A GENERATING TREE APPROACH TO k-NONNESTING PARTITIONS AND PERMUTATIONS

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Abstract. We describe a generating tree approach to the enumeration and exhaustive generation of k-nonnesting set partitions and permutations. Unlike previous work in the literature using the connections of these objects to Young tableaux and restricted lattice walks, our approach deals directly with partition and permutation diagrams. We provide explicit functional equations for the generating functions, with k as a parameter.

1. Introduction

Arc annotated sequences, also called *arc diagrams*, are structures that have proved particularly useful for encoding a variety of combinatorial classes. An arc annotated sequence is a row of increasingly labelled vertices, from 1 to n, with some collection of arcs between them, restricted according to the object. Many combinatorial objects, including matchings, set partitions, labelled graphs, RNA substructures and permutations, have straightforward descriptions using arc annotated sequences [6, 7, 8, 9, 11]. Two important patterns that occur in arc annotated sequences, and thus can be defined for each of the combinatorial objects that these sequences represent, are nestings and crossings. A set of k arcs forms a *k-nesting* (respectively *k-crossing*) if each of the $\binom{k}{2}$ pairs of arcs nest (resp. cross). Figure 1 illustrates a 3-nesting and a 3-crossing. Precise definitions of k-nestings in set partitions and permutations are given in Sections 2.4 and 5, respectively.

Figure 1. A 3-nesting and a 3-crossing.

A diagram is *k-nonnesting* if it contains no k-nesting, and it is *k-noncrossing* if it contains no k-crossing. Chen, Deng, Du, Stanley and Yan [6] proved that k-nonnesting partitions are equinumerous to k-noncrossing partitions bijectively using Young tableaux as an intermediate object. In fact, they proved stronger distribution results, both for partitions and for matchings. This work was extended by de Mier [10] to show that for embedded labelled graphs with certain restrictions, the number of k-noncrossing ones is the same as the number of k-nonnesting ones. Most recently, Burrill, Mishna and Post [5] proved analogous results for permutations.

For most classes described by an arc annotated sequence, the following general enumerative question is open:

How many elements of class $\mathcal{C}$ of size n do not contain any k-nesting?

For all of the above classes, this question is equivalent to counting the number of k-noncrossing objects. The matching case was resolved by Chen *et al.* [6], giving rise to lovely formulas. The set partition case was addressed by Bousquet-Mélou and Xin [4], who found explicit results in the case of $k = 3$. Their method passes through Young tableaux and restricted lattice paths to deliver functional equations, which are then analyzed using the kernel method. They showed that the generating function of 3-noncrossing set partitions

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is D-finite —i.e., it satisfies a linear differential equation with polynomial coefficients—, and hypothesized that this is not the case for larger k:

Conjecture 1.1 ([4]). *For every $k > 3$, the generating function of k-noncrossing set partitions is not D-finite.*

Mishna and Yen [13] determined functional equations for k-nonnesting set partitions, and described a process for isolating coefficients, giving additional evidence for Conjecture 1.1. To date in the literature, there is only very limited enumerative information on k-nonnesting permutations for $k \geq 3$. The case $k = 2$ is easy, since 2-nonnesting permutations are those that avoid the pattern 321, that is, those without decreasing subsequences of length 3, and thus they are counted by the Catalan numbers. Note that, in general, k-nonnesting permutations are not closed under pattern containment: the permutation 564312 is 3-nonnesting, but it contains the pattern 54312, which is not a 3-nonnesting permutation.

1.1. **Main results.** Our main contribution is a generating tree approach which passes through neither Young tableaux nor lattice paths, and can be used to describe both k-nonnesting set partitions and k-nonnesting permutations, for arbitrary k. The generating tree construction leads to a functional equation for the generating function which provides us with information about the corresponding series. We describe generic functional equations in a number of variables which grows with k.

The key innovation in this study is a new class of structures that are essentially arc diagrams "under construction", which we call *open arc diagrams*. These are arc diagrams in which we allow semi-arcs with a single end-point. Usual arc annotated sequences form the subclass of diagrams with no semi-arcs. Aside from nestings, it is relevant to consider *future* nestings in open arc diagrams. We describe how to identify both patterns in a generating tree construction.

We also find an interesting connection between open arc diagrams related to 3-nonnesting set partitions and Baxter permutations.

1.2. **Plan of paper.** We describe open arc diagrams and their generating trees in Section 2. In Section 3 we give the first application of this construction to determine a set of functional equations to enumerate 3-nonnesting set partitions. In Section 4 we give the construction for k-nonnesting partitions for general k, and we also consider partitions that avoid a related pattern called an enhanced k-nesting.

In Section 5, using similar ideas and combining the descriptions for usual and enhanced nestings in set partitions, we construct a generating tree for k-nonnesting permutations and obtain analogous functional equations.

Finally, in Section 6 we discuss future research and possible extensions of our work in connection with the original motivating problem, the study of RNA secondary structures.

2. Open arc diagrams: a generalization of set partitions

2.1. **Arc diagrams.** An arc diagram of size n is an embedded graph with vertices 1 to n drawn in an increasing row. We take the convention of labelling our diagrams from left to right, so we can refer to left and right end-points of an arc. We first focus on the case of arc diagrams representing set partitions, which we call *partition diagrams* for short. In this case, the arcs are always drawn above the vertices, and the partition block $\{a_1, a_2, \ldots, a_j\}$, where $a_1 < a_2 < \cdots < a_j$, is represented by the arcs $(a_1, a_2), (a_2, a_3), \ldots, (a_{j-1}, a_j)$. Figure 2 shows the diagram of a partition of $\{1, \ldots, 9\}$.

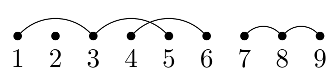

Figure 2. The arc diagram representation of the partition $\{1, 3, 5\}\{2\}\{4, 6\}\{7, 8, 9\}$.

The vertices of a partition diagram can be of four types: *fixed points*, *openers*, *closers* and *transitories*. A fixed point has no incident edges, an opener has degree one and is the left end-point of an arc, a closer has degree one and is the right end-point of an arc, and a transitory vertex has degree two and is the right end-point of one arc and the left end-point of another. In Figure 2, the vertex labelled 2 is a fixed point; the vertices labelled 1, 4, and 7 are openers; vertices 5, 6, and 9 are closers; and 3 and 8 are transitory.

2.2. **Open arc diagrams.** We generalize partition diagrams by allowing two additional vertex types: *semi-openers* and *semi-transitories*. These vertices are incident to *semi-arcs* with no right end-point, which we also call *open arcs*. We call these generalized diagrams *open partition diagrams*. To ensure a standard representation, we shall continue the semi-arcs to a vertical line to the right of vertex n, and retain their order, not allowing the semi-arcs to intersect. We denote the semi-arc with left end-point i by $(i, *)$. Figure 3 displays an example. We view open partition diagrams as ancestors to proper set partitions, in a process which incrementally adds vertices in numerical order, possibly closing semi-arcs and/or opening new ones. The open partition diagram π in Figure 3 is an ancestor to the two set partitions represented in Figure 4, among infinitely many others.

$\pi =$

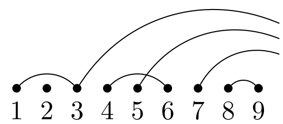

FIGURE 3. An example of an open partition diagram, π.

FIGURE 4. Two set partitions with π as an ancestor.

An open partition diagram could also be viewed as a set partition in which each block is coloured one of two colours: one for proper blocks, and another for blocks ending with a semi-arc. For example, π above represents the bi-coloured partition $\{\mathbf{1}, \mathbf{3}\}, \{\mathbf{5}\}, \{\mathbf{7}\}, \{2\}, \{4, 6\}, \{8, 9\}$. As such, it is easy to determine that the exponential generating function for open partition diagrams is e^{2e^z-2}.

2.3. **Generating tree for open partition diagrams.** To construct a generating tree, we describe how to create open partition diagrams of size n from one of size $n-1$. There are four different possibilities for the type of vertex n when it is added to an open partition diagram of size $n-1$:

(1) fixed point;
(2) semi-opener;
(3) semi-transitory (provided there is an available semi-arc);
(4) closer (provided there is an available semi-arc).

For example, the diagram in Figure 3 generates the 8 diagrams in Figure 5, which we call its children.

The number of children that can be generated by adding a vertex to the diagram depends only upon the number of semi-arcs. Suppose a diagram has m semi-arcs (coming from either semi-opener or semi-transitory vertices). Here are the possible ways to generate its children, depending on the type of the vertex that is added:

(1) fixed point: one child with m semi-arcs;
(2) semi-opener: one child with $m+1$ semi-arcs;
(3) semi-transitory: m children, each with m semi-arcs;
(4) closer: m children, each with $m-1$ semi-arcs.

This sums to a total of $2m+2$ children for any diagram with m semi-arcs. We remark that, in the notation of [1], this generating tree is described by the system

$$[(2) : (2\ell) \to (2\ell)^{\ell}(2\ell+2)(2\ell-2)^{\ell-1}],$$

where the label of a diagram equals twice its number of semi-arcs. From this generating tree, one can recover the exponential generating function e^{2e^z-2} for open partition diagrams.

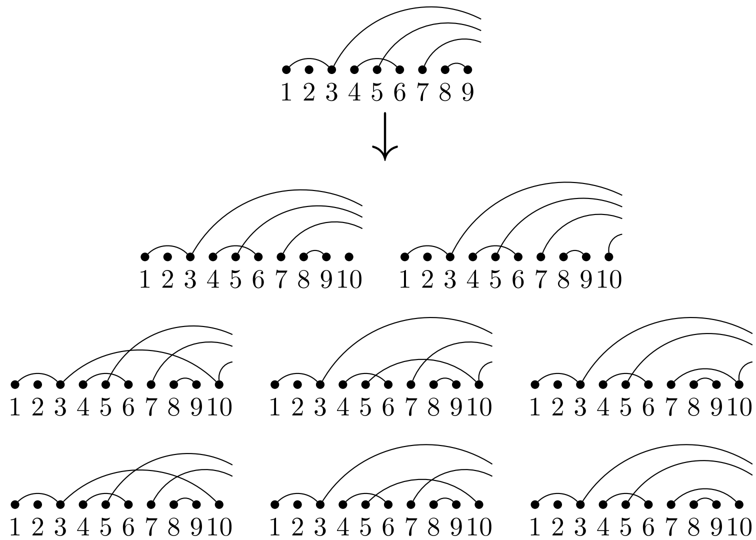

FIGURE 5. The open partition diagram for π and its children.

In Sections 3 and 4 we will modify this generating tree in order to incorporate the k-nonnesting constraint. We will use the notion of future nestings, which we define next.

2.4. **Nestings and future nestings.** Recall that a k-nesting in a partition diagram is a set of k mutually nesting arcs, that is, arcs $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1.$$

To incorporate the nesting constraint to the generating tree, we need to generalize the notion of k-nestings to open partition diagrams. We define a *(regular) k-nesting* in an open partition diagram to be a set of k mutually nesting (closed) arcs. To handle semi-arcs, we introduce the concept of a *future k-nesting*, which is a set of $k-1$ arcs that form a regular $k-1$-nesting together with a single semi-arc whose left end-point is to the left of first end-point of the $k-1$-nesting (see Figure 6). As its name suggests, any proper set partition that is a descendant of a diagram with a future k-nesting necessarily contains a k-nesting, because the semi-arc in the future k-nesting eventually has to close. Conversely, every partition diagram with a k-nesting has an ancestor with a future k-nesting, namely one where all but the outermost arc in the k-nesting have been closed. Note that having multiple semi-arcs above a $k-1$-nesting does not guarantee ℓ-nestings in its descendants for $\ell > k$. For example, the three semi-arcs in Figure 3, together with the arc under all three of them, become a 4-nesting in the partition on left of Figure 4, but they become three separate 2-nestings in partition on the right.

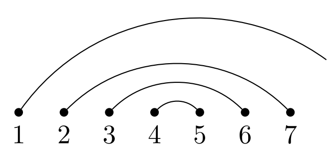

FIGURE 6. An example of a future 4-nesting.

We say that an open partition diagram is *k-nonnesting* if it contains neither regular nor future k-nestings. To enumerate k-nonnesting partitions, our strategy is to generate k-nonnesting open partition diagrams. Of these diagrams, k-nonnesting set partitions correspond to the ones that have no semi-arcs. As such, we are generating more objects than we require, but we can specialize the generating function for the more general class of objects to obtain the generating function for the subset of k-nonnesting partitions. Mishna and Yen [13] gave a construction which generates only this class, but our construction has two important advantages over theirs: it can handle the enhanced case (see Section 4.2), and it can be extended to k-nonnesting permutations (see Section 5).

3. 3-nonnesting set partitions

We warm up with the case of 3-nonnesting set partitions, as it forms the template for the general case for both partitions and permutations. In Section 3.1 we describe the tree in the 3-nonnesting case, and in Section 3.2 we derive a functional equation for the corresponding generating function, which we manipulate in Section 3.3 using the kernel method to obtain a recurrence for its coefficients.

3.1. **Generating tree.** Let us construct a generating tree for 3-nonnesting open partition diagrams as a subtree of the one in Section 2.3. To each 3-nonnesting open partition diagram, we associate a label containing two pieces of information: the number of semi-arcs in the diagram, and the number of semi-arcs which belong to a future 2-nesting. The latter value is the number of semi-arcs above at least one closed arc. For example, in Figure 7, the arrows indicate the semi-arcs which are in future 2-nestings. Since among the four semi-arcs there are two with this property, we associate to this diagram the label $[4, 2]$. Now consider a child of this diagram where the added vertex 12 is a closer or semi-transitory. If the added vertex closed the semi-arc originating from vertex 7, then $(7, 12)$, $(8, 9)$ and $(3, *)$ would form a future 3-nesting. This is precisely what we are trying to avoid. On the other hand, vertex 12 can close any of the other semi-arcs without creating a future 3-nesting. Additionally, no such nesting is created if the added vertex 12 is a fixed point or a semi-opener.

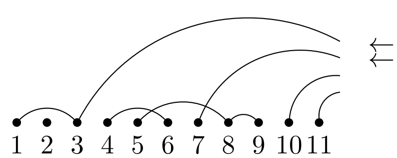

Figure 7. An open partition diagram with label $[4, 2]$.

Suppose now that π is an arbitrary 3-nonnesting open partition diagram. To avoid creating a future 3-nesting when we add a new vertex to π, the only restriction is that if the added vertex is a closer or semi-transitory, it must not close a semi-arc of a future 2-nesting of π, except if it is the very top semi-arc. Note also that since π does not contain future 3-nestings, the addition of a vertex cannot create a regular 3-nesting.

Next we show how to do the bookkeeping on the labels to incorporate this restriction. A fixed point or a semi-opener added to a diagram π has no effect on whether or not a semi-arc is in a future 2-nesting. Suppose now that a closer vertex is added to π. If this vertex closes a semi-arc not belonging to a future 2-nesting of π, all the semi-arcs above it are now in a future 2-nesting, even if they were not in one of π, and the semi-arcs below it are unaffected. If the new vertex closes a semi-arc belonging to a future 2-nesting of π, then it must be the top semi-arc, and in this case, the number of semi-arcs is reduced by one, as is the number of semi-arcs in a 2-nesting. A semi-transitory vertex added to π behaves exactly like a closer, except that a new semi-arc, not belonging to any future 2-nesting, is created.

Applying these considerations to the example from Figure 7, the different types of vertices that can be added and the resulting labels are described below.

(1) Fixed point: one child with label $[4, 2]$.
(2) Semi-opener: one child with label $[5, 2]$, as one semi-arc not contained in future 2-nestings is added.
(3) Semi-transitory: the number of semi-arcs is preserved, and the number of those belonging to future 2-nestings depends on the semi-arc that is closed as follows.
  - (11,*), the three arcs above it become part of future 2-nestings, so the child has label $[4, 3]$;
  - (10,*), the two arcs above it remain part of future 2-nestings, so the child has label $[4, 2]$;
  - (3,*), the number of semi-arcs in future 2-nestings is reduced by one, so the child has label $[4, 1]$.

(4) Closer: the number of semi-arcs is reduced by one, but otherwise it is analogous to the semi-transitory case, hence the three labels are $[3,3], [3,2], [3,1]$.

Remark that the number of children and their labels depend only upon the label of the parent. Indeed, suppose the parent is a diagram with label $[i,j]$. Adding a fixed point produces a child with label $[i,j]$, while adding a semi-opener we get a child labelled $[i+1,j]$. Assuming that $i \geq 1$, adding a closer yields $i-j$ children with labels ranging from $[i-1,j]$ to $[i-1,i-1]$, corresponding to closing the semi-arcs not belonging to future 2-nestings, and, if $j > 0$, an additional child with label $[i-1,j-1]$, corresponding to closing the top semi-arc in a future 2-nesting. Adding a semi-transitory vertex has a similar effect to adding a closer, but the first part of the label in the children is i instead of $i-1$. This description leads to the following theorem about the generating tree scheme, whose first few levels are shown in Figure 8.

Theorem 3.1. *Let $\Pi^{(2)}$ be the set of 3-nonnesting open partition diagrams. To each diagram, associate the label $\ell(\pi) = [i,j]$ if π has i semi-arcs, j of which belong to some future 2-nesting. Then the number of diagrams in $\Pi^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label $[0,0]$, and succession rule given by*

$$(1) \qquad [i,j] \to \begin{cases} [i,j], & & \textit{(fixed point)} \\ [i+1,j], & & \textit{(semi-opener)} \\ [i,j],[i,j+1],\ldots,[i,i-1], & \textit{if } i \geq 1 & \textit{(semi-transitory)} \\ [i-1,j],[i-1,j+1],\ldots,[i-1,i-1], & \textit{if } i \geq 1 & \textit{(closer)} \\ [i,j-1],[i-1,j-1]. & \textit{if } i \geq 1 \textit{ and } j > 0 & \textit{(semi-transitory and closer when } j>0\textit{)} \end{cases}$$

The number of 3-nonnesting set partitions of $\{1,\ldots,n\}$ is equal to the number of nodes with label $[0,0]$ at level n.

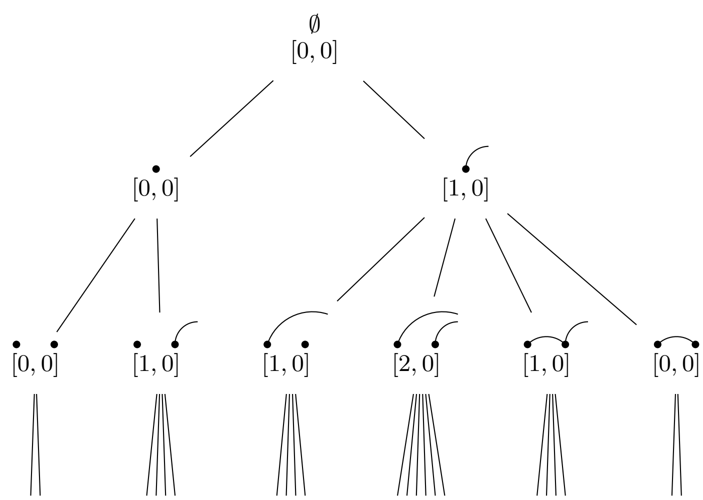

FIGURE 8. Generating tree for 3-nonnesting open partition diagrams. The first few levels agree with the generating tree for all open partition diagrams, since the 3-nonnesting restriction only influences levels further down the tree.

3.2. **Functional equation.** Here we translate the generating tree from Theorem 3.1 into a functional equation for the ordinary generating function enumerating 3-nonnesting open partition diagrams by size and label. The generating function for 3-nonnesting set partitions will then be obtained by a simple evaluation.

Consider the generating function $A(u,v;z) = \sum A_{i,j}(n) u^i v^j z^n$, where $A_{i,j}(n)$ is the number of 3-nonnesting open partition diagrams at level n of the generating tree with label $[i,j]$. For the sake of simplicity, we henceforth use $A(u,v)$ to denote $A(u,v;z)$. The root of the tree, the empty set partition, has label $[0,0]$ and translates to the term $u^0v^0z^0=1$. Next we consider the recursive step that generates each level from the previous one by adding vertices of one of four different types.

From the addition of a fixed point or a semi-opener, that is, the first two lines of Equation (1), we get

$$z(A(u,v)+uA(u,v)).$$

Let us now consider the addition of a semi-transitory vertex. The addition of a closer is analogous substituting the term u^{i-1} for u^i. If $j>i$, we have that $A_{i,j}(n)=0$. The rules for adding a semi-transitory in Equation (1) translate to

$$\begin{aligned} &z\sum_{i,n} A_{i,0}(n)u^i(1+v+v^2+\cdots+v^{i-1})z^n && \text{if } j=0,\\ &z\sum_{i,n} A_{i,j}(n)u^i(v^{j-1}+v^j+\cdots+v^{i-1})z^n && \text{if } 0<j\le i. \end{aligned}$$

Adding these, multiplying by z^n, and using the formula for finite a geometric sum, we get

$$z\Big(\sum_{i,j,n} A_{i,j}(n)u^i\frac{v^j-v^i}{1-v}z^n \;+\; \sum_{i,j\ge 1,n} A_{i,j}(n)u^iv^{j-1}z^n\Big) \;=\; z\left(\frac{A(u,v)-A(uv,1)}{1-v}+\frac{A(u,v)-A(u,0)}{v}\right).$$

Combining the expressions for all four types of vertices, we obtain the following functional equation.

Corollary 3.2. *The generating function $A(u,v)$ for 3-nonnesting open partition diagrams, with variables u and v marking values i and j in the label, respectively, and z marking number of vertices, satisfies the functional equation*

$$A(u,v)=1+z\left((1+u)A(u,v)+\left(1+\frac{1}{u}\right)\left(\frac{A(u,v)-A(uv,1)}{1-v}+\frac{A(u,v)-A(u,0)}{v}\right)\right). \tag{2}$$

3.3. **Manipulating the equation.** We use two variable transformations to simplify the functional equation (2), in preparation for the application of the algebraic kernel method in order to extract coefficients $A_{0,0}(n)$, the number of set partitions without a 3-nesting. The method we follow for the analysis is parallel to the methods in [3, 13].

To get an expression for $A_{0,0}(n)$ in terms of the coefficients $A_{i,j}(k)$, for $k\le n-1$, we generate functional equations equivalent to Equation (2), which are then taken in an alternating sum. The resulting cancellations yield a new functional equation from where we extract the sub-series $\sum_n A_{0,0}(n)z^n$, which is the generating function for 3-nonnesting set partitions.

3.3.1. *A simplifying change of variables.* First, to remove the restriction that the exponent of u is always greater than or equal to the exponent of v, we replace v by $t=v/u$ to get

$$A(u,t)=\sum_{i,j,n} A_{i,j}(n)u^it^jz^n=\sum_{i,j,n} A_{i,j}(n)u^{i-j}v^jz^n,$$

renamed as $A^*(u,v):=\sum_{i,j,n} A_{i,j}(n)u^{i-j}v^jz^n$. Then the various forms of A are transformed as follows:

$$A(u,v)=A^*(u,uv),\quad A(uv,1)=A^*(uv,uv),\quad A(u,0)=A^*(u,0).$$

Equation (2) in terms of A^* becomes

$$A^*(u,uv) \;=\; 1+z\Big((1+u)A^*(u,uv)+\left(1+\frac{1}{u}\right)\left(\frac{A^*(u,uv)-A^*(uv,uv)}{1-v}+\frac{A^*(u,uv)-A^*(u,0)}{v}\right)\Big),$$

which, letting $w=uv$, simplifies to

$$\begin{aligned} A^*(u,w)&=1+z\Big((1+u)A^*(u,w)\quad+\left(1+\frac{1}{u}\right)\Big(\frac{A^*(u,w)-A^*(w,w)}{1-\frac{w}{u}}+\frac{A^*(u,w)-A^*(u,0)}{\frac{w}{u}}\Big)\Big)\\ &=1+z(1+u)\Big(A^*(u,w)+\frac{A^*(u,w)-A^*(w,w)}{u-w}+\frac{A^*(u,w)-A^*(u,0)}{w}\Big). \end{aligned}$$

By collecting all the terms with $A^*(u,w)$, we get the functional equation in its kernel form

$$\tilde{K}(u,w)A^*(u,w) = 1 + z(1+u)\left(\frac{A^*(w,w)}{w-u} - \frac{A^*(u,0)}{w}\right),$$

where

$$\tilde{K}(u,w) = 1 - z(1+u)\left(1 + \frac{1}{u-w} + \frac{1}{w}\right).$$

3.3.2. *A symmetrizing substitution and an alternating sum.* To achieve symmetry in the kernel, we let $u = l + w$ to get

$$K(l,w) = \tilde{K}(l+w,w) = 1 - z(1+w+l)\left(1 + \frac{1}{l} + \frac{1}{w}\right)$$

which is symmetric in l and w. Now, we introduce a multiplicative factor $M(l,w) = lw^2$ on both sides of

$$K(l,w)A^*(l+w,w) = 1 + z(1+l+w)\left(-\frac{A^*(w,w)}{l} - \frac{A^*(l+w,0)}{w}\right)$$

to get

$$K(l,w)lw^2A^*(l+w,w) = lw^2 + z(1+l+w)\left(-w^2A^*(w,w) - lwA^*(l+w,0)\right). \tag{3}$$

Note that the last term

$$-z(1+l+w)lwA^*(l+w,0)$$

is invariant when l and w are interchanged. Now we apply the algebraic kernel method where we take the orbit sum of Equation (3) over the symmetric group of order 2, $\mathfrak{S}_2$.

The signed orbit sum of Equation (3) divided by $K(l,w)M(l,w)$ is

$$\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\frac{\sigma(lw^2A^*(l+w,w))}{lw^2} = \frac{1}{K(l,w)}\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\frac{\sigma(lw^2)}{lw^2} + \frac{z(1+l+w)}{K(l,w)}\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\frac{\sigma(-w^2A^*(w,w))}{lw^2}.$$

Next, we expand the orbit sum to get

$$A^*(l+w,w) - \frac{l}{w}A^*(l+w,l) = \frac{1}{K(l,w)}\left(1 - \frac{l}{w}\right) + \frac{z(1+l+w)}{K(l,w)}\left(\frac{l}{w^2}A^*(l,l) - \frac{1}{l}A^*(w,w)\right). \tag{4}$$

3.3.3. *Series extraction.* Our goal is to extract $A_{0,0}(n)$, the coefficient of $u^0v^0z^n$. This is equivalent to obtaining the series $A(0,0) = A^*(0,0)$. Let us define a linear operator $\mathcal{L}$ for this purpose:

$$\mathcal{L}(l^iw^j) = \begin{cases} 1 & \text{if } i=j=0;\\ 0 & \text{otherwise.}\end{cases}$$

When $\mathcal{L}$ is applied to Equation (4), the left hand side gives

$$\begin{aligned}
&\mathcal{L}\left(A^*(l+w,w) - \frac{l}{w}A^*(l+w,l)\right)\\
&= \mathcal{L}\Bigg(\sum_{i,j,n}A_{i,j}(n)(l+w)^{i-j}w^jz^n - \frac{l}{w}\sum_{i,j,n}A_{i,j}(n)(l+w)^{i-j}l^jz^n\Bigg)\\
&= \sum_n A_{0,0}(n)z^n,
\end{aligned}$$

because the second expression with l/w always contains terms with at least one l, thus $\mathcal{L}$ maps it to 0. Applying $\mathcal{L}$ to the right hand side of Equation (4), we obtain

$$\mathcal{L}\left(\frac{1}{K(l,w)}\Big(1-\frac{l}{w}\Big)+\frac{z(1+l+w)}{K(l,w)}\Big(\frac{l}{w^2}A^*(l,l)-\frac{1}{l}A^*(w,w)\Big)\right)$$
$$=\sum_{n\geq 0}\left(\sum_{\substack{i,j,k\\ i+j+k=n}}\binom{n}{i,j,k}^2\Big(1-\frac{k}{j+1}\Big)\right.$$
$$+\sum_{0\leq i,j}\sum_{k=0}^{n-1}A_{i,j}(k)\Big(\sum_{q,r}\binom{n-k}{p,q,r}\binom{n-k-1}{p-i,i+1-q,r-2}\Big)$$
$$\left.-\sum_{0\leq i,j}\sum_{k=0}^{n-1}A_{i,j}(k)\Big(\sum_{q,r}\binom{n-k}{p,q,r}\binom{n-k-1}{p-i,q-1,r+i}\Big)\right)z^n.$$

Theorem 3.3. *The number of* 3*-nonnesting set partitions of* $\{1,\dots,n\}$ *is*

$$A_{0,0}(n)=\sum_{\substack{i,j,k\\ i+j+k=n}}\binom{n}{i,j,k}^2\left(1-\frac{k}{j+1}\right)$$
$$+\sum_{k=0}^{n-1}\sum_{0\leq j\leq i\leq k}A_{i,j}(k)\left(\sum_{p,q,r}\binom{n-k}{p,q,r}\binom{n-k-1}{p-i,q+i,r-1}\left(\frac{r-1}{q+i+1}-1\right)\right).$$

The sequence defined by the first sum has exponential generating function $e^{2z}(1-z)^{-1}$. Ideally, we would like to turn the latter two sums into a tighter recurrence solely in terms of $A_{0,0}(k)$, $k<n$.

4. k-nonnesting set partitions

In Section 4.1, we generalize the construction of the generating tree and the functional equation from Section 3 to arbitrary k. In Section 4.2, we consider a variation of the notion of nestings, and derive analogous results for this case. In Section 4.3, we present enumerative data that can be obtained with our equations, and a surprising connection to Baxter permutations.

4.1. **Generating tree and functional equation.** It will be convenient to shift the index and consider $k+1$-nonnesting partitions. Suppose that π is an $k+1$-nonnesting open partition diagram. To avoid creating a future $k+1$-nesting when adding a new vertex to π, the only restriction is that, if the added vertex is a closer or semi-transitory, it is not allowed to close a semi-arc belonging a future k-nesting of π, unless it is the very top semi-arc. Note also that since π does not contain future $k+1$-nestings, the addition of a vertex cannot create a regular $k+1$-nesting.

In order to keep track of future k-nestings, we also need to keep track of future j-nestings for $j<k$. We define the *nesting index* of a semi-arc to be the maximum j such that there is a j-nesting beneath it. Equivalently, the nesting index of a semi-arc is the largest j such that the semi-arc is in a future $j+1$-nesting. We will keep track of the distribution of nesting indices on the semi-arcs, updating this distribution every time we add a vertex, and avoiding the appearance of future $k+1$-nestings.

To each $k+1$-nonnesting open partition diagram π we associate a label with k components $\ell(\pi)=[s_0,\dots,s_{k-1}]$, where s_i is defined to be the number of semi-arcs of nesting index greater than or equal to i. Figure 9 contains an example. Remark that s_0 is the total number of semi-arcs, and s_{k-1} is the number of semi-arcs in future k-nestings. For $k=2$, this labelling is consistent with Section 3.1. Furthermore, note that, by definition, $s_0\geq s_1\geq\cdots\geq s_{k-1}\geq 0$. The label of the empty partition is $[0,0,\dots,0]$, since it contains no semi-arcs.

As before, we can predict the labels of the children of a given node from its label alone, based on an analysis of the four types of nodes that we can add. When a semi-arc is closed, the effect on the labels is determined by the fact that the nesting index of those semi-arcs above it which had the same nesting index

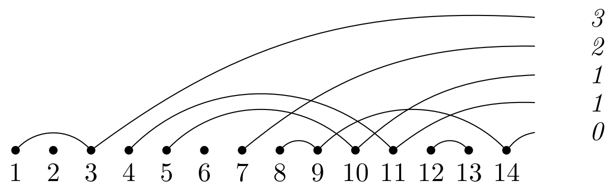

FIGURE 9. An open partition diagram with label $[5,4,2,1]$. The nesting index of each semi-arc is labelled in italics.

increases by one. This allows us to describe a succession rule for the generating tree, summarized in the following theorem, which generalizes Theorem 3.1.

Theorem 4.1. *Let $\Pi^{(k)}$ be the set of $k+1$-nonnesting open partition diagrams. To each diagram, associate the label $\ell(\pi) = [s_0, \dots, s_{k-1}]$, where s_i is the number of semi-arcs with nesting index $\geq i$. Then, the number of diagrams in $\Pi^{(k)}$ of size n is the number of nodes at level n in the generating tree with root label $[0, 0, \dots, 0]$, and succession rule given by*

$$
\begin{aligned}
&[s_0, s_1, \dots, s_{k-1}] \rightarrow \\
&\qquad [s_0, s_1, \dots, s_{k-1}], && (1)\\
&\qquad [s_0+1, s_1, \dots, s_{k-1}], && (2)\\
&\qquad [s_0, s_1-1, \dots, s_{j-1}-1, i, s_{j+1}, \dots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1, \text{ and } s_j \leq i \leq s_{j-1}-1 && (3)\\
&\qquad [s_0-1, s_1-1, \dots, s_{j-1}-1, i, s_{j+1}, \dots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1, \text{ and } s_j \leq i \leq s_{j-1}-1 && (4)\\
&\qquad [s_0, s_1-1, \dots, s_{k-1}-1], [s_0-1, s_1-1, \dots, s_{k-1}-1], \quad \text{if } s_{k-1} > 0. && (5)
\end{aligned}
$$

Proof. The labels arise from adding the following kinds of vertices:

(1) a fixed point;
(2) a semi-opener;
(3) a semi-transitory;
(4) a closer;
(5) a semi-transitory or a closer that closes the top semi-arc, if the diagram had a future k-nesting.

□

Corollary 4.2. *The generating function for $k+1$-nonnesting open partition diagrams, with variable v_i marking value s_i in the label, and z marking number of vertices, denoted $Q = Q(v_0, v_1, \dots, v_{k-1}) = Q(v_0, v_1, \dots, v_{k-1}; z)$, satisfies the functional equation*

$$
\begin{aligned}
Q = 1 + z(1+v_0)\Bigg(Q &+ \frac{1}{v_0 v_1 \dots v_{k-1}}(Q - Q(v_0, v_1, \dots, v_{k-2}, 0)) \\
&+ \sum_{j=1}^{k-1} \frac{1}{v_0 v_1 \dots v_{j-1}(1-v_j)}(Q - Q(v_0, \dots, v_{j-2}, v_{j-1}v_j, 1, v_{j+1}, \dots, v_{k-2}, v_{k-1}))\Bigg).
\end{aligned}
$$

Setting $v_0 = 0$, we obtain the generating function for $k+1$-nonnesting partitions, $Q(0, v_1, \dots, v_{k-1}; z)$. Note that this is a function of z only, because of the restriction $s_0 \geq s_1 \geq \dots \geq s_{k-1} \geq 0$. The functional equation in Corollary 4.2, which specializes to Corollary 3.2 for $k = 2$, is amenable to series generation, and it remains to be seen what can be extracted using a kernel method analysis.

4.2. **Enhanced nestings.** Another relevant pattern in partition diagrams is the *enhanced k-nesting.* An enhanced k-nesting is either a k-nesting, or a set of $k-1$ arcs $(i_1, j_1), \dots, (i_{k-1}, j_{k-1})$ and a fixed point vertex i_k (a singleton block in the corresponding partition) such that

$$i_1 < i_2 < \dots < i_{k-1} < i_k < j_{k-1} < \dots < j_1,$$

that is, a $k-1$-nesting with a fixed point inside the innermost arc. With a comparable definition for enhanced k-crossings, the symmetry between nesting and crossing patterns for different structures has been shown to hold in the enhanced version of the patterns as well. Set partitions are one such structure. Bousquet-Mélou and Xin [4] considered both enhanced and usual crossings in set partitions. Our construction that generates k-nonnesting partitions can be easily modified to generate partitions with no enhanced k-nesting. For this purpose, we define a *future enhanced k-nesting* as an enhanced $k-1$-nesting together with a semi-arc beginning to its left, and we let the *enhanced nesting index* of a semi-arc be the largest j such that it is in a future enhanced $j+1$-nesting. The labels of the nodes in the generating tree now keep track of semi-arcs according to their enhanced nesting index. The only difference in the construction of the tree is that the addition of a fixed point to an open partition diagram π can create future enhanced 2-nestings. Indeed, the semi-arcs that had enhanced nesting index 0 in π have enhanced nesting index 1 after the fixed point is added. This results in the following variation of Theorem 4.1.

Theorem 4.3. *Let $\widetilde{\Pi}^{(k)}$ be the set of open partition diagrams with neither enhanced $k+1$-nestings nor future enhanced $k+1$-nestings. To each diagram, associate the label $\ell(\pi) = [s_0, \ldots, s_{k-1}]$, where s_i is the number of semi-arcs with enhanced nesting index $\geq i$. Then the number of diagrams in $\widetilde{\Pi}^{(k)}$ of size n is the number of nodes at level n in the generating tree with root label $[0, 0, \ldots, 0]$, and succession rule given by*

$$[s_0, s_1, \ldots, s_{k-1}] \rightarrow$$

$$[s_0, s_0, s_2, \ldots, s_{k-1}], \tag{1}$$

$$[s_0+1, s_1, \ldots, s_{k-1}], \tag{2}$$

$$[s_0, s_1-1, \ldots, s_{j-1}-1, i, s_{j+1}, \ldots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1, \text{ and } s_j \leq i \leq s_{j-1}-1 \tag{3}$$

$$[s_0-1, s_1-1, \ldots, s_{j-1}-1, i, s_{j+1}, \ldots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1, \text{ and } s_j \leq i \leq s_{j-1}-1 \tag{4}$$

$$[s_0, s_1-1, \ldots, s_{k-1}-1], [s_0-1, s_1-1, \ldots, s_{k-1}-1], \quad \text{if } s_{k-1} > 0. \tag{5}$$

Proof. The labels arise from adding the following kind of vertices:

(1) a fixed point;
(2) a semi-opener;
(3) a semi-transitory;
(4) a closer;
(5) a semi-transitory or a closer that closes the top semi-arc, if the diagram had a future enhanced k-nesting.

□

This succession rule can again be translated into a functional equation for the corresponding generating function.

Corollary 4.4. *The generating function for open partition diagrams with neither regular nor future enhanced $k+1$-nestings, with variable v_i marking value s_i in the label, and z marking number of vertices, denoted $P = P(v_0, v_1, \ldots, v_{k-1}) = P(v_0, v_1, \ldots, v_{k-1}; z)$, satisfies the functional equation*

$$\begin{aligned} P = 1 + z \Bigg(& v_0 P + \frac{1+v_0}{v_0 v_1 \ldots v_{k-1}} \left(P - P(v_0, v_1, \ldots, v_{k-2}, 0)\right) \\ & + \sum_{j=2}^{k-1} \frac{1+v_0}{v_0 v_1 \ldots v_{j-1}(1-v_j)} \left(P - P(v_0, \ldots, v_{j-2}, v_{j-1}v_j, 1, v_{j+1}, \ldots, v_{k-1})\right) \\ & + \frac{(1+v_0)P - (1+v_0 v_1) P(v_0 v_1, 1, v_2, \ldots, v_{k-1})}{v_0(1-v_1)} \Bigg). \end{aligned}$$

Again, $P(0, v_1, \ldots, v_{k-1}; z)$, which is a function of z only, is the generating function for partitions avoiding enhanced $k+1$-nestings.

4.3. **Enumerative data, and a connection with Baxter permutations.** Corollaries 4.2 and 4.4 allow us to generate data for the number of set partitions of size n avoiding $k+1$-nestings and avoiding enhanced $k+1$-nestings for small k and n. Tables 1 and 2 present the initial counting sequences, and relevant references to the On-line Encyclopedia of Integer Sequences [12]. We are able to generate more many terms than are listed.

$k+1$	OEIS	Initial terms
3	A108304	1, 2, 5, 15, 52, 202, 859, 3930, 19095, 97566, 520257, 2877834, 16434105, 96505490, 580864901, 3573876308, 22426075431, 143242527870, 929759705415, 6123822269373, 40877248201308
4	A108305	1, 2, 5, 15, 52, 203, 877, 4139, 21119, 115495, 671969, 4132936, 26723063, 180775027, 1274056792, 9320514343, 70548979894, 550945607475, 4427978077331, 36544023687590, 309088822019071
5	A192126	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115974, 678530, 4212654, 27627153, 190624976, 1378972826, 10425400681, 82139435907, 672674215928, 5712423473216, 50193986895328, 455436027242590
6	A192127	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213596, 27644383, 190897649, 1382919174, 10479355676, 82850735298, 681840170501, 5828967784989, 51665915664913, 473990899143781
7	A192128	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899321, 1382958475, 10480139391, 82864788832, 682074818390, 5832698911490, 51723290618772, 474853429890994

Table 1. Counting sequences for $k+1$-nonnesting set partitions.

$k+1$	OEIS	Initial terms
3	A108307	1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, 348889, 1768483, 9220655, 49286863, 269346822, 1501400222, 8519796094, 49133373040, 287544553912, 1705548000296, 10241669069576
4	A192855	1, 2, 5, 15, 52, 203, 876, 4120, 20883, 113034, 648410, 3917021, 24785452, 163525976, 1120523114, 7947399981, 58172358642, 438300848329, 3391585460591, 26898763482122, 218263920521938
5	A192865	1, 2, 5, 15, 52, 203, 877, 4140, 21146, 115945, 678012, 4205209, 27531954, 189486817, 1365888674, 10278272450, 80503198320, 654544093035, 5511256984436, 47950929125540, 430240226306346
6	A192866	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678569, 4213555, 27643388, 190878823, 1382610179, 10474709625, 82784673008, 680933897225, 5816811952612, 51505026270176, 471875801114626
7	A192867	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644436, 190899266, 1382956734, 10480097431, 82863928963, 682058946982, 5832425824171, 51718812364549, 474782378367618

Table 2. Counting sequences for set partitions avoiding enhanced $k+1$-nestings.

In our computations, we remarked that the number of open partition diagrams on n vertices with neither regular nor future enhanced 3-nestings coincides with the number of Baxter permutations of size n for up to at least 200 terms. Specializing the functional equation from Corollary 4.4 gives

$$\begin{aligned} B(u,v) &= 1 \\ &+ z\left(uB(u,v) + \frac{B(u,v)-vB(uv,1)}{1-v} + \frac{B(u,v)-B(u,0)}{v} + \frac{B(u,v)-B(uv,1)}{u(1-v)} + \frac{B(u,v)-B(u,0)}{uv}\right). \end{aligned}$$

The initial series development is

$$\begin{aligned} B(u,v) = 1 &+ (u+1)z + (uv+u^2+2u+2)z^2 + (v^2u^2+2vu^2+4uv+6u+3u^2+u^3+5)z^3 \\ &+ (u^3v^3+7v^2u^2+2v^2u^3+16uv+11vu^2+3vu^3+20u+12u^2+4u^3+u^4+15)z^4 + \dots. \end{aligned}$$

Evaluated at $u=v=1$, it seems to coincide with the generating function for Baxter permutations:

$$B(1,1) = 1 + 2z + 6z^2 + 22z^3 + 92z^4 + 422z^5 + 2074z^6 + 10754z^7 + 58202z^8 + 326240z^9 + 1882960z^{10} + \dots.$$

This leads us to the following conjecture.

Conjecture 4.5. *Let $\mathcal{B}_n$ be the set of open partition diagrams on n vertices with neither regular nor future enhanced 3-nestings. There exists a bijection between $\mathcal{B}_n$ and the set of Baxter permutations of length $n+1$.*

The above sequence is also known to count plane bipolar orientations [2]. This raises the question of whether open partition diagrams avoiding regular and future enhanced k-nestings are in bijection with some family of maps. The generating trees known for Baxter permutations and plane bipolar orientations are very different from the trees that we get for open partition diagrams.

5. k-NONNESTING PERMUTATIONS

There are some similarities between k-nonnesting permutations and k-nonnesting set partitions, yet none of the techniques that have been applied in the literature to the enumeration of k-nonnesting set partitions [4, 13] works for permutations. On the other hand, our method can be modified to take advantage of these similarities and deal with k-nonnesting permutations as well. This gives rise to the first substantial set of enumerative data on k-nonnesting permutations. Previously, only data up to $n = 12$ was known, and its obtaining required extensive computation.

First, we recall how to represent a permutation as an arc diagram. This representation was first used by Corteel [8], and in a modified form by Elizalde [11]. It is essentially a drawing of the cycle structure of the permutation. Given $\sigma \in \mathfrak{S}_n$, the diagram of σ has an arc between i and $\sigma(i)$ for each i from 1 to n, and the arc is drawn above the vertices (and is called an *upper arc*) if $i \le \sigma(i)$, and below the vertices (and is called a *lower arc*) if $i > \sigma(i)$. We call such a representation a *permutation diagram* of size n. In this notation, a subset of arcs is a *k-nesting* if either

(1) all k arcs are upper arcs and form an enhanced k-nesting with the definition from Section 4.2 (considering fixed points (i, i));
(2) all k arcs are lower arcs and form a k-nesting with the definition from Section 1.

We call the two above possibilities *upper enhanced k-nestings* and *lower k-nestings*, respectively. The reason behind the slight dissymmetry in the definition comes from the original paper of Corteel [8], where this was necessary for bijections between certain classes of permutations. Burrill *et al.* [5] maintained this dissymmetry. In any case, the construction we provide can be easily adapted to make a uniform treatment of upper and lower arcs. For simplicity, we do not draw an arc from i to itself when $\sigma(i) = i$, although such a fixed point contributes to enhanced k-nestings of upper arcs. Figure 10 shows an example of a permutation diagram.

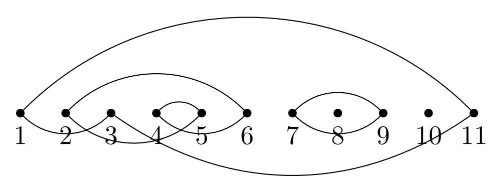

FIGURE 10. The permutation $\sigma = (1\,11\,3)(2\,6\,4\,5)(7\,9)(8)(10)$ and its arc diagram representation. This diagram has two 3-nestings: $\{(1, 11), (2, 6), (4, 5)\}$ and $\{(1, 11), (7, 9), (8, 8)\}$.

As in the case of set partitions, there is a related definition of k-crossings in permutations. Burrill *et al.* proved that k-noncrossing and k-nonnesting permutations are equinumerous. However, they present very limited enumerative results, which we significantly improve upon here.

We remark that the subset of upper arcs forms a set partition, and so does the subset of lower arcs. However, these two partitions are not independent. Their relationship is explicitly described in [5]. This interpretation as a pair of partitions allows us to extend the construction from Section 4 quite naturally.

The vertices of a permutation diagram can be of five types: *fixed points*, *openers*, *closers*, *upper transitories*, and *lower transitories*. A fixed point has no incident edges, an opener (resp. closer) is the left (resp. right) end-point of an upper and a lower arc, and an upper (resp. lower) transitory vertex is the right end-point of an upper (resp. lower) arc and the left end-point of another.

5.1. **Generating tree for open permutation diagrams.** Similarly to what we did for partitions, we consider a more general class of objects that we call *open permutation diagrams*, by allowing three additional vertex types: *semi-openers*, *upper semi-transitories*, and *lower semi-transitories*. A semi-opener is the left end-point of an upper and a lower semi-arc, and an upper (resp. lower) transitory vertex is the right end-point of an upper (resp. lower) arc and the right end-point of an upper (resp. lower) semi-arc. Note that in an open permutation diagram, the number of upper semi-arcs equals the number of lower semi-arcs.

An open permutation diagram can also be viewed as a permutation where each cycle of length i can be coloured one of $i+1$ possible colours: one to indicate that the cycle has no semi-arcs, and the other i for the possible choices of an arc in the cycles to be converted into two semi-arcs. It follows that the exponential generating function for open permutation diagrams is

$$\frac{1}{1-z}\exp\left(\frac{z}{1-z}\right).$$

To create an open permutation diagram of size n from one of size $n-1$, we add a vertex labelled n which can be of any of the five following types:

(1) fixed point;
(2) semi-opener;
(3) upper semi-transitory (provided there is an available upper semi-arc);
(4) lower semi-transitory (provided there is an available lower semi-arc);
(5) closer (provided there are available upper and lower semi-arcs).

We can easily make a generating tree for the class of open permutation diagrams by keeping track of the number ℓ of upper semi-arcs:

$$[(0) : (2\ell) \to (2\ell)(2\ell+2)(2\ell)^{2\ell}(2\ell-2)^{\ell^2}].$$

To incorporate the nesting constraint to the tree, we define a notion of future nestings in open permutation diagrams. A *future enhanced upper k-nesting* is an upper enhanced $k-1$-nesting together with an upper semi-arc beginning to its left, and a *future lower k-nesting* is a lower $k-1$-nesting together with a lower semi-arc beginning to its left. An example of each is in Figure 11. The enhanced nesting index of an upper semi-arc is the largest j such that the semi-arc is in a future enhanced upper $j+1$-nesting. The nesting index of a lower semi-arc is the largest j such that the semi-arc is in a future lower $j+1$-nesting. An open permutation diagram is called *k-nonnesting* if it has no regular or future enhanced upper k-nestings, and no regular or future lower k-nestings.

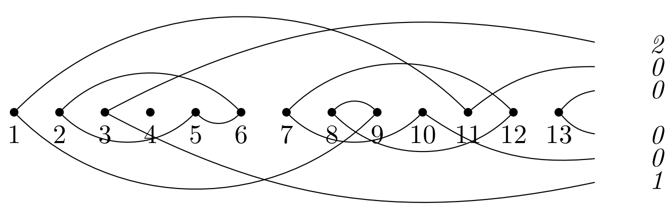

FIGURE 11. An open permutation diagram. The upper arcs $(7,12)$, $(8,9)$ and the upper semi-arc $(3,*)$ form a future enhanced upper 3-nesting. The lower arc $(7,10)$ and the lower semi-arc $(3,*)$ form a future lower 2-nesting. The nesting index of each semi-arc is labelled in italics.

Again, it will be convenient to shift the index and consider $k+1$-nonnesting permutations. To each $k+1$-nonnesting open permutation diagram, we associate a label consisting of a 3-tuple, $[h;\mathbf{r};\mathbf{s}]$. Here, h is the number of upper semi-arcs (and hence also the number of lower semi-arcs), $\mathbf{r}=[r_1,\ldots,r_{k-1}]$ is a vector such that r_i is the number of upper semi-arcs of enhanced nesting index greater than or equal to i, and $\mathbf{s}=[s_1,\ldots,s_{k-1}]$ is a vector such that s_i is the number of lower semi-arcs of nesting index greater than or equal to i. The label of the 4-nonnesting diagram in Figure 11 is $[3;1,1;1,0]$.

We note that in an open permutation diagram σ, if we consider loops as fixed points, the upper (resp. lower) arcs and semi-arcs form an open partition diagram σ^+ (resp. σ^-) on the vertices $\{1,\dots,n\}$. If the label of σ is $[h;\mathbf{r};\mathbf{s}]$, then the label of σ^+ as described in Theorem 4.1 is $[h,\mathbf{r}]$, and the label of σ^+ as described in Theorem 4.3 is $[h,\mathbf{s}]$. In particular, $h \geq r_1 \geq \cdots \geq r_{k-1} \geq 0$ and $h \geq s_1 \geq \cdots \geq s_{k-1} \geq 0$.

5.2. **3-nonnesting permutations.** Again, the case of 3-nonnesting permutations is sufficiently insightful for the general method without being overly complicated. In this section we describe the generating tree for 3-nonnesting open permutation diagrams, and we determine a functional equation for the generating function.

5.2.1. *Generating tree.* The label of a 3-nonnesting open permutation diagram is $[h,r_1,s_1]$ (we use commas instead of semicolons in this section). Here, $2h$ is the total number of semi-arcs, r_1 is the number of upper semi-arcs that belong to a future enhanced upper 2-nesting, and s_1 is the number of lower semi-arcs that belong to a future lower 2-nesting. An example of the labelling is given in Figure 12, where the numbers on the vertices are omitted. The arrows indicate semi-arcs that are in future 2-nestings: two upper semi-arcs and one lower semi-arc, hence the label of the diagram is $[4,2,1]$.

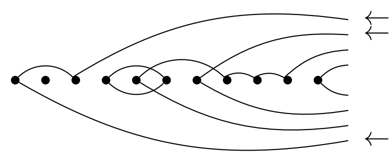

FIGURE 12. An arc diagram with label $[4,2,1]$.

To predict the labels of the children of a 3-nonnesting open permutation diagram, we consider the different types of vertices that can be added. To avoid future 3-nestings, we are not allowed to add a closer or semi-transitory vertex that closes a semi-arc belonging to a future enhanced upper 2-nesting or a future lower 2-nesting, unless it is an outermost semi-arc, i.e., the top upper semi-arc or the bottom lower semi-arc. As an example, the labels of the children of the diagram in Figure 12 are generated by adding vertices of different types as follows.

(1) Fixed point: one child with label $[4,4,1]$, since the upper semi-arcs belong now to future enhanced upper 2-nestings.
(2) Semi-opener: one child with label $[5,2,1]$.
(3) Upper semi-transitory: closing the upper semi-arcs that are not in future enhanced 2-nestings gives the labels $[4,2,1]$ and $[4,3,1]$; closing the top semi-arc (the only one in a 2-nesting that we are allowed to close) removes one future upper 2-nesting, giving the label $[4,1,1]$.
(4) Lower semi-transitory: all lower semi-arcs can be closed, and the four resulting labels are $[4,2,0]$, $[4,2,1]$, $[4,2,2]$ and $[4,2,3]$.
(5) Closer: we simultaneously and independently close an upper and a lower semi-arc, among those that we are allowed to close. There are three choices for the former and four for the latter, giving twelve children with labels $[3,1,0]$, $[3,1,2]$, $[3,1,1]$, $[3,1,3]$, $[3,2,0]$, $[3,2,2]$, $[3,2,1]$, $[3,2,3]$, $[3,3,0]$, $[3,3,2]$, $[3,3,1]$, $[3,3,3]$.

The succession rule for the generating tree is described in Theorem 5.1. Another example is drawn in Figure 13, and the first few levels of the tree appear in Figure 14.

Theorem 5.1. *Let $\Sigma^{(2)}$ be the set of 3-nonnesting open permutation diagrams. To each diagram σ associate the label $\ell(\sigma) = [h,r,s]$, where $2h$ is the total number of semi-arcs, r is the number of semi-arcs in a future enhanced upper 2-nesting and s is the number of semi-arcs in a future lower 2-nesting. Then, the number of diagrams in $\Sigma^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label $[0,0,0]$,*

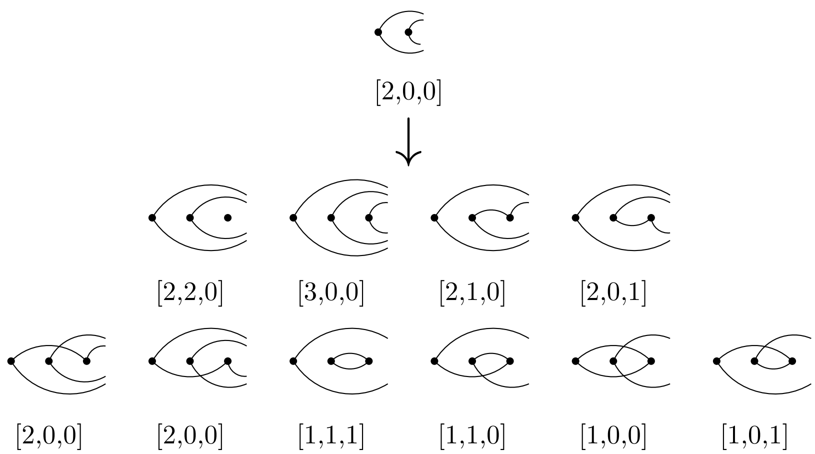

FIGURE 13. A 3-nonnesting open permutation diagram and its children.

and succession rule given by

$$
[h,r,s] \to \begin{array}{lll}
[h,h,s] & & (1) \\
[h+1,r,s] & & (2) \\
[h,i,s] & \textit{for } \max\{0,r-1\} \leq i \leq h-1 & (3) \\
[h,r,j] & \textit{for } \max\{0,s-1\} \leq j \leq h-1 & (4) \\
[h-1,i,j] & \textit{for } \max\{0,r-1\} \leq i \leq h-1 \textit{ and } \max\{0,s-1\} \leq j \leq h-1. & (5)
\end{array}
$$

The number of 3*-nonnesting permutations of size* n *is equal to the number of nodes with label* $[0,0,0]$ *at the* n*-th level of this generating tree.*

Proof. The labels correspond to the addition of the following nodes to a diagram σ:

(1) a fixed point, which results in all the upper semi-arcs becoming part of future enhanced 2-nestings;
(2) a semi-opener, which produces a new upper semi-arc and a new lower one, neither of which is in a future 2-nesting;
(3) an upper semi-transitory closing a semi-arc not belonging to a future enhanced upper 2-nesting or, if $r > 0$, possibly closing the top semi-arc;
(4) a lower semi-transitory closing a semi-arc not belonging to a future lower 2-nesting or, if $s > 0$, possibly closing the bottom semi-arc;
(5) a closer, which can close any combination of an upper and a lower semi-arc among those allowed to close in parts (3) and (4).

□

5.2.2. *Functional equation.* We translate the generating tree from Theorem 5.1 into a functional equation for the multivariate generating function $F(u,v,w) = \sum F_{h,r,s}(n)u^h v^r w^s z^n$ where $F_{h,r,s}(n)$ is the number of open permutation arc diagrams at level n with label $[h,r,s]$. The coefficient $F_{0,0,0}(n)$ is the number of 3-nonnesting permutations of $\{1,2,\ldots,n\}$.

The translation process to derive a functional equation is analogous to the set partition case. We consider the five types of vertices in turn to analyze their contribution. The form of the functional equation is

$$F(u,v,w) = 1 + z(\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5),$$

where Ψ_i is the contribution for adding a vertex of type (i), which we compute next.

(1) Fixed point. Note that case (1) in the succession rule can alternatively be included by extending the range of i in case (3) to include h. Thus, it will be simpler to compute $\Psi_1 + \Psi_3$ in item (3) below.
(2) Opener. $\Psi_2 = uF(u,v,w)$.

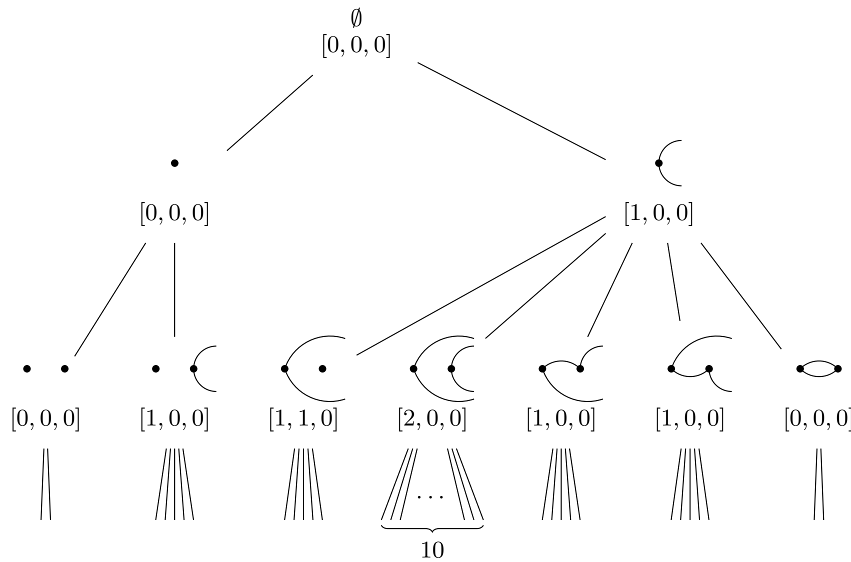

Figure 14. Generating tree for 3-nonnesting open permutation diagrams. The first few levels coincide with those of the tree for all open permutation diagrams.

(3) Upper transitory and fixed point. $\Psi_1 + \Psi_3 = \frac{F(u,v,w)-vF(uv,1,w)}{1-v} + \frac{F(u,v,w)-F(u,0,w)}{v}$, found using the formula for a finite geometric sum in the expressions below:

$$z \sum_{h,s,n} F_{h,0,n}(n) u^h (1 + v + v^2 + \ldots + v^h) w^s z^n \qquad \text{if } j = 0,$$
$$z \sum_{h,s,n} F_{h,r,s}(n) u^i h (v^{r-1} + v^j + \ldots + v^h) w^s z^n \qquad \text{if } 0 < j \le i.$$

(4) Lower transitory. $\Psi_4 = \frac{F(u,v,w)-F(uw,v,1)}{1-w} + \frac{F(u,v,w)-F(u,v,0)}{w}$.

(5) Closer. The addition of a closer to a diagram with label $[h, r, s]$ contributes

$$\sum_{h,n} F_{h,r,s}(n) u^{h-1} (v^{\max\{r-1,0\}} + \ldots + v^{h-1})(w^{\max\{s-1,0\}} + \cdots + w^{h-1}) z^n,$$

which can be simplified using finite geometric sum formulas, and separating the case when $r = 0$ or $s = 0$:

$$\begin{aligned}\Psi_5 = \frac{F(u,v,w) - F(uw,1,w) - F(uw,v,1) + F(uvw,1,1)}{u(1-v)(1-w)} \\ + \frac{F(u,v,w) - F(u,0,w) - F(uw,v,1) + F(uw,0,1)}{uv(1-w)} \\ + \frac{F(u,v,w) - F(u,v,0) - F(uv,1,w) + F(uv,1,0)}{uw(1-v)} \\ + \frac{F(u,v,w) - F(u,0,w) - F(u,v,0) + F(u,0,0)}{uvw}.\end{aligned}$$

Adding all five contributions, we get the following corollary.

Corollary 5.2. *The multivariate generating function for 3-nonnesting open partition diagrams, denoted*

$$F(u, v, w; z) = \sum_{h,r,s,n} F_{h,r,s}(n) u^h v^r w^s z^n,$$

where $F_{h,r,s}(n)$ is the number of diagrams of size n with label $[h,r,s]$, satisfies the functional equation

$$
\begin{aligned}
F(u,v,w) = 1 + z\Bigg(& uF(u,v,w) \\
& + \frac{F(u,v,w) - v(uv,1,w)}{1-v} + \frac{F(u,v,w) - F(u,0,w)}{v} + \frac{F(u,v,w) - F(uw,v,1)}{1-w} \\
& + \frac{F(u,v,w) - F(u,v,0)}{w} + \frac{F(u,v,w) - F(uw,1,w) - F(uw,v,1) + F(uvw,1,1)}{u(1-v)(1-w)} \\
& + \frac{F(u,v,w) - F(u,0,w) - F(uw,v,1) + F(uw,0,1)}{uv(1-w)} \\
& + \frac{F(u,v,w) - F(u,v,0) - F(uv,1,w) + F(uv,1,0)}{uw(1-v)} \\
& + \frac{F(u,v,w) - F(u,0,w) - F(u,v,0) + F(u,0,0)}{uvw}\Bigg).
\end{aligned}
$$

We have found this equation useful to generate terms in the sequence, but we have been unable to solve it, or to find an explicit expression for $F_{0,0,0}(n)$, the number of 3-nonnesting permutations.

5.3. **The general case: $k+1$-nonnesting permutations.** The construction in Section 5.2 can easily be generalized to this case. Recall that to each $k+1$-nonnesting open permutation diagram we assign a label $[h;\mathbf{r};\mathbf{s}] = [h; r_1, r_2, \ldots, r_{k-1}; s_1, s_2, \ldots, s_{k-1}]$. To describe the succession rule of the corresponding generating tree, we think of $[h,\mathbf{r}]$ as the label of the upper set partition, where we consider enhanced nestings, and of $[h,\mathbf{s}]$ as the label of the lower set partition, where we consider usual nestings. We use $\mathbf{r}-1$ as a shorthand for $r_1-1, r_2-1, \ldots, r_{k-1}-1$, and similarly for $\mathbf{s}-1$. When the parameters r_0 and s_0 are used below in $(3b)$, $(4b)$, etc., they are defined to be equal to h.

Theorem 5.3. *Let $\Sigma^{(k)}$ be the set of $k+1$-nonnesting open permutation diagrams. To each diagram σ associate the label $\ell(\sigma) = [h;\mathbf{r};\mathbf{s}] = [h; r_1, r_2, \ldots, r_{k-1}; s_1, s_2, \ldots, s_{k-1}]$, where $2h$ is the number of semi-arcs, and r_i (resp. s_i) is the number of open upper (resp. lower) semi-arcs of enhanced nesting index (resp. nesting index) greater than or equal to i. Then the number of diagrams in $\Sigma^{(k)}$ of size n is the number of nodes at level n in the generating tree with root label $[0;\mathbf{0};\mathbf{0}]$, and succession rule given by*

$$[h;\mathbf{r};\mathbf{s}] \longrightarrow$$

$$
\begin{aligned}
& [h; h, r_2, \ldots, r_{k-1}; \mathbf{s}], && (1)\\
& [h+1; \mathbf{r}; \mathbf{s}], && (2)\\
& [h; \mathbf{r}-1; \mathbf{s}], \quad \textit{if } r_{k-1} \geq 1, && (3a)\\
& [h; r_1-1, \ldots, r_{j-1}-1, i, r_{j+1}, \ldots, r_{k-1}; \mathbf{s}], \quad \textit{for } 1 \leq j \leq k-1 \textit{ and } r_j \leq i \leq r_{j-1}-1, && (3b)\\
& [h; \mathbf{r}; \mathbf{s}-1], \quad \textit{if } s_{k-1} \geq 1, && (4a)\\
& [h; \mathbf{r}; s_1-1, \ldots, s_{\jmath-1}-1, \imath, s_{\jmath+1}, \ldots, s_{k-1}], \quad \textit{for } 1 \leq \jmath \leq k-1 \textit{ and } s_\jmath \leq \imath \leq s_{\jmath-1}-1, && (4b)\\
& [h-1; \mathbf{r}-1; \mathbf{s}-1], \quad \textit{if } r_{k-1} \geq 1 \textit{ and } s_{k-1} \geq 1 && (5a)\\
& [h-1; \mathbf{r}-1; s_1-1, \ldots, s_{\jmath-1}-1, \imath, s_{\jmath+1}, \ldots, s_{k-1}], \\
& \qquad \textit{if } r_{k-1} \geq 1, \textit{ for } 1 \leq \jmath \leq k-1 \textit{ and } s_\jmath \leq \imath \leq s_{\jmath-1}-1 && (5b)\\
& [h-1; r_1-1, \ldots, r_{j-1}-1, i, r_{j+1}, \ldots, r_{k-1}; \mathbf{s}-1], \\
& \qquad \textit{if } s_{k-1} \geq 1, \textit{ for } 1 \leq j \leq k-1 \textit{ and } r_j \leq i \leq r_{j-1}-1 && (5c)\\
& [h-1; r_1-1, \ldots, r_{j-1}-1, i, r_{j+1}, \ldots, r_{k-1}; s_1-1, \ldots, s_{\jmath-1}-1, \imath, s_{\jmath+1}, \ldots, s_{k-1}], \\
& \textit{for } 1 \leq \jmath \leq k-1 \textit{ and } s_\jmath \leq \imath \leq s_{\jmath-1}-1, \textit{ and for } 1 \leq j \leq k-1 \textit{ and } r_j \leq i \leq r_{j-1}-1. && (5d)
\end{aligned}
$$

Proof. The labels correspond to the addition of the following nodes to a diagram σ:

(1) a fixed point;

(2) a semi-opener;
(3a) an upper semi-transitory closing the top semi-arc, if σ has a future enhanced upper $k-1$-nesting;
(3b) an upper semi-transitory;
(4a) a lower semi-transitory closing the bottom semi-arc, if σ has a future lower $k-1$-nesting;
(4b) a lower semi-transitory;
(5a) a closer that closes both the top and the bottom semi-arcs, if σ has both a future enhanced upper $k-1$-nesting and a future lower $k-1$-nesting;
(5b) a closer that closes the top semi-arc and a lower semi-arc that is not the bottom one, if σ has a future enhanced upper $k-1$-nesting;
(5c) a closer that closes the bottom semi-arc and an upper semi-arc that is not the top one, if σ has a future lower $k-1$-nesting;
(5d) a closer that closes an upper and a lower semi-arc, neither of which is an outermost one.

□

We have also translated the succession rules of the generating tree for $k+1$-nonnesting open permutation diagrams into a functional equation for the corresponding generating function, which has $2k$ variables. We omit it here due to space constraints.

$k+1$	OEIS	Initial terms
3	A193938	1, 2, 6, 24, 118, 675, 4333, 30464, 230615, 1856336, 15738672, 139509303, 1285276242, 12248071935, 120255584181, 1212503440774, 12519867688928, 132079067871313
4	A193935	1, 2, 6, 24, 120, 720, 5034, 40087, 356942, 3500551, 37343168, 428886219, 5257753614, 68306562647, 934747457369, 13404687958473, 200554264435218, 3118638648191005
5	A193936	1, 2, 6, 24, 120, 720, 5040, 40320, 362856, 3627385, 39864333, 477407104, 6183182389, 86033729930, 1278515941177, 20185987771091
6	A193937	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916680, 478991641, 6226516930, 87157924751, 1306945300264

Table 3. Counting sequences for $k+1$-nonnesting permutations.

5.4. **Enumerative data.** We used the *gfun* package of Maple (version 3.53) to try to fit the counting sequences for k-nonnesting permutations (with $3 \leq k \leq 6$) into a differential equation using 80 terms, with no success. Thus, we make the following conjecture.

Conjecture 5.4. *The ordinary generating function for k-nonnesting permutations is not D-finite for any $k > 2$.*

6. Perspectives

Let us finish by describing some possible future directions of research extending our work. Having found functional equations for the generating functions of partitions and permutations avoiding k-nestings (which are equinumerous to those avoiding k-crossings), a natural extension would be to describe the distribution of the number of k-crossings and k-nestings on partitions and permutations. More generally, given $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r$, what is the number of partitions (resp. permutations) of size n with i_k k-crossings and i_ℓ ℓ-nestings for $1 \leq k, \ell \leq r$? Is this number the same if the words "crossings" and "nestings" are switched? A reasonable first step towards describing this distribution would be to study the number of ℓ-nestings (for $\ell < k$) in k-nonnesting partitions (resp. permutations). Perhaps probabilistic arguments could be used to obtain information about the expected number of ℓ-nestings. Also, it may be worth investigating whether a notion of future k-crossings can be used to translate the k-noncrossing condition on open partition diagrams into certain restrictions on some appropriately defined labels, from where a generating tree for k-noncrossing partitions could be constructed. This might allow us to impose simultaneously noncrossing and nonnesting restrictions.

In a different direction, we expect that our generating trees and functional equations can be used to code random generation schemes for k-nonnesting partitions and permutations. Finally, it would be interesting to generalize our open diagram construction to study k-nestings in other combinatorial structures, such as tangled diagrams and other objects used to predict RNA secondary structures.

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