SUPERPOSITION'S PROPERTIES OF LOGARITHMIC GENERATING FUNCTIONS

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Abstract

The paper discusses logarithmic generating functions and their properties. The theorem which is based on compositions of positive numbers and its conclusion are proved. Examples are given.

Key words: Logarithmic generating functions, superposition of generating functions, composition of positive number.

Introduction

Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory. One of the advantages of the generating function is that infinite number sequence can be represented in the form of single expression. Generating functions are divided into different classes: ordinary, exponential, Dirichlet, Poisson, etc. In this paper one more class is considered - logarithmic generating functions.

1 Logarithmic generating functions

Definition 1. Power series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} x^n, \tag{1}$$

where $a(n)$ – is integer sequence, is a logarithmic generating function.

Logarithmic generating function differs from an ordinary one because elements $a(n)$ divided by order number, i.e. $\frac{a(n)}{n}$, are used as power series coefficients. In many cases elements $a(n)$ equal 1 and numbers like $\frac{1}{n}$ are used as coefficients. One more difference is that there is no constant term.

2 Superposition of logarithmic generating functions

Superposition of logarithmic generating functions can be found in the same way as for ordinary generating functions.

Let there be functions $f(n)$, $r(n)$ and their generating functions $F(x) = \sum_{n\geq 1} f(n)x^n$, $R(x) = \sum_{n\geq 0} r(n)x^n$ accordingly. Generating function $Z(x)$) is a superposition of generating functions $F(x)$ and $R(x)$:

$$Z(x) = R(F(x)), \tag{2}$$

$$z(n) = \sum_{k=1}^{n} \sum_{\pi_k \in C_n} f(\lambda_1)f(\lambda_2)\dots f(\lambda_k)r(k) = \sum_{k=1}^{n} F^{\Delta}(n,k)r(k), \tag{3}$$

where $F^{\Delta}(n,k) = \sum_{\pi_k \in C_n} f(\lambda_1)f(\lambda_2)\dots f(\lambda_k)$ – is a compositae of generating function $F(x) = \sum_{n\geq 1} f(n)x^n$ [1].

Due to the source [1] estimation evaluation of $F^{\Delta}(n,k)$ is of paramount importance for obtaining the superposition of generating functions because formula (3) can be used for the calculation of superposition.

Superposition of logarithmic generating functions has several properties that distinguish it from the others. They are expressed in the following statements.

Statement 2.1. The value of the derivative superposition of logarithmic generating functions is integer for any n.

$$\acute{z}(n) = n \sum_{k=1}^{n} F^{\Delta}(n,k)r(k) = \sum_{k=1}^{n} \frac{n}{k} F^{\Delta}(n,k)a(k), \tag{4}$$

where $r(k)$ – coefficients of logarithmic generating function $R(x) = \sum_{n\geq 0} \frac{a(n)}{n} x^n$, $F(x)$ – generating function with integer coefficients.

Statement 2.2. The value of superposition of logarithmic generating functions without $n-th$ term for a prime integer n is integer. The converse is false.

$$z(n) = \sum_{k=1}^{n-1} F^{\Delta}(n,k)r(k) = \sum_{k=1}^{n-1} \frac{a(k)}{k} F^{\Delta}(n,k). \tag{5}$$

These properties are based on the following theorem and its corollary.

3 Theorem about the sum, which is based on the positive number's compositions

Theorem 2. *The sum*

$$\sum_{k=1}^{n} \frac{n}{k} \sum_{\substack{\lambda_i > 0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k = n}} a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_k} \tag{6}$$

is integer for any integer sequence $a_1, a_2, \dots, a_n$.

Proof. We prove the theorem in two ways.

1) Compositions of a positive number n with exactly k parts are the basis for formula (6). Consider the properties of these compositions. Let there be a multiset of positive integers $L = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, all greater than zero, and their sum equals n. Then, according to the source [2], the number of compositions for n is equal to

$$b(L) = \binom{k}{j_1, j_2, \ldots, j_m},$$

where j_i – number of equal λ_l in the multiset L.

Consider the following options.

1. Let us assume that $j_z = 1$, $z = \overline{1, m}$, and if $j_1 \neq 1$, then swap j_1 and j_z, and finally we get

$$\binom{k}{1}\binom{k-1}{j_2, \ldots, j_m}.$$

Therefore $b(L)$ is divisible by k.

2. Let us assume that n and k are relatively prime. Then we can find j_z such that k and j_z are relatively prime and $\binom{k}{j_z}$ is divisible by k. Hence $b(L)$ is divisible by k.

3. Let us assume that $NOD(n, k) > 1$. Then according to the 2nd option $b(L)$ is divisible by $k/NOD(n, k)$. Hence $nb(L)/k$ is integer.

The theorem is proved.

2) Let us construct a generating function $F(x) = a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$. Then

$$F^{\Delta}(n, k) = \sum_{\substack{\lambda_i > 0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k=n}} a_{\lambda_1}a_{\lambda_2}\ldots a_{\lambda_k}.$$

is the compositae of this generating function according to its definition.

Hence coefficients of superposition of generating functions $G(x) = \ln\left(\frac{1}{1-F(x)}\right)$ are given by

$$g(n) = \sum_{k=1}^{n} \frac{F^{\Delta}(n, k)}{k},$$

$$G(x) = \sum_{n>0} g(n)x^n.$$

If consider derivative $G'(x) = \left[\ln\left(\frac{1}{1-F(x)}\right)\right]'$ we can obtain the following expression

$$\left(\frac{F'(x)}{1-F(x)}\right) = g_1 + 2g_2x^1 + \ldots + ng_nx^{n-1} + \ldots$$

Consider the left part as product of generating functions $F'(x)$ and $\left(\frac{1}{1-F(x)}\right)$. Coefficients of $F'(x)$ are integers. Coefficients of superposition of generating functions $H(x) = \left(\frac{1}{1-F(x)}\right)$

are also integers by virtue of the fact that

$$h(n) = \sum_{k=1}^{n} F^{\Delta}(n,k).$$

Product of functions with integer coefficients also has integer coefficients. Hence the expression for the coefficients

$$ng(n) = n\sum_{k=1}^{n} \frac{F^{\Delta}(n,k)}{k}$$

are integers.

The theorem is proved. □

Consider some simple examples.

Example 3.1. Let a_n be prime integers and $a_1 = 1$. Then for $n = 6$ we have:

$$n\sum_{k=1}^{n} \frac{1}{k} \sum_{\substack{\lambda_i>0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k=n}} a_{\lambda_1}a_{\lambda_2}\ldots a_{\lambda_k} = 6\left(11 + \frac{14+20+9}{2} + \frac{15+36+8}{3} + \frac{12+24}{4} + \frac{10}{5} + \frac{1}{6}\right) = 380$$

Example 3.2. $a_i = 1$, $i = \overline{1,n}$ then

$$\sum_{\substack{\lambda_i>0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k=n}} a_{\lambda_1}a_{\lambda_2}\ldots a_{\lambda_k} = \binom{n-1}{k-1}.$$

because it accounts the number of n compositions that have k parts, $n-$ positive number. Hence

$$\sum_{k=1}^{n} \frac{n}{k}\binom{n-1}{k-1} = 2^n - 1.$$

4 Corollary of theorem 2

Corollary 3. *For any integer sequence $a_1, a_2, \ldots, a_n$, where n is a prime number, sum*

$$\sum_{k=1}^{n-1} \frac{1}{k} \sum_{\substack{\lambda_i>0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k=n}} a_{\lambda_1}a_{\lambda_2}\ldots a_{\lambda_k} \tag{7}$$

is integer.

Proof. Refer to proof 1) of theorem 2. By virtue of the fact that k and n are relatively prime for any k (with exception of $k = n$), then $b(L)$ is divisible by k. □

Hence the value of expression

$$\sum_{k=1}^{n-1} \frac{F^{\Delta}(n,k)}{k} \tag{8}$$

is integer for any n, that are prime numbers. The converse is false, i.e. if n is not prime, then proper value (8) may be either integer or not.

Consider example in the general case. Let us take for simplicity not large prime $n = 5$.

$$\sum_{k=1}^{n-1} \frac{1}{k} \sum_{\substack{\lambda_i>0 \\ \lambda_1+\lambda_2+\ldots+\lambda_k=n}} a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_k} =$$

$$= \left(a_5 + \frac{b(1,4)a_1a_4 + b(2,3)a_2a_3}{2} + \frac{b(1,1,3)a_1a_1a_3 + b(1,2,2)a_2a_2a_1}{3} + \frac{b(1,1,1,2)a_1a_1a_1a_2}{4} \right) =$$

$$= \left(a_5 + \frac{2a_1a_4 + 2a_2a_3}{2} + \frac{3a_1a_1a_3 + 3a_2a_2a_1}{3} + \frac{4a_1a_1a_1a_2}{4} \right) =$$

$$= \left(a_5 + a_1a_4 + a_2a_3 + a_1a_1a_3 + a_2a_2a_1 + a_1a_1a_1a_2 \right).$$

The result is integer because the sum of integer numbers' product is integer.

It is obvious from the above that corollary of the theorem is a special case of the statement 2.2, if $a_n = 1$. Statement 2.2 is also realized in the general case because divisibility comes about from the multinomial coefficient and doesn't depend on integer sequence a_n.

Consider some specific examples.

Example 4.1. Refer to the example above.

$$\sum_{k=1}^{n} \frac{n}{k} \binom{n-1}{k-1} = 2^n - 1.$$

Hence the value of expression

$$\sum_{k=1}^{n-1} \frac{1}{k} \binom{n-1}{k-1} = \frac{2^n - 2}{n}$$

is integer for prime n.

Example 4.2. Let us have a generating function $F(x) = x + x^2$ and its compositae,according to the source [1],$F^{\Delta}(n,k) = \binom{k}{n-k}$, then coefficients of superposition $\ln\left(\frac{1}{1-x-x^2}\right)$ are given by

$$g_n = \sum_{k=1}^{n} \binom{k}{n-k} \frac{1}{k},$$

$$ng_n = [1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571].$$

This formula generates Lukas numbers (A000032)[3]. Hence the value of expression

$$\frac{L_n - 1}{n}$$

is integer for prime numbers, where L_n - Lukas numbers.

$$L(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

or

$$L(n) = Fib(n) + 2Fib(n-1) = Fib(n+1) + Fib(n-1).$$

Example 4.3. Let us have the generating function for Catalan numbers $F(x) = \frac{1-\sqrt{1-4x}}{2x}$ and its compositae, due to the source [1], $F^{\Delta}(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$, then coefficients of superposition $\ln\left(\frac{1}{1-F(x))}\right)$ are given by

$$g_n = \sum_{k=1}^{n} \frac{k}{n}\binom{2n-k-1}{n-1}\frac{1}{k} = \frac{1}{n}\sum_{k=1}^{n}\binom{2n-k-1}{n-1},$$

$$ng_n = [1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378].$$

This formula induces sequence of integers A001700[3], wherefrom

$$ng_n = \binom{2n-1}{n-1}.$$

Hence value of expression

$$\frac{1}{n}\left(\binom{2n-1}{n-1} - 1\right)$$

is integer for prime numbers.

Conclusion

Based on Theorem 2 and its corollary some of the superposition's properties of the logarithmic generating functions are obtained. They are expressed in statements 2.1 and 2.2. This result allows us to construct algorithms which are based on superposition of the logarithmic generating functions for verification of the positive numbers' simplicity.

References

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