A GEOMETRIC AND COMBINATORIAL VIEW OF WEIGHTED VOTING

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ABSTRACT. A natural partial ordering exists on all weighted games and, more broadly, on all linear games. We describe several properties of the partially ordered sets formed by these games and utilize this perspective to enumerate proper linear games with one generator. We introduce a geometric approach to weighted voting by considering the convex polytope of all possible realizations of a weighted game and connect this geometric perspective to the weighted games poset in several ways. In particular, we prove that generic vertical lines in $\mathcal{C}_n$, the union of all weighted n-voter polytopes, correspond to maximal saturated chains in the poset of weighted games, i.e., the poset is a blueprint for how the polytopes fit together to form $\mathcal{C}_n$. Finally, we describe the facets of each polytope, from which we develop a method for determining the weightedness of any linear game that covers or is covered by a weighted game.

1. INTRODUCTION

Weighted voting refers to the situation where n players, each with a certain weight, vote on a yes or no motion. For one side to win, the weights of its voters must reach a certain fixed quota q. A natural example is a corporation: each stockholder is a voter with weight equal to the shares of stock he or she owns. The goal of this article is to describe certain combinatorial and geometric structures of weighted voting and to detail the connections between these viewpoints.

Weighted voting forms an important class of *simple games* (cf. Definition 2.1), whose framework offers several different interpretations. Simple games may be viewed as a type of hypergraphs [9, 12] and also as logic gates [19, 20]; in these situations, weighted voting corresponds to threshold graphs and threshold functions, respectively. We suggest the excellent book [24] as a first reference on simple games for the nonexpert reader.

When the voters in a simple game are totally ordered, the game is called *linear* [7] (or *directed* [15] or *complete* [8]). All weighted games are linear (but not vice-versa), since the voters' weights provide a natural ordering. Taylor and Zwicker [23] have characterized linear games via swap robustness and weighted games via trade robustness; we utilize this trading approach in section 3.

A total ordering on the voters in a linear game naturally leads to a partial ordering on coalitions. For n voters, the coalitions form the well-known poset $M(n)$. Stanley has shown [22] that this poset is rank-unimodal and exhibits the Sperner property and a generalized Sperner property. Linear games are in one-to-one correspondence with the filters of $M(n)$, which form the filtration poset, denoted $J(M(n))$ or J_n. The generators of the filter corresponding to a game are its shift-minimal winning coalitions. Krohn and Sudhölter [15] introduce this partial ordering on simple linear games and weighted games and

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investigate several consequences of the Sperner property. They then use linear programming methods to obtain efficient algorithms which test whether or not a linear game is weighted. As a subposet of J_n, we construct the posets $\mathcal{W}_n$ of all weighted games and Π_n of all proper linear games in the style of Krohn and Sudhölter. All of these posets are symmetric, ranked lattices. Figure 1 depicts $M(3), M(4)$, and the top half of J_4.

We also introduce a geometric approach to weighted voting systems. By scaling the weights to sum to 1, we define the simplex of normalized weights Δ_n and the configuration region $\mathcal{C}_n = (0,1] \times \Delta_n$, which depicts all realizations of n-player weighted games in quota-weight space. Each coalition corresponds to a half-space intersecting $\mathcal{C}_n$; weighted games correspond to the polytopes constructed by the hyperplanes bounding these spaces. We show in section 4.1 that each polytope is convex, n-dimensional, closed on the top and side facets, and open on the bottom facets.

This geometric approach is quite different from the classical 'separating hyperplanes' approach, in which coalitions represent vertices of the n-dimensional unit cube. A linear game is weighted if the sets of vertices which correspond to its winning coalitions may be separated by appropriate hyperplanes from the remaining vertices (the losing coalitions) of the cube. See [2, 20, 24] for details.

Each possible hierarchy of voters is associated to a subsimplex of Δ_n. We show in Theorem 4.7 that the hierarchy for a weighted game corresponds to the smallest subsimplex onto which its polytope projects. As a corollary, we characterize symmetric games as the only ones that project onto corners of Δ_n.

The first of two main results of this article, Theorem 4.12, connects the above geometric approach to the poset of weighted games. We show that for a generic choice of weights, *moving vertically through the configuration region traverses a maximal saturated chain in $\mathcal{W}_n$.* In other words, we may view the polytopes for n voters as building blocks, and the ordering in $\mathcal{W}_n$ provides instructions on how to stack them so as to construct $\mathcal{C}_n$.

In Theorem 4.13, we describe the correlation between facets, hierarchies, and posets; furthermore, we prove that a weighted game's polytope has $n - k + d$ facets, where k is the number of nontrivial symmetry classes of voters in the game and d is the degree of the game as a vertex in $\mathcal{W}_n$.

The second main result of this article, Theorem 4.17, provides a method for determining the weightedness of a linear game covering or covered by a weighted game in J_n. This method reduces to a linear programming problem which is different and possibly simpler than the standard linear programming approaches (cf. [5, 15]) for determining weightedness.

This article is organized as follows. Section 2 describes the relevant background on weighted and simple games and assumes little expertise with voting theory. Section 3 contains our combinatorial approach via the partial orderings on coalitions, linear games, and weighted games. In Section 4, we detail the geometry of weighted voting and its connections to the aforementioned posets. We conclude by describing relevant open problems and future work in Section 5.

2. Background

2.1. **Weighted voting systems.** Weighted voting systems, also known as *weighted games*, belong to a much larger class known as *simple games*. To understand them, we require some preliminary definitions.

Throughout this paper we restrict to a finite set of n voters (or players), who vote yes or no on a motion. The set of voters who vote the same on a given motion is known as a *coalition*. The set $N = \{1, 2, \ldots, n\}$ of all voters is called the *grand coalition*. In weighted voting, a coalition is *winning* if the sum of its weights is greater than or equal to the quota. A *minimal winning coalition* is one possessing no winning coalition as a proper subset; if any voter leaves such a coalition the resulting coalition will no longer be winning. A *dictator* has weight greater than or equal to the quota. A *dummy* is a voter appearing in no minimal winning coalitions.

A first, straightforward observation is that the weights of voters can be misleading in understanding weighted voting. The winning coalitions determine everything about a weighted voting system. For instance, the sets of weights $w_3 = w_2 = 0.49, w_1 = 0.02)$ and $w'_3 = w'_2 = w'_1 = 1/3$ are vastly different but at a quota of 0.51 produce the same winning coalitions. Both of these represent a simple majority system in which any two voters can win by voting together.

Definition 2.1. A *simple game* g is a pair (N, W_g) in which $N = \{1, 2, \ldots, n\}$ is a finite set of players and W_g is a collection of subsets of N which represent the winning coalitions for game g, that satisfies $N \in W_g, \emptyset \notin W_g$, and the *monotonicity property*: ($S \in W_g$ and $S \subseteq R \subseteq N) \Rightarrow R$ imply W_g.

Henceforth, we assume that all games considered are simple. Also, we note that many authors choose to omit the requirement that $N \in W_g, \emptyset \notin W_g$. Our choice is advantageous from a voting and a geometric perspective but not from a combinatorial perspective, as we discuss in Remark 3.4.

Definition 2.2. A simple game is *weighted* if there exist weights $w_i \in [0, \infty)$ and a quota $q \in (0, \sum w_i]$ such that coalition A is winning if and only if the sum w_A of its weights is greater than or equal to the quota.

The vector $(q : \mathbf{w}) = (q : w_n, \ldots, w_2, w_1)$ is said to *realize* (or *represent*, we use these terms synonymously) v as a weighted game.

Contrary to much of the voting literature, we enumerate voters by *increasing* weight, in order to easily determine the rank of a coalition in the poset $M(n)$ (cf. Section 3.1). We shall refer to voters by the corresponding ordinals: the n^{th} voter $\mathbf{n}$ has the greatest weight, voter $\mathbf{n-1}$ has the next greatest, ..., voter $\mathbf{1}$ has the lowest weight.

Note that weighted voting is scale-invariant: multiplying each of the weights and the quota by a positive constant does not change the winning coalitions. Therefore, we may normalize the weights so that they sum to 1. Define a *normalized weight* to be a vector $\mathbf{w} = (w_n, \ldots, w_1)$ that satisfies

$$w_n \geq w_{n-1} \geq \ldots \geq w_1 \geq 0, \qquad \sum_{i=1}^{n} w_i = 1. \tag{1}$$

We denote the set of normalized weights in $\mathbf{R}^n$ as Δ_n, since it forms an $(n-1)$-dimensional simplex with vertices $p_1 = (1,0,0,0,\ldots)$, $p_2 = (\frac{1}{2},\frac{1}{2},0,0,\ldots)$, $p_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,\ldots)$, $\ldots$, $p_n = (\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$.

Remark 2.3. Geometrically, ordering the weights restricts their geometry from the orthant[1] $\mathbf{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$ to the closure (in the subspace topology) of one particular component of the configuration space $\mathtt{C}_n((0,\infty))$. (The configuration space $\mathtt{C}_n(X)$ consists of all ordered n-tuples of distinct $x_i \in X$.) This closure produces an unbounded polytope of infinite rays, diffeomorphic to $\Delta_n \times (0,\infty)$. Normalizing the weights deformation retracts this space onto the compact simplex Δ_n. $\diamond$

On Δ_n, we use coordinates $\{w_n, w_{n-1}, \ldots, w_2\}$ and view w_1 as a dependent variable equal to $1 - w_n - \ldots - w_2$.

Definition 2.4. For n voters, the *configuration region* $\mathcal{C}_n$ is the space of all realizations $(q : \mathbf{w})$ of weighted games, that is,

$$\mathcal{C}_n := (0,1] \times \Delta_n \subset \mathbf{R}^{n+1}. \tag{2}$$

In Section 4, we study the geometry of $\mathcal{C}_n$ as an approach toward understanding weighted voting. We note here that weighted voting systems may be equivalently defined as the nonempty equivalence classes of points in $\mathcal{C}_n$ where two points are equivalent if they produce the same winning coalitions.

2.2. **Background on simple games.**

We will require a few definitions regarding simple games, which we provide here. We begin with the desirability relation on voters, which was introduced in [12] and generalized in [18]. (See also [20].)

Definition 2.5. Let (N,W) be a simple game. We say that voter $\mathbf{i}$ is *more desirable* than voter $\mathbf{j}$ (denoted $\mathbf{i} \succeq \mathbf{j}$) in (N,W) if

$$S \cup \{\mathbf{j}\} \in W \Rightarrow S \cup \{\mathbf{i}\} \in W, \qquad \text{for all } \mathrm{S} \subseteq \mathrm{N} \setminus \{\mathbf{i},\mathbf{j}\}.$$

We say that voters $\mathbf{i}$ and $\mathbf{j}$ are *equally desirable* (denoted $\mathbf{i} \sim \mathbf{j}$) in (N,W) if

$$S \cup \{\mathbf{j}\} \in W \iff S \cup \{\mathbf{i}\} \in W, \qquad \text{for all } \mathrm{S} \subseteq \mathrm{N} \setminus \{\mathbf{i},\mathbf{j}\}.$$

Any simple game with a totally ordered desirability relation is called *linear* (or *directed* or *complete*.)

Each linear game breaks the voters into equivalence classes of equally desirable voters; this decomposition is called a *hierarchy*. For example, suppose for a linear game v on 7 voters that $\mathbf{7} \succ \mathbf{6} \sim \mathbf{5} \succ \mathbf{4} \succ \mathbf{3} \sim \mathbf{2} \sim \mathbf{1}$; then there are four classes of voters. We may express its hierarchy either as the string $(\succ\sim\succ\succ\sim\sim)$ or using its *power composition* (1,2,1,3), frequently denoted in the literature by $\overline{n}$. If a game has n_0 dummies, n_1 voters in its strongest class, n_2 in the next strongest, down to n_k in its weakest nontrivial class, then it has power composition $(n_1, n_2, \ldots, n_k)$, which is a composition of $n - n_0$ into k parts. See Carreras and Freixas [1] or Freixas and Molinero [6] for several important uses of the power composition as a vector. We note that any power index which is monotone, i.e., respects the desirability ordering, must distribute power according to this composition.

[1] An *orthant* is the n-dimensional analogue of a quadrant or octant.

One may view any linear game for m players as inducing a linear game for $n > m$ players by simply adding $n - m$ dummies. The mth strongest voter remains in this position, which means in our notation we add $n - m$ to each voter's number. For example, the game $\langle 321, 42\rangle$ (generated by coalitions 321 and 42, see section 3.2 for notation) for $m = 4$ voters induces the game $\langle 543, 64\rangle$ (generated by coalitions 543 and 64) for $n = 6$ voters; the induced game has two dummy voters. Both games have power composition $(2, 2)$. Linear games share most properties with their induced games, including power compositions, weightedness and properness.

We will study partial orderings of both linear and weighted games in section 3. For now, let us consider the ordering on voters in a weighted game.

For a weighted game v, the desirability ordering weakly respects the ordering by weights. If $w_i = w_j$, then $\mathbf{i} \sim \mathbf{j}$. If $w_i > w_j$, then $\mathbf{i} \succeq \mathbf{j}$, since the weight of $\{j\} \cup S$ $(i, j \notin S)$ will be strictly less than the weight of $\{i\} \cup S$, and hence $\{j\} \cup S \in W_v$ implies $\{i\} \cup S \in W_v$.

Thus, a weighted game possesses a total ordering on the voters, and hence all weighted games are linear. Not all linear games are weighted though. The first examples occur for $n = 6$ voters, where 60 of the 1171 linear games fail to be weighted; for reference, we list these in Appendix A.

In many voting contexts, there is a restriction that at most one side may win, i.e., no coalition and its complement are both winning. One rationale for this is that if opposing sides (complementary coalitions) could both win, a decision would not be reachable and the result of the process would be a stalemate.

Definition 2.6. A simple game is said to *proper* if for each complementary pair A, A^c of coalitions, at most one is winning; otherwise it is *improper*. A simple game is said to be *strong* if for each complementary pair A, A^c of coalitions, at least one is winning. A simple game is said to be *self-dual* if for each complementary pair A, A^c of coalitions, precisely one is winning, i.e., if it is both proper and strong.

Note that the terminology has changed somewhat substantially as the literature on this subject has evolved from many different viewpoints; we largely follow [24]. For example, self-dual games have been referred to as *constant-sum* games by von Neumann and Morgenstern [25], *strong* games by Isbell [13], and *zero-sum* games by Krohn and Sudhölter [15]. The term *self-dual* is most descriptive in our context since the notion of duality plays an important role in our approach.

Definition 2.7. For a simple game v, we define its *dual game* v^* by specifying its winning coalitions:

$$\text{coalition } A \text{ is winning in } v^* \quad \Leftrightarrow \quad \text{coalition } A^c \text{ is losing in } v.$$

Thus our choice of terminology is intuitive: a linear game v is self-dual if and only if v equals its dual game v^*.

3. Weighted and linear games as partially ordered sets

A *partially ordered set*, or *poset*, P, is a set equipped with a binary ordering $\leq$ which is reflexive, antisymmetric, and transitive. The ordering is partial (as opposed to total) since not all elements of the set must be comparable under the ordering. An element $y \in P$ is said to *cover* another element $x \in P$ if $y > x$ and there is no z such that $y > z > x$.

A poset P is said to be *ranked* (or *graded*) if there exists a *rank function* $\rho : P \to \mathbf{N}$ compatible with the partial ordering such that $\rho(y) = \rho(x) + 1$ if y covers x. The value $\rho(x)$ is called the *rank* of the element x. The numbers of elements of each rank can be organized into a *rank-generating function* given by

$$\sum_{r=0}^{m} a_r q^r,$$

where m is the maximal rank and a_r is the number of elements of rank r.

In this section, we describe four different posets associated to weighted voting and linear games. We begin in section 3.1 with the well-known poset $M(n)$ which represents the ordering on coalitions within an ordered set of n voters. Then we argue that the linear games on n voters are in bijection with the filters of $M(n)$. These filters possess a partial ordering, from which we form a poset of linear games J_n. We also construct subposets representing all weighted games and all proper linear games.

3.1. **An ordering on coalitions.** We label a coalition of voters by listing the indices of the voters represented in the coalition as a decreasing sequence. For example, the coalition formed by voters **5**, **4**, and **2** is denoted $\{5, 4, 2\}$ or merely 542 when clear.

Definition 3.1. The coalition $B = \{b_1, b_2, \ldots, b_j\}$ is greater than or equal to the coalition $A = \{a_1, a_2, \ldots, a_k\}$ (written $B \geq A$) if and only if $k \leq j$ and for each $1 \leq i \leq k$ there exists a nonnegative integer x_i such that $b_i = a_i + x_i$.

Definition 3.1 produces a partial ordering on the set of all coalitions formed by n voters since it satisfies the reflexivity, antisymmetry, and transitivity conditions. Krohn and Sudhölter [15] use this ordering to count weighted and linear games. We call the resulting partially ordered set the *coalitions poset* and denote it by $M(n)$. Figure 1 depicts the coalitions posets $M(3)$ and $M(4)$.

We use the notation $M(n)$ to emphasize the fact that this poset appears in several other settings and has many interesting combinatorial properties. In particular, the rank-generating function of $M(n)$ is $\prod_{i=1}^{n}(1 + q^i)$, which was proven to have unimodal coefficients by Hughes [10]. The structure of $M(n)$ was shown by Lindström [17] to be related to a conjecture of Erdös and Moser [3, 4, 21]. Stanley uses the Coxeter system structure of type B_n to obtain a different construction of the poset $M(n)$ and uses this new description to show that the poset exhibits property S (which is a stronger property than the Sperner property) and to give a new proof that the poset is rank-unimodal [22]. The poset $M(n)$ is also known to be a distributive lattice, a fact that will assist us herein.

3.2. **A poset for linear games.** Having defined a poset of coalitions, we now define posets for linear, proper, and weighted games.

Consider two coalitions $B \geq A$; if A is winning in simple game v, then B must also be winning in v. This means that the set W_v of winning coalitions in game v extends down from the top of $M(n)$ to one or more lowest elements. The set W_v forms what is called a *filter* of $M(n)$; its lowest elements are the generators of this filter. Since each linear game v is determined by its set of winning coalitions, each one can be described uniquely as a filter of $M(n)$.

The *shift-minimal winning coalitions* (or *generators*) for a linear game are the generators of the filter it represents in $M(n)$. We observe that every shift-minimal winning coalition is minimal, but not vice-versa. We will use the shift-minimal winning coalitions to denote a linear game, e.g., if A and B are the generators for game v, we write $v = \langle A, B\rangle$. (For singly generated systems, we often drop the brackets.)

Example 3.2. Consider the weighted game on 4 voters realized by $(0.6 : 0.35, 0.25, 0.2, 0.2)$. Its winning coalitions are $321, 421, 43, 431, 432, 4321$; of these, the first three are minimal. There are two shift-minimal winning coalitions, namely 321 and 43. We denote this game as $\langle 321, 43\rangle$. ◇

Linear games on n voters are in one-to-one correspondence with filters of $M(n)$. By choosing any set of incomparable coalitions in $M(n)$, we are specifying the shift-minimal winning coalitions for some unique linear game, and we are uniquely specifying the generators of a filter of $M(n)$. It is worth noting that not every filter of $M(n)$ produces a weighted voting system. We may thus construct a poset of linear games using the natural partial ordering (containment) on filters as follows.

Definition 3.3. A linear game v is *stronger than* another linear game u, denoted $v \succeq u$, if every winning coalition in v also wins in u, i.e., if $W_v \subset W_u$. This is a partial ordering on the set of linear games with n players. We refer to it as the *linear games poset* and denote it by $J(M(n))$ or J_n.

The linear games poset is a distributive lattice – for any finite poset P, the poset $J(P)$ of filters of P under the containment ordering is known to be a distributive lattice. Also, J_n is ranked by the number of losing coalitions in each game. For example, the linear game $\langle 653, 5432\rangle$ in J_6 has rank 48 since there are 16 winning coalitions in this linear game, 64 total coalitions in $M(6)$, and therefore $64 - 16 = 48$ losing coalitions.

Remark 3.4. Formally, the linear games poset is only a *subposet* of a ranked lattice, since we have excluded games of rank 0 (where every coalition is winning) and rank 2^n (where every coalition is losing) from our Definition 2.1 of *simple games*. (A ranked poset must have minimal rank 0.) Many authors choose to include these games. Were we to extend J_n to include them, then it would have minimal rank 0 and maximal rank 2^n. From a voting perspective, these two games are somewhat unnatural as they represent situations where the voters have no control over the outcome. Worse, some of our results in section 4 are not valid for these two games. ◇

Let us now consider three subposets of J_n. Denote by J_n^+ the 'top half' of J_n, that is all games of rank exceeding or equal to half the maximal rank 2^n. We frequently consider proper linear games; these also form an induced subposet of J_n, which we denote as Π_n. Recall that in a proper linear game, if a coalition A is winning then its complement A^c cannot be winning. Therefore at most 2^{n-1} coalitions can be winning in a proper linear game. This means that the elements of Π_n must contain at least 2^{n-1} losing coalitions and therefore must have rank at least 2^{n-1}; thus Π_n lies in J_n^+. However, there do exist improper linear games of rank greater than 2^{n-1}, as we discuss in Theorem 3.8.

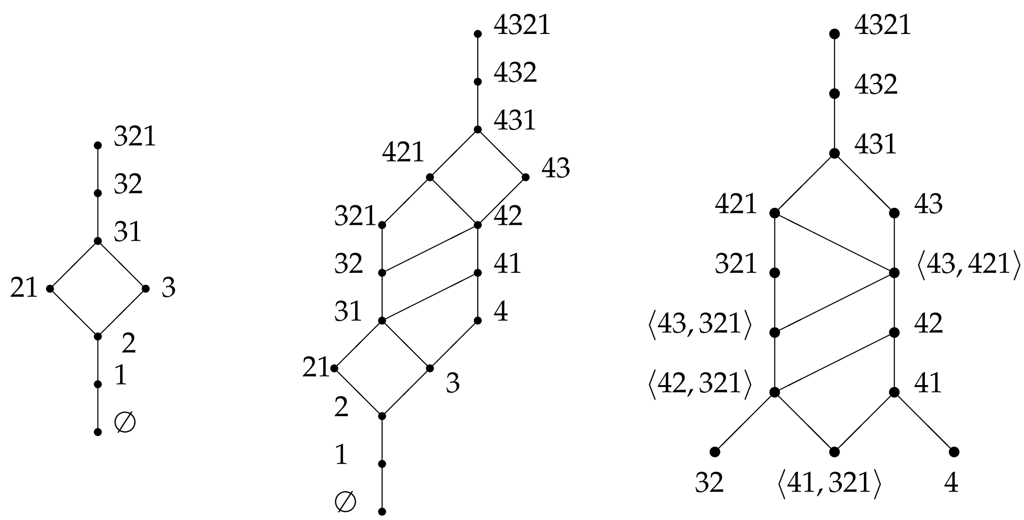
In poset J_n (and Π_n), game v covers game u if $W_u = W_v \cup \{A\}$ for some coalition $A \notin W_v$.

Definition 3.5. The *weighted games poset* $\mathcal{W}_n$ is the set of all weighted voting systems on n voters along with the partial ordering that arises from this covering relation.

It is important to note that as defined $\mathcal{W}_n$ forms a subposet of the linear games poset J_n, but we do not know whether it is an induced subposet. This question of inducement can be rephrased as follows: if $v \succ u$ in J_n for two weighted games, must there exist a saturated chain in J_n from u to v comprised only of weighted games? We conjecture that yes, there must be.

Conjecture 3.6. *The weighted games poset $\mathcal{W}_n$ is an induced subposet of J_n.*

We at times refer to the induced subposet of proper weighted games, denoted $\mathcal{W}_n^+$. Just as $\mathcal{W}_n$ is a (not necessarily induced) subposet of J_n, the poset $\mathcal{W}_n^+$ is a (not necessarily induced) subposet of J_n^+. Figure 1 shows the poset J_4^+, which equals $\mathcal{W}_4^+$ and Π_4.



(a) Coalitions Poset $M(3)$ (b) Coalitions Poset $M(4)$ (c) Proper linear games poset $J_4^+ = \Pi_4 = \mathcal{W}_4^+$

FIGURE 1. Poset examples for 3 and 4 voters

3.3. **Comparing posets.** In this section, we ask when the posets $\mathcal{W}_n, \Pi_n$, and J_n^+ are equal. We prove that all weighted games in $\mathcal{W}_n^+$ are proper, some proper games are unweighted, and some linear games in J_n^+ are improper.

In general, determining if a linear game is weighted can be difficult. One characterization of weighted games that will be useful below was given by Taylor and Zwicker [23] in terms of general trading. In the following, a trade is not restricted to a one-for-one exchange of voters. Any number of voters can be moved among coalitions arbitrarily, provided none of the resulting coalitions contains more than one copy of any voter. A simple game G is said to be *trade robust* if for every collection $X = \{X_1, X_2, \ldots, X_j\}$ of (not necessarily disjoint) winning coalitions in G, it is not possible to trade members among the coalitions to produce a collection $Y = \{Y_1, Y_2, \ldots, Y_j\}$ such that the coalitions in Y are all losing.

Theorem 3.7. [23] *A game G is weighted if and only if it is trade robust.*

Consider the linear game on 9 voters generated by the single shift-minimal coalition 8741, denoted $\langle 8741 \rangle$. Let $X = \{9741, 8752\}$ and trade 8 for 41 to form $Y = \{987, 75421\}$. The coalitions in X are all winning coalitions in $\langle 8741 \rangle$ and the coalitions in Y are all losing in $\langle 8741 \rangle$. Therefore the linear game $\langle 8741 \rangle$ is unweighted since it fails to be trade robust.

Theorem 3.8. *The following inclusions hold*

$$\mathcal{W}_n^+ \subset \Pi_n \subset J_n^+. \tag{3}$$

The first inclusion is strict precisely for $n \geq 7$ voters and the second is strict precisely for $n \geq 6$.

Proof. The second inclusion, $\Pi_n \subset J_n^+$, is immediate since every proper linear game has at least half of the coalitions losing. Thus its rank is at least 2^{n-1}, and it lies in the top half of J_n. We use the following lemma to prove the first inclusion.

Lemma 3.9. *Let $v \in J_n$ be an improper weighted game. Then W_v, the set of winning coalitions for v, must include at least one coalition from each complement pair.*

Proof of Lemma 3.9. We assume v is weighted and improper, so it includes a pair of complementary winning coalitions, A and A^c. We may trade voters between these coalitions to form any desired complement pair of coalitions, B and B^c. Trade robustness guarantees that at least one of these coalitions is winning. □

Now we show the first inclusion in (3). Suppose the weighted game v is improper. By the above lemma, W_v includes at least one coalition from each complement pair. As v is improper, W_v also includes both coalitions of some complement pair, so the rank of v is less than 2^{n-1}, i.e., $v \notin \mathcal{W}_n^+$. Hence, $\mathcal{W}_n^+ \subset \Pi_n$.

The following examples establish when these inclusions are strict. Note that for $n \leq 5$ players, all linear games are weighted, so both inclusions above are equalities. For 6 voters, there are 60 unweighted games: 20 have rank less than 32, while 20 have rank equal to 32, and 20 have rank greater than 32; we list these in Appendix A. None of them are proper.

(1) The game $\langle 6531 \rangle \in \Pi_7$ is not weighted since it fails trade robustness.
(2) Of the 40 improper games in J_6^+, the one with the highest rank, 37, is $\langle 65, 4321 \rangle$. It is generated by two complementary coalitions and is thus improper.

To obtain examples for larger n values, simply add dummy voters to these games; the induced games share the weightedness and properness of the original game. Thus we have established the strictness criteria above. □

3.4. **Poset properties.** We now consider various properties of our posets of games, including their ranks, covers, and inclusions.

Recall that the filters in the poset J_n are ranked by the cardinality of their losing coalitions, and since the cardinalities vary from zero to 2^n, the poset J_n has rank 2^n.

Proposition 3.10. *The weighted games poset $\mathcal{W}_n$ achieves each rank from 1 to $2^n - 1$. The posets $\mathcal{W}_n^+, \Pi_n$, and J_n^+ achieve each rank from 2^{n-1} to $2^n - 1$.*

Proof. By construction $\mathcal{W}_n \subset J_n$; Theorem 3.8 showed that $\mathcal{W}_n^+ \subset \Pi_n \subset J_n^+$. So, it suffices to construct a weighted game for each rank.

Let voter $\mathbf{i}$ have unnormalized weight 2^{i-1}. Under these weights, the coalitions are totally ordered in the sense that no two coalitions have the same weight. Choosing a quota of 1 produces the weighted game $\langle 1 \rangle$ which has rank 1; choosing a quota of 3 produces the weighted game $\langle 21 \rangle$ which has rank 3. In general, choosing a quota of r produces a weighted game of rank r. Quotas of 0 and 2^n correspond to the situations where all coalitions and no coalitions, respectively, are winning; we exclude these cases from our definition of simple games (cf. Remark 3.4). □

Corollary 3.11. *A proper linear game v has minimal rank 2^{n-1} in Π_n if and only if v is self-dual.*

Proof. First note that a game v is self-dual if and only if for each winning coalition A in v, A^c must be losing in v. Therefore every self-dual game has rank 2^{n-1} and in particular, a proper self-dual linear game has rank 2^{n-1}.

Conversely, let v be a proper linear game with minimal rank 2^{n-1} in Π_n. Then precisely one coalition from each complement pair A, A^c is winning in v, since at most one of A, A^c can be winning in v. Therefore v is self-dual. □

This statement is false for linear games in general; of the 41 games with rank 32 in J_6, only 21 are self-dual.

Proposition 3.12. *A (proper) linear game with k shift-minimal winning coalitions is covered by precisely k elements in J_n (repectively, in Π_n).*

Proof. Let v be a linear game with k shift-minimal winning coalitions. The linear games covering v in J_n are the filters of $M(n)$ obtained by removing exactly one of the shift-minimal winning coalitions from W_v. Since there are k generators which can be removed, there are k different filters covering v in J_n.

If v is proper, removing a winning coalition will retain properness, so proper games are only covered by proper games. Thus our result holds in Π_n as well. □

We observe that Proposition 3.12 is not true for weighted voting systems. For example, the weighted voting system $\langle 987, 8741\rangle$ in $\mathcal{W}_9^+$ is weighted with two shift-minimal winning coalitions. It is covered only by $\langle 987, 9741, 8751, 8742\rangle$ and not by $\langle 8741\rangle$ since $\langle 8741\rangle$ is not weighted.

We now describe a useful inclusion of J_n into J_{n+1}. Recall from the end of section 2 that the game $v = \langle A_1, A_2, \ldots\rangle$ on n voters induces the game $\tilde{v}$ by adding a dummy voter. So we have a map $J_n \hookrightarrow J_{n+1}$ that sends v to $\tilde{v}$. The rank of $\tilde{v}$ is twice that of v. Thus, we may conclude that if a game has k dummies, its rank must be a multiple of 2^k.

Since induced games preserve weightedness and properness, this map also produces the inclusions $\mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1}$ and $\Pi_n \hookrightarrow \Pi_{n+1}$.

3.5. **Enumerating linear games.** The tasks of counting linear and weighted games are difficult since the number of each grows rapidly as the number of voters increases; full results are known only for $n \leq 9$ voters [15, 19]. The enumeration of simple games has been studied by mathematicians for over a century, beginning with Dedekind's 1897 work in which he determined the number of simple games with four or fewer voters. Recently Freixas and Puente [8] investigated linear simple games with one shift-minimal winning coalition, Kurz and Tautenhahn [16] have enumerated linear simple games with two shift-minimal winning coalitions, and Freixas and Kurz [5] have provided a formula for the number of weighted games with one shift-minimal winning coalition and two types of voters. We extend this research to proper linear games by counting those with one shift-minimal coalition.

Posets provide a natural tool for ensuring that we have enumerated all linear games for n voters. For a linear game v, its set of winning coalitions W_v is the filter in $M(n)$ generated by the shift-minimal winning coalitions in v. A new linear game can be obtained by either removing a generator from W_v or adding a new coalition to W_v which is covered only by elements of W_v but is not in W_v, i.e., a *shift-maximal losing coalition*. This procedure either adds one or subtracts one, respectively, to the rank of the linear game. To obtain a new linear game with the same rank as v, perform both operations: remove a winning coalition A from W_v and add a new winning coalition which is not in $W_v \setminus \{A\}$ but is covered only by elements of $W_v \setminus \{A\}$.

Theorem 3.13. *For n voters, the number of proper linear games generated by exactly one shift-minimal winning coalition is*

$$2^n - \binom{n}{\lfloor n/2 \rfloor}. \tag{4}$$

The coalitions A for which $\langle A \rangle$ is a proper game are precisely the subsets of N which contain k of the largest $2k-1$ numbers in $[n]$ for some $k \leq n$.

Proof. We consider the 2^n different games $\langle A \rangle$, where $A \subset N$. Such a game is proper if and only if $A^c \not\succeq A$, i.e., if the complement of A does not lie in the filter $\langle A \rangle$. By Definition 3.1, $A^c \not\succeq A$ is equivalent to having the kth element of A be greater than the kth element of A^c for some $k \leq n$. Thus, $\langle A \rangle$ is proper if and only if A contains k of the largest $2k-1$ numbers in N (for some $k \leq n$).

This is equivalent to the number of ways to flip a fair coin n times so that a majority of heads had occurred at some point. This is sequence A045621 in the Online Encyclopedia of Integer Sequences [11] and is given by formula (4) above. □

4. The geometry of weighted voting representations

We now study the geometry of realizations of weighted games. Recall from Section 2 that since weighted voting is scale invariant, we may normalize the weights so that they sum to 1. Also, Δ_n denotes the $(n-1)$-dimensional simplex of normalized weights for weighted, n-voter games and $\mathcal{C}_n = (0,1] \times \Delta_n$ denotes the space of all realizations of such games. We envision $\mathcal{C}_n \subset \mathbf{R}^n$ depicted with coordinate q pointing upwards (in the vertical direction) and will refer to 'top' and 'bottom' features based on appropriate q values.

Consider all of the realizations in $\mathcal{C}_n$ for a weighted game $v \in \mathcal{W}_n$; these points define a polytope P_v in $\mathcal{C}_n$. The polytopes P_v encode a rich amount of information about weighted games; the goal of this section is to describe the geometry of weighted voting and its connections with posets and hierarchies.

By a *polytope*, we mean the generalization of a polygon or polyhedron to bounded k-dimensional objects. In $\mathbf{R}^k$, each polytope is bounded by a finite number of hyperplanes; these define $(k-1)$-dimensional subpolytopes called *facets*. We do not assume that all polytopes are convex; convex polytopes may be viewed as the convex hull of a finite set of points. Each P_v is in fact convex, as we demonstrate in Proposition 4.2.

Let us begin by examining the geometry of the configuration regions: $\mathcal{C}_1$ is a line segment of quotas above the point p_1 (where $w_1 = 1$); $\mathcal{C}_2$ is the rectangle $(0,1] \times \overline{p_2 p_1}$. Notice that $\mathcal{C}_1$ embeds naturally into $\mathcal{C}_2$. Figure 2 depicts $\mathcal{C}_3$, which is a triangular prism. Notice that

$\mathcal{C}_2$ embeds into $\mathcal{C}_3$ as the back facet, and $\mathcal{C}_1$ embeds as the rightmost edge. This is true in general: every $\mathcal{C}_k$ naturally embeds into $\mathcal{C}_n$ for $k < n$.

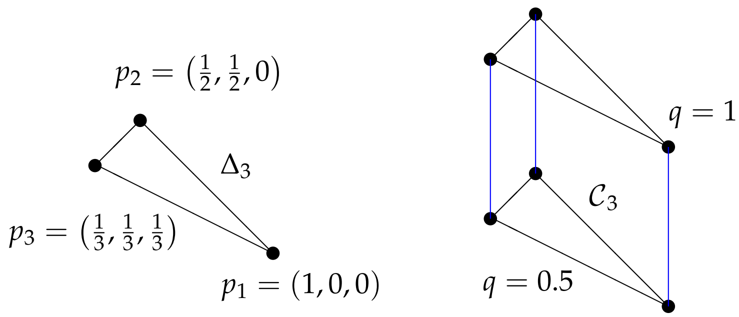


FIGURE 2. For 3 voters, Δ_3, the region of normalized weights, and $\mathcal{C}_3$, the configuration region of quotas and weights

For a weighted game w on m players induced from a game v on $n < m$ players, the polytope P_v is the projection of P_w under the natural projection $\mathcal{C}_m \to \mathcal{C}_n$. Thus the geometry of $\mathcal{C}_m$ completely determines the geometry of $\mathcal{C}_n$.

4.1. **Polytope structure.** Let us now describe how the polytopes in the configuration region $\mathcal{C}_n$ are formed. First, let w_A equal the sum of the weights of voters in coalition A. Consider the set of points in $\mathcal{C}_n$ where $q = w_A$, i.e., where A has precisely enough weight to win; the set of these points lies in a hyperplane h_A. Unless A is empty or equal to the grand coalition N, the hyperplane h_A intersects $\mathcal{C}_n$ in a codimension one subset that slants – its normal vector is neither horizontal nor vertical. Observe that h_N forms the top facet of $\mathcal{C}_n$ and $h_\emptyset$ the bottom facet; the latter is not actually contained in $\mathcal{C}_n$.

Remark 4.1. These hyperplanes respect the ordering on coalitions in $M(n)$; we have $A > B$ if and only if h_A lies strictly above h_B on the interior of $\mathcal{C}_n$. Equivalently, A and B are incomparable coalitions if and only if their hyperplanes intersect on the interior of $\mathcal{C}_n$.

Coalition A is winning at a realization $(q : \mathbf{w})$ if $q \leq w_A$, so we may visualize the points which have A winning as the closed subset X_A of $\mathcal{C}_n$ bounded above by hyperplane h_A. Similarly the points which have coalition B losing form the open subset $(X_B)^c = \mathcal{C}_n \setminus X_B$ which is bounded below by h_B. Thus, for a weighted game v, we may view its polytope P_v as the intersection of all 'winning subsets' such as X_A and all 'losing subsets' such as $(X_B)^c$:

$$P_v = \left(\bigcap_{A \in W_v} X_A \right) \bigcap \left(\bigcap_{B \notin W_v} (X_B)^c \right).$$

This formulation demonstrates that each polytope P_v is closed on top and open on bottom and is convex.

Proposition 4.2. *Each polytope P_v associated to a weighted voting system v is convex.*

We will be interested in what occurs by moving along a vertical line in $\mathcal{C}_n$ from a realization $Q = (q : \mathbf{w})$ in P_v. These motions are equivalent to changing the quota while fixing the weights.

- Moving upwards from Q (increasing the quota) guarantees that each losing coalition will remain losing. A coalition A winning at Q remains winning until after the line crosses the hyperplane h_A.
- Moving downwards from Q (decreasing the quota) guarantees that each winning coalition will remain winning. A coalition B losing at Q remains losing until the line intersects the hyperplane h_B.

Proposition 4.3. *Each polytope P_v is n-dimensional.*

Proof. Since v is weighted, there exists some representation $Q_0 = (q_0, \mathbf{w})$ for the game, so each polytope P_v includes at least one point. Move 'upwards' by fixing the weight vector $\mathbf{w}$ and increasing the quota until reaching the top boundary of P_v at some point $Q = (q_1, \mathbf{w})$; n.b., Q might equal Q_0.

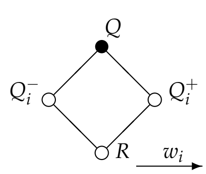
We first show that Q itself lies in the polytope P_v. Points in polytope P_v all satisfy the same inequalities: $q \leq w_A$ for any winning coalition A and $q > w_B$ for any losing coalition B. Moving upwards from Q_0, we first encounter the boundary of P_v at the lowest point where one or more of the winning inequalities becomes an equality $q = w_A$. The point Q satisfies the same inequalities as Q_0 does (if $Q_0 \neq Q$, all inequalities at Q_0 are strict, whereas if $Q_0 = Q$ then at least one at Q is weakly satisfied). Thus, Q is a realization of v.

Assume at Q there are k inequalities that are weakly satisfied, corresponding to coalitions $A_1, \ldots, A_k$, with weight equal to quota q_1. The remaining $2^n - k$ hyperplanes lie either above or below point Q; let δ be the minimum distance down to the next highest hyperplane(s) and let $R = (q_1 - \delta : \mathbf{w})$. As we travel downwards from Q, no coalition will ever change from winning to losing; only when we encounter the next hyperplane does some coalition(s) change from losing to winning. Hence, all points moving down from Q are in P_v until we reach a quota of $q_1 - \delta$ at R.

Recall that we are using coordinates $\{w_n, \ldots, w_2\}$ on Δ_n with $w_1 = 1 - w_n - \cdots - w_2$. Writing out the hyperplane equation $q = w_A$, we see that its slope in direction w_i, for $i > 1$, is one of $\{+1, 0, -1\}$:

- $+1$ if $\mathbf{i} \in A$, but $1 \notin A$,
- -1 if $i \notin A$, but $1 \in A$,
- 0 if both $\mathbf{i}$ and 1 are in A or neither is in A.

Thus we know that the interior of the diamond depicted below formed by points Q, R, and $Q_i^{\pm} = (q, w_n, \ldots, w_i \pm \delta/2, w_{i-1}, \ldots)$ lies in P_v.



Thus, the convexity of P_v implies that it contains the interior of the n-dimensional polytope spanned by points $Q, R, Q_2^{\pm}, \ldots, Q_n^{\pm}$. So we have shown P_v is n-dimensional. $\square$

The boundary of polytope P_v is comprised of three different types of facets. We count these in Theorem 4.13 using the poset $\mathcal{W}_n$ and the hierarchy of v.

(1) *top facets* – each is associated to a hyperplane h_A for some winning coalition A; its interior is contained in P_v;
(2) *bottom facets* – each is associated to a hyperplane h_A for some losing coalition A; it is disjoint from P_v;
(3) *vertical facets* – each lies above a codimension one subsimplex of Δ_n; its interior is contained in P_v.

Higher codimension elements of P_v are formed by the intersection of two or more facets. They are included in P_v if and only if no bottom facets are part of the intersection.

Now we turn our attention back to dual games. A game and its dual share many properties, including power compositions, weightedness, properness, and congruent polytope interiors.

Theorem 4.4. *The interior of P_v, the polytope associated to the weighted game v, is the reflection of the interior of P_{v^*} in $\mathcal{C}_n$ about the hyperplane $q = 0.5$.*

Proof. Let $(q : \mathbf{w})$ be a point in the interior of P_v. We prove that $(1 - q : \mathbf{w})$ lies in the interior of P_{v^*}. Consider a coalition A for game v. Then A is winning in v if and only if A^c is losing in v^*, so the weight of A satisfies $w_A > q$ if and only if $w_{A^c} < 1 - q$. (This establishes the well-known fact that v^* is weighted if and only if v is.) □

As an immediate corollary, we gain another characterization of self-dual weighted games, which we showed in Corollary 3.11 are precisely the ones that lie in the middle rank 2^{n-1} in $\mathcal{W}_n$.

Corollary 4.5. *The weighted game $v \in \mathcal{W}_n$ is self-dual (i.e., $v = v^*$) if and only if its polytope P_v is symmetric about the hyperplane $q = 0.5$.*

4.2. **Polytopes and hierarchies.** In this section, we describe how the hierarchy of voters in a weighted game v is related to the polytope P_v. We begin with several lemmas which will be useful in future proofs and then specify a one-to-one correspondence between $(k-1)$-dimensional subsimplices of Δ_n and compositions into k parts of natural numbers less than or equal to n. Let $\pi(P_v)$ be the vertical projection of the polytope P_v onto Δ_n, which we call the *footprint* of v. It is comprised of all weights that are part of some realization of v.

Lemma 4.6. *There exists a realization of v in which all voters in the same symmetry class have the same weight.*

Proof. Given a realization $(q : \mathbf{w})$ of v, our strategy is to replace the weight w_i by the average a_i of the weights of all voters in the same symmetry class $[i]$ as voter $\mathbf{i}$. We claim this operation preserves the set of winning coalitions W_v. Assuming the claim, we perform this operation for each symmetry class and thereby constructively prove the lemma.

Consider a coalition $B \in W_v$ which contains m voters from class $[i]$. By the definition of voter symmetry, B will remain winning if we replace these voters by the weakest m voters in class $[i]$. Furthermore, B will remain winning if we reassign weights to the voters in class $[i]$, so long as the new weights do not cause the total weight of the weakest m voters in $[i]$ to decrease. The new weights must also respect the ordering on voters. Replacing each weight by the average a_i accomplishes both conditions, and thus causes B to remain winning.

A similar argument shows that losing coalition C will remain losing if we average weights within a symmetry class. Thus, W_v is preserved by the averaging operation and the claim is proven. □

We note that we can average the weights of any two (or more) consecutive voters in the same symmetry class and still obtain a realization of v using the same quota.

Let σ be a $(k-1)$-dimensional subsimplex whose vertices are $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, where $i_1 < i_2 < \ldots < i_k$. Recall that $p_i = \left(\frac{1}{i}, \ldots, \frac{1}{i}, 0, \ldots, 0\right) \in \Delta_n$ has i nonzero coordinates. This implies that for any point $\mathbf{w}$ in σ, the first i_1 coordinates of $\mathbf{w}$ are equal, as are the next $i_2 - i_1$ coordinates, and so forth. Thus, the coordinates of any point $\mathbf{w}$ in σ can be grouped into k different classes of equal values; if $i_k \neq n$, there is one additional class of $n - i_k$ coordinates, which are all 0. Let $m_1 = i_1$ and $m_j = i_j - i_{j-1}$ for $j > 1$. We refer to the composition $\underline{m} := (m_1, m_2, \ldots, m_k)$ as the *composition associated to* σ. This establishes a bijection between $(k-1)$-dimensional subsimplices of Δ_n and compositions of $n - m_0$ into k parts. By appending m_0 to $\underline{m}$, we form the *extended composition associated to* σ, denoted $\underline{m}'$.

We observe that simplex $\sigma \subset \Delta_n$ contains simplex $\tau \subset \Delta_n$ if and only if composition $\underline{m}'{}_\sigma$ refines $\underline{m}'_\tau$.

The following theorem establishes a relation between polytopes and power compositions.

Theorem 4.7. *For weighted game v, let $\sigma \subset \Delta_n$ be the smallest dimensional subsimplex that intersects the footprint of v. The power composition of v is the composition associated to σ.*

The theorem implies that the power composition of a weighted game v can be obtained directly from its polytope P_v. While the converse is untrue, the power composition does tell us precisely which subsimplices of Δ_n intersect the polytope, namely those which contain σ.

To prove Theorem 4.7, we first need a definition and a lemma. We obtain the *extended power composition* $\overline{n}' = (n_1, n_2, \ldots, n_k, n_o)$ of n for v by appending n_0, the number of dummies in v, to the power composition.

Lemma 4.8. *Let v be a weighted game. A subsimplex $\tau \subset \Delta_n$ intersects the footprint of v if and only if the extended composition associated to τ is a refinement of the extended power composition $\overline{n}'$. Furthermore, if τ intersects the footprint, then the horizontal projection of polytope P_v onto the element $\epsilon_\tau := (0,1] \times \tau \subset \mathcal{C}_n$ is contained in P_v, i.e., it equals $P_v \cap \epsilon_\tau$.*

Proof. Let v be a weighted game on n voters and let τ be an arbitrary subsimplex of Δ_n (of any dimension) given by vertices $p_{i_1}, p_{i_2}, \ldots, p_{i_j}$. We first assume that τ intersects $\pi(P_v)$ and show that the extended composition $\underline{m}'$ associated to τ refines $\overline{n}'$. If τ intersects $\pi(P_v)$, then there is a realization $(q : \mathbf{w}) \in P_v$ such that $\mathbf{w} \in \tau$. In this realization, the first $m_1 := i_1$ voters have the same weights, the next $m_2 := i_2 - i_1$ voters have the same weights, and so forth. Voters with equal weights must lie in the same symmetry class. This means the first m_1 voters must lie in the same symmetry class in v, the next m_2 voters must lie in the same symmetry class, which is possibly the same symmetry class as the first m_1 voters, and so forth. Therefore the extended composition $\underline{m}' = (m_1, m_2, \ldots m_j, m_0)$ associated to τ refines the extended power composition.

Now we prove the reverse implication. We assume $\underline{m}'$ refines $\overline{n}'$ and show that τ must intersect $\pi(P_v)$. First, there exists a realization $Q = (q : \mathbf{w})$ of v for which all voters in the same symmetry class have the same weight by Lemma 4.6. Since the polytope is n-dimensional, we may travel a short distance away from Q (remaining inside P_v) along some vector which changes the weights of voters from different parts of $\underline{m}$ while fixing the weights of the voters in the same part of $\underline{m}$. We arrive at a point $Q' \in P_v$ lying above τ. Hence τ intersects the footprint of v, which establishes the first statement of the lemma.

Now we prove the second statement. From the proof of Lemma 4.6, we concluded that if we replace the weights of any number of consecutive voters in the same class by their average, we stay within P_v. Since the quota remains fixed, this operation corresponds to a horizontal motion within polytope P_v from an arbitrary realization of v to a point on some element ϵ_τ in the boundary of $\mathcal{C}_n$. Thus the horizontal projection of P_v onto ϵ_τ is already inside P_v. □

Proof of Theorem 4.7. We first prove that the statement of the theorem is well-defined, that is, that there exists a unique smallest-dimensional subsimplex $\sigma \subset \Delta_n$ which intersects the footprint of v. Consider two subsimplices σ_1 and σ_2 that both intersect $\pi(P_v)$. The vertices p_i which lie in both σ_1 and σ_2 determine the subsimplex $\sigma_1 \cap \sigma_2$; note that this intersection is necessarily nonempty, else all voters become dummies. Furthermore, the composition associated to $\sigma_1 \cap \sigma_2$ is the common refinement of the compositions associated to σ_1 and σ_2 and respects the symmetry classes of the voters in v. By Lemma 4.8, $\sigma_1 \cap \sigma_2$ must intersect $\pi(P_v)$ as well. Thus, there must exist a unique smallest subsimplex σ which intersects $\pi(P_v)$. Furthermore, σ is contained in every subsimplex that intersects $\pi(P_v)$.

From Lemma 4.8 we know that the extended composition $\underline{m}'$ associated to σ refines the extended power composition $\overline{n}'$ of v. We will prove that $\underline{m} = \overline{n}$. Note that $\overline{n}'$ is associated to some subsimplex τ. Lemma 4.8 implies that τ intersects P_v. Thus $\sigma \subseteq \tau$, which implies that the vertices of σ are a subset of the vertices of τ. Thus, the extended composition $\overline{n}'$ associated to τ refines the extended composition $\underline{m}'$ associated to σ. Therefore $\overline{n}'$ refines $\underline{m}'$ and $\underline{m}'$ refines $\overline{n}'$. So $\underline{m}' = \overline{n}'$ and the proof is complete. □

This theorem leads to several interesting results. Some concerning power distributions are described in our upcoming work [14]. Others are more immediate, such as the following result about symmetric games.

A *symmetric game* (or *collegium*) is one for which the winning coalitions are precisely those containing at least k out of the $n - n_0$ nondummy voters; it is symmetric in that each nondummy voter has the same role. For n voters, there are $\binom{n+1}{2}$ symmetric games, all weighted. Of these, $\frac{n^2+2n}{4}$ are proper if n is even, and $\frac{n^2+2n+1}{4}$ are proper if n is odd.

Corollary 4.9. *The only weighted systems that lie above the corners of Δ_n are the symmetric games. Above point p_j lie j different symmetric games: $\langle n-j+1\rangle$, $\langle (n-j+2)(n-j+1)\rangle, \ldots, \langle (n-1)\cdots(n-j+1)\rangle, \langle n(n-1)\cdots(n-j+1)\rangle$. Each, in order, occupies quotas of length $1/j$ above p_j.*

Example 4.10. Let us consider the 8 weighted games for $n = 3$ voters. Of these, 6 are symmetric games. Games $\langle 1\rangle, \langle 21\rangle, \langle 321\rangle$ each have power composition (3). These are the only games which lie above the 0-dimensional subsimplex $\{p_3\}$. Representations $(q : p_3)$ lie in $\langle 1\rangle$ for $q \in (0, 1/3]$, lie in $\langle 21\rangle$ for $q \in (1/3, 2/3]$, and lie in $\langle 321\rangle$ for $q \in (2/3, 1]$.

Systems $\langle 2\rangle$ and $\langle 32\rangle$ each have power composition (2). These are the only systems which lie above the 0-dimensional subsimplex $\{p_2\}$. Representations $(q : p_2)$ lie in $\langle 2\rangle$ for $q \in (0, 1/2]$ and lie in $\langle 32\rangle$ for $q \in (1/2, 1]$.

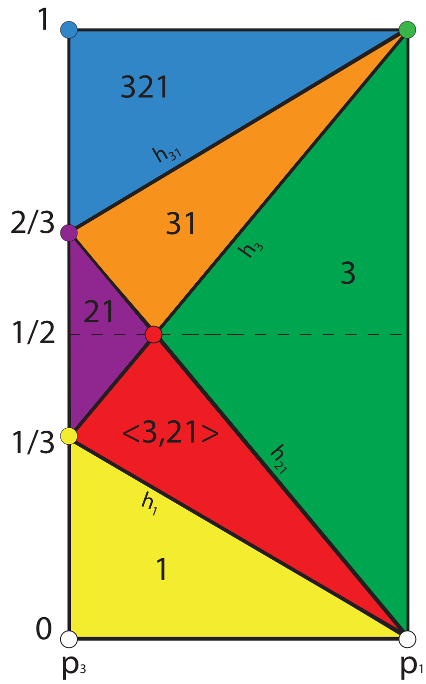


FIGURE 3. From Example 4.10, the face $\overline{p_1 p_3}$ of $\mathcal{C}_3$ is shown along with all polytopes which intersect it (labeled by their corresponding game). Notice (cf. Theorem 4.4) that the interiors of dual systems are reflected about the hyperplane $q = 1/2$.

The only game in any $\mathcal{C}_n$ that lies above point p_1 is the dictator system $\langle n \rangle$.

The simplest nonsymmetric games occur for 3 voters; they are $\langle 31 \rangle$ and its dual $\langle 3, 21 \rangle$. Each has power composition $(1, 2)$, which is the composition associated with the one-dimensional subsimplex $\overline{p_1 p_3} \subset \Delta_3$. We depict the face $\overline{p_1 p_3}$ of $\mathcal{C}_3$ and the polytopes which intersect it in Figure 3.

4.3. **A geometric view of weighted voting posets.** Our last results demonstrate that the geometric viewpoint of weighted games via their polytope is highly correlated to both hierarchies and the poset of weighted games. In this section we prove that the polytopes in $\mathcal{C}_n$ are situated according to poset $\mathcal{W}_n$. Furthermore, we show that the facets of P_v correspond to covering relations in $\mathcal{W}_n$ and to the hierarchy of voters in v.

Definition 4.11. A weight vector is called *generic* if the sums w_A are distinct for each coalition A.

Theorem 4.12. *Given a generic weight vector* $\mathbf{w}$, *consider the vertical line in* $\mathcal{C}_n$ *above* $\mathbf{w}$. *As the quota increases, the voting systems traversed form a saturated chain in* $\mathcal{W}_n$, *the poset of weighted voting systems. Moreover, the chains*

(1) are maximal: each one begins with system $\langle 1 \rangle$ (of unique minimal rank 1) and finishes with consensus rule $\langle N \rangle$ (of unique maximal rank $2^n - 1$);
(2) are self-dual: if system v is in the chain, so too is v^*.

We note that not every saturated chain corresponds to a vertical line segment above a generic point. Obstructions beyond self-duality exist, but these are not well understood. Every saturated chain does correspond to some piecewise-linear motion through polytopes, as we see in Corollary 4.14. In section 5, we discuss some approaches for studying saturated chains in posets $\mathcal{W}_n$ and J_n.

Proof. Given a generic weight vector $\mathbf{w}$, all representations $(q : \mathbf{w})$ lie in system $\langle 1 \rangle$ for $q \in (0, w_1]$. Similarly for $q \in (1 - w_1, 1]$, the representations lie in the consensus rule system.

Suppose we are at a representation Q in system v. Moving upwards from Q, we remain in v until we encounter the next lowest hyperplane h_A at $q = w_A$. As we cross this hyperplane, coalition A changes from winning to losing, which means we move into a system u with winning coalitions $W_u = W_v \setminus \{A\}$. This is precisely what it means to say that u covers v in $\mathcal{W}_n$.

Since the weight vector $\mathbf{w}$ is generic, we will never encounter two hyperplanes at once (else $q = w_A = w_B$). Thus, as we cross each of the $2^n - 1$ hyperplanes h_A corresponding to nonempty coalitions, one coalition switches from winning to losing and the rank increases by one, until finally we arrive in the consensus rule system. Thus our vertical line corresponds to a maximal saturated chain. The duality of the chain follows immediately from Theorem 4.4. $\square$

The hierarchy of v and its position within poset $\mathcal{W}_n$ determine which facets occur for polytope P_v.

Theorem 4.13. *Let v be a weighted game whose n voters form k nontrivial symmetry classes. Let d represent the degree of v in the (Hasse diagram of) poset $\mathcal{W}_n$. Then, its polytope has $n - k + d$ facets.*

1. *The top facets of polytope P_v are in one-to-one correspondence with the weighted games u_i that cover v in $\mathcal{W}_n$, unless $v = \langle N \rangle$, which has one top facet $\{q = 1\}$. The facet is a subset of hyperplane h_A, where A is the one coalition winning in v but not u_i. Coalition A is shift-minimal for v.*
2. *The bottom facets of polytope P_v are in one-to-one correspondence with the weighted games g_i that are covered by v in $\mathcal{W}_n$, unless $v = \langle 1 \rangle$, which has one bottom facet $\{q = 0\}$. The facet is a subset of hyperplane h_B, where B is the one coalition winning in g_i but not v. Set B is a shift-maximal losing coalition for v, i.e., it is a generator of the order ideal of losing coalitions for v in $M(n)$.*
3. *There exist $n - k$ vertical facets in polytope P_v. Each lies above a subsimplex of Δ_n given by $w_{i+1} = w_i$ or $w_1 = 0$.*

Proof. We begin with the first statement. The case when $v = \langle N \rangle$ is immediate, so assume $v \neq \langle N \rangle$. Let F be a top facet of P_v; we will describe the unique game corresponding to F. There is a hyperplane h_A which contains F; it is unique since hyperplanes arising from distinct coalitions have distinct normal vectors. Consider a point Q in the interior of F;

without loss of generality we may assume the weights in Q are generic. Moving vertically from below Q to above Q changes coalition A from winning to losing. If we travel upwards by a small enough amount so as to not cross any other hyperplane, we ensure that no other coalition changes its status. Thus the points immediately above Q lie in the game whose winning coalitions are $W_v \setminus \{A\}$; such a game covers v in $\mathcal{W}_n$.

Now suppose we have a weighted game u which covers v in $\mathcal{W}_n$, i.e., $W_u = W_v \setminus \{A\}$. The corresponding polytopes P_u and P_v lie on the same side of all other hyperplanes h_B ($B \neq A$). These polytopes are top-dimensional and distinct, so they must be separated by the hyperplane h_A. We are guaranteed that h_A intersects P_v in a facet by the existence of generic points in their intersection. Thus, the game u corresponds to a unique facet of P_v.

Observe that the second statement in the theorem is merely the corresponding restatement of the first one from the point of view of the greater coalition rather than the lesser coalition. The arguments above prove this statement as well.

Now we consider the third statement. The region of allowable weights Δ_n is an $(n-1)$-dimensional simplex, so it has n facets, which are given by equations $w_{i+1} = w_i$ ($1 \leq i \leq n-1$) and $w_1 = 0$. In proving Lemma 4.6 we concluded that polytope P_v contains points where any two consecutive voters $\mathbf{i}$ and $\mathbf{i+1}$ in the same symmetry class have equal weights. Further, these points may be chosen so that the weights are otherwise generic. Indeed we get an $(n-1)$-dimensional set of such points, all of which lie in the interior of the corresponding vertical facet of P_v.

Thus, if voters $\mathbf{i}$ and $\mathbf{i+1}$ lie in the same symmetry class, then P_v contains a vertical facet over $w_i = w_{i+1}$; the converse is clearly also true. Similarly, voter $\mathbf{1}$ is a dummy if and only if P_v contains points over $w_1 = 0$. For k different nontrivial symmetry classes, there are $n-k$ of these voters and hence $n-k$ vertical facets.

The degree d of v in $\mathcal{W}_n$ is equal to the number of covers of v plus the number of weighted games covered by v. Therefore d is equal to the number of top facets plus the number of bottom facets. These facets together with the $n-k$ vertical facets comprise the $n-k+d$ facets of the polytope P_v. □

This theorem tells us a great deal about how the structure of $\mathcal{W}_n$ arises in $\mathcal{C}_n$.

Corollary 4.14. *Every saturated chain of games in $\mathcal{W}_n$ may be achieved by some piecewise linear motion through $\mathcal{C}_n$.*

Remark 4.15. Since every n-dimensional polytope has at least $n+1$ facets, we may immediately conclude that the degree d of $v \in \mathcal{W}_n$ is greater than the number of symmetry classes k. When $d = k+1$, polytope P_v is a simplex. We note that all games of 4 or fewer voters have a simplex as their polytope; so too do 101 out of 117 games with 5 voters. The exceptions are the following 8 proper games and their duals: $\langle 541, 5321\rangle$, $\langle 541, 4321\rangle$, $\langle 541, 532, 4321\rangle$, $\langle 531, 4321\rangle$, $\langle 54, 531, 4321\rangle$, $\langle 521, 4321\rangle$, $\langle 54, 521, 4321\rangle$, $\langle 53, 521, 4321\rangle$. ◇

An n-dimensional polytope is *simple* if it has n facets meeting at every vertex, e.g., a cube is simple; an octahedron is not. Every simplex is simple. However, not all polytopes P_v are simple. As an example, the polytope for game $v = \langle 521, 4321\rangle$ has seven facets (two top, two bottom, three vertical); six of them meet at the vertex $\left(\frac{3}{5} : \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0\right)$.

4.4. **When unweighted games cover weighted games.** From Theorem 4.13, we obtain a geometric understanding of the distinction between weighted and unweighted games. After considering another corollary of this theorem, we detail a method for determining the weightedness of linear games which either cover or are covered by a weighted game in J_n.

Corollary 4.16. *If a weighted game v has k more shift-minimal winning coalitions than top facets in its polytope, then k of the games covering v in J_n are unweighted. Similarly, if v covers ℓ more games in J_n than it has bottom facets in its polytope, then ℓ of the covered games are unweighted.*

Proof. Theorem 4.13 states that every weighted game which covers a weighted game v corresponds to a unique top facet of P_v. Each cover of v in J_n is obtained by removing a shift-minimal winning coalition; see Proposition 3.12. Therefore, shift-minimal winning coalitions can be partitioned into two classes; those which correspond to a top facet and those which do not. The k shift-minimal winning coalitions which do not correspond to a top facet correspond to the k unweighted games which cover v.

Similarly, the shift-maximal losing coalitions are in one-to-one correspondence with games in J_n covered by v. Thus, they can be partitioned into two classes; those which correspond to a bottom facet and those which do not. Therefore the ℓ shift-maximal losing coalitions which do not correspond to a bottom facet correspond to the ℓ unweighted games covered by v. □

Geometrically, an unweighted game u covers a weighted game v in J_n precisely when one generator fails to be an active constraint in defining the polytope.

Theorem 4.17. *Let u be a linear game covering a weighted game v in J_n and assume $W_u = W_v \setminus \{A\}$. Then u is weighted if and only if there exists a weight $\mathbf{w}$ in the footprint $\pi(P_v)$ such that $w_A < w_B$ at $\mathbf{w}$ for all other winning coalitions B of v.*

Similarly, if a weighted game u covers a linear game v in J_n in such a way that $W_u = W_v \setminus A$, then v is weighted if and only if there exists a weight $\mathbf{w}$ in the footprint $\pi(P_u)$ such that $w_A > w_C$ at $\mathbf{w}$ for all other losing coalitions C of u.

We note that it suffices to check only the shift-minimal winning coalitions in the first statement and only the shift-maximal losing coalitions in the second.

Proof. Assume u is weighted. This means u covers v in $\mathcal{W}_n$, so by Theorem 4.13, hyperplane h_A forms a top facet of the polytope P_v. We claim that for any point $Q = (q : \mathbf{w})$ on the interior of this top facet, the weight w_A must be less than the weight of all other winning coalitions in v. Note that point Q is a realization of v. If $w_B < w_A$ at Q, where $q = w_A$, then the weight of B is less than the quota, so B is losing in v. If $w_B = w_A$ at Q, then the point Q is not actually on the interior of the facet; rather, Q lies on the face where P_v intersects $h_A \cap h_B$. Thus, the claim holds and we have proven one direction of the first statement.

To prove the other direction, assume there exists a weight $\mathbf{w} \in \pi(P_v)$ where $w_A < w_B$ for all other winning coalitions B of v. Since $\mathbf{w}$ lies under P_v, there exists some realization $(q : \mathbf{w})$ of v. Choosing a quota q' greater than w_A and less than the minimum of all weights w_B for $B \in W_v \setminus \{A\}$ guarantees that we have realized the game whose winning coalitions are precisely $W_v \setminus \{A\}$, namely game u. Thus we have shown u is weighted, which finishes the proof of the first statement.

The proof of the second statement in the theorem is directly analogous. □

Example 4.18. Consider the weighted game $v = \langle 987, 8741 \rangle$ mentioned in section 3.4. It has two generators, but if 987 becomes a losing coalition, then the unweighted game $u = \langle 8741 \rangle$ results. By Theorem 4.13, its polytope P_v has only one top facet. Though we know (by trade robustness) that $\langle 8741 \rangle$ is unweighted, we reprove it here using Theorem 4.17.

We argue that if the voters' weights are restricted to lie in the footprint of v, then the generators of v are actually comparable there: coalition 987 is stronger than 8741 among all weights in $\pi(P_v)$. The weights of these two coalitions are equal along the set $S = \{\mathbf{w} \mid w_9 = w_4 + w_1\}$. Our strategy is to show S is disjoint from $\pi(P_v)$. For any choice of weights $\mathbf{w}$ in S, coalitions 9752 and 75421 must have the same weight. The former is winning in v, while the latter is losing, so no weights from S can form a represention of v. Thus, 987 and 8741 are comparable above the footprint $\pi(P_v)$ of v; the representation $(22 : 9, 9, 9, 3, 3, 3, 1, 1, 1)$ of v shows that 987 is stronger than 8741 there.

Had we no *a priori* knowledge of the weightedness of $\langle 8741 \rangle$, observe that the preceding paragraph would be sufficient to determine that it is unweighted. ◇

We might ask, is Theorem 4.17 a useful method for determining weightedness? Quite possibly, for the relevant games. Understanding relevancy raises the question of determining how many games in J_n either cover or are covered by a weighted game. Though this footprint method still results in a linear programming (LP) problem, it is one that is different and possibly easier than the traditional LP problem of determining the existence of weights so that all generators are greater than all shift-maximal losing coalitions. One slight drawback is the reliance upon knowledge of the poset J_n; while outputting this poset is computationally infeasible for more than a small number of voters, obtaining local knowledge of how game v sits inside J_n is more straightforward. A challenge for future work is to efficiently implement this as an algorithm for computation.

5. Future directions

Our hope is that the combinatorial (poset) and geometric (polytope) approaches to linear games that we describe herein will lead to a greater understanding of linear games and weighted voting. Many natural questions remain to be answered about these structures, some of which we have already mentioned (e.g., Conjecture 3.6, saturated chains, implementing the method of Theorem 4.17).

One direction for further study is the connection between the geometry of $\mathcal{C}_n$ and power distributions for weighted games. In our upcoming paper [14], we define a geometrically-based, monotonic power index on all weighted games which has several useful properties.

Another avenue for further investigation is the classification of the maximal saturated chains. Every saturated chain γ in poset J_n produces an ordering on all shift-minimal winning coalitions for the games in γ. This is a linear ordering, except for the generators of the highest game g in γ; these are strictly greater than all other generators but incomparable to each other.

For example, consider the following chain in the linear games poset J_5^+:

$$\langle 54, 531 \rangle < \langle 54, 532 \rangle < \langle 541, 532 \rangle < \langle 532 \rangle < \langle 542, 5321 \rangle < \langle 543, 5321 \rangle.$$

This chain places the following linear ordering on the generators that it contains:

$$531 < 54 < 541 < 532 < 542 < 5321.$$

This ordering is inconsistent, as the first inequality implies $31 < 4$ whereas the last one implies $4 < 31$; no set of weights could possibly accomplish this. Thus, this particular chain does not correspond to a vertical line within the weighted voting polygon.

Definition 5.1. A chain γ in J_n is *inconsistent* if its ordering on shift-minimal winning coalitions reduces to an inconsistent set of inequalities. Otherwise, γ is *consistent*.

Determining the consistency of a saturated chain is a linear programming exercise. Let us note a necessary condition for consistency. Each poset $M(n)$ has 2^{n-k} copies of $M(k)$ ($k < n$) naturally embedded into it. Traveling up a maximal saturated chain γ in J_n imparts a total order on the coalitions in $M(n)$. For γ to be consistent, the order endowed on the coalitions in the first copy of $M(k)$ must be preserved in all other embedded copies of $M(k)$. As an example, if the saturated chain γ in J_5 declares $3 < 21$ in $M(3)$, then the following must hold: $43 < 421$, $53 < 521$, and $543 < 5421$. This is not a sufficient condition for consistency however.

We would like to understand which saturated chains in $\mathcal{W}_n$ correspond to vertical line segments in $\mathcal{C}_n$. Clearly, such a chain must be consistent and self-dual. We conjecture that this is also a sufficient condition.

Conjecture 5.2. *Every maximal saturated chain that is both consistent and self-dual is associated to some vertical line segment in $\mathcal{C}_n$ lying above a generic weight vector.*

In addition, among proper games, inconsistency first arises with 5 voters. For Π_4 (which equals J_4^+ and $\mathcal{W}_4^+$), there are 14 distinct maximal saturated chains, all of which are consistent and correspond to vertical line segments in $\mathcal{C}_4$.

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Appendix A. Unweighted linear games with 6 voters

Table 1 lists all unweighted linear games with 6 voters in J_6^+. The left-hand side lists 20 games with minimal rank 32 in J_6^+, while the right-hand-side lists 20 games with rank greater than 32. The remainder of the unweighted linear games with 6 voters are obtained by taking the duals of the unweighted linear games on the right-hand-side. We note that all 60 unweighted games in J_6 are improper.

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Rank 32 games	Higher rank games	Rank
621, 542	621, 543, 5421	33
621, 543, 5321	631, 542	33
63, 5421	632, 541	33
631, 541	64, 4321	33
631, 542, 5321	64, 543, 5321	33
632, 541, 5321	65, 542, 4321	33
64, 542, 5321	65, 621, 543	33
64, 543, 4321	65, 632, 5321	33
64, 621, 543	65, 632, 543, 4321	33
64, 631, 5321	65, 641, 543, 4321	33
64, 632, 4321	65, 641, 632, 4321	33
64, 632, 543, 5321	621, 543	34
641, 532	64, 5321	34
65, 621, 543, 5421	65, 632, 4321	34
65, 631, 4321	65, 641, 4321	34
65, 631, 542	65, 642, 543, 4321	34
65, 632, 541	65, 543, 4321	35
65, 632, 542, 4321	65, 642, 4321	35
65, 641, 542, 4321	65, 643, 4321	36
65, 641, 632, 543, 4321	65, 4321	37

TABLE 1. This table lists, in terms of their generators, 40 of the 60 unweighted linear games with 6 voters. The remaining 20 are the dual games of the ones in the second column.