$B_{Lq,2}^{(n)}$	$B_{Lq,3}^{(n)}$	$B_{Lq,4}^{(n)}$
2	2	$-9/2$	–	–	–
3	12	51	6	4	–
4	$\frac{236}{3}$	$\frac{1976}{3}$	46	$\frac{122}{3}$	$\frac{92}{3}$
5	$\frac{1610}{3}$	$\frac{19777}{3}$	329	379	446
6	$\frac{56356}{15}$	$\frac{915601}{15}$	$\frac{11768}{5}$	$\frac{17228}{5}$	$\frac{14560}{3}$

Table 8: Upper part: the first sixteen  $N$ -space coefficients in Eqs. (5.2) and (5.3) for the LL and NLL small- $x$  resummed the gluon coefficient function for the longitudinal fragmentation function. Lower part: the first five (NLL and NNLL) coefficients for the corresponding quark coefficient function defined in Eqs. (5.4) and (5.5).

$n$	$A_{L,q}^{(n)}$	$B_{L,q,1}^{(n)}$	$B_{L,q,2}^{(n)}$	$B_{L,q,3}^{(n)}$	$B_{L,q,4}^{(n)}$
7	$\frac{401944}{15}$	$\frac{8167748}{15}$	$\frac{84996}{5}$	$\frac{154428}{5}$	$\frac{236236}{5}$
8	$\frac{6784088}{35}$	$\frac{499053868}{105}$	$\frac{868164}{7}$	$\frac{28758068}{105}$	$\frac{45616904}{105}$
9	$\frac{148855862}{105}$	$\frac{4294474801}{105}$	$\frac{6390999}{7}$	$\frac{84243073}{35}$	$\frac{135167864}{35}$
10	$\frac{3295405924}{315}$	$\frac{36585726017}{105}$	$\frac{142228288}{21}$	$\frac{734599784}{35}$	$\frac{2115778496}{63}$
11	$\frac{24496904632}{315}$	$\frac{103121715842}{35}$	$\frac{1061967908}{21}$	$\frac{286309749296}{1575}$	$\frac{151010702344}{525}$
12	$\frac{2015422894136}{3465}$	$\frac{85829821660568}{3465}$	$\frac{87708776636}{231}$	$\frac{9038620655308}{5775}$	$\frac{6030487800584}{2475}$
13	$4.3730248 \cdot 10^9$	$2.0729526 \cdot 10^{11}$	$2.8644635 \cdot 10^9$	$1.3405311 \cdot 10^{10}$	$2.0469392 \cdot 10^{10}$
14	$3.3024706 \cdot 10^{10}$	$1.7282691 \cdot 10^{12}$	$2.1699543 \cdot 10^{10}$	$1.1428789 \cdot 10^{11}$	$1.7086228 \cdot 10^{11}$
15	$2.5036749 \cdot 10^{11}$	$1.4363931 \cdot 10^{13}$	$1.6497736 \cdot 10^{11}$	$9.7038944 \cdot 10^{11}$	$1.4189917 \cdot 10^{12}$
16	$1.9045398 \cdot 10^{12}$	$1.1906480 \cdot 10^{14}$	$1.2582746 \cdot 10^{12}$	$8.2093343 \cdot 10^{12}$	$1.1736262 \cdot 10^{13}$

Table 9: Continuation of the part Table 8 for the quark coefficient function for  $F_L$  to order  $\alpha_s^{16}$ .

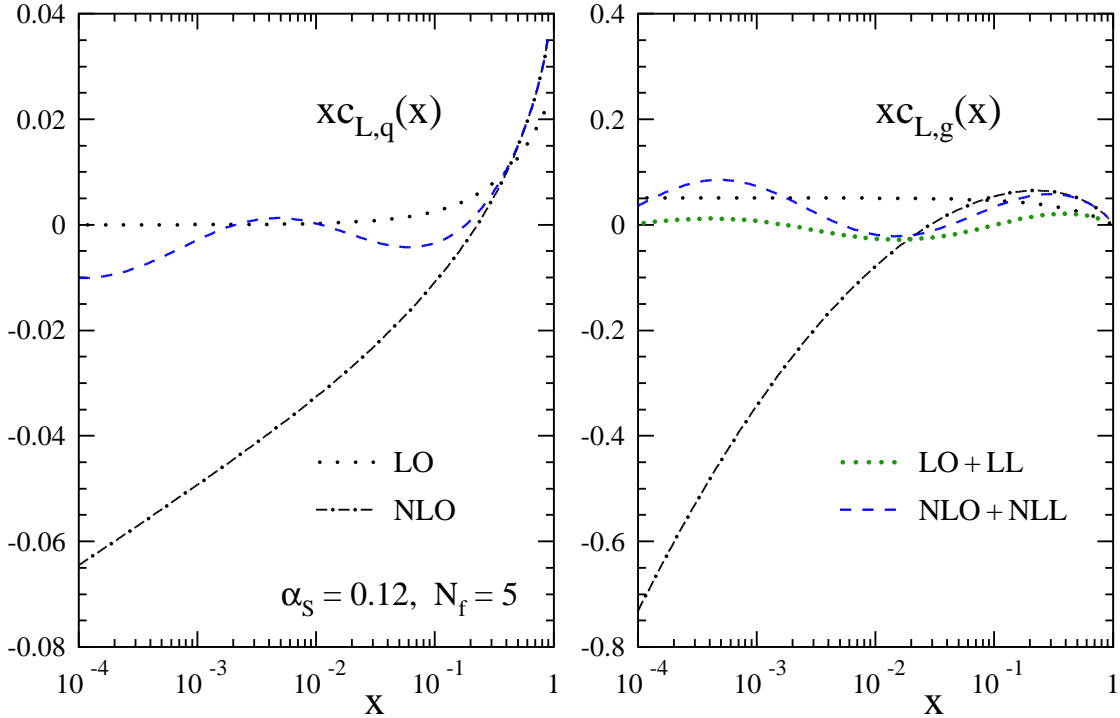


Figure 6: The quark and gluon coefficient functions for  $F_L$  at a typical value of  $\alpha_s$ . Shown are the LO and NLO fixed-order approximations, and the matched LL (for  $c_{L,g}$ ) and NLL resummed results obtained by adding the respective small- $x$  terms at all relevant higher orders.

## 6 Summary and Outlook

We have derived the all-order resummation of the highest three small- $x$  double logarithms,

$$\alpha_s^n x^{-1} \ln^{2n-\ell_0-\ell} x \quad \text{with} \quad \ell = 0, 1, 2, \quad (6.1)$$

for all four flavour-singlet timelike splitting functions – with  $\ell_0 = 2$  for  $P_{\text{gq}}^T$  and  $P_{\text{gg}}^T$  and  $\ell_0 = 3$  for  $P_{\text{qq}}^T$  and  $P_{\text{qg}}^T$  – and for both singlet coefficient functions for the transverse fragmentation function  $F_T$  in semi-inclusive electron-positron annihilation (SIA) – with  $\ell_0 = 2$  for  $c_{T,\text{q}}$  and  $\ell_0 = 1$  for  $c_{T,\text{g}}$  – together with the corresponding results for SIA via an intermediate scalar  $\phi$  like the Higgs boson in the heavy top-quark limit. For the longitudinal fragmentation function  $F_L$  present fixed-order results, which serve as input quantities for the resummation, allow only the determination of the highest two logarithms, i.e.,  $\ell = 0, 1$  in Eq. (6.1) with  $\ell_0 = 3$  for  $c_{L,\text{q}}$  and  $\ell_0 = 2$  for  $c_{L,\text{g}}$ .

The coefficients of the above logarithms have been calculated explicitly to order  $\alpha_s^{16}$  which is not the highest computationally feasible order, but sufficient for numerically accurate results down to  $x = 10^{-4}$ , a range in  $x$  that should be more than sufficient for all foreseeable analyses of data. These calculations have been performed in Mellin- $N$  space, using the latest versions of FORM and TFORM [34, 35] at all stages. The results agree with the leading logarithmic (LL) result of Refs. [30] for the splitting functions  $P_{\text{gq}}^T$  and  $P_{\text{gg}}^T$ , and with the only additional result so far derived in the  $\overline{\text{MS}}$  scheme, the recent LL contributions to the coefficient function  $c_{T,\text{g}}$  [32].

The resummation has been derived by decomposing the unfactorized partonic fragmentation functions  $\widehat{F}_{a,p}(x, \alpha_s, \varepsilon)$  in dimensional regularization at any order  $\alpha_s^n$  into  $n$  (or  $n - 1$  in the quark cases) contributions of the form

$$\varepsilon^{-2n+n_0} x^{-1-2k\varepsilon} (A + B\varepsilon + C\varepsilon^2 + \dots) \quad \text{with} \quad k = 1, 2, \dots, n \quad (6.2)$$

and  $n_0 = 1$  for  $a = T, \phi$  and  $p = \text{g}$ ,  $n_0 = 2$  for  $a = T, \phi$  and  $p = \text{q}$  and for  $a, p = L, \text{g}$ , and  $n_0 = 3$  for  $a, p = L, \text{q}$ , with the  $k = 1$  contributions missing in the quark cases. The KLN-related cancellations between the contributions in Eq. (6.2), together with the powers of  $\varepsilon$  fixed by fixed-order calculations [7–17, 24], lead to overconstrained systems of equations for the leading logarithmic, next-to-leading logarithmic (NLL) [and next-to-next-to-leading logarithmic (NNLL)] expansion parameters  $A, B$  [and  $C$ ] in the decomposition (6.2) which can be solved to (in principle) any order  $n$ . Given the large number of extra constraints and checks – including the correct predictions of the respective highest two small- $x$  logarithms in the third-order timelike splitting functions [17, 24] and the non-trivial all-order agreement with the known LL results [30, 32] – there is no need for an additional derivation of the decomposition (6.2) from the structure of higher-order Feynman diagrams and phase-space integrations.

Whilst the setup of the resummation is elegant and simple, most of the new results are not, as we have not succeeded to find the general expressions and generating functions for the resulting series of coefficients, with the exception of the NLL corrections to the splitting functions  $C_F^{-1} P_{\text{gq}}^T$  and  $P_{\text{gg}}^T$  in the limit  $C_F = 0$ . The results have therefore been presented via detailed  $N$ -space tables which, hopefully, will be used for finding some of the now unknown general expressions. The most interesting target in this respect are the non-integer coefficients in Table 1, as the solution of any one of these three series would be sufficient to clarify the analytic structure of all NLL ( $\alpha_s^n x^{-1} \ln^{2n-3} x$ ) contribution to the matrix of the timelike splitting functions.



The small- $x$  resummation has a striking effect on the numerical behaviour of the splitting functions and coefficient functions in the region  $x \lesssim 10^{-2}$ . All fixed-order spikes for  $x \rightarrow 0$ , which dwarf their single-logarithmic counterparts in the spacelike splitting functions and deep-inelastic scattering (DIS) [25–29], are removed by forming the N<sup>n</sup>LO + N<sup>n</sup>LL combinations of fixed-order and higher-order resummed results, mostly leaving small and apparently oscillating functions. This behaviour is qualitatively similar to the LL results of Ref. [30, 32] which are known in a closed form and thus can be evaluated down to extremely small values of  $x$ . While some theoretical questions remain that can only be clarified by future third- and fourth-order calculations, the present resummation should prove sufficient for analyses of SIA data in the foreseeable future.

We have verified that the present approach can be extended to the non- $x^{-1}$  double logarithms in the (even- $N$  based) DIS structure functions  $F_2$  and  $F_L$  (recall that there are no ‘genuine’  $x^{-1}$  double logarithms in DIS; those encountered in the  $\phi$ -exchange coefficient functions in Refs. [51, 52] are artifacts of using the heavy-top approximation outside its domain of validity). These double-logarithmic terms form the leading small- $x$  contributions in the non-singlet cases, see Refs. [53] for the LL resummation of the spacelike non-singlet splitting functions; they can be relevant at intermediate values of  $x$  also in flavour-singlet quantities, see Ref. [54]. The corresponding NNLL resummations will be presented in a subsequent publication.

One may expect that, analogous to the large- $x$  cases in Refs. [51, 55], the resummation of the small- $x$  double logarithms can be extended to (all) higher powers of the prefactor  $x$  in Eq. (6.1) for the quantities considered here (and their even- $N$  spacelike counterparts) – but not for the asymmetric fragmentation function  $F_A$  which is related to the odd- $N$  structure function  $F_3$  known to receive additional contributions with  $1/n_c$  and higher group factors [53, 56]. We have explicitly checked the direct generalization of our approach to the LL and NLL  $x^a$  contributions in singlet SIA for  $a = 0, \dots, 6$ . It works, but only for  $a = 0$  and even values, and with the form (6.2) replaced by

$$\varepsilon^{-2n+1} x^{a-k\varepsilon} (A + B\varepsilon + C\varepsilon^2 + \dots) \quad \text{with} \quad k = 2, \dots, n+1 \quad (6.3)$$

which, in fact, is what one may have ‘naively’ expected from Refs. [9] also for the  $x^{-1}$  terms. The predictions resulting from Eq. (6.3) should be useful in the context of future third- and fourth-order calculations. Conceivably also all small- $x$  double logarithms in the timelike and spacelike higher-order singlet splitting functions (and the corresponding SIA and DIS coefficient functions) could turn out to be ‘inherited’ from lower-order quantities. This issue deserves further studies including the case of  $\mathcal{N}=4$  Super Yang-Mills theory addressed, for example, in Ref. [23, 57].

A FORM file of our results presented in Sections 3 – 5 can be obtained by downloading the source of this article from the arXiv servers or from the author upon request.

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