COMBINATORIAL ANALOGUES OF AD-NILPOTENT IDEALS FOR UNTWISTED AFFINE LIE ALGEBRAS

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Abstract. We study certain types of ideals in the standard Borel subalgebra of an untwisted affine Lie algebra. We classify these ideals in terms of the root combinatorics and give an explicit formula for the number of such ideals in type A. The formula involves various aspects of combinatorics of Dyck paths and leads to a new interesting integral sequence.

1. Introduction, motivation and description of the results

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ the corresponding Borel subalgebra. An ideal $\mathfrak{i}$ of $\mathfrak{b}$ is called *ad-nilpotent* if the adjoint action on $\mathfrak{b}$ of every element from $\mathfrak{i}$ is nilpotent.

It is easy to see that ad-nilpotent ideals of $\mathfrak{b}$ are all contained in $\mathfrak{n}_+$. Furthermore, as $\mathfrak{h}$ acts semi-simply on $\mathfrak{n}_+$ with one-dimensional roots spaces, any ad-nilpotent ideal $\mathfrak{i}$ decomposes into a direct sum of (some of these) root spaces. It follows that the number of such ideals is finite. Various classification and enumeration problems related to ad-nilpotent ideals in the above situation were studied in [CP1, CP2, CDR, KOP, AKOP, Pa], see also references therein.

The aim of the present paper is to generalize the problem described above to the situation of affine Kac-Moody Lie algebras in a sensible and interesting way. There are several natural obstructions which we have to deal with. So, let now $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be an affine Kac-Moody Lie algebra with a fixed *standard* triangular decomposition and $\hat{\mathfrak{b}} = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ the corresponding Borel subalgebra. First of all, it is easy to observe that the only ad-nilpotent ideal of $\hat{\mathfrak{b}}$ is the zero ideal. Hence we propose instead to consider *strong ideals* of $\hat{\mathfrak{b}}$, that is ideals in $\hat{\mathfrak{b}}$ of finite codimension, which are contained in $\hat{\mathfrak{n}}_+$. For finite-dimensional $\mathfrak{g}$ this notion coincides with that of an ad-nilpotent ideal of $\mathfrak{b}$. It is easy to see that the number of strong ideals in $\hat{\mathfrak{b}}$ is infinite.

Let $\mathfrak{i}$ be a strong ideal in $\hat{\mathfrak{b}}$. Then, similarly to the finite dimensional case, the ideal $\mathfrak{i}$ can be written as a direct sum of its intersections with the root subspaces of $\hat{\mathfrak{g}}$. However, unlike the finite dimensional case, in the case of affine Lie algebras some root spaces of $\hat{\mathfrak{g}}$ might have

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dimension bigger than one. An ideal $\mathfrak{i}$ in $\hat{\mathfrak{b}}$ is called *thick* provided that the following condition is satisfied: for any root α of $\hat{\mathfrak{g}}$ the fact that $\mathfrak{i}\cap\hat{\mathfrak{g}}_\alpha \neq 0$ implies $\hat{\mathfrak{g}}_\alpha \subset \mathfrak{i}$. An ideal $\mathfrak{i}$ in $\hat{\mathfrak{b}}$ is called *combinatorial* provided that it is strong and thick. Unfortunately, the number of combinatorial ideals is still infinite.

Let $\mathfrak{i}$ be a combinatorial ideal. Then the set of all roots α for which $\hat{\mathfrak{g}}_\alpha \subset \mathfrak{i}$ is called the *support* of $\mathfrak{i}$ and is denoted by $\mathrm{supp}(\mathfrak{i})$ (the corresponding root space of $\mathfrak{i}$ is denoted $\mathfrak{i}_\alpha$). Let δ denote the indivisible positive imaginary root of $\hat{\mathfrak{g}}$. Two combinatorial ideals $\mathfrak{i}$ and $\mathfrak{j}$ will be called *equivalent* provided that there exists $k \in \mathbb{Z}$ such that $\mathrm{supp}(\mathfrak{i}) = k\delta + \mathrm{supp}(\mathfrak{j})$. We will show that for untwisted affine Lie algebras the number of equivalence classes of combinatorial ideals in $\hat{\mathfrak{b}}$ is finite, which naturally leads to the problems of their classification and enumeration.

We give a general answer to the classification problem in the case of untwisted affine Lie algebras in terms of certain root combinatorics. For the affine $\mathfrak{sl}_n$ we also answer the enumeration problem in some nice combinatorial terms involving the combinatorics of Catalan numbers (via Dyck paths) and their generalizations. For $n \in \mathbb{N}$ we will define an $n \times n$-matrix $\mathbf{C}_n$ with nonnegative integer entries (related to the Dyck path combinatorics), and a linear transformation ω on the linear space of all $n \times n$-matrices (over some commutative ring) such that for the standard scalar product $A \cdot B = \sum_{i,j} a_{i,j} b_{i,j}$ we have the following:

Theorem 1. *The number of equivalence classes of combinatorial ideals in the case* $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_n$ *equals* $\mathbf{b}_n := \mathbf{C}_n \cdot \omega\, \mathbf{C}_n$.

The integral sequence $\{\mathbf{b}_n : n \geqslant 1\}$ seems to be new. We note that appearance of Catalan and related numbers in this problem is expected, as Catalan numbers enumerate ad-nilpotent ideals in the Borel subalgebra of $\mathfrak{sl}_n$ (a very natural bijection of such ideals with Dyck paths can be found in [Pa], a less natural bijection appears in [AKOP], however, the latter one has the advantage that it controls the nilpotency degree of the ideal in terms of some combinatorial parameters of the corresponding Dyck path).

We show that the Dyck path combinatorics controls various algebraic properties of combinatorial ideals in the affine case as well. In particular, we describe the number of generators for combinatorial ideals in terms of valleys and peaks of the corresponding Dyck paths. We also propose a combinatorial generalization of the notion of nilpotency degree, in particular, of that of an abelian ideal (which we call quasi-abelian). We describe quasi-abelian ideals combinatorially using intervals in the poset of Dyck paths. Enumeration of quasi-abelian combinatorial ideals for $\hat{\mathfrak{sl}}_n$ leads to yet another new integral sequence. However, for this one we do not have any explicit formula.

Finally, we also address the problem of studying arbitrary $\hat{\mathfrak{b}}$-ideals in $\hat{\mathfrak{n}}_+$ and describe possible supports for such ideals. The above notion of equivalence for ideals lifts to arbitrary ideals resulting in what we call *support equivalent* ideals. This leads to yet another finite classification problem for $\hat{\mathfrak{g}}$. In the case of $\hat{\mathfrak{sl}}_n$ we describe equivalence classes for the latter problem combinatorially using quadruples of Dyck paths.

The paper is organized as follows: in Section 2 we consider the Lie theoretic part of the problem, studying various types of ideals in the Borel subalgebra of an untwisted affine Lie algebra. In the first subsection we collect necessary preliminaries, in the second subsection we introduce our main objects, which we call combinatorial ideals, and in the last subsection we classify equivalence classes of combinatorial ideals in terms of root combinatorics. Section 3 is mostly combinatorial. We start by defining some constructions with matrices which we use to develop some aspects of combinatorics of Dyck paths. We then use these combinatorics to prove Theorem 1 answering the enumeration problem for combinatorial ideals in type A. In the last part of the section we relate various algebraic properties of combinatorial ideals to combinatorics of Dyck paths. In Section 4 we consider all $\hat{\mathfrak{b}}$-ideals in $\hat{\mathfrak{n}}_+$ and describe their supports in terms of root combinatorics, showing that this leads to a new finite problem. In type A we relate these combinatorics to that of Dyck paths.

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2. Combinatorial ideals for untwisted affine Lie algebras

As usual, we denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of positive and non-negative integers, respectively.

2.1. **Untwisted affine Lie algebras.** Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Lie algebra. Then $\hat{\mathfrak{g}}$ is the universal central extension of the extension of the loop algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, the Lie bracket in which is given by $[x\otimes t^k, y\otimes t^l]=[x,y]\otimes t^{k+l}$ for $x,y\in\mathfrak{g}$ and $k,l\in\mathbb{Z}$, by the derivation d such that $[d,x\otimes t^k]=k\,x\otimes t^k$ for $x\in\mathfrak{g}$ and $k\in\mathbb{Z}$, see [Ca, Chapter 17] for details.

Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be a triangular decomposition of $\mathfrak{g}$. Then the standard triangular decomposition $\hat{\mathfrak{g}}=\hat{\mathfrak{n}}_-\oplus\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}}_+$ is defined as follows:

$\hat{\mathfrak{h}}$ is spanned by $\mathfrak{h}$, d and the center; and $\hat{\mathfrak{n}}_\pm$ is spanned by $\mathfrak{n}_\pm$ and all elements of the form $x \otimes t^{\pm k}$, where $x \in \mathfrak{g}$ and $k > 0$. Set $\hat{\mathfrak{b}} := \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$.

Let $\Delta \subset \mathfrak{h}^*$ and $\hat{\Delta} \subset \hat{\mathfrak{h}}^*$ denote the root system of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$, respectively. Then the triangular decompositions above induce partitions $\Delta = \Delta_+ \cup \Delta_-$ and $\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_-$ of Δ and $\hat{\Delta}$, respectively, into positive and negative roots. Abusing notation we identify Δ with a subset of $\hat{\Delta}$ by viewing $\mathfrak{g}$ as a subalgebra of $\hat{\mathfrak{g}}$ via the map $x \mapsto x \otimes 1$.

We denote by $\hat{\Sigma}_+$ (resp. Σ_+) the submonoid of the additive monoid $\hat{\mathfrak{h}}^*$ (resp. $\mathfrak{h}^*$) generated by $\hat{\Delta}_+$ (resp. Δ_+). Recall the usual partial order $\leqslant$ on $\hat{\mathfrak{h}}^*$ defined as follows: for $\lambda, \mu \in \hat{\mathfrak{h}}^*$ we have $\lambda \leqslant \mu$ if and only if $\mu - \lambda \in \hat{\Sigma}_+$. Then the simple roots in $\hat{\Delta}_+$ are the simple roots of Δ together with the root $\alpha_0 := -\alpha_{\max} + \delta$, where $\alpha_{\max}$ is the unique maximal (with respect to $\leqslant$) root in Δ_+.

We denote by D the set of roots of all elements of the form $x \otimes 1$, $x \in \mathfrak{n}_+$, and $x \otimes t$, $x \in \mathfrak{h} \oplus \mathfrak{n}_-$. Then $\hat{\Delta}_+$ is the disjoint union of $D = D_0$ and $D_k := D + k\delta$ for $k > 0$. Further, for $\alpha, \beta \in D$ such that $\alpha \neq \beta$, the difference $\alpha - \beta$ is not of the form $k\delta$ for $k \in \mathbb{Z}$. Note that for every ideal $\mathfrak{i}$ in $\hat{\mathfrak{n}}_+$ of finite codimension, the support of $\mathfrak{i}$ contains the union of all D_k for $k \gg 0$.

2.2. **Basic ideals.** A combinatorial ideal $\mathfrak{i}$ of $\mathfrak{b}$ is called *basic* provided that $\mathrm{supp}(\mathfrak{i}) \cap D \neq \varnothing$.

Theorem 2. (*a*) *We have* $\mathrm{supp}(\mathfrak{i}) \supset D' := \bigcup_{k>0} D_k$ *for any basic* $\mathfrak{i}$.

(*b*) *Every equivalence class of combinatorial ideals contains a unique basic ideal.*

(*c*) *The number of equivalence classes of combinatorial ideals is finite.*

Proof. Let $\mathfrak{i}$ be a basic ideal. First we claim that $\delta \in \mathrm{supp}(\mathfrak{i})$. Indeed, if $\mathrm{supp}(\mathfrak{i})$ contains some $\alpha \in \Delta_+$, then, commuting with positive root elements from $\mathfrak{n}_+$, we get that $\mathrm{supp}(\mathfrak{i})$ contains $\alpha_{\max}$. Commuting a nonzero root element for the latter root with a nonzero root element for the root α_0 we get a nonzero element for the root δ, as required. If $\mathrm{supp}(\mathfrak{i})$ does not contain any $\alpha \in \Delta_+$, then it either contains δ (in which case we have nothing to prove) or a root of the form $\alpha + \delta$ for some $\alpha \in \Delta_-$. In the latter case, commuting any nonzero root element for the root $\alpha + \delta$ with any nonzero root element for the positive root $-\alpha$ we get a nonzero element for the root δ, as required.

As $\mathfrak{i}$ is thick, we then have $\mathfrak{g}_\delta \subset \mathfrak{i}$. Commuting $\mathfrak{g}_\delta$ with $\hat{\mathfrak{n}}_+$ we obtain that $\mathrm{supp}(\mathfrak{i})$ contains all real roots from D'. That $\mathrm{supp}(\mathfrak{i})$ contains all $k\delta$ for $k > 1$ follows from this similarly to the previous paragraph. This proves claim (a).

Let $\mathfrak{i}$ be a combinatorial ideal. We claim that the equivalence class of $\mathfrak{i}$ contains a basic ideal. We prove this by induction on the minimal possible k such that $\mathrm{supp}(\mathfrak{i}) \cap D_k \neq \varnothing$. If $k = 0$, then $\mathfrak{i}$ is basic and we have nothing to prove. If $k > 0$, define

$$\mathfrak{j} := \bigoplus_{\alpha\in\mathrm{supp}(\mathfrak{i})} \hat{\mathfrak{g}}_{\alpha-\delta}.$$

As $k > 0$, the support of $\mathfrak{j}$ is contained in $\cup_{i\geqslant k-1}D_i$ (and intersects D_{k-1} non-trivially) and hence $\mathfrak{j} \subset \hat{\mathfrak{n}}_+$. As $\mathrm{supp}(\mathfrak{i}) \supset \cup_{i\geqslant m}D_i$ for some m big enough, the same is true for $\mathrm{supp}(\mathfrak{j})$, which means that $\mathfrak{j}$ has finite codimension.

Let $\alpha \in \mathrm{supp}(\mathfrak{j})$, $x \otimes t^l \in \mathfrak{j}_\alpha$ for some $x \in \mathfrak{g}$ and $y \otimes t^m \in \hat{\mathfrak{n}}_+$ be a root element for a root β. Assume that $[x \otimes t^l, y \otimes t^m] \neq 0$. Then $\alpha + \beta \in \hat{\Delta}_+$. Moreover, $[x \otimes t^{l+1}, y \otimes t^m] \neq 0$. However, the root of $x \otimes t^{l+1}$ equals $\alpha + \delta \in \mathrm{supp}(\mathfrak{i})$ and thus $x \otimes t^{l+1} \in \mathfrak{i}$ as $\mathfrak{i}$ is thick. Therefore $[x \otimes t^{l+1}, y \otimes t^m] \neq 0$ also belongs to $\mathfrak{i}$, which implies that $\alpha + \delta + \beta \in \mathrm{supp}(\mathfrak{i})$. This yields that $\alpha + \beta \in \mathrm{supp}(\mathfrak{j})$. Hence $[x \otimes t^l, y \otimes t^m] \in \mathfrak{j}$ by the definition of $\mathfrak{j}$. This shows that $\mathfrak{j}$ is an ideal.

By construction, the ideals $\mathfrak{i}$ and $\mathfrak{j}$ are equivalent. Moreover, as already mentioned above, the support of $\mathfrak{j}$ intersects D_{k-1} non-trivially. Hence, by the inductive assumption, we get that the equivalence class of $\mathfrak{i}$ contains a basic ideal.

Let now $\mathfrak{i}$ and $\mathfrak{j}$ be two equivalent basic ideals. Then both $\mathrm{supp}(\mathfrak{i})$ and $\mathrm{supp}(\mathfrak{j})$ contain D' by claim (a). Further, $\mathrm{supp}(\mathfrak{i})\cap D = \mathrm{supp}(\mathfrak{j})\cap D$ as for any different $\alpha, \beta \in D$ the difference $\alpha - \beta$ is not of the form $k\delta$, $k \in \mathbb{Z}$. This implies $\mathrm{supp}(\mathfrak{i}) = \mathrm{supp}(\mathfrak{j})$ and hence $\mathfrak{i} = \mathfrak{j}$ as both ideals are thick, proving claim (b).

Finally, from the above it follows that a basic ideal is uniquely determined by the intersection of its support with D. As D is a finite set, it follows that the number of basic ideals is finite. Hence claim (c) follows from claim (b). □

Remark 3. Using arguments similar to those used in the proof of Theorem 2(a) one can show that every non-zero ideal of $\hat{\mathfrak{n}}_+$ has finite codimension, see Corollary 22.

2.3. **Classification of basic ideals.** For $\alpha \in D$ construct a subset $\overline{\alpha}$ of D recursively as follows: set $\overline{\alpha}_0 = \{\alpha\}$, and for $i > 0$ put

$$\overline{\alpha}_i = \{\gamma \in D : \text{ there is } \alpha \in \hat{\Delta}_+ \text{ and } \beta \in \overline{\alpha}_{i-1} \text{ such that } \gamma = \alpha + \beta\}.$$

Then $\overline{\alpha}_0 \subset \overline{\alpha}_1 \subset \overline{\alpha}_2 \subset \dots$ by construction and, as D is finite, there is i_0 such that $\overline{\alpha}_i = \overline{\alpha}_{i+1}$ for all $i \geqslant i_0$. Set $\overline{\alpha} := \overline{\alpha}_{i_0}$ (cf. [CP1, Section 2]).

Define the partial order $\preceq$ on D as follows: for $\alpha, \beta \in D$ set $\alpha \preceq \beta$ if and only if $\beta \in \overline{\alpha}$ (which is equivalent to $\overline{\beta} \subset \overline{\alpha}$). It is easy to see that δ is the unique maximal element of D with respect to $\preceq$. For $\alpha \in D$ define

$$\mathfrak{i}(\alpha) := \bigoplus_{\beta\in\overline{\alpha}\cup D'} \hat{\mathfrak{g}}_{\beta}.$$

Theorem 4. (*a*) *For any $\alpha \in D$ the space $\mathfrak{i}(\alpha)$ is a basic ideal in $\hat{\mathfrak{b}}$.*

(*b*) *If* $\mathfrak{j}$ *is a basic ideal in* $\hat{\mathfrak{b}}$ *such that* $\mathfrak{j}_\alpha \neq 0$ *for some* $\alpha \in D$*, then* $\mathfrak{j} \supset \mathfrak{i}(\alpha)$.

(*c*) *If* $\mathfrak{j}$ *is a basic ideal in* $\hat{\mathfrak{b}}$*, then*

$$\mathfrak{j} = \sum_{\alpha \in D \cap \mathrm{supp}(\mathfrak{j})} \mathfrak{i}(\alpha).$$

(*d*) *There is a bijection between the set of basic ideals in* $\hat{\mathfrak{b}}$ *and nonempty anti-chains of the finite poset* $(D, \preceq)$.

(*e*) *There is a bijection between the set of basic ideals in* $\hat{\mathfrak{b}}$ *and submodules of the* $\hat{\mathfrak{b}}$*-module* $\hat{\mathfrak{n}}_+/\mathfrak{i}(\delta)$.

Proof. Claim (a) follows directly from the definitions. To prove claim (b), let $\mathfrak{j}$ be a basic ideal in $\hat{\mathfrak{b}}$ such that $\mathfrak{j}_\alpha \neq 0$. Then $D' \subset \mathrm{supp}(\mathfrak{j})$ by Theorem 2(a). Let $\beta \in D$ be such that $\alpha \preceq \beta$. To prove claim (b) it is enough to show that $\beta \in \mathrm{supp}(\mathfrak{j})$. If $\beta = \delta$, then $\beta \in \mathrm{supp}(\mathfrak{j})$ as was shown in the proof of Theorem 2(a), hence we may assume $\beta \neq \delta$ (i.e. β is a real root). As $\alpha \preceq \beta$, there is a sequence of real roots $\gamma_0 = \alpha, \gamma_1, \dots, \gamma_k = \beta$ in D such that $\xi_i := \gamma_i - \gamma_{i-1} \in \hat{\Delta}_+$ for all $i = 1, 2, \dots, k$. Note that each ξ_i is real and hence gives rise to an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}_i$ of $\hat{\mathfrak{g}}$. Then $\dim \hat{\mathfrak{g}}_{\gamma_i} = 1$ for all i as all γ_i are real, and the classical $\mathfrak{sl}_2$-theory applied to the adjoint action of $\mathfrak{s}_i$ on $\hat{\mathfrak{g}}$ implies $[\hat{\mathfrak{g}}_{\xi_i}, \hat{\mathfrak{g}}_{\gamma_{i-1}}] \neq 0$. The latter yields $\beta \in \mathrm{supp}(\mathfrak{j})$ and claim (b) follows.

Claim (c) follows directly from claim (b) and Theorem 2(a). To prove claim (d), let $\mathfrak{j}$ be a basic ideal. Denote by $B_\mathfrak{j}$ the set of all minimal (with respect to $\preceq$) elements in $D \cap \mathrm{supp}(\mathfrak{j})$. Then $B_\mathfrak{j}$ is a nonempty anti-chain in D. Conversely, given a nonempty anti-chain B in D, define $\overline{B}$ to be the coideal of D generated by B and set

$$\mathfrak{i}_B := \bigoplus_{\beta \in \overline{B} \cup D'} \mathfrak{g}_\beta.$$

Then it is easy to check that $\mathfrak{i}_B$ is a basic ideal and that the maps $\mathfrak{i} \mapsto B_\mathfrak{i}$ and $B \mapsto \mathfrak{i}_B$ are mutually inverse bijections. Claim (d) follows.

If $\mathfrak{i}$ is a basic ideal, then $\mathfrak{i} \supset \mathfrak{i}(\delta)$ and hence the image of $\mathfrak{i}$ in the $\hat{\mathfrak{b}}$-module $\hat{\mathfrak{n}}_+/\mathfrak{i}(\delta)$ is a submodule and this map from the set of basic ideals to the set of submodules in $\hat{\mathfrak{n}}_+/\mathfrak{i}(\delta)$ is injective. It is also easily seen to be surjective as the full preimage of a submodule in $\hat{\mathfrak{n}}_+/\mathfrak{i}(\delta)$ is a basic ideal (here it is important that all $\hat{\mathfrak{h}}$-weight spaces of $\hat{\mathfrak{n}}_+/\mathfrak{i}(\delta)$ are one-dimensional as they correspond to real roots). This completes the proof. □

The ideal $\mathfrak{i}(\alpha)$ can be considered as a kind of "principal" basic ideal generated by $\hat{\mathfrak{g}}_\alpha$. The next proposition relates the natural order $\leqslant$ to the order $\preceq$ on D defined above.

Proposition 5. *The orders* $\leqslant$ *and* $\preceq$ *coincide on* D.

Proof. Obviously, $\preceq$ is a subset of $\leqslant$ (as a binary relation), so we only have to prove the reverse inclusion. Let $\alpha, \beta \in D$ be such that $\alpha \leqslant \beta$. We have to show that $\alpha \preceq \beta$. If $\beta = \delta$, then $\alpha \preceq \delta$ is clear, so in the following we may assume that $\beta \neq \delta$, that is that both α and β are real roots.

Case 1. Assume first that $\beta \in \Delta_+$, then $\alpha \in \Delta_+$ as well. Let $\gamma_1, \dots, \gamma_k$ be simple roots such that $\beta - \alpha \in \sum_{i=1}^{k} \mathbb{N}\, \gamma_i$ and $\mathfrak{a}$ be the semi-simple Lie subalgebra of $\hat{\mathfrak{g}}$ which these roots (and their negatives) generate. Consider the finite-dimensional $\mathfrak{a}$-module

$$V := \bigoplus_{\xi \in \alpha + \sum_{i=1}^{k} \mathbb{Z}\gamma_i} \hat{\mathfrak{g}}_\xi.$$

It is enough to show that $\alpha + \gamma_i$ is a weight of V for some i. Indeed, if this is the case, then $\alpha \preceq \alpha + \gamma_i$, furthermore, $\alpha + \gamma_i \leqslant \beta$ and the claim follows by induction on the height of $\beta - \alpha$. Assume that $\alpha + \gamma_i$ is not a weight of V for every i. Then α is an $\mathfrak{a}$-highest weight of V. Let $\beta' \geqslant \beta$ be the highest weight of the unique simple subquotient V' of V, which intersects the one-dimensional space $\hat{\mathfrak{g}}_\beta$ non-trivially. Then $\alpha \leqslant \beta'$ are two $\mathfrak{a}$-dominant weights. By [Hu, Proposition 21.3], α is a weight of V'. Since $\hat{\mathfrak{g}}_\alpha$ is one-dimensional, and α is a highest weight of V, we get $\alpha = \beta' = \beta$ and we are done.

Case 2. Assume that $\alpha \notin \Delta_+$, then $\beta \notin \Delta_+$ as well. Then $\alpha \leqslant \beta$ implies $-\beta + \delta \leqslant -\alpha + \delta$ and $-\beta + \delta, -\alpha + \delta \in \Delta_+$. From Case 1 we have $-\beta + \delta \preceq -\alpha + \delta$, which implies $\alpha \preceq \beta$.

Case 3. Finally, assume that $\alpha \in \Delta_+$ while $\beta \notin \Delta_+$. In this case we will need the following auxiliary lemma:

Lemma 6. *There does not exist a decomposition of the maximal root of Δ_+ of the form*

$$\alpha_{\max} = \xi + \zeta + \eta \qquad (2.1)$$

such that the following conditions are satisfied:

(*i*) $\xi, \zeta \in \Delta_+$ *while* $\xi + \zeta \notin \Delta_+$*;*
(*ii*) η *is a linear combination of some simple roots* $\eta_1, \dots, \eta_k$*,* $k \geqslant 1$*, with positive integer coefficients;*
(*iii*) $\xi + \eta_i \notin \Delta_+$ *and* $\zeta + \eta_i \notin \Delta_+$ *for all* i*.*

Given Lemma 6, the proof of Case 3 goes as follows: As $\alpha \in \Delta_+$, $\beta \notin \Delta_+$ and $\alpha \leqslant \beta$, we can write $\beta = \alpha + \alpha_0 + \gamma$ for some $\gamma \in \Sigma_+$. Let $\gamma_1, \dots, \gamma_k$ denote all simple roots which appear in the decomposition of γ as a linear combination of simple roots with positive integer coefficients. If $\alpha + \gamma_i$ is a root for some i, then $\alpha \preceq \alpha + \gamma_i$ and $\alpha + \gamma_i \leqslant \beta$ by construction, so we can complete the argument by induction on the height of $\beta - \alpha$. Similarly, if $\beta - \gamma_i$ is a root for some i. Finally, if $\alpha - \beta + \delta \in \Delta_+$, then $\beta - \alpha \in \hat{\Delta}_+$ and hence $\alpha \preceq \beta$ by definition. If none of the above is satisfied, then, taking $\xi = \alpha$, $\zeta = -\beta + \delta$ and $\eta = \gamma$, we see that these elements satisfy conditions (i)–(iii) of Lemma 6, which is a contradiction by Lemma 6. This completes the proof. □

Proof of Lemma 6. We prove Lemma 6 by a brute force case-by-case analysis of all reduced irreducible finite root systems. Before we start this analysis we make some remarks. Assume that the decomposition of the form (2.1) exists. Let Δ_η denote the root system generated by η_i, $i = 1, \dots, k$. We have $\varnothing \neq \Delta_\eta \subsetneq \Delta$ (the latter inequality follows from conditions (i) and (iii)).

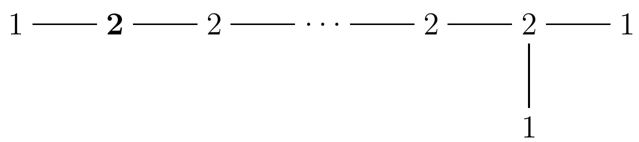
From condition (iii) it follows that reflection with respect to each η_i either does not effect ξ and ζ or decreases them (in this proof by "decreases" we always mean "with respect to $\leqslant$"). The element η satisfies $0 < \eta < \alpha_{\max}$ and thus must be decreased by the reflection with respect to at least one of the η_i's. This and (2.1) imply that every decrease of η automatically gives a decrease of $\alpha_{\max}$. The coordinate of a vector $v \in \mathfrak{h}^*$ (with distinguished basis of simple roots), reflection with respect to which decreases v, will be called *decreasable*. We will also say that the corresponding simple root *decreases* v.

Type A. In this case $\alpha_{\max}$ is as follows, with decreasable coordinates in bold (here and in the rest of the proof we use [Hu, Chapter III] as a reference):

$$\mathbf{1} \text{ —— } 1 \text{ —— } 1 \text{ —— } \cdots \text{ —— } 1 \text{ —— } 1 \text{ —— } \mathbf{1}$$

The system Δ_η must contain at least one simple root which decreases $\alpha_{\max}$. Without loss of generality we thus may assume that Δ_η contains the leftmost simple root. Then η's coordinate at it must be 1 and, since this coordinate is decreasable, the next from the left coordinate must be 1 as well. Since this coordinate is not decreasable, its right neighbor must be 1 again and so on. We get $\eta = \alpha_{\max}$, which is not possible.

Type D. In this case $\alpha_{\max}$ is as follows, with the unique decreasable coordinate in bold:



The system Δ_η must contain the simple root which decreases $\alpha_{\max}$. The corresponding coordinate of η can thus be 1 or 2. If we assume that it is 2, then, using the uniqueness of decreasable coordinate in η one shows that $\eta = \alpha_{\max}$, a contradiction. Hence this coordinate is 1. The coordinate to the left cannot be 1 as in this case it would be decreasable, hence it is 0. This forces the coordinate to the right to be 1. The latter is not decreasable which forces the next coordinate to the right to be 1 or 2. But a 2 gives a decreasable coordinate, and hence the only possibility is 1. In this way we show that all nonzero coordinates of η have to be 1 and thus the last nonzero coordinate which we get will be decreasable, a contradiction.

Type B. In this case $\alpha_{\max}$ is as follows, with the unique decreasable coordinate in bold:

$$1 \text{ —— } \mathbf{2} \text{ —— } 2 \text{ —— } \cdots \text{ —— } 2 \text{ —— } 2 \Longrightarrow 2$$

Here the argument is similar to the one we used in type D.

Type C. In this case $\alpha_{\max}$ is as follows, with the unique decreasable coordinate in bold:

$$\mathbf{2} \text{ —— } 2 \text{ —— } 2 \text{ —— } \cdots \text{ —— } 2 \text{ —— } 2 \Longleftarrow 1$$

Here again the argument is similar to the one we used in type D with the difference that we do not need to bother about what happens to the left of the decreasable root.

Type E_6. In this case $\alpha_{\max}$ is as follows, with the unique decreasable coordinate in bold:

$$\begin{array}{ccccccccc} 1 & \text{——} & 2 & \text{——} & 3 & \text{——} & 2 & \text{——} & 1 \\ & & & & | & & & & \\ & & & & \mathbf{2} & & & & \end{array}$$

Similarly to the previous cases we get that the coordinate of η at the unique decreasable root of $\alpha_{\max}$ cannot be 2 for the latter forces $\eta = \alpha_{\max}$, so this coordinate is 1. So the coordinate of η in its unique neighbor must be also 1. Now the next neighbor to the right can have coordinates 0, 1 or 2. The value 2 is not possible as the coordinate will be decreasing. The value 1 forces the value 1 in the next right neighbor by the "type A"-argument, and this last coordinate becomes decreasing. Hence the only possibility is 0. By symmetry, the value of the coordinate to the left from the triple point is also zero. This forces the triple point to be decreasing and we again have a contradiction.

Types $E_{7,8}$. Are similar to E_6 (but with many more cases to go through) and left to the reader.

Type F_4. In this case $\alpha_{\max}$ is as follows, with the unique decreasable coordinate in bold:

$$\mathbf{2} \text{ —— } 3 \Longrightarrow 4 \text{ —— } 2$$

Similarly to the previous cases we see that the leftmost coordinate of η must have value 1, which also forces value 1 or its right neighbor. Now the third coordinate cannot be decreasing and hence can only have values $1, 2$. The value 2 forces the rightmost value to be 2 and this rightmost coordinate becomes decreasing, a contradiction. Hence the third coordinate has value 1. The last coordinate must then be zero as $\Delta \neq \Delta_\eta$. But so far we could have

$$\eta \quad = \quad 1 \text{ —— } 1 \Longrightarrow 1 \text{ —— } 0$$

However, in this case $\xi+\zeta$ is a positive root which contradicts condition (i).

Type G_2. This is a short direct computation which is left to the reader. □

Remark 7. The special case of Proposition 5 in the classical case of finite dimensional $\mathfrak{g}$ is implicit when comparing the first paragraph of [CP1, Section 2] with the first paragraph of [AKOP, Section 2]. However, we did not manage to find any explicit proof in the literature. An alternative proof of this special case follows, for example, from [CDR, Lemma 1.1(ii)].

Let $\mathfrak{i}$ be a basic ideal. Then $\mathfrak{i}_+ := \mathfrak{i} \cap \mathfrak{n}_+$ is an ad-nilpotent ideal in $\mathfrak{b}$ (or, equivalently, a $\mathfrak{b}$-submodule of $\mathfrak{n}_+$). Consider the $\mathfrak{b}$-module $\mathfrak{g}/\mathfrak{b}$ which is identified, both as a vector space and an $\mathfrak{h}$-module, with $\mathfrak{n}_-$ in the natural way. Denote by $\mathfrak{i}_-$ the $\mathfrak{b}$-submodule of $\mathfrak{g}/\mathfrak{b}$ which under this identification corresponds to the direct sum of root spaces of negative roots β such that $\beta + \delta \in \mathrm{supp}(\mathfrak{i})$. From Theorem 4 it follows that the pair $(\mathfrak{i}_+, \mathfrak{i}_-)$ determines $\mathfrak{i}$ uniquely.

Denote by $\mathfrak{b}_-$ the opposite Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_-$. It is easy to see that the $\mathfrak{h}$-complement of $\mathfrak{i}_-$ in $\mathfrak{n}_-$ is in fact a $\mathfrak{b}_-$-submodule of $\mathfrak{n}_-$. Applying the Chevalley involution we obtain that $\mathfrak{i}$ is uniquely determined by a pair of ad-nilpotent ideals in $\mathfrak{b}$. This implies the following:

Corollary 8. *The number of basic ideals in* $\hat{\mathfrak{b}}_+$ *does not exceed the square of the number of ad-nilpotent ideals in* $\mathfrak{b}$*.*

Later on we will see that not every pair of ad-nilpotent ideals in $\mathfrak{b}$ corresponds in this way to a basic ideals in $\hat{\mathfrak{b}}_+$. In the next section we exploit this connection to enumerate basic ideals in $\hat{\mathfrak{b}}_+$ for the affine Lie algebra $\hat{\mathfrak{sl}}_n$.

3. Enumeration of basic ideals in type A

3.1. **Some matrix combinatorics.** Let $\Bbbk$ be a commutative ring with 1. For $n \in \mathbb{N}$ consider the $\Bbbk$-algebra $\mathrm{M}_n(\Bbbk)$ of all $n \times n$ matrices with coefficients from $\Bbbk$. We will denote elements of $\mathrm{M}_n(\Bbbk)$ in the following standard way:

$$A = (A_{i,j}) = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$$

and for such notation the indices i and j always run through the set $\{1, 2, \dots, n\}$.

For $A, B \in \mathrm{M}_n(\mathbb{k})$ set $A \cdot B = \sum_{i,j} a_{i,j} b_{i,j}$. Then $\cdot : \mathrm{M}_n(\mathbb{k}) \times \mathrm{M}_n(\mathbb{k}) \to \mathbb{k}$ is a bilinear form on $\mathrm{M}_n(\mathbb{k})$. Define the linear operator ω on $\mathrm{M}_n(\mathbb{k})$ by setting

$$(\omega\, A)_{i,j} = \sum_{k=n-i}^{n} \sum_{m=n-j}^{n} a_{k,m}$$

(here we assume that $a_{0,m} = a_{k,0} = 0$). For example, we have

$$\omega \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 28 & 33 & 33 \\ 39 & 45 & 45 \\ 39 & 45 & 45 \end{pmatrix}.$$

We will also need another linear operator τ on $\mathrm{M}_n(\mathbb{k})$ defined as follows:

$$(\tau\, A)_{i,j} = \sum_{s=i-1}^{n} a_{s,j} \tag{3.1}$$

(here we assume $a_{0,j} = 0$). For example, we have

$$\tau \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 12 & 15 & 18 \\ 12 & 15 & 18 \\ 11 & 13 & 15 \end{pmatrix}.$$

3.2. **Some Dyck path combinatorics.** In this subsection we collect some necessary elementary combinatorics of Dyck paths (most of which we failed to find in the literature, but we are not going to be surprised if it exists). For $n \in \mathbb{N}$ define the integral matrix $\mathbf{C}_n \in \mathrm{M}_n(\mathbb{Z})$ recursively as follows:

$$\mathbf{C}_1 := (1); \quad \mathbf{C}_{n+1} = \big(\tau\, \mathbf{C}_n\big) \oplus \mathbf{C}_1. \tag{3.2}$$

For small values of n we have:

$$\mathbf{C}_1 := \begin{pmatrix} 1 \end{pmatrix}; \quad \mathbf{C}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{C}_3 := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\mathbf{C}_4 := \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathbf{C}_5 := \begin{pmatrix} 5 & 5 & 3 & 1 & 0 \\ 5 & 5 & 3 & 1 & 0 \\ 3 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From the definition it is clear that all $\mathbf{C}_n$ have non-negative coefficients. What is much less clear, but suggested by the above examples, is that every matrix $\mathbf{C}_n$ is symmetric. We will prove this later in Corollary 10. We denote the coefficient $(\mathbf{C}_n)_{i,j}$ by $\mathbf{c}^{(n)}_{i,j}$.

Recall that for $n \in \mathbb{N}_0$ a *Dyck path* of semilength n is a path in the first quadrant of the coordinate (x, y)-plane from $(0, 0)$ to $(2n, 0)$ with steps along $(1, 1)$, called *rises*, and along $(1, -1)$, called *falls*. A *peak* of a Dyck path is the end point of a rise followed by a fall. A *valley* of a Dyck path is the end point of a fall followed by a rise. The y-coordinate of a point is usually called the *height*. A Dyck path is called *primitive* if $(0,0)$ and $(2n,0)$ are the only points from the x-axis which belong to the path. For a Dyck path p we denote by $\mathbf{v}(p)$ and $\mathbf{p}(p)$ the number of valleys and peaks in p, respectively. By $\mathbf{p}_i(p)$, $i=1,\dots,n$, we also denote the number of peaks of height at least $i+1$. By $\mathbf{v}_{(i)}(p)$, $i=0,\dots,n-1$, we denote the number of valleys of height i in p. By $\mathtt{v}_{(i)}(p)$, $i=0,\dots,n-1$, we denote the set of valleys of height i in p, in particular, $\mathbf{v}_{(i)}(p)=|\mathtt{v}_{(i)}(p)|$. Rises, falls, peaks and valleys are counted from the left to the right, e.g. the first peak is the leftmost peak and the last valley is the rightmost valley. Note that every Dyck path has at least one peak, but it may contain no valleys. Further, every Dyck path starts with a rise and ends with a fall. We denote by $\mathcal{D}_n$ the set of all Dyck paths of semilength n. It is well-known (see for example [St, Chapter 6]) that $|\mathcal{D}_n|$ is the n-th Catalan number $C_n=\frac{1}{n+1}\binom{2n}{n}$. The set $\mathcal{D}_n$ is partially ordered in the natural way with respect to the relation $p\leqslant q$ defined as follows: path q never goes below path p.

A natural way to encode Dyck paths is by the corresponding sequence of rises and falls, e.g. $rrfrffrf$ (the condition is that the sequence contains n rises and n falls and each prefix of the sequence contains at least as many rises as falls). See an example in Figure 1. We denote by $*$ the usual involution on Dyck paths defined by reversing them and swapping rises and falls (which corresponds to reflecting the Dyck path at the vertical line given by the equation $x=n$), for example

$$(rrrfrrffff)^*=rrrrffrfff.$$

We denote by $\mathbf{p}$ the unique Dyck paths with no valleys, that is the path of the form $rr\dots rff\dots f$. We also denote by $\mathbf{q}$ the unique Dyck paths with n peaks, that is the path of the form $rfrf\dots rf$. The paths $\mathbf{p}$ and $\mathbf{q}$ are the maximum and the minimum elements of $\mathcal{D}_n$ with respect to the partial order $\leqslant$, respectively.

The height of a peak of a Dyck path of semilength n is a positive integer between 1 and n. For $i,j\in\{1,2,\dots,n\}$ denote by $\mathcal{D}_n(i,j)$ the set of all Dyck paths for which i is the height of the first peak and j is the height of the last peak (note that some of the $\mathcal{D}_n(i,j)$ might be empty). We also set $\mathcal{D}_n(0,j)=\varnothing$ for all j.

Theorem 9. *For all $n\in\mathbb{N}$ and $i,j\in\{1,2,\dots,n\}$ we have*

$$|\mathcal{D}_n(i,j)|=\mathbf{c}^{(n)}_{i,j}.$$

Proof. We proceed by induction on n. The basis $n=1$ is trivial. Further, note that $\mathcal{D}_n(n,j)$ is non-empty only in the case $j=n$, moreover $\mathcal{D}_n(n,n)=\{\mathbf{p}\}$. Similarly for $\mathcal{D}_n(i,n)$. Hence the statement of our theorem is true for all n in all cases where $i=n$ or $j=n$ as by construction of $\mathbf{C}_n$ we have $\mathbf{c}^{(n)}_{n,n}=1$ and $\mathbf{c}^{(n)}_{i,n}=0=\mathbf{c}^{(n)}_{n,j}$ for all $i,j\neq n$.

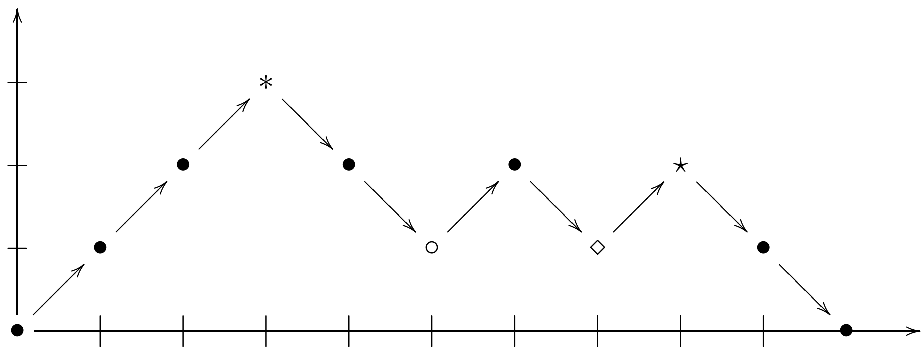


FIGURE 1. The Dyck path $rrrffrfrff \in \mathcal{D}_5(3,2)$, the first peak is $*$ (of height 3), the last peak is $\star$ (of height 2), the first valley is $\circ$, the last valley is $\diamond$ (both valleys have height 1)

Define the map

$$F : \mathcal{D}_n(i,j) \to \bigcup_{s=i-1}^{n-1} \mathcal{D}_{n-1}(s,j)$$

as follows: given a Dyck path in $\mathcal{D}_n(i,j)$, delete the first peak in this Dyck path, that is the leftmost occurrence of rf. Define the map

$$G : \bigcup_{s=i-1}^{n-1} \mathcal{D}_{n-1}(s,j) \to \mathcal{D}_n(i,j)$$

as follows: given a Dyck path in $\cup_{s=i-1}^{n-1}\mathcal{D}_{n-1}(s,j)$, insert a peak after the first $i-1$ rises, that is an rf after the first $i-1$ letters r of the path. It is straightforward to verify that F and G are mutually inverse bijections, and hence

$$\left|\bigcup_{s=i-1}^{n-1} \mathcal{D}_{n-1}(s,j)\right| = |\mathcal{D}_n(i,j)| \tag{3.3}$$

by the bijection rule. As the union on the left hand side is disjoint, from the inductive assumption we have

$$\begin{aligned} |\mathcal{D}_n(i,j)| &\overset{(3.3)}{=} \left|\bigcup_{s=i-1}^{n-1} \mathcal{D}_{n-1}(s,j)\right| \\ &\overset{\text{(by induction)}}{=} \sum_{s=i-1}^{n-1} \mathbf{c}_{s,j}^{(n-1)} \\ &\overset{(3.2)\text{ and }(3.1)}{=} \mathbf{c}_{i,j}^{(n)}. \end{aligned}$$

The proof is complete. □

Various enumeration problems involving Dyck paths were considered in [De], in particular, there one can find a formula for enumeration with respect to the height of the first peak.

Corollary 10. *The matrix* $\mathbf{C}_n$ *is symmetric for every* $n \in \mathbb{N}$ *and the sum of all entries in* $\mathbf{C}_n$ *equals* C_n.

Proof. The second claim follows directly from Theorem 9. The first claim follows from Theorem 9 applying the involution $*$. □

The entries of $\mathbf{C}_n$ are directly related to several classical combinatorial objects associated with the combinatorics of Catalan numbers. Our first observation is a relation to what is known as the *Catalan triangle* (sequence A009766 in [Sl]), see [Ba]. It is defined, in analogy with Pascal's triangle, as the following triangular array of integers with the property that every entry equals the sum of the entry above and the entry to the left:

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 2 & & & \\ 1 & 3 & 5 & 5 & & \\ 1 & 4 & 9 & 14 & 14 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The entry $c_{i,j}$ in the i-th row and the j-th column equals

$$c_{i,j} = \frac{(i+j)!(i-j+1)}{j!(i+1)!}.$$

From the definition of $\mathbf{C}_n$ it follows that the first two rows of $\mathbf{C}_n$ coincide. These coinciding rows have the following interpretation in terms of the Catalan triangle:

Proposition 11. *For* $n \geqslant 2$ *and any* $j \in \{1, 2, \ldots, n-1\}$ *we have* $\mathbf{c}_{1,j}^{(n)} = c_{n-2,n-1-j}$.

Proof. For $j = n-1$ the claim follows from the definitions. Hence it is enough to show that the entries of the first row of $\mathbf{C}_n$ satisfy (with respect to the first row of $\mathbf{C}_{n-1}$) the same recursion. By Theorem 9, for this it is enough to produce a bijection between $\mathcal{D}_n(2, j)$ and

$$\mathcal{D}_n(1, j+1) \cup \mathcal{D}_{n-1}(1, j-1).$$

Denote by X the set of all primitive Dyck paths in $\mathcal{D}_n(2, j)$ and set $Y := \mathcal{D}_n(2, j) \setminus X$. Every $p \in X$ has the form rqf for some path $q \in \mathcal{D}_{n-1}(1, j-1)$ and vice versa, for every $q \in \mathcal{D}_{n-1}(1, j-1)$ the path rqf belongs to X giving us a bijection between X and $\mathcal{D}_{n-1}(1, j-1)$.

Every Dyck path $p \in Y$ has the form $r(rfrxf)\mathbf{fr}yrf^j$, where the bold valley $\mathbf{fr}$ is the first return of p to the diagonal. Define the map

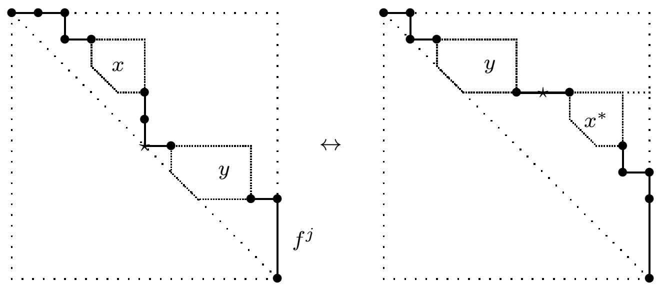


FIGURE 2. Second bijection in the proof of Proposition 11, for compactness both Dyck paths are rotated clockwise by $\frac{\pi}{4}$, key points used in the proof are marked with $\star$

$\varphi : Y \to \mathcal{D}_n(1, j+1)$ via

$$r(rfrxf)fryrf^j \mapsto rfryr(rfrxf)^* f^j.$$

We claim that φ is a bijection, the inverse of which is defined as follows: Every path $q \in \mathcal{D}_n(1, j+1)$ has the form $rfrv\mathbf{rr}wfrf^{j+1}$, where the bold $\mathbf{rr}$ is the last crossing of height j. The map $\varphi^{-1} : Y \to \mathcal{D}_n(1, j+1)$ is then defined via:

$$rfrvrrwfrf^{j+1} \mapsto r(rwfrf)^* frvrf^j.$$

This bijection is illustrated in Figure 2. The claim follows. □

Our next observation is related to the combinatorics of Pascal triangle.

Proposition 12. *Let $n \in \mathbb{N}$ and $i, j \in \{1, 2, \ldots, n\}$. Then we have:*

(*a*) $\mathbf{c}^{(n)}_{i,j} + \sum_{s=1}^{n} \mathbf{c}^{(n)}_{s,i+j+1} = \sum_{s=i}^{n} \mathbf{c}^{(n)}_{s,j+1}$;

(*b*) $\mathbf{c}^{(n)}_{i,j} + \mathbf{c}^{(n)}_{1,i+j} = \mathbf{c}^{(n)}_{i+1,j} + \mathbf{c}^{(n)}_{i,j+1}$.

Proof. Claim (b) reduces to claim (a) using the recursion from the construction of $\mathbf{C}_n$ given by (3.1).

By Theorem 9, to prove claim (a) we have to construct a bijection between the set

$$A := \mathcal{D}_n(i, j) \cup \bigcup_{s=1}^{n} \mathcal{D}_n(s, i+j+1)$$

and the set

$$B := \bigcup_{s=i}^{n} \mathcal{D}_n(s, j+1).$$

We denote by A' the set $\bigcup_{s=1}^{n} \mathcal{D}_n(s, i+j+1)$. Let X denote the set of primitive Dyck paths in B and set $Y := B \setminus X$.

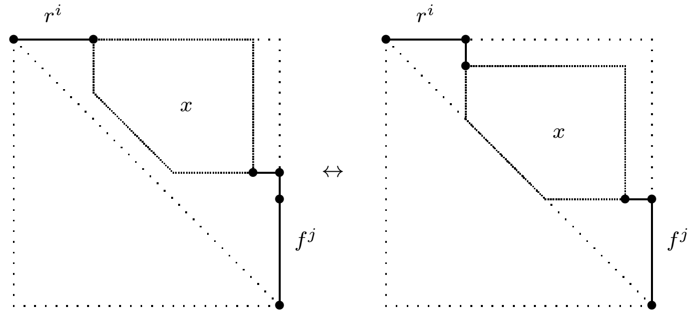


FIGURE 3. First bijection in the proof of Proposition 12, for compactness both Dyck paths are rotated clockwise by $\frac{\pi}{4}$

First we construct a bijection between $\mathcal{D}_n(i,j)$ and X. Every $p \in \mathcal{D}_n(i,j)$ can be written in the form $r^i fxrf^j$. Transforming this path into $r^i xrf^{j+1}$ produces a path from X. This defines a bijection with the inverse defined as follows: Every $q \in X$ can be written in the form $r^i yrf^{j+1}$. As q is primitive, transforming this path into $r^i fyrf^j$ produces a path $\mathcal{D}_n(i,j)$ which gives the desired inverse (see Figure 3).

It is left to construct a bijection between A' and Y. Every $p \in Y$ can be written in the form $r^i x\mathbf{fr}yrf^{j+1}$, where the bold **fr** marks the first return of the imprimitive path p to the x-axis. It is easy to show that transforming

$$r^i xfryrf^{j+1} \mapsto ryrx^* f^* (r^i)^* f^{j+1}$$

defines a map from Y to A'. The inverse of this map is defined as follows: every path $q \in A'$ can be written in the form $ru\mathbf{rr}wrf^{i+j+1}$, where the bold **rr** marks the last crossing of the height $j+1$. It is easy to show that transforming

$$rurrwrf^{i+j+1} \mapsto (f^i)^* w^* r^* rurf^{j+1}$$

defines the desired inverse from A' to Y (see Figure 4), completing the proof. □

Denote by $\mathbf{C}'_n$ the principal minor of $\mathbf{C}_n$ corresponding to the first $n-1$ rows and columns. Proposition 12 implies that the part of $\mathbf{C}'_n$ lying below and on the opposite diagonal coincides with the Pascal triangle (appropriately rotated). The part above the opposite diagonal follows almost the same recursion (every element is the sum of the right and below neighbors) but with a kind of a "correction term".

Here is an explicit formula for $\mathbf{c}^{(n)}_{i,j}$:

Proposition 13. *For $n \in \mathbb{N}$ and $i,j \in \{1,2,\dots,n-1\}$ we have:*

$$\mathbf{c}^{(n)}_{i,j} = \binom{(n-1-i)+(n-1-j)}{n-1-i} - \binom{(n-1-i)+(n-1-j)}{n-i-j-1}.$$

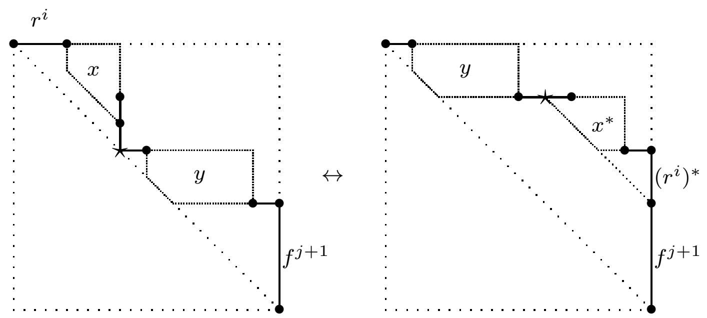


FIGURE 4. Second bijection in the proof of Proposition 12, for compactness both Dyck paths are rotated clockwise by $\frac{\pi}{4}$, key points used in the proof are marked with $\star$

Proof. We have to count the number of paths in the plane from the point $(i+1, i-1)$ to the point $(2n-j-1, j-1)$ along $(1,1)$ and $(1,-1)$ which do not go below the x-axis. This is a slight generalization of the classical ballot problem (see [An]), which can be solved using André's reflection principle (see e.g. [Gr, Section 1.6] for application of this principle to Catalan combinatorics). The number of all paths from $(i+1, i-1)$ to $(2n-j-1, j-1)$ along $(1,1)$ and $(1,-1)$ equals $\binom{(n-1-i)+(n-1-j)}{n-1-i}$ (the first summand of our formula), since we have to do $(n-1-i)+(n-1-j)$ steps, and arbitrary $n-1-i$ of them can be chosen to be rises.

Let p be a path from $(i+1, i-1)$ to $(2n-j-1, j-1)$ which goes below the x-axis. Then p has the form $xffy$ where ff indicates the first crossing of the x axis. Denote by y' the path obtained from y by swapping all rises and falls. Then $xffy'$ is a path from $(i+1, i-1)$ to $(2n-j-1, -j-1)$. Conversely, every path from $(i+1, i-1)$ to $(2n-j-1, -j-1)$ crosses the x-axis. Then we can write this path in the form $uffw$ where ff indicates the first crossing of the y axis. Then the path $uffw'$, where w' is obtained from w by swapping all rises and falls, is a path from $(i+1, i-1)$ to $(2n-j-1, j-1)$ which goes below the x-axes.

Now the claim follows from the observation that the number of paths from $(i+1, i-1)$ to $(2n-j-1, -j-1)$ equals $\binom{(n-1-i)+(n-1-j)}{n-i-j-1}$ (the second summand in our formula). $\square$

3.3. **Proof of Theorem 1.** For $n > 1$ there is an obvious bijection between strong ideals of $\hat{\mathfrak{sl}}_n$ and $\hat{\mathfrak{gl}}_n$. Therefore it is convenient to use the convention that $\hat{\mathfrak{sl}}_1$ is a codimension one subalgebra of $\hat{\mathfrak{gl}}_1$.

We denote by $\mathfrak{B}_n$ the (finite) set of basic ideals for the affine Lie algebra $\hat{\mathfrak{sl}}_n$ and set $\mathbf{b}_n := |\mathfrak{B}_n|$. The aim of this subsection is to show that $\mathbf{b}_n = \mathbf{C}_n \cdot \omega\, \mathbf{C}_n$ as stated in Theorem 1.

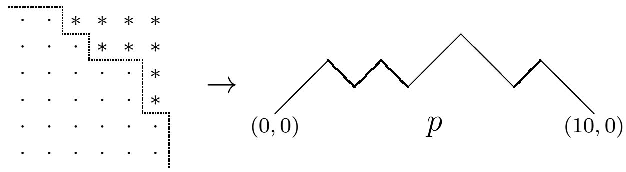


FIGURE 5. The Dyck path p associated to $\mathfrak{i}$, the elements of $\mathbf{s}_+(\mathfrak{i})$ are given by $*$

With every $\mathfrak{i} \in \mathfrak{B}_n$ we associate a pair $\Phi(\mathfrak{i}) := (p, q)$ of Dyck paths of semilength n in the following way: View an $n \times n$ matrix as a square in the coordinate plane with vertexes $(0, 0)$, $(n, 0)$, $(0, -n)$ and $(n, -n)$ in the natural way (the entries of the matrix are 1×1 boxes of this square). Our convention is that the simple root α_i, $i = 1, \dots, n-1$, of Δ_+ corresponds to the box with the north-west coordinate $(i, 1-i)$ and the opposite root $-\alpha_i$ corresponds to the box with the north-west coordinate $(i-1, -i)$. Fill in all boxes corresponding to the roots in $\mathbf{s}_+(\mathfrak{i}) := \Delta_+ \cap \operatorname{supp}(\mathfrak{i})$. They all belong to the part of our matrix above the diagonal. Since $\mathfrak{i}$ is an ideal, if some box is filled, then both its right and upper neighbors are filled as well (provided that they belong to our square). Consider the line l starting at $(0, 0)$ and ending at $(n, -n)$ which goes along the vectors $(1, 0)$ and $(0, -1)$ separating the filled boxed from the ones which are not filled. Rotate this line counterclockwise around the origin by $\frac{\pi}{4}$ and scale it appropriately to obtain the Dyck path p (of semilength n). An example is shown in Figure 5.

The construction of q is similar. Fill in all boxes corresponding to the roots $\alpha \in \Delta_-$ such that $\alpha + \delta \in \operatorname{supp}(\mathfrak{i})$ (we denote the set of all such α by $\mathbf{s}_-(\mathfrak{i})$). They all belong to the part of our matrix below the diagonal. Since $\mathfrak{i}$ is an ideal, if some box is filled, then both its right and upper neighbors are filled as well (provided that they belong to the part of our square below the diagonal). Consider the line l starting at $(0, 0)$ and ending at $(n, -n)$ which goes along the vectors $(1, 0)$ and $(0, -1)$ separating the filled boxed below the diagonal from the ones which are not filled. Rotate this line counterclockwise by $\frac{\pi}{4}$, reflect it in the x-axis and scale it appropriately to obtain the Dyck path q (or semilength n). An example is shown in Figure 6.

From Theorem 4(d) it follows that different ideals in $\mathfrak{B}_n$ give different pairs of Dyck paths. However, as we will see below, not every pair of Dyck paths can be obtained in such way. A pair of Dyck paths which corresponds in this way to some ideal in $\mathfrak{B}_n$ will be called *admissible*. We will prove Theorem 1 by counting the number of admissible pairs of Dyck paths. Denote by $\mathfrak{P}_n$ the set of all admissible pairs of Dyck paths and for $k, m \in \{1, 2, \dots, n\}$ let $\mathfrak{P}_n(k, m)$ denote the set of all

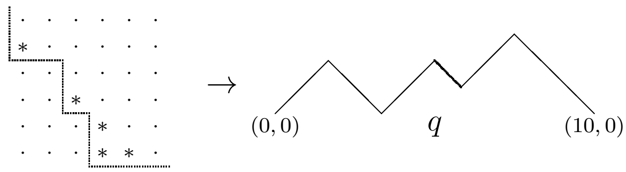


FIGURE 6. The Dyck path q associated to $\mathfrak{j}$, the elements of $\mathbf{s}_-(\mathfrak{j})$ are given by $*$

pairs (p, q) from $\mathfrak{P}_n$ for which k and m are the heights of the first and the last peak of p, respectively. Then $\mathfrak{P}_n$ is the disjoint union of the $\mathfrak{P}_n(k, m)$'s.

Lemma 14. *We have $(p, q) \in \mathfrak{P}_n(k, m)$ if and only if the first peak of q has height at least $n - m$ and the last peak of q has height at least $n - k$.*

Proof. Let $\mathfrak{i}$ be the basic ideal corresponding to (p, q). Then, by Proposition 5, for any $\alpha \in \Delta_+ \cap \operatorname{supp}(\mathfrak{i})$ every element in $\gamma \in D$ such that $\gamma \geqslant \alpha$ must belong to $\operatorname{supp}(\mathfrak{i})$. This means that for every $\beta \in \Delta_-$ with the property $\beta \geqslant \alpha - \alpha_{\max}$, for some $\alpha \in \Delta_+ \cap \operatorname{supp}(\mathfrak{i})$, the root $\beta + \delta$ belongs to $\operatorname{supp}(\mathfrak{i})$.

If $\alpha = \alpha_s + \alpha_{s+1} + \cdots + \alpha_r$ for some s, r such that $1 \leqslant s \leqslant r \leqslant n-1$, then every $\beta \in \Delta_-$ with the property $\beta \geqslant \alpha - \alpha_{\max}$ has the form $-(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j)$ either for some $j < s$ or some $i > r$. The maximal value of s is achieved by the root which corresponds to the box of our square for which the south-west corner is the last valley of p. The last peak of p has height $n - s = m$. This implies that the first peak of q has height at least $n - m$, see Figure 7.

The minimal value of r is achieved by the root which corresponds to the box of our square for which the south-west corner is the first valley of p. The first peak of p has height $r = k$. This implies that the last peak of q has height at least $n - k$.

Conversely, let (p, q) be a pair of Dyck paths. Assume that if the first peak of q has height at least $n - m$ and the last peak of q has height at least $n - k$. Consider the set I of all roots in Δ_+ which correspond to the boxes to the north-east of p in the matrix setup together will all roots of the form $\beta + \delta$, where β corresponds to the boxes to the north-east of q in the matrix setup (i.e. between q and the diagonal). Then $I \cup \{\delta\} \subset D$ is a non-empty subset and the argument from above implies that $\alpha \in I$, $\beta \in D$ and $\beta \geqslant \alpha$ implies $\beta \in I \cup \{\delta\}$. This means that (p, q) is admissible. $\square$

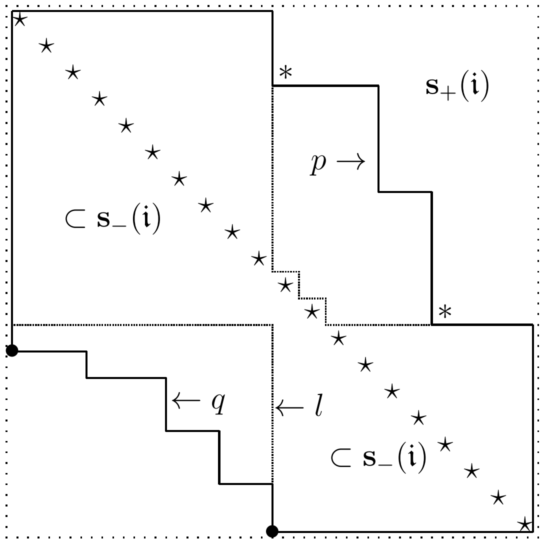


FIGURE 7. Argument in the proof of Lemma 14: asterisks show the first and the last valleys of p which determine the dashed line l, the region between l and the diagonal must belong to $\mathbf{s}_-(\mathfrak{i})$ and hence the path q must be below l, the first and the last peaks of q are marked by $\bullet$

Proof of Theorem 1. From the definitions we have:

$$\mathbf{b}_n = |\mathfrak{P}_n| = \left| \bigcup_{k,m=1}^{n} \mathfrak{P}_n(k,m) \right| = \sum_{k,m=1}^{n} |\mathfrak{P}_n(k,m)| \,.$$

From Lemma 14 it follows that there is a bijection between $\mathfrak{P}_n(k,m)$ and the set

$$\bigcup_{i=n-m}^{n} \bigcup_{j=n-k}^{n} \mathcal{D}_n(i,j) \times \mathcal{D}_n(k,m).$$

The assertion of Theorem 1 follows now from Theorem 9 and the definitions. □

Remark 15. The integral sequence $\{\mathbf{b}_n : n \geqslant 1\}$ seems to be new (at least we could not find anything similar in [Sl]). The values of this sequence for small n are:

1, 4, 18, 82, 370, 1648, 7252, 31582, 136338, 584248...

Corollary 16. *For $n \in \mathbb{N}$ we have*

$$\mathbf{b}_n = \sum_{i,j=1}^{n} \mathbf{c}_{i,j}^{(n)} \sum_{k=n-i}^{n} \sum_{m=n-j}^{n} \mathbf{c}_{k,m}^{(n)}.$$

Combining this with Proposition 13 gives and explicit formula for $\mathbf{b}_n$.

3.4. **Algebraic properties of combinatorial ideals.**

Let $\mathfrak{j}$ be a combinatorial ideal. Following [Pa], a root $\alpha \in \Delta_+$ is said to be a *generator* of $\mathfrak{j}$ if $\alpha \in \mathrm{supp}(\mathfrak{j})$ and

$$\bigoplus_{\beta \in \mathrm{supp}(\mathfrak{j}) \setminus \{\alpha\}} \hat{\mathfrak{g}}_\beta$$

is a combinatorial ideal. It is easy to see that the number of generators is in fact constant on the equivalence classes of combinatorial ideals. Hence it is enough to consider basic ideals.

Let $\mathfrak{i}$ be a basic ideal for $\hat{\mathfrak{sl}}_n$ and $\Phi(\mathfrak{i}) = (p, q)$. In the classical case (studied in [Pa]) the number of generators of an ad-nilpotent ideal in the Borel subalgebra of $\mathfrak{sl}_n$ equals the number of valleys in the associated Dyck path. In our case we have:

Proposition 17. *Assume that $p \in \mathcal{D}_n(a, b)$ and $q \in \mathcal{D}_n(c, d)$. Then the number of generators of $\mathfrak{i}$ equals 1 if $D \cap \mathrm{supp}(\mathfrak{i}) = \{\delta\}$ and*

$$\mathbf{v}(p) + \mathbf{p}_1(q) - \delta_{d,n-a} - \delta_{c,n-b},$$

where $\delta_{i,j}$ is the Kronecker symbol, otherwise.

Proof. The claim is obvious in the case $D \cap \mathrm{supp}(\mathfrak{i}) = \{\delta\}$. In all other cases the proof is best understood by looking at Figure 7. The generators of $\mathbf{s}_+(\mathfrak{i})$ are boxes whose south-west corner is a valley of p (this is the result from [Pa]). The region between l and the diagonal belongs to $\mathfrak{i}$ automatically and thus contains no generators. The remaining generators of $\mathbf{s}_-(\mathfrak{i})$ are boxes whose south-west corner is a peak of q. Hence we have to count all peaks of q with three exceptions: we should not count the first peak of q if it coincides with the first peak of l and similarly for the last peak; we should not count peaks of height one as the corresponding boxes are on the diagonal and hence do not correspond to any roots (or rather correspond to the imaginary root which is not a generator by the proof of Theorem 4). The claim follows. □

Set

$$\mathfrak{a} = \bigoplus_{\alpha \in D'} \hat{\mathfrak{g}}_\alpha.$$

A basic ideal $\mathfrak{i}$ will be called *quasi-abelian* provided that the image of $\mathfrak{i}$ in $\hat{\mathfrak{n}}_+/\mathfrak{a}$ is an abelian ideal. In the finite dimensional case it is known that the number of abelian ad-nilpotent ideals in $\mathfrak{b}$ equals $2^{\dim \mathfrak{h}}$, see [CP1].

For $p \in \mathcal{D}_n(i, j)$ denote by $\overline{p}$ the unique minimal (with respect to $\leqslant$) path in $\mathcal{D}_n(n-j, n-i)$ (here we set $\mathcal{D}_n(0, 0) := \mathcal{D}_n(1, 1)$). Note that Lemma 14 can then be reformulated as follows: $(p, q) \in \mathfrak{P}_n(k, m)$ if and only if $\overline{p} \leqslant q$.

Proposition 18. *The map Φ induces a bijection between the set of quasi-abelian basic ideals in $\hat{\mathfrak{b}}$ and the set of all pairs (p, q) of Dyck paths satisfying the condition $\overline{p} \leqslant q \leqslant p$.*

Proof. Let $\mathfrak{i}$ be a basic ideal and $(p, q) = \Phi(i)$. Then $\overline{p} \leqslant q$. The condition $q \leqslant p$ is equivalent to the following condition: for every $\alpha \in \mathbf{s}_+(\mathfrak{i})$ we have $-\alpha \notin \mathbf{s}_-(\mathfrak{i})$, which is obviously necessary for the ideal $\mathfrak{i}$ to be quasi-abelian.

On the other hand, $\overline{p} \leqslant p$ implies $(n - i) + (n - j) \leqslant n$. This yields that there exist a simple root α_m in Δ_+ such that every root in $\mathbf{s}_+(\mathfrak{i})$ has the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for some $i \leqslant m \leqslant j$. Hence $\mathfrak{i} \cap \mathfrak{n}_+$ is an abelian ideal of $\mathfrak{n}_+$. Furthermore, the condition

$$\alpha \in \mathbf{s}_+(\mathfrak{i}) \quad \text{implies} \quad -\alpha \notin \mathbf{s}_-(\mathfrak{i})$$

also means that for any $\alpha \in \mathbf{s}_+(\mathfrak{i})$ and $\beta \in \mathbf{s}_-(\mathfrak{i})$ we have $[\hat{\mathfrak{g}}_\alpha, \hat{\mathfrak{g}}_\beta] \in \mathfrak{a}$, which forces the ideal $\mathfrak{i}$ to be quasi-abelian. The claim follows. □

Problem 19. Find an explicit formula for the number of quasi-abelian ideals in $\hat{\mathfrak{sl}}_n$.

For small values of n the answer to Problem 19 (computed using a dull examination of all possible cases on a computer) is as follows:

$$1,\ 3,\ 11,\ 44,\ 183,\ 774,\ 3294,\ 14034,\ 59711,\ 253430, \ldots$$

This sequence again seems to be new and does not appear in [Sl].

As usual, for an ideal $\mathfrak{i}$ consider the lower central series $\mathfrak{i}^0 := \mathfrak{i}$ and $\mathfrak{i}^{m+1} := [\mathfrak{i}^m, \mathfrak{i}]$, $m \geqslant 0$. The ideal $\mathfrak{i}$ is nilpotent if $\mathfrak{i}^m = 0$ for some m and the minimal such m is called the *nilpotency degree* of $\mathfrak{i}$ and denoted $\mathrm{nd}(\mathfrak{i})$. In the classical case of the Lie algebra $\mathfrak{sl}_n$ there is a beautiful bijection between the ad-nilpotent ideals of $\mathfrak{b}$ and Dyck paths, constructed in [AKOP], which leads to a combinatorial interpretation of the nilpotency degree of an ad-nilpotent ideal in terms of the height of the corresponding Dyck path.

To generalize this, for a basic ideal $\mathfrak{i}$ define the *quasi-nilpotency degree* $\mathrm{qnd}(\mathfrak{i})$ of $\mathfrak{i}$ as the minimal non-negative integer m such that $\mathfrak{i}^m \subset \mathfrak{a}$. In other words, $\mathrm{qnd}(\mathfrak{i})$ is the nilpotency degree of the image of $\mathfrak{i}$ in $\hat{\mathfrak{n}}_+/\mathfrak{a}$. Then $\mathfrak{i}$ is quasi-abelian if and only if $\mathrm{qnd}(\mathfrak{i}) = 1$. This invariant can be interpreted in combinatorial terms as follows:

Proposition 20. *Let $\mathfrak{i}$ be a basic ideal and $m = \mathrm{nd}(\mathfrak{i}_+)$.*

(*a*) *The number $\mathrm{qnd}(\mathfrak{i})$ equals either m or $m + 1$.*
(*b*) *If $m = 0$ then $\mathrm{qnd}(\mathfrak{i}) = 1$.*
(*c*) *If $m > 0$, then $\mathrm{qnd}(\mathfrak{i}) = m$ if and only if there does not exist $\beta \in \mathbf{s}_-(\mathfrak{i})$ such that $-\beta \in \mathrm{supp}(\mathfrak{i}_+^{m-1})$.*

Proof. Claim (b) is clear, so in the rest of the proof we may assume $m > 0$.

Obviously, $\mathrm{qnd}(\mathfrak{i}) \geqslant m$. First we observe that if $\beta \in \Delta_-$ is such that $-\beta \notin \mathbf{s}_+(\mathfrak{i})$, then $[\mathfrak{i}_+, \hat{\mathfrak{g}}_{\beta+\delta}] \subset \mathfrak{a}$. Hence, if $\beta \in \Delta_-$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbf{s}_+(\mathfrak{i})$ are such that

$$[\hat{\mathfrak{g}}_{\alpha_k}, [\hat{\mathfrak{g}}_{\alpha_{k-1}}, \dots [\hat{\mathfrak{g}}_{\alpha_1}, \hat{\mathfrak{g}}_{\beta+\delta}] \dots]] \not\subset \mathfrak{a},$$

then all the following elements:

$$-\beta, -(\beta+\alpha_1), \dots, -(\beta+\alpha_1+\alpha_2+\dots+\alpha_{k-1})$$

belong to $\mathbf{s}_+(\mathfrak{i})$. This means that $\hat{\mathfrak{g}}_\beta \in \mathfrak{i}_+^{k-1}$, implying (a).

Conversely, if $\beta \in \Delta_-$ is such that $-\beta \in \mathrm{supp}(\mathfrak{i}_+^{m-1})$, then there exist $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbf{s}_+(\mathfrak{i})$ with the following property:

$$\hat{\mathfrak{g}}_{-\beta} = [\hat{\mathfrak{g}}_{\alpha_1}, [\hat{\mathfrak{g}}_{\alpha_2}, \dots [\hat{\mathfrak{g}}_{\alpha_{m-1}}, \hat{\mathfrak{g}}_{\alpha_m}] \dots]].$$

This implies

$$[\hat{\mathfrak{g}}_{\alpha_m}, [\hat{\mathfrak{g}}_{\alpha_{m-1}}, \dots [\hat{\mathfrak{g}}_{\alpha_1}, \hat{\mathfrak{g}}_{\beta+\delta}] \dots]] \not\subset \mathfrak{a}$$

and claim (c) follows, completing the proof. □

It would be interesting to have an explicit formula for the number of basic ideal $\mathfrak{i}$ satisfying $\mathrm{qnd}(\mathfrak{i}) = m$, where $m = 1, 2, 3, \dots$. This extends Problem 19 which covers the case $m = 1$.

4. Arbitrary ideals and their supports

4.1. **Supports of arbitrary ideals.** Let again $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra. In this subsection we describe the support of an arbitrary nonzero $\hat{\mathfrak{b}}$-ideal in $\hat{\mathfrak{n}}_+$. We start with adjusting our previous definition of the support to this more general situation. Let $\mathfrak{i}$ be a $\hat{\mathfrak{b}}$-ideal in $\hat{\mathfrak{n}}_+$. The *support* $\mathrm{supp}(\mathfrak{i})$ is the set of all $\alpha \in \hat{\Delta}_+$ such that $\mathfrak{i}_\alpha := \mathfrak{i} \cap \hat{\mathfrak{g}}_\alpha \neq 0$. The *level* of a nonzero ideal $\mathfrak{i}$ is the minimal positive integer l such that $l\delta \in \mathrm{supp}(\mathfrak{i})$. Obviously, the level is well-defined for every nonzero ideal. Our first observation is the following:

Proposition 21. *Let $\mathfrak{i}$ be an ideal of level l. Then we have $\hat{\mathfrak{g}}_{(l+1)\delta} \subset \mathfrak{i}$ and for every $i > l$ and $\alpha \in D_i$ we have $\hat{\mathfrak{g}}_\alpha \subset \mathfrak{i}$.*

Proof. Let $h \otimes t^l$ be a nonzero element in $\mathfrak{i}_{l\delta}$. Then $h \neq 0$ and hence there exists a simple root $\beta \in \Delta_+$ such that $\beta(h) \neq 0$. As $\mathfrak{i}$ is a $\hat{\mathfrak{b}}$-ideal and the root $l\delta+\beta$ is real, this implies $\hat{\mathfrak{g}}_{\beta+l\delta} \subset \mathfrak{i}$. Commuting $\hat{\mathfrak{g}}_{\beta+l\delta}$ with $\hat{\mathfrak{g}}_\gamma$ for suitable $\gamma \in \Delta_+$, we obtain $\hat{\mathfrak{g}}_{\xi+l\delta} \subset \mathfrak{i}$ for any $\xi \in \Delta_+$, $\xi \geqslant \beta$. The complement to the set of all such ξ belongs to the hyperplane spanned by all simple roots of Δ_+ different from β. This implies that the set of all such ξ spans $\mathfrak{h}^*$. It follows that the commutants of the form

$$[\hat{\mathfrak{g}}_{\xi+l\delta}, \hat{\mathfrak{g}}_{-\xi+(l+1)\delta}],$$

where ξ is as above, span $\hat{\mathfrak{g}}_{(l+1)\delta}$. Hence $\hat{\mathfrak{g}}_{(l+1)\delta} \subset \mathfrak{i}$. This implies that $\hat{\mathfrak{g}}_{\beta+(l+1)\delta} \subset \mathfrak{i}$ for every real $\beta \in D$ and the proof is completed by induction. □

As an immediate corollary we obtain:

Corollary 22. *Every nonzero $\hat{\mathfrak{b}}$-ideal in $\hat{\mathfrak{n}}_+$ has finite codimension.*

For a nonzero $\hat{\mathfrak{b}}$-ideal $\mathfrak{i}$ of level l in $\hat{\mathfrak{n}}_+$ set:

$$\begin{array}{rcl} \mathsf{a}_+(\mathfrak{i}) & := & \{\alpha \in \Delta_+ : \alpha + (l-1)\delta \in \mathrm{supp}(\mathfrak{i})\}; \\ \mathsf{a}_-(\mathfrak{i}) & := & \{\alpha \in \Delta_- : \alpha + l\delta \in \mathrm{supp}(\mathfrak{i})\}; \\ \mathsf{a}'_+(\mathfrak{i}) & := & \{\alpha \in \Delta_+ : \alpha + l\delta \in \mathrm{supp}(\mathfrak{i})\}; \\ \mathsf{a}'_-(\mathfrak{i}) & := & \{\alpha \in \Delta_- : \alpha + (l+1)\delta \in \mathrm{supp}(\mathfrak{i})\}. \end{array}$$

These sets determine the support of $\mathfrak{i}$ in the following sense:

Proposition 23. *Let $\mathfrak{i}$ be an ideal of level l. Then we have:*

$$\begin{aligned} \mathrm{supp}(\mathfrak{i}) = \{l\delta, (l+1)\delta\} \cup \{\alpha + (l-1)\delta : \alpha \in \mathsf{a}_+(\mathfrak{i})\} \cup \\ \cup \{\alpha + l\delta : \alpha \in \mathsf{a}_-(\mathfrak{i}) \cup \mathsf{a}'_+(\mathfrak{i})\} \cup \\ \cup \{\alpha + (l+1)\delta : \alpha \in \mathsf{a}'_-(\mathfrak{i})\} \cup \bigcup_{i \geqslant l+1} D_i. \end{aligned}$$

Proof. That $\mathrm{supp}(\mathfrak{i})$ contains the right hand side follows from Proposition 21 and definitions. To prove the reverse inclusion it is enough to show that $\mathrm{supp}(\mathfrak{i})$ does not intersect D_i for $i < l-1$. Assume that $\beta \in \mathrm{supp}(\mathfrak{i}) \cap D_i$ for some $i < l-1$. Then β is a real root as $\mathfrak{i}$ has level l. If $\beta = \gamma + i\delta$ for some $\gamma \in \Delta_+$, then, commuting $\hat{\mathfrak{g}}_\beta$ with $\hat{\mathfrak{g}}_{-\beta+\delta}$ we get $(i+1)\delta \in \mathrm{supp}(\mathfrak{i})$, a contradiction. If $\beta = \gamma + (i+1)\delta$ for some $\gamma \in \Delta_-$, then, commuting $\hat{\mathfrak{g}}_\beta$ with $\hat{\mathfrak{g}}_{-\gamma}$ we again get $(i+1)\delta \in \mathrm{supp}(\mathfrak{i})$, a contradiction. The claim follows. □

For a nonzero $\hat{\mathfrak{b}}$-ideal $\mathfrak{i}$ in $\hat{\mathfrak{n}}_+$ set $\mathfrak{i}_+ := \oplus_{\alpha \in \mathsf{a}_+(\mathfrak{i})} \hat{\mathfrak{g}}_\alpha$ and define $\mathfrak{i}'_+$ similarly (using $\mathsf{a}'_+(\mathfrak{i})$). Then both $\mathfrak{i}_+$ and $\mathfrak{i}'_+$ are $\mathfrak{b}$-ideals of $\mathfrak{n}_+$. Denote by $\mathfrak{i}_-$ the $\mathfrak{b}$-submodule of $\mathfrak{g}/\mathfrak{b}$ which is canonically identified with $\oplus_{\alpha \in \mathsf{a}_-(\mathfrak{i})} \hat{\mathfrak{g}}_\alpha$ and define $\mathfrak{i}'_-$ similarly (using $\mathsf{a}'_-(\mathfrak{i})$). From Proposition 23 it follows that the quadruple

$$\underline{\mathfrak{i}} := (\mathfrak{i}_+, \mathfrak{i}_-, \mathfrak{i}'_+, \mathfrak{i}'_-)$$

determines $\mathrm{supp}(\mathfrak{i})$. Taking the $\mathfrak{h}$-complements to $\mathfrak{i}_-$ and $\mathfrak{i}'_-$ in $\hat{\mathfrak{n}}_-$ and applying the Chevalley involution we obtain that $\mathrm{supp}(\mathfrak{i})$ is uniquely determined by a quadruple of $\mathfrak{b}$-ideals in $\mathfrak{n}$. This implies:

Corollary 24. *The cardinality of the set of all possible supports for nonzero $\hat{\mathfrak{b}}$-ideals of level l in $\hat{\mathfrak{n}}_+$ does not exceed the fourth power of the number of $\mathfrak{b}$-ideals in $\mathfrak{n}_+$.*

As expected, not every quadruple can appear as $\underline{\mathfrak{i}}$ for some nonzero $\hat{\mathfrak{b}}$-ideal of level l in $\hat{\mathfrak{n}}_+$. Here are some natural restrictions:

Proposition 25. *Let $\mathfrak{i}$ be a nonzero $\hat{\mathfrak{b}}$-ideal of level l in $\hat{\mathfrak{n}}_+$.*

(*a*) *If $\mathfrak{i}_+ \neq 0$, then $\mathfrak{i}'_- = \mathfrak{g}/\mathfrak{b}$.*
(*b*) *Both $\mathfrak{i}'_+$ and $\mathfrak{i}'_-$ are nonzero.*
(*c*) *If $\mathsf{a}_-(\mathfrak{i})$ contains $-\alpha$ for all simple $\alpha \in \Delta_+$, then $\mathfrak{i}'_+ = \mathfrak{n}_+$.*

(*d*) *If* $\mathfrak{i}_+ = \mathfrak{n}_+$*, then* $\mathtt{a}_-(\mathfrak{i})$ *equals either* Δ_- *or* $\Delta_- \setminus \{-\alpha_{\max}\}$*. If* $\mathfrak{i}'_+ = \mathfrak{n}_+$*, then* $\mathtt{a}'_-(\mathfrak{i})$ *equals either* Δ_- *or* $\Delta_- \setminus \{-\alpha_{\max}\}$*.*

Proof. If $\mathfrak{i}_+ \neq 0$, then $\alpha_{\max} \in \mathtt{a}_+(\mathfrak{i})$. Applying twice the adjoint action of $\hat{\mathfrak{g}}_{-\alpha_{\max}+\delta}$ to $\hat{\mathfrak{g}}_{\alpha_{\max}+(l-1)\delta}$ we get a nonzero space and thus $-\alpha_{\max} \in \mathtt{a}'_-(\mathfrak{i})$. The latter implies $\mathtt{a}'_-(\mathfrak{i}) = \Delta_-$ and claim (a) follows.

Let $h \otimes t^l$ be a nonzero element in $\mathfrak{i}_{l\delta}$. Then there exists $\alpha \in \Delta_+$ such that $\alpha(h) \neq 0$, which implies $\alpha \in \mathtt{a}'_+(\mathfrak{i})$ and $-\alpha \in \mathtt{a}'_-(\mathfrak{i})$. Claim (b) follows.

If $\mathtt{a}_-(\mathfrak{i})$ contains $-\alpha$ for all simple α, then commutations with the corresponding $\hat{\mathfrak{g}}_\alpha$'s imply $\hat{\mathfrak{g}}_{l\delta} \subset \mathfrak{i}$. Now claim (c) follows using commutation with $\hat{\mathfrak{g}}_\beta$, $\beta \in \Delta_+$.

If $\beta \in \Delta_+$ is not maximal, then there exists a simple root $\alpha \in \Delta_+$ such that $\alpha + \beta \geqslant \beta$. Commuting $\hat{\mathfrak{g}}_{\alpha+(l-1)\delta}$ with $\hat{\mathfrak{g}}_{-\alpha-\beta+\delta}$ gives that $\hat{\mathfrak{g}}_{-\beta+l\delta} \subset \mathfrak{i}$. This proves the first part of claim (d) and the second part is proved similarly. □

A more detailed description of possible supports in type A is given in Subsection 4.3 below.

4.2. **Support equivalent ideals.** Two nonzero $\hat{\mathfrak{b}}$-ideals $\mathfrak{i}$ and $\mathfrak{j}$ in $\hat{\mathfrak{n}}_+$ will be called *support equivalent* provided that $\mathrm{supp}(\mathfrak{i}) = \mathrm{supp}(\mathfrak{j}) + k\delta$ for some $k \in \mathbb{Z}$. This notion is motivated by the following statement:

Proposition 26. *For* $l \in \mathbb{N}$ *let* $\mathcal{S}_l$ *denote the set of all possible supports of* $\hat{\mathfrak{b}}$*-ideals of level* l *in* $\hat{\mathfrak{n}}_+$*. Then the map* $X \mapsto X + \delta$ *is a bijection between* $\mathcal{S}_l$ *and* $\mathcal{S}_{l+1}$*.*

Proof. If $\mathfrak{i}$ is an ideal of level l, define the ideal $\mathfrak{j}$ of level $l+1$ as follows: $\mathfrak{j}$ is generated, as a vector space, by all elements of the form $x \otimes t^{i+1}$ whenever $x \in \mathfrak{g}$ is such that and $x \otimes t^i \in \mathfrak{i}$.

Conversely, if $\mathfrak{j}$ is an ideal of level $l+1$, define the ideal $\mathfrak{i}$ of level l as follows: $\mathfrak{i}$ is generated, as a vector space, by all elements of the form $x \otimes t^i$ whenever $x \in \mathfrak{g}$ is such that and $x \otimes t^{i+1} \in \mathfrak{j}$.

It is easy to check that these maps are mutually inverse bijections between the sets of ideals of level l and level $l+1$. By construction, we also have $\mathrm{supp}(\mathfrak{j}) = \mathrm{supp}(\mathfrak{i}) + \delta$. The claim follows. □

From Corollary 24 it follows that the number of equivalence classes of support equivalent ideals is finite. It would be interesting to have an explicit combinatorial formula for this number for all untwisted affine Lie algebras.

4.3. **Support equivalent ideals in type** A**.** We go back to the special case $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_n$. From Proposition 26 it follows that in order to understand supports of $\hat{\mathfrak{b}}$-ideal in $\hat{\mathfrak{n}}_+$ it is enough to understand elements of $\mathcal{S}_1$. We will need the following definition: for a Dyck path

$p \in \mathcal{D}_n$ denote by $\mathtt{m}(p)$ the set of all points $(2m, 0)$, $m = 1, 2, \dots, n-1$, for which

$$\{(2m-2,0),(2m,0),(2m+2,0)\} \setminus (\mathtt{v}_{(0)}(p) \cup \{(0,0),(2n,0)\}) \neq \varnothing.$$

Recall that $\alpha_1, \dots, \alpha_{n-1}$ are the usual simple roots of $\mathfrak{sl}_n$ (the root α_i corresponds to the matrix unit $e_{i,i+1}$). Let $\{h_1, \dots, h_{n-1}\}$ be the basis of $\mathfrak{h}$, dual to $\{\alpha_1, \dots, \alpha_{n-1}\}$, that is $\alpha_j(h_i) = \delta_{i,j}$. With every ideal $\mathfrak{i}$ of level 1 we associate the quadruple $\Psi(\mathfrak{i}) = (p, q, p', q')$ of Dyck paths as follows: p and p' are associated to $\mathfrak{i}_+$ and $\mathfrak{i}'_+$ (or, rather, $\mathtt{a}_+(\mathfrak{i})$ and $\mathtt{a}'_+(\mathfrak{i})$), respectively, and q and q' are associated to $\mathfrak{i}_-$ and $\mathfrak{i}'_-$ (or, rather, $\mathtt{a}_-(\mathfrak{i})$ and $\mathtt{a}'_-(\mathfrak{i})$), respectively, as described in Subsection 3.3. The quadruple (p, q, p', q') defines the support equivalence class of the ideal $\mathfrak{i}$ uniquely, however, not every quadruple of Dyck paths appears as $\Psi(\mathfrak{i})$ for some $\mathfrak{i}$. For $n = 1$ we have a unique quadruple $(p, q, p', q') = (\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p})$ and the corresponding unique equivalence class of support equivalent ideals. Our main result in this subsection is the following:

Theorem 27. *Assume that $n > 1$. A quadruple (p, q, p', q') of Dyck paths has the form $\Psi(\mathfrak{i})$ for some $\hat{\mathfrak{b}}$-ideal of level 1 in $\hat{\mathfrak{n}}_+$ if and only if one of the following (mutually excluding) conditions is satisfied:*

(*i*) $p = q' = \mathbf{p}$, $q = \mathbf{q}$ *and* $\mathbf{v}_{(0)}(p') = 1$;
(*ii*) $p = \mathbf{p}$, $q = \mathbf{q}$, $\mathbf{v}_{(0)}(p') > 1$ *and* $q' \geqslant \overline{p'}$;
(*iii*) $p = \mathbf{p}$, $q \neq \mathbf{q}$, $\mathtt{m}(q) \subset \mathbf{v}_{(0)}(p')$, $q' \geqslant \overline{p'}$ *and, additionally,* $q' = \mathbf{p}$ *if* $(2, 0) \notin \mathtt{v}_{(0)}(q)$ *or* $(2n-2, 0) \notin \mathtt{v}_{(0)}(q)$;
(*iv*) $p \neq \mathbf{p}$, $q \geqslant \overline{p}$, $\mathtt{m}(q) \cup \{(2,0),(2n-2,0)\} \subset \mathbf{v}_{(0)}(p')$, *and* $q' = \mathbf{p}$.

Proof. If $\Psi(\mathfrak{i}) = (\mathbf{p}, \mathbf{q}, p', q')$, then $h \otimes t \in \mathfrak{i}$ for some nonzero $h \in \mathfrak{h}$ and hence $\alpha_i(h) \neq 0$ for some i, implying $\alpha_i \in \mathtt{a}'_+(\mathfrak{i})$ and so $\mathtt{v}_{(0)}(p') \neq \varnothing$. Assume first that $\Psi(\mathfrak{i}) = (p, q, p', q')$ and that

$$p = \mathbf{p},\ q = \mathbf{q},\ \mathtt{v}_{(0)}(p') = \{(2m, 0)\}, m \in \{1, 2, \dots, n-1\}. \tag{4.1}$$

Then we have $\mathfrak{i}_+ = \mathfrak{i}_- = 0$, moreover, $\mathfrak{i}_\delta$ has to be one-dimensional and generated by $h_m \otimes t$. As $\alpha_{\max}(h_m) = 1 \neq 0$, it follows that $-\alpha_{\max} \in \mathtt{a}'_-(\mathfrak{i})$, which implies $q' = \mathbf{p}$. On the other hand, given (p, q, p', q') satisfying (i), let $\mathfrak{j}$ denote the subspace of $\hat{\mathfrak{n}}_+$, spanned by $h_m \otimes t$ and all $\hat{\mathfrak{g}}_\alpha$ such that $\alpha \in D_i$, $i > 1$, or $\alpha \in D_1$ is such that α does not have the form $\beta + \delta$ for some $\beta \in \Delta_+$ for which the corresponding box is to the north-east of p' in the matrix setup of Subsection 3.3 (see Figure 5). It is easy to see that $\mathfrak{j}$ is an ideal and that $\Psi(\mathfrak{j}) = (p, q, p', q')$, which proves our theorem for quadruples satisfying (4.1). The number of such quadruples equals the number of Dyck paths of semilength n having a unique valley of height 0. It is well-known that this number is C_{n-1}.

Assume now that $\Psi(\mathfrak{i}) = (p, q, p', q')$ and that

$$p = \mathbf{p},\ q = \mathbf{q},\ \mathbf{v}_{(0)}(p') > 1. \tag{4.2}$$

Then from the proof of Lemma 14 we get $q' \geqslant \overline{p'}$. On the other hand, assume that (p, q, p', q') satisfies (ii). Let $(2m, 0)$ and $(2k, 0)$ be two different valleys of p'. Denote by $\mathfrak{j}$ the subspace of $\hat{\mathfrak{n}}_+$, spanned by $(h_m - h_k) \otimes t$ and all $\hat{\mathfrak{g}}_\alpha$ such that $\alpha \in D_i$, $i > 1$, or $\alpha \in D_1$ is such that either $\alpha = \beta + \delta$ for some $\beta \in \Delta_+$ for which the corresponding box is to the north-east of p' in the matrix setup of Subsection 3.3 or $\alpha = \beta + 2\delta$ for some $\beta \in \Delta_-$ for which the corresponding box is to the north-east of q'. Note that if $\beta \in \Delta_-$ corresponds to a box lying to the south-west of $\overline{p}$, then $\beta(h_m - h_k) = 0$. Using this it is easy to check that $\mathfrak{j}$ is an ideal and $\Psi(\mathfrak{j}) = (p, q, p', q')$, which proves our theorem for quadruples satisfying (4.2).

Assume now that $\Psi(\mathfrak{i}) = (p, q, p', q')$ and that

$$p = \mathbf{p},\ q \neq \mathbf{q}. \tag{4.3}$$

From the proof of Lemma 14 we get $\overline{p'} \leqslant q'$. If $(2m, 0) \notin \mathtt{v}_{(0)}(q)$ for some $m \in \{1, \dots, n-1\}$, then $\hat{\mathfrak{g}}_{-\alpha_m+\delta} \subset \mathfrak{i}$. Applying twice the adjoint action of $\hat{\mathfrak{g}}_{\alpha_m}$ we get that $\hat{\mathfrak{g}}_{\alpha_m+\delta} \subset \mathfrak{i}$, which yields $(2m, 0) \in \mathtt{v}_{(0)}(p')$. If $(2m-2, 0) \notin \mathtt{v}_{(0)}(q)$ for some $m \in \{2, \dots, n-1\}$, then $\hat{\mathfrak{g}}_{-\alpha_{m-1}+\delta} \subset \mathfrak{i}$. Commuting this with $\hat{\mathfrak{g}}_{\alpha_{m-1}}$ and then with $\hat{\mathfrak{g}}_{\alpha_m}$ we get that $\hat{\mathfrak{g}}_{\alpha_m+\delta} \subset \mathfrak{i}$, which yields that $(2m, 0) \in \mathtt{v}_{(0)}(p')$. If $(2m+2, 0) \notin \mathtt{v}_{(0)}(q)$ for some $m \in \{1, \dots, n-2\}$, then $\hat{\mathfrak{g}}_{-\alpha_{m+1}+\delta} \subset \mathfrak{i}$. Commuting this with $\hat{\mathfrak{g}}_{\alpha_{m+1}}$ and then with $\hat{\mathfrak{g}}_{\alpha_m}$ we get that $\hat{\mathfrak{g}}_{\alpha_m+\delta} \subset \mathfrak{i}$, which yields that $(2m, 0) \in \mathtt{v}_{(0)}(p')$. If $(2, 0) \notin \mathtt{v}_{(0)}(q)$, then $\mathfrak{i}$ contains $\hat{\mathfrak{g}}_{-\alpha_1+\delta}$ and hence also

$$[\hat{\mathfrak{g}}_{-\alpha_1+\delta}, \hat{\mathfrak{g}}_{-\alpha_{\max}+\alpha_1+\delta}] = \hat{\mathfrak{g}}_{-\alpha_{\max}+2\delta},$$

which implies $q' = \mathbf{p}$. Similarly for $(2n-2, 0) \notin \mathtt{v}_{(0)}(q)$. This establishes necessity of condition (iii).

On the other hand, to prove sufficiency assume that (p, q, p', q') satisfies (iii). Denote by $\mathfrak{j}$ the subspace of $\hat{\mathfrak{n}}_+$, spanned by

- the space V_1 generated by $\hat{\mathfrak{g}}_{\alpha+\delta}$ for every $\alpha \in \Delta_-$ corresponding to a box lying to the north-east of q in the matrix setup of Subsection 3.3;
- the space V_2 generated by $[\hat{\mathfrak{g}}_{\alpha+\delta}, \hat{\mathfrak{g}}_{-\alpha}]$ for every $\alpha \in \Delta_-$ corresponding to a box lying to the north-east of q;
- the space V_3 generated by $\hat{\mathfrak{g}}_{\alpha+\delta}$ for every $\alpha \in \Delta_+$ corresponding to a box lying to the north-east of p';
- the space V_4 generated by $\hat{\mathfrak{g}}_{\alpha+2\delta}$ for every $\alpha \in \Delta_-$ corresponding to a box lying to the north-east of q';
- the space V_5 generated by $\hat{\mathfrak{g}}_{2\delta}$ and all $\hat{\mathfrak{g}}_\beta$, $\beta \in D_i$, $i > 1$.

We claim that $\mathfrak{j}$ is an ideal. The essential combinatorics of the proof is shown in Figure 8. That $V_3 \oplus V_4 \oplus V_5$ is an ideal follows from the proof of Lemma 14. By construction of V_2, the adjoint action of $\mathfrak{n}_+$ applied to V_1 or $V_1 \oplus V_2$ generates the same subspace of $\hat{\mathfrak{g}}$, call it X. Let $\mathtt{a}(X)$ be the set of all $\alpha \in \Delta_+$ such that $\hat{\mathfrak{g}}_{\alpha+\delta} \subset X$. Then $\mathtt{a}(X)$ is the support of an ad-nilpotent ideal in $\mathfrak{b}$. Let w be the corresponding Dyck path. Using a computation similar to the one used in the previous paragraph one checks that w is the maximum (with respect to $\leqslant$) path

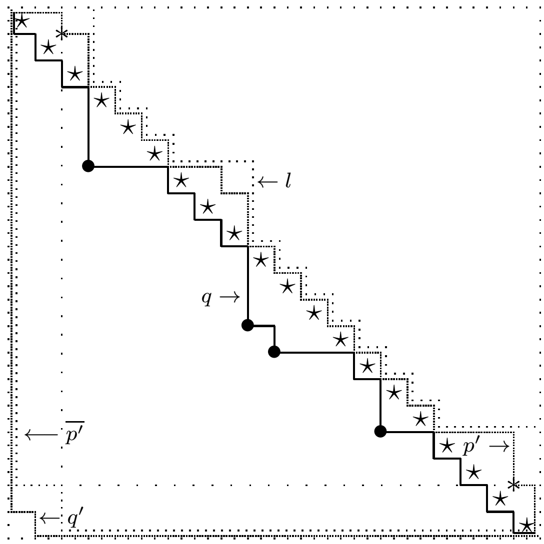


FIGURE 8. Combinatorics of Theorem 27(iii): the path p' must contain the area on the upper right side of l (which is determined by $\bullet$ of q) and the path q' must contain the area on the upper right side of $\overline{p'}$ (which is determined by $*$ of p')

satisfying $\mathtt{m}(q) \subset \mathbf{v}_{(0)}(w)$ (this is shown by the line l in Figure 8). As $\mathtt{m}(q) \subset \mathbf{v}_{(0)}(p')$, we have $p' \leqslant w$ and it follows that $V_1 \oplus V_2 \oplus V_3$ is stable under the adjoint action of $\mathfrak{b}$. It remains to show that for $\alpha \in \Delta_-$ the adjoint action of $\hat{\mathfrak{g}}_{\alpha+\delta}$ maps V_1 to $V_4 \oplus V_5$. This is clear in the cases $(2,0) \in \mathtt{v}_{(0)}(q)$ or $(2n-2,0) \in \mathtt{v}_{(0)}(q)$. In other cases this follows from Lemma 14 using the condition $\overline{p'} \leqslant q'$. By construction, $\Psi(\mathfrak{j}) = (p,q,p',q')$. This proves our theorem for all quadruples satisfying (4.3).

Finally, assume that $\Psi(\mathfrak{i}) = (p,q,p',q')$ and that

$$p \neq \mathbf{p}. \tag{4.4}$$

Then $\overline{p} \leqslant q$ by Lemma 14 and $q' = \mathbf{p}$ by Proposition 25(a). Further, $\hat{\mathfrak{g}}_{\alpha_{\max}} \subset \mathfrak{i}$, which implies that

$$[\hat{\mathfrak{g}}_{\alpha_{\max}}, \hat{\mathfrak{g}}_{-\alpha_{\max}+\alpha_1+\delta}] = \hat{\mathfrak{g}}_{\alpha_1+\delta} \subset \mathfrak{i}$$

and

$$[\hat{\mathfrak{g}}_{\alpha_{\max}}, \hat{\mathfrak{g}}_{-\alpha_{\max}+\alpha_{n-1}+\delta}] = \hat{\mathfrak{g}}_{\alpha_{n-1}+\delta} \subset \mathfrak{i}.$$

Hence $\{(2,0),(2n-2,0)\} \subset \mathtt{v}_{(0)}(p')$. That $\mathtt{m}(q) \subset \mathtt{v}_{(0)}(p')$ is proved similarly to the previous case. This establishes necessity of (iv).

To prove sufficiency assume that (p,q,p',q') satisfies (iv). Denote by $\mathfrak{j}$ the subspace of $\hat{\mathfrak{n}}_+$, spanned by

- the space V_1 generated by $\hat{\mathfrak{g}}_\alpha$ for every $\alpha \in \Delta_+$ corresponding to a box lying to the north-east of p in the matrix setup of Subsection 3.3;

- the space V_2 generated by $\hat{\mathfrak{g}}_{\alpha+\delta}$ for every $\alpha \in \Delta_-$ corresponding to a box lying to the north-east of q;
- the space V_3 generated by $[\hat{\mathfrak{g}}_{\alpha+\delta}, \hat{\mathfrak{g}}_{-\alpha}]$ for every $\alpha \in \Delta_-$ corresponding to a box lying to the north-east of q and by $[\hat{\mathfrak{g}}_{-\alpha+\delta}, \hat{\mathfrak{g}}_{\alpha}]$ for every $\alpha \in \Delta_+$ corresponding to a box lying to the north-east of p;
- the space V_4 generated by $\hat{\mathfrak{g}}_{\alpha+\delta}$ for every $\alpha \in \Delta_+$ corresponding to a box lying to the north-east of p';
- the space V_5 generated by $\hat{\mathfrak{g}}_{\beta}$, $\beta \in D_i$, $i > 1$, or $\beta \in D_1$, $\beta \neq \gamma + \delta$, $\gamma \in \Delta_+$.

Similarly to the previous case one shows that $\mathfrak{j}$ is an ideal. By construction, $\Psi(\mathfrak{j}) = (p, q, p', q')$. This proves our theorem for all quadruples satisfying (4.4) and completes the proof. □

For small values of n the number of equivalence classes of support equivalent ideals for $\hat{\mathfrak{sl}}_n$ (obtained by a direct calculation) is as follows:

$$1,\ 4,\ 21,\ 100,\ 455, \ldots$$

This sequence again seems to be new and does not appear in [Sl].

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