The Wigner $3n$-j Graphs up to 12 Vertices

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The 3-regular graphs representing sums over products of Wigner $3-jm$ symbols are drawn on up to 12 vertices (complete to $18j$-symbols), and the irreducible graphs on up to 14 vertices (complete to $21j$-symbols). The Lederer-Coxeter-Frucht notations of the Hamiltonian cycles in these graphs are tabulated to support search operations.

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I. WIGNER SYMBOLS AND CUBIC GRAPHS

A. Wigner Sums

We consider sums of the form

$$\sum_{m_{01},m_{02},\ldots} \begin{pmatrix} j_{01} & j_{02} & j_{03} \\ m_{01} & m_{02} & m_{03} \end{pmatrix} \begin{pmatrix} j_{01} & j_{..} & j_{..} \\ -m_{01} & m_{..} & m_{..} \end{pmatrix} \cdots \tag{1}$$

over products of Wigner $3jm$-symbols which are closed in the sense (i) that the sum is over all tupels of magnetic quantum numbers $m_{..}$ admitted by the standard spectroscopic multiplicity of the factors, (ii) that for each column designed by $j_{..}$ and $m_{..}$ another column with sign-reversed m appears in another factor [17, 39, 40].

Each term contains n factors—each factor a $3jm$-symbol—and $3n/2$ independent variables $j_{..}$ for which a pair of distinct indices in the interval 0 to $n-1$ will be used in this script. The numerical value of each factor, internal symmetries or selection rules are basically irrelevant for most of this work.

B. Yutsis Reduction

The Yutsis method maps the product structure of a Wigner $3n$-j symbol onto a labeled 3-regular (also known as cubic) digraph [3, 26, 31, 45]. Each factor is represented by a vertex. An edge is drawn between each pair of vertices which share one of the $j_{..}$; an edge *is* a j-value associated with a "bundle" of m-values. Since each factor comprises three $j_{..}$, the graph becomes 3-regular, i.e., in-degree and out-degree are both 3. The graphs are directed (i.e., digraphs) where head and tail of the edge denote which of the factors carries which of the two signs of the m-value. We shall enumerate vertices from 0 to $n-1$ further below; the two indices of the $j_{..}$ and its associated $m_{..}$ are just the two labels of the two vertices that are connected by the edge.

Once an undirected unlabeled connected graph is set up, adding a sign label and a direction to the edges (i.e., an order and sign of the three quantum numbers in the Wigner symbol) adds no information besides phase factors.

A related question is whether and which cuts through the edges exist that split any of these graphs into vertex-induced binary trees. The two trees generated by these means represent recoupling schemes [1, 4, 14, 19, 29, 41, 42]. The association generalizes the relation between Clebsch-Gordan coefficients (connection coefficients between sets of orthogonal polynomials [21, 27]) and the Wigner $3j$ symbols to higher numbers of coupled angular momenta.

C. Connectivity

The rules of splitting the sum (1) into sums of lower vertex count depend on the edge-connectivity of the cubic graph, i.e., the minimum number of edges that must be removed to cut the graph into at least two disconnected parts. Cubic graphs are at most 3-connected because removal of the three edges that run into any vertex turns that vertex into a singleton.

Wigner sums can be hierarchically decomposed for 1-connected, 2-connected and those 3-connected diagrams which are separated by cutting 3 lines into subgraphs with more than 1 vertex left [5, 45]. These will be plotted subsequently with one to three red edges to illustrate this property. Focus is therefore shifted to the remaining, "irreducible" graphs. Every cycle (closed path along a set of edges) in those consists of at least 4 edges, because a cycle of 3 edges can clearly be disconnected cutting the external 3 edges. All of their edges are kept black; they define "classes" of j-symbols [34, 45].

D. LCF notation

In the majority of our cases, simple cubic graphs are Hamiltonian, which means they support at least one Hamiltonian cycle, a closed path along the edges which visits each vertex exactly once and uses each edge at most once [7]. (See A001186 and A164919 in the Encyclopedia of Integer Sequences for a statistics of this feature [38].)

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The structure of the graph is in essence caught by arranging the vertices of such a walk on a circle—which uses already two third of the edges to complete the cycle—and then specifying which chords need to be drawn to account for the remaining one third of the edges. The chords are potentially crossing. Whether the graph is planar or not (i.e., whether it could be drawn on a flat sheet of paper without crossing lines) is not an issue.

The Lederberg-Coxeter-Frucht (LCF) notation is an ASCII representation of these chords (diagonals) in cubic Hamiltonian graphs [12, 20]. For each vertex visited, starting with the first, the distance to the vertex is noted where the chord originating there re-joins the cycle. The distance is an integer counting after how many additional steps along the cycle that opposite vertex of the chord will be visited, positive for a forward direction along the cycle, negative for a backward direction. The direction is chosen to minimize the absolute value of this distance, and to use the positive value if there is a draw. This generates a comma-separated list of n integers in the half-open interval $(-n/2, n/2]$, where n is the number of vertices. The values 0 or ± 1 do not appear because we are considering only simple graphs (loopless, without multiple edges).

Because the choice of the starting vertex of a Hamiltonian cycle is arbitrary, and because one may reverse the walking direction, two LCF strings may be trivially equivalent in two ways: (i) a cyclic permutation or (ii) reverting the order while flipping all signs (unless the entry is $n/2$) is an irrelevant modification.

There are two notational contractions that are accompanied by some symmetry of the graph:

- If the vector of n distances is a repeated block of numbers of the form $[a, b, c, \ldots x, a, b, c, \ldots x, \ldots]$, the group is written down once with an exponent counting the frequency of occurrences, $[a, b, c, \ldots x]^f$.
- If the distance vector has an inverted palindromic symmetry of the form $[a, b, c, \ldots x, -x, \ldots - c, -b, -a]$, the repeated part is replaced by a semicolon and dash $[a, b, c, \ldots x; -]$.

If more than one Hamiltonian cycle exist in the graph, non-trivial but equivalent LCF notations appear. In the following chapters, lines

```
LCF ... = ...
```

with one or more equal signs signal graphs which support more than one cycle.

The structure of the graph may also be visualized as a carbon or silicate molecule with some graphic viewers if this information is encoded as a SMILES string [43]. The Hamiltonian cycle defines the backbone of a ring, and the cords are enumerated and serve as indices to the atoms to recover the missing bonds.

The Wiener index of the undirected graph (sum of the distances of unordered pairs of vertices) will be reported

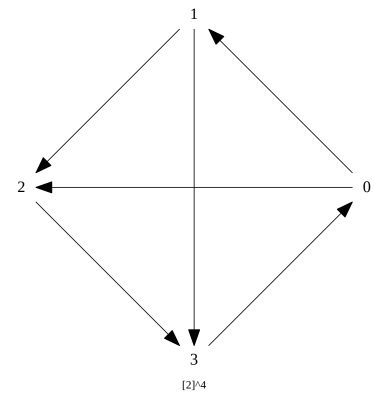


FIG. 1. The graph on $n = 4$ vertices, defining the $6j$-symbol.

as an integer number attached to a `W` [22]. The diameter of the undirected graph (largest distance between any two vertices) is written down attached to a `d`, and the girth of the undirected graph (length of the shortest cycle) is attached to a `g`. Finally, the Estrada index (sum of the exponentials of the eigenvalue spectrum of the adjacency matrix) follows after a `EE` [23]. (These numbers are rounded to 10^{-5}, the minimum precision to generate unique indices for the graphs on 14 nodes.)

II. 4 AND 6 VERTICES

The main part of the manuscript shows the nonequivalent (up to a permutation of the vertex labels) simple cubic graphs, sorted along increasing number of vertices and increasing edge-connectivity.

The labels are an indication of at least one Hamiltonian cycle through the graphs where one was found. In the applications, the labels are replaced by the two sign labels of the node's orientation, i.e., basically a phase label which relates to the ordering of the j-symbols in the Wigner $3jm$-symbol at that vertex [6, 28].

The directions of the edges are an almost arbitrary choice as well, pointing from the vertex labeled with the lower number to the vertex labeled with the higher number.

On 4 vertices we find the planar version of a tetrahedron, Figure 1.

6 vertices support the two graphs in Figure 2. Their LCF notations are:

```
LCF [3,-2,2]^2  W21 d2 g3 EE25.07449
LCF [3]^6  W21 d2 g4 EE24.13532
```

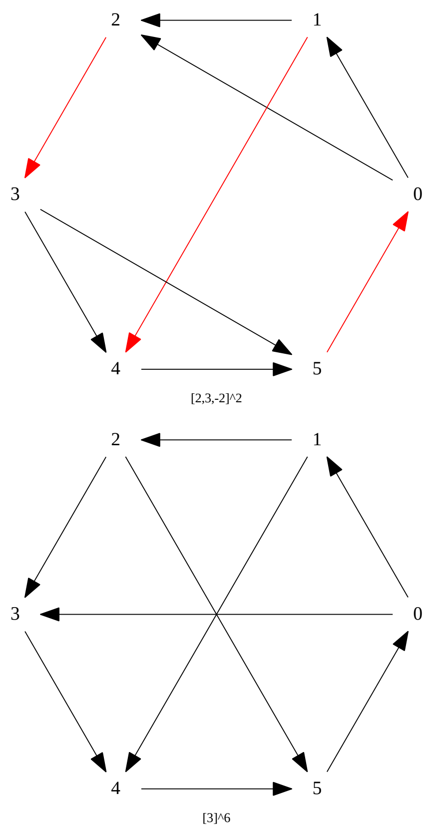


FIG. 2. The 2 graphs on $n = 6$ vertices. `[3]^6` (called the utility graph if undirected, unlabeled) defines the $9j$-symbol [25].

III. 8 VERTICES

All 5 cubic graphs on 8 vertices are shown in Figure 3. Their representations by LCF strings are:

```
LCF [2,-2,-2,2]^2  W50 d3 g3 EE33.73868
LCF [2,3,-2,3;-]  = [4,-2,4,2]^2 W46 d3 g3 EE30.97135
LCF [3,3,4,-3,-3,2,4,-2]  W44 d2 g3 EE30.03607
LCF [-3,3]^4  W48 d3 g4 EE29.39381
LCF [4]^8  = [4,-3,3,4]^2 W44 d2 g4 EE29.09522
```

The two Hamiltonian cycles indicated by the first two LCF representations for the graph `[2,3,-2,3;-]` in Figure 3 are: Walking along the vertices labeled $0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 0$ generates the LCF name `[2,3,-2,3;-]`. The alternative Hamiltonian cycle $0 \to 2 \to 3 \to 6 \to 7 \to 5 \to 4 \to 1 \to 0$ is described by the name `[4,-2,4,2]^2`.

The last graph in Figure 3 is another example hosting two cycles, equivalent to switching between Figures 19.1a and 19.1b in the Yutsis–Levinson–Vanagas book [45]: The notation `[4]^8` describes a Hamiltonian Path along the vertices $0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 0$. The alternative `[4,-3,3,4]^2` corresponds to the path $7 \to 0 \to 1 \to 2 \to 6 \to 5 \to 4 \to 3 \to 7$.

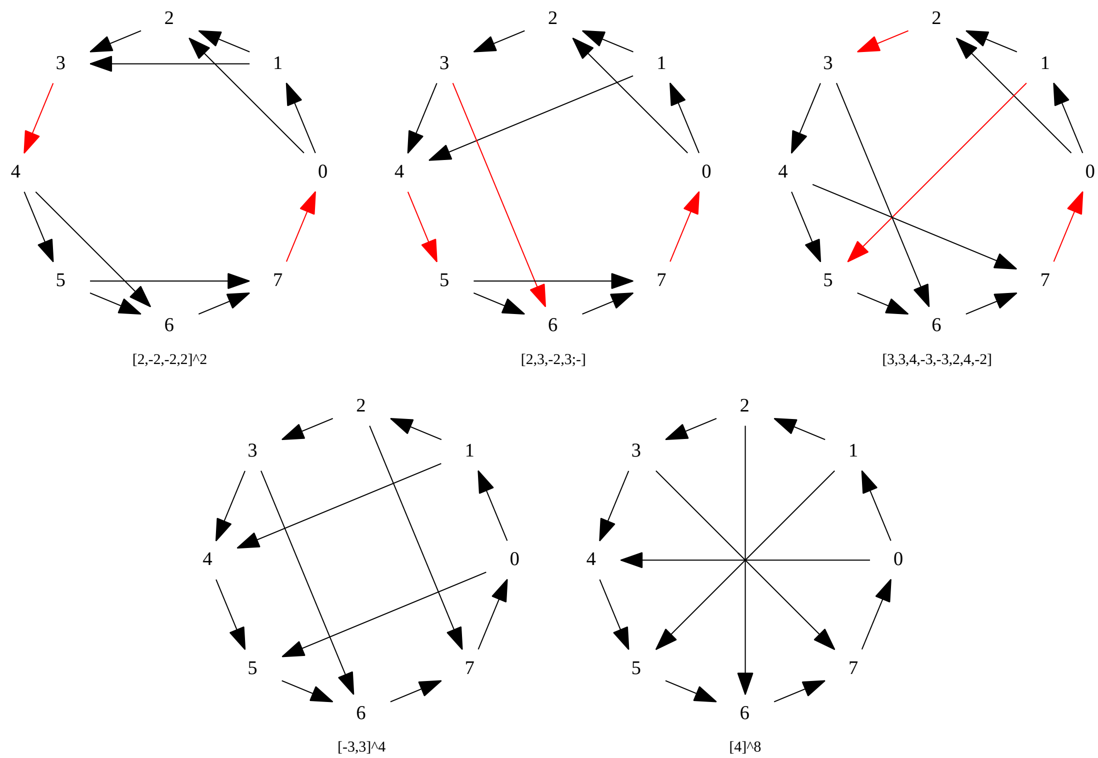


FIG. 3. Graphs on $n = 8$ vertices. The two which are cyclically 4-connected define the two $12j$-symbols [2, 35]. The undirected, unlabeled version of `[-3,3]^4` is the cubical graph.

IV. 10 VERTICES

The 19 graphs with 10 vertices (15 edges) are shown in Figure 4 if 1- or 2-connected, Figure 5 if 3-connected reducible, and Figure 6 if irreducible [24].

Two of the 19 graphs, one in Figure 4 (`W111 d5 g3 EE42.60094`) and one in Figure 6 (`W75 d2 g5 EE34.21829`), are not Hamiltonian, so we are left with 17 lines of LCF strings of their Hamiltonian cycles:

Figure 4:

```
LCF [3,-2,-4,-3,2,2,-2,-2,4,2]  W91 d4 g3 EE39.41746
LCF [-2,-2,3,3,3;-]  W90 d3 g3 EE38.90980
LCF [2,-3,-2,2,2;-]  W90 d3 g3 EE40.39508
LCF [-2,5,2,2,-2]^2  W93 d4 g3 EE40.69426
```

Figure 5:

```
LCF [3,-2,5,-3,2]^2  = [3,-2,4,-3,4,2,-4,-2,-4,2] W85 d3 g3 EE37.44960
LCF [-3,5,2,5,-2,4,5,3,5,-4]  = [-4,2,5,-2,4,4,4,5,-4,-4] = [-3,2,4,-2,4,4,-4,3,-4,-4] W82 d3 g3 EE36.00
LCF [-4,3,3,5,-3,-3,4,2,5,-2]  = [3,-4,-3,-3,2,3,-2,4,-3,3] W85 d3 g3 EE36.68162
LCF [3,-3,5,-3,2,4,-2,5,3,-4]  W84 d3 g3 EE36.25442
LCF [-4,-2,5,2,4,-2,4,5,-4,2]  W81 d3 g3 EE36.77120
LCF [2,3,-2,3,-3;-]  = [-4,4,2,5,-2]^2 W87 d3 g3 EE37.69671
LCF [4,-2,5,2,-4,-2,2,5,-2,2]  W84 d3 g3 EE38.01880
LCF [2,4,-2,3,4;-]  = [2,5,-2,5,5]^2 W83 d3 g3 EE37.01785
LCF [-3,3,3,5,-3]^2  W85 d3 g4 EE35.83204
```

Figure 6:

```
LCF [5,-4,4,-4,4]^2  = [5,-4,-3,3,4,5,-3,4,-4,3] W79 d3 g4 EE34.72233
LCF [5,5,-4,4,5]^2  = [-3,4,-3,3,4;-] = [4,-3,4,4,-4;-] = [-4,3,5,5,-3,4,4,5,5,-4] W81 d3 g4 EE34.97449
LCF [5]^10  = [-3,3]^5 = [5,5,-3,5,3]^2 W85 d3 g4 EE35.40679
LCF [3,-4,4,-3,5]^2  W85 d3 g4 EE35.47908
```

In terms of the standard nomenclature

- `[5]^10` is the $15j$-coefficient of the first kind (the Möbius ladder graph for that vertex count),
- `[3,-4,4,-3,5]` is the second kind,
- `[5,-4,4,-4,4]^2` the third,
- `[5,5,-4,4,5]^2` the fourth
- and the Petersen Graph (which has no Hamiltonian cycle [11, 37]) the fifth [45].

The number of classes of $3n-j$ symbols for even $n = 4, 6, 8, \ldots$ grows as 1, 1, 2, 5, 18, 84, 607, 6100, 78824, 1195280, 20297600, 376940415,... [10]. (An apparently erroneous 576 is sometimes quoted instead of 607 [16, 44]).

The volume of such lists grows with the number of vertices, which leads to the main objective of this work. Starting from a Wigner product of the form (1), its cubic graph is quickly drawn, but whether the graph is the same as (in our geometric mathematical framework isomorphic to) another one needs a kind of signature or classification. One might build a frequency statistics of the number of shortest cycles in the spirit of finding faces of the the polytope of a 3-dimensional ball-and-stick model of the graph, or count the number of cut sets and compare these.

Another approach is supported here: find at least one Hamiltonian cycle, generate the LCF string, and use a reverse lookup in the LCF table to see whether any two strings occur in the same line. The common idea is to replace strenuous visual recognition of graphs by a comparison of ASCII representations.

The ancillary files contain the source code of a small Java program which supports the detection of Hamiltonian cycles. Its input is an edge list of a simple cubic Hamiltonian graph. The cycles are computed by walking from the first node of the first edge in all three directions and generating a tree of non-interfering walks recursively [30]. The output is a LCF string and a vertex chain along each cycle found, and optionally a representation in `dot` format which can be plotted by the `graphviz` commands.

The ancilliary files contain also cage-type graphs detailed as sets of `gnuplot` commands and molfiles [13]. These graphs can be rotated interactively which helps to decipher the cycle structure and symmetries.

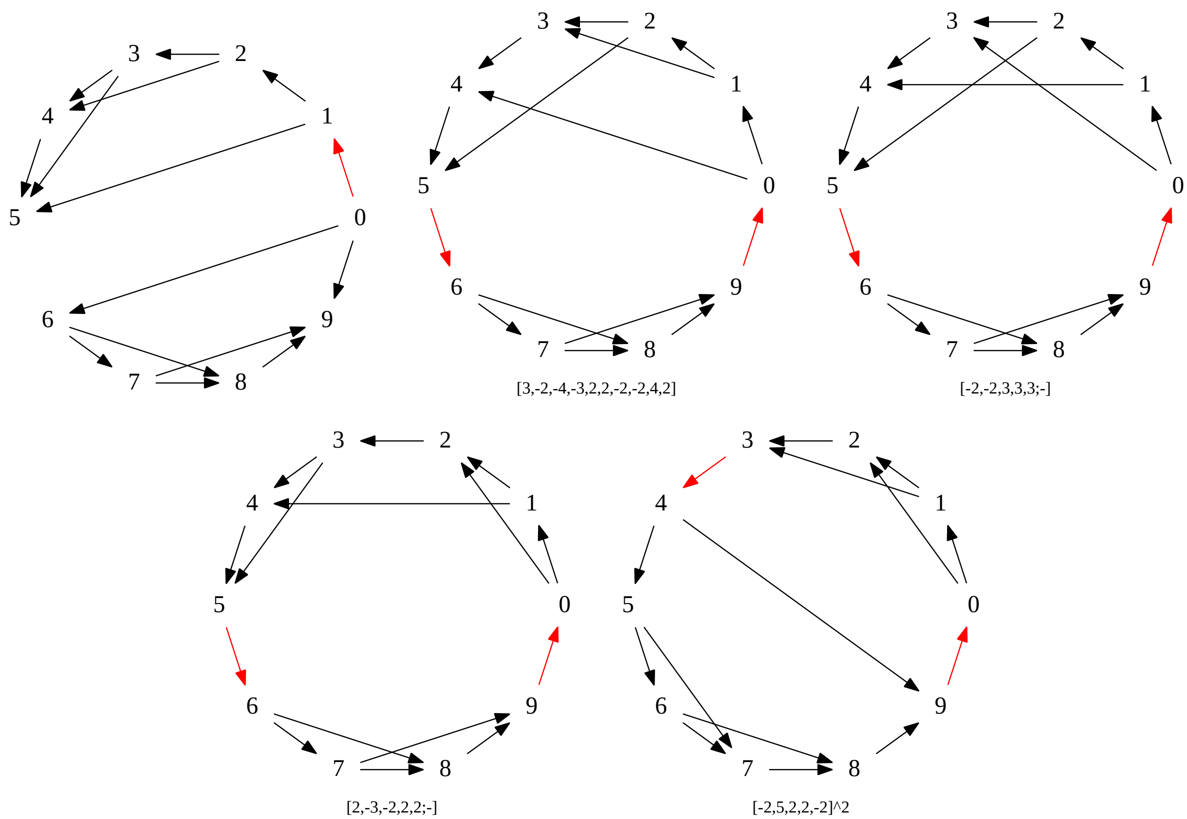


FIG. 4. Graphs on $n = 10$ vertices which disconnect on 1 or 2 edges.

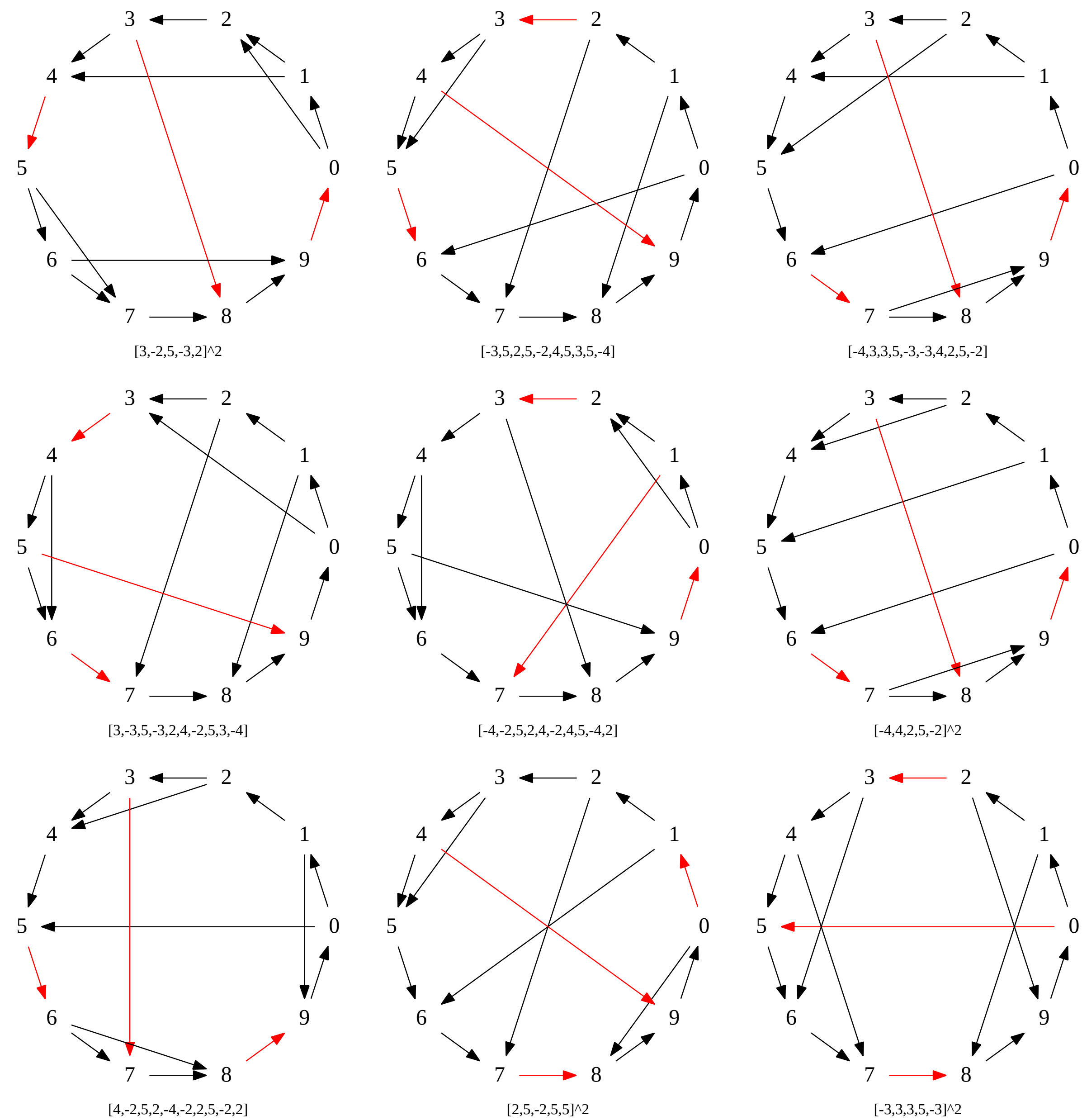


FIG. 5. The 3-connected reducible graphs on $n = 10$ vertices.

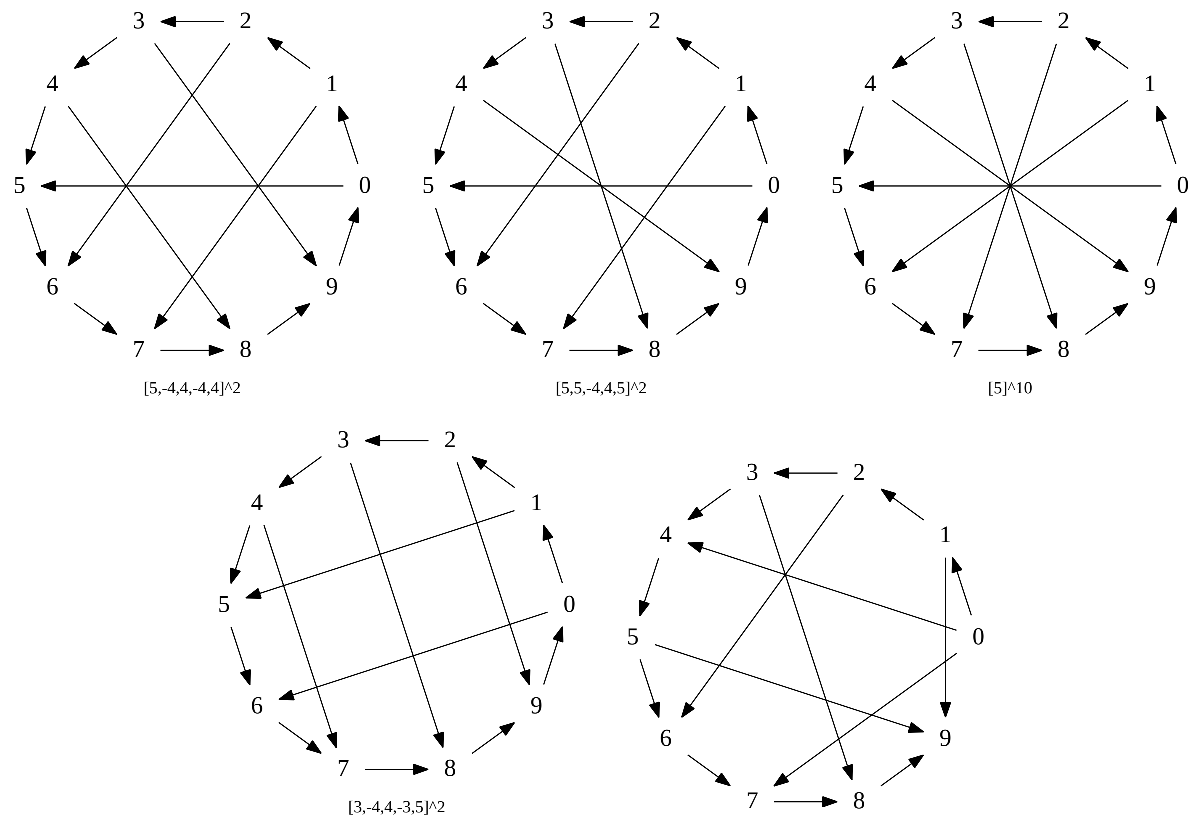


FIG. 6. Graphs on $n = 10$ vertices which define the $15j$-symbols. Four are cyclically 4-connected, one is cyclically 5-connected. The one without a LCF name is the Petersen graph.

V. 12 VERTICES

The 85 graphs with 12 vertices (18 edges) are 1-connected (Figure 7), 2-connected (Figures 8–9) 3-connected reducible (Figures 10–13) and cyclically 4- or 5-connected (Figures 14–15).

The four graphs in Figure 7 and one graph in Figure 13 are not Hamiltonian, which leaves us with 80 lines of LCF notations:

Figure 8:

```
   LCF [3,-2,-4,-3,4,2]^2  = [4,2,3,-2,-4,-3;-] W150 d5 g3 EE45.12486
   LCF [3,-2,-4,-3,3,3,3,-3,-3,-3,4,2]  W149 d4 g3 EE44.63116
   LCF [-3,2,3,-2,2,-3,-2,4,2,3,-2,-4]  W149 d4 g3 EE46.12066
   LCF [3,3,-3,-3,-3,3]^2  W152 d4 g4 EE44.14446
   LCF [2,-3,-2,3,3,3;-]  W152 d4 g3 EE45.63732
   LCF [3,-2,2,-3,-2,2]^2  W152 d4 g3 EE47.13249
   LCF [-2,3,6,3,-3,2,-3,-2,6,2,2,-2]  = [4,2,-4,-2,-4,6,2,2,-2,-2,4,6] W149 d4
g3 EE45.89062
   LCF [3,4,-3,-3,6,-4,2,2,-2,-2,6,3]  W146 d4 g3 EE44.94265
   LCF [3,-2,-4,-3,5,2,2,-2,-2,-5,4,2]  W154 d5 g3 EE46.30261
   LCF [-3,-3,-3,5,2,2;-]  W153 d4 g3 EE45.76519
   LCF [2,-3,-2,5,2,2;-]  W153 d4 g3 EE47.22986
   LCF [2,4,-2,3,-5,-4,-3,2,2,-2,-2,5]  = [5,2,-4,-2,-5,-5,2,2,-2,-2,4,5] W143
d4 g3 EE45.58501
```

Figure 9:

```
   LCF [-2,-2,4,4,4,4;-]  = [3,-4,-4,-3,2,2;-] =
[5,3,4,4,-3,-5,-4,-4,2,2,-2,-2] W145 d4 g3 EE44.90052
   LCF [4,-2,4,2,-4,-2,-4,2,2,-2,-2,2]  = [5,-2,2,3,-2,-5,-3,2,2,-2,-2,2] W148
d4 g3 EE46.95537
   LCF [2,2,-2,-2,-5,5]^2  W160 d5 g3 EE47.72073
   LCF [-2,-2,4,5,3,4;-]  W141 d4 g3 EE44.63910
   LCF [5,2,-3,-2,6,-5,2,2,-2,-2,6,3]  W146 d4 g3 EE45.63214
   LCF [4,-2,3,3,-4,-3,-3,2,2,-2,-2,2]  W150 d4 g3 EE46.28096
   LCF [-2,-2,5,3,5,3;-]  = [-2,-2,3,5,3,-3;-] W147 d4 g3 EE45.05416
   LCF [2,2,-2,-2,6,6]^2  W158 d5 g3 EE47.35563
   LCF [-3,2,-3,-2,2,2;-]  W152 d4 g3 EE47.39504
   LCF [-2,-2,5,2,5,-2;-]  W143 d4 g3 EE46.51523
   LCF [6,-2,2,2,-2,-2,6,2,2,-2,-2,2]  W153 d4 g3 EE48.40271
   LCF [-2,2,2,-2]^3  W162 d4 g3 EE50.42874
```

Figure 10:

```
   LCF [2,3,-2,3,-3,3;-]  = [-4,6,4,2,6,-2]^2 W144 d4 g3 EE44.66589
   LCF [-4,6,3,3,6,-3,-3,6,4,2,6,-2]  = [-2,3,-3,4,-3,3,3,-4,-3,-3,2,3] W140 d4
g3 EE43.61888
   LCF [-5,2,-3,-2,6,4,2,5,-2,-4,6,3]  = [-2,3,-3,4,-3,4,2,-4,-2,-4,2,3] =
[3,-2,3,-3,5,-3,2,3,-2,-5,-3,2] W142 d4 g3 EE44.32053
   LCF [-5,-5,4,2,6,-2,-4,5,5,2,6,-2]  = [4,-2,3,4,-4,-3,3,-4,2,-3,-2,2] W136
d3 g3 EE44.01162
   LCF [-5,-5,3,3,6,-3,-3,5,5,2,6,-2]  = [2,4,-2,3,5,-4,-3,3,3,-5,-3,-3] W136
d3 g3 EE43.11500
   LCF [2,4,-2,3,6,-4,-3,2,3,-2,6,-3]  = [2,4,-2,3,5,-4,-3,4,2,-5,-2,-4] =
[-5,2,-3,-2,5,5,2,5,-2,-5,-5,3] W138 d4 g3 EE43.87324
   LCF [-5,2,-3,-2,6,3,3,5,-3,-3,6,3]  = [4,-2,-4,4,-4,3,3,-4,-3,-3,4,2] =
[-3,3,3,4,-3,-3,5,-4,2,3,-2,-5] W139 d4 g3 EE43.30141
   LCF [2,3,-2,4,-3,6,3,-4,2,-3,-2,6]  = [-4,5,-4,2,3,-2,-5,-3,4,2,4,-2] W139
d4 g3 EE44.05952
   LCF [6,3,-4,-4,-3,3,6,2,-3,-2,4,4]  = [-5,-4,4,2,6,-2,-4,5,3,4,6,-3] =
[3,4,4,-3,4,-4,-4,3,-4,2,-3,-2] = [4,5,-4,-4,-4,3,-5,2,-3,-2,4,4] =
```

[4,5,-3,-5,-4,3,-5,2,-3,-2,5,3] W136 d4 g3 EE42.91096
LCF [4,6,-4,-4,-4,3,3,6,-3,-3,4,4] = [-5,-4,3,3,6,-3,-3,5,3,4,6,-3] = [4,-3,5,-4,-4,3,3,-5,-3,-3,3,4] W135 d3 g4 EE42.08576
LCF [3,3,4,-3,-3,4;-] = [3,6,-3,-3,6,3]^2 W136 d3 g4 EE42.58760
LCF [4,-2,5,2,-4,-2,3,-5,2,-3,-2,2] = [5,-2,2,4,-2,-5,3,-4,2,-3,-2,2] = [2,-5,-2,-4,2,5,-2,2,5,-2,-5,4] W139 d4 g3 EE44.95991

Figure 11:

LCF [-2,6,2,-4,-2,3,3,6,-3,-3,2,4] = [-2,2,5,-2,-5,3,3,-5,-3,-3,2,5] W139 d4 g3 EE44.12975
LCF [2,4,-2,6,2,-4,-2,4,2,6,-2,-4] = [2,5,-2,2,6,-2,-5,2,3,-2,6,-3] W139 d4 g3 EE44.87532
LCF [6,3,-3,-5,-3,3,6,2,-3,-2,5,3] = [3,5,3,-3,4,-3,-5,3,-4,2,-3,-2] = [-5,-3,4,2,5,-2,-4,5,3,-5,3,-3] W140 d4 g3 EE43.12097
LCF [3,-3,5,-3,-5,3,3,-5,-3,-3,3,5] W142 d4 g4 EE42.31141
LCF [4,2,4,-2,-4,4;-] = [3,5,2,-3,-2,5;-] = [6,2,-3,-2,6,3]^2 W141 d4 g3 EE44.00528
LCF [3,6,4,-3,6,3,-4,6,-3,2,6,-2] = [4,-4,5,3,-4,6,-3,-5,2,4,-2,6] = [-5,5,3,-5,4,-3,-5,5,-4,2,5,-2] W137 d4 g3 EE42.72638
LCF [6,-5,2,6,-2,6,6,3,5,6,-3,6] = [6,2,-5,-2,4,6,6,3,-4,5,-3,6] = [5,5,6,4,6,-5,-5,-4,6,2,6,-2] = [-4,4,-3,3,6,-4,-3,2,4,-2,6,3] = [6,2,-4,-2,4,4,6,4,-4,-4,4,-4] = [-3,2,5,-2,-5,3,4,-5,-3,3,-4,5] = [-5,2,-4,-2,4,4,5,5,-4,-4,4,-5] W133 d3 g3 EE42.37675
LCF [2,6,-2,5,6,4,5,6,-5,-4,6,-5] = [5,6,-4,-4,5,-5,2,6,-2,-5,4,4] = [2,4,-2,-5,4,-4,3,4,-4,-3,5,-4] = [2,-5,-2,4,-5,4,4,-4,5,-4,-4,5] W131 d3 g3 EE42.19745
LCF [2,4,-2,-5,5,-4]^2 = [-5,2,4,-2,6,3,-4,5,-3,2,6,-2] W135 d4 g3 EE43.48153
LCF [-4,-4,4,2,6,-2,-4,4,4,4,6,-4] = [-4,-3,4,2,5,-2,-4,4,4,-5,3,-4] = [-3,5,3,4,-5,-3,-5,-4,2,3,-2,5] W137 d4 g3 EE42.85630
LCF [2,5,-2,4,4,5;-] = [2,4,-2,4,4,-4;-] = [-5,5,6,2,6,-2]^2 = [5,-2,4,6,3,-5,-4,-3,2,6,-2,2] W134 d3 g3 EE43.48061
LCF [3,6,-4,-3,5,6,2,6,-2,-5,4,6] = [2,-5,-2,4,5,6,4,-4,5,-5,-4,6] = [5,-4,4,-4,3,-5,-4,-3,2,4,-2,4] W131 d3 g3 EE42.11275

Figure 12:

LCF [6,-5,2,4,-2,5,6,-4,5,2,-5,-2] = [-2,4,5,6,-5,-4,2,-5,-2,6,2,5] = [5,-2,4,-5,4,-5,-4,2,-4,-2,5,2] W133 d4 g3 EE43.16541
LCF [2,-5,-2,6,3,6,4,-3,5,6,-4,6] = [6,3,-3,4,-3,4,6,-4,2,-4,-2,3] = [5,-4,6,-4,2,-5,-2,3,6,4,-3,4] = [5,-3,5,6,2,-5,-2,-5,3,6,3,-3] = [-5,2,-5,-2,6,3,5,5,-3,5,6,-5] = [-3,4,5,-5,-5,-4,2,-5,-2,3,5,5] = [5,5,5,-5,4,-5,-5,-5,-4,2,5,-2] W134 d4 g3 EE42.32276
LCF [5,-3,6,3,-5,-5,-3,2,6,-2,3,5] = [2,6,-2,-5,5,3,5,6,-3,-5,5,-5] = [5,5,5,6,-5,-5,-5,-5,2,6,-2,5] = [4,-3,5,2,-4,-2,3,-5,3,-3,3,-3] = [5,5,-3,-5,4,-5,-5,2,-4,-2,5,3] W135 d3 g3 EE42.67156
LCF [2,4,-2,5,3,-4;-] = [5,-3,2,5,-2,-5;-] = [3,6,3,-3,6,-3,2,6,-2,2,6,-2] W138 d4 g3 EE43.74286
LCF [6,2,-4,-2,-5,3,6,2,-3,-2,4,5] = [2,3,-2,4,-3,4,5,-4,2,-4,-2,-5] = [-5,2,-4,-2,-5,4,2,5,-2,-4,4,5] W136 d4 g3 EE43.61258
LCF [5,2,5,-2,5,-5;-] = [6,2,-4,-2,4,6]^2 = [2,-5,-2,6,2,6,-2,3,5,6,-3,6] = [-5,-2,6,6,2,5,-2,5,6,6,-5,2] W134 d3 g3 EE43.34214
LCF [-3,4,5,-5,2,-4,-2,-5,3,3,5,-3] W134 d3 g3 EE42.79794
LCF [6,-4,3,4,-5,-3,6,-4,2,4,-2,5] = [-4,6,-4,2,5,-2,5,6,4,-5,4,-5] = [5,-5,4,-5,3,-5,-4,-3,5,2,5,-2] W131 d3 g3 EE42.05815
LCF [-5,2,4,-2,-5,4;-] W135 d4 g3 EE43.25057
LCF [2,5,-2,5,3,5;-] = [6,-2,6,6,6,2]^2 = [5,-2,6,6,2,-5,-2,3,6,6,-3,2] W136 d3 g3 EE43.60342

```
   LCF [6,-2,6,4,6,4,6,-4,6,-4,6,2]  = [5,6,-3,3,5,-5,-3,6,2,-5,-2,3] W133 d3
g3 EE42.23739
   LCF [4,-2,4,6,-4,2,-4,-2,2,6,-2,2]  = [5,-2,5,6,2,-5,-2,-5,2,6,-2,2] W135 d3
g3 EE44.43130
```

Figure 13:

```
   LCF [6,-2,2]^4  W138 d3 g3 EE45.76235
   LCF [2,6,-2,6]^3  W135 d3 g3 EE44.26200
```

Figure 14:

```
   LCF [-3,3]^6  = [3,-5,5,-3,-5,5]^2 W144 d4 g4 EE42.27027
   LCF [6,-3,6,6,3,6]^2  = [6,6,-5,5,6,6]^2 = [3,-3,4,-3,3,4;-] =
[5,-3,6,6,3,-5]^2 = [5,-3,-5,4,4,-5;-] = [6,6,-3,-5,4,4,6,6,-4,-4,5,3] W134 d3
g4 EE41.69366
   LCF [-4,4,4,6,6,-4]^2  = [6,-5,5,-5,5,6]^2 = [4,-3,3,5,-4,-3;-] =
[-4,-4,4,4,-5,5]^2 W132 d3 g4 EE41.28733
   LCF [-4,6,3,6,6,-3,5,6,4,6,6,-5]  = [-5,4,6,6,6,-4,5,5,6,6,6,-5] =
[5,-3,4,6,3,-5,-4,-3,3,6,3,-3] = [4,-4,6,4,-4,5,5,-4,6,4,-5,-5] =
[4,-5,-3,4,-4,5,3,-4,5,-3,-5,3] W132 d3 g4 EE41.34305
   LCF [3,4,5,-3,5,-4;-]  = [3,6,-4,-3,4,6]^2 = [-4,5,5,-4,5,5;-] =
[3,6,-4,-3,4,4,5,6,-4,-4,4,-5] = [4,-5,5,6,-4,5,5,-5,5,6,-5,-5] =
[4,-4,5,-4,-4,3,4,-5,-3,4,-4,4] W130 d3 g4 EE41.02128
   LCF [4,-4,6]^4  = [3,6,3,-3,6,-3]^2 = [-3,6,4,-4,6,3,-4,6,-3,3,6,4] W134 d3
g4 EE41.66461
   LCF [6,-5,5]^4  = [3,4,-4,-3,4,-4]^2 W130 d3 g4 EE41.16056
   LCF [-3,5,-3,4,4,5;-]  = [4,-5,5,6,-4,6]^2 = [-3,4,-3,4,4,-4;-] =
[5,6,-3,-5,4,-5,3,6,-4,-3,5,3] = [5,6,4,-5,5,-5,-4,6,3,-5,5,-3] W132 d3 g4
EE41.28805
   LCF [4,-3,4,5,-4,4;-]  = [4,5,-5,5,-4,5;-] = [-5,-3,4,5,-5,4;-] W128 d3 g4
EE40.61559
   LCF [6,-4,6,-4,3,5,6,-3,6,4,-5,4]  = [6,-4,3,-4,4,-3,6,3,-4,4,-3,4] =
[5,6,-4,3,5,-5,-3,6,3,-5,4,-3] = [5,-5,4,6,-5,-5,-4,3,5,6,-3,5] =
[5,5,-4,4,5,-5,-5,-4,3,-5,4,-3] W130 d3 g4 EE40.93704
   LCF [6,-3,5,6,-5,3,6,-5,-3,6,3,5]  = [3,-4,5,-3,4,6,4,-5,-4,4,-4,6] W130 d3
g4 EE40.99207
   LCF [6,-4,5,-5,4,6,6,-5,-4,4,5,6]  W128 d3 g4 EE40.72559
```

Figure 15:

```
   LCF [4,-5,4,-5,-4,4;-]  W126 d3 g5 EE40.34891
   LCF [6,4,6,6,6,-4]^2  = [-3,4,-3,5,3,-4;-] = [-5,3,6,6,-3,5,5,5,6,6,-5,-5] =
[-3,3,6,4,-3,5,5,-4,6,3,-5,-5] W134 d3 g4 EE41.55455
   LCF [3,5,5,-3,5,5;-]  = [-3,5,-3,5,3,5;-] = [5,-3,5,5,5,-5;-] W136 d4 g4
EE41.45861
   LCF [-5,5]^6  = [5,-5,-3,3]^3 W132 d3 g4 EE41.05212
   LCF [6]^12  = [6,6,-3,-5,5,3]^2 W138 d3 g4 EE42.25614
   LCF [6,-5,-4,4,-5,4,6,-4,5,-4,4,5]  W126 d3 g5 EE40.40388
```

A. 1-connected

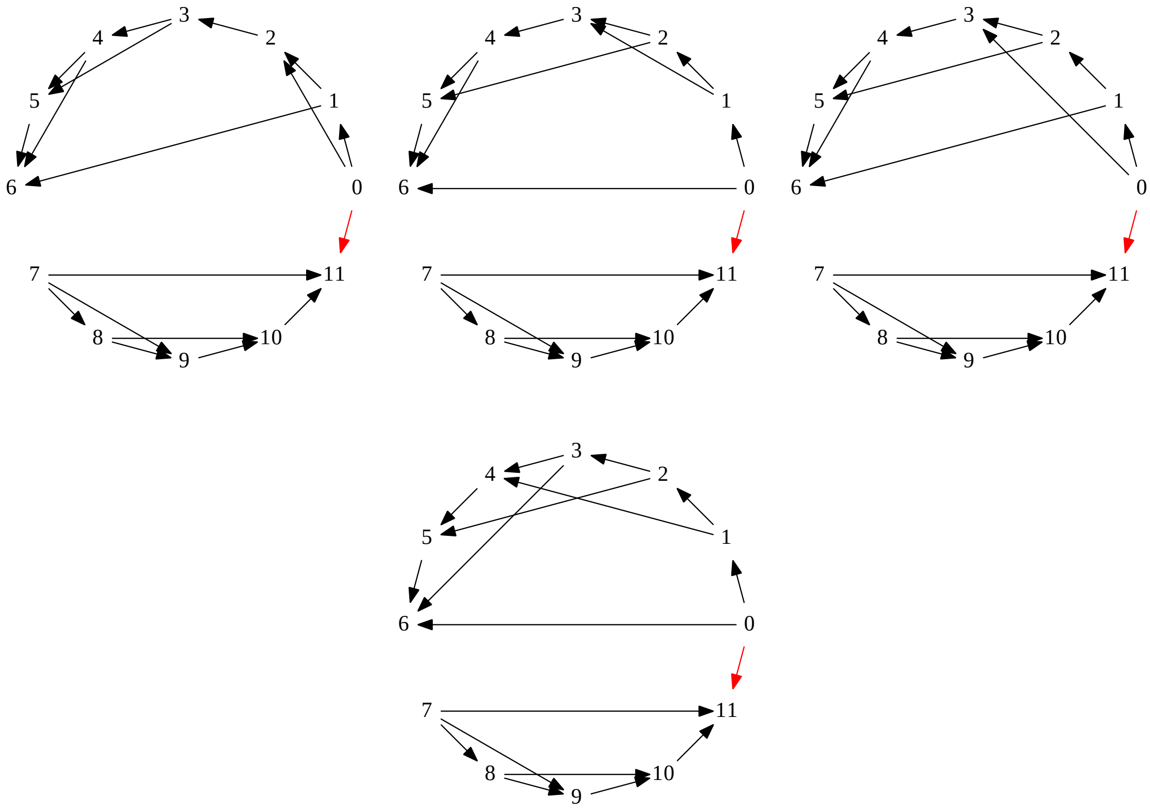


FIG. 7. 1-connected graphs on $n = 12$ vertices. `W184 d6 g3 EE49.84524`, `W172 d5 g3 EE48.45339`, `W178 d6 g3 EE47.78916`, and `W172 d5 g3 EE47.10611` in that order.

B. 2-connected

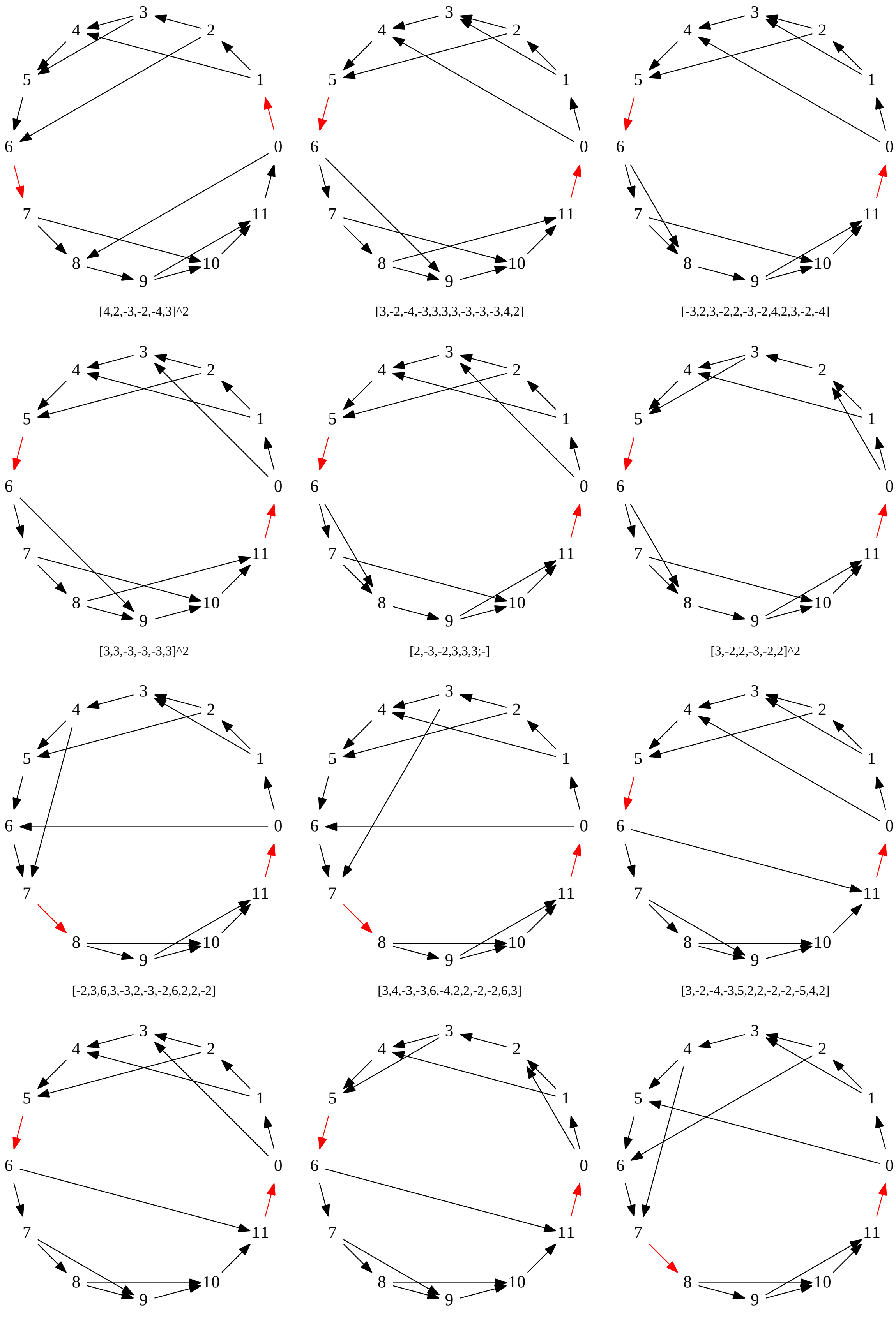


FIG. 8. 2-connected graphs on $n = 12$ vertices (start).

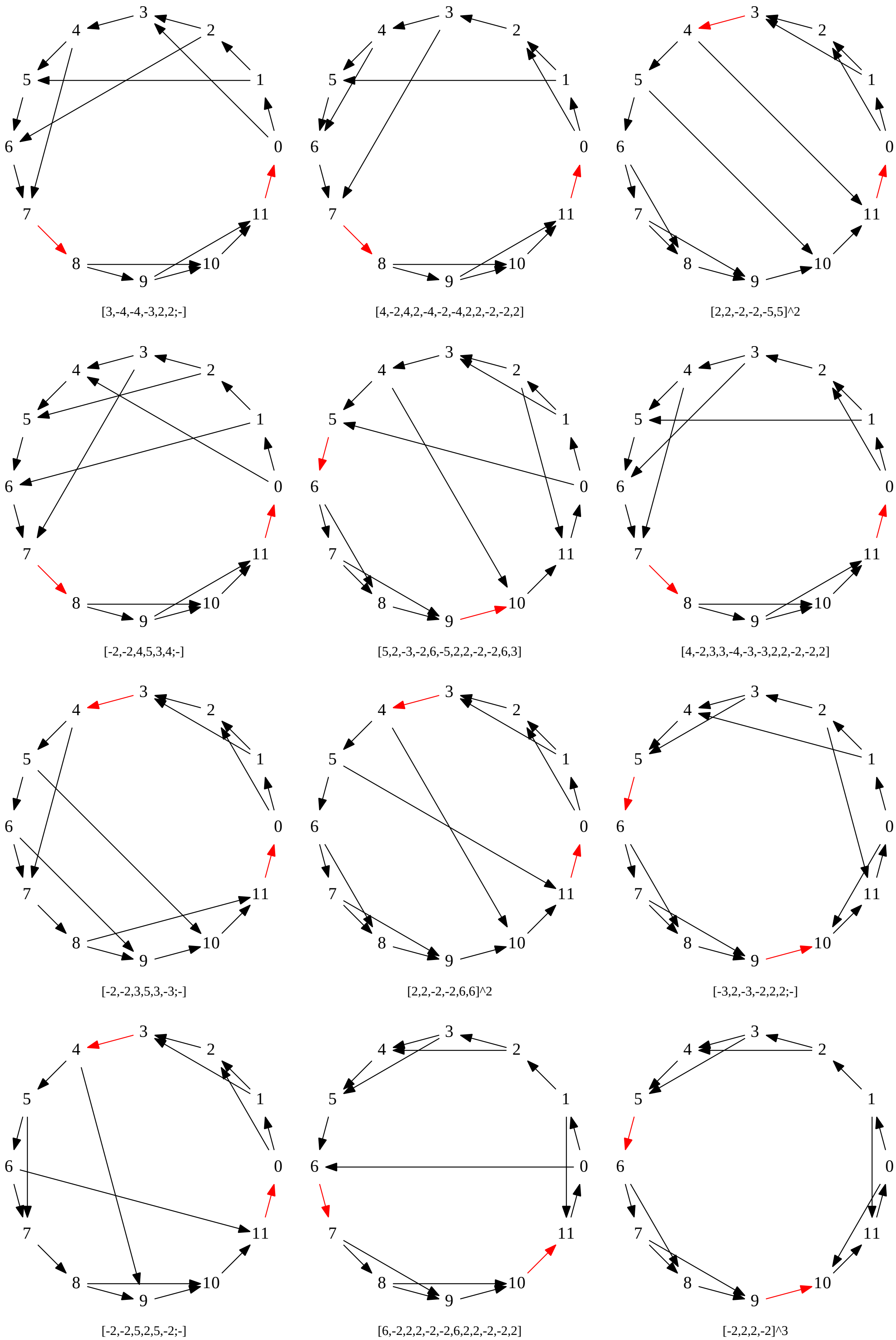


FIG. 9. 2-connected graphs on $n = 12$ vertices (end).

C. 3-connected reducible

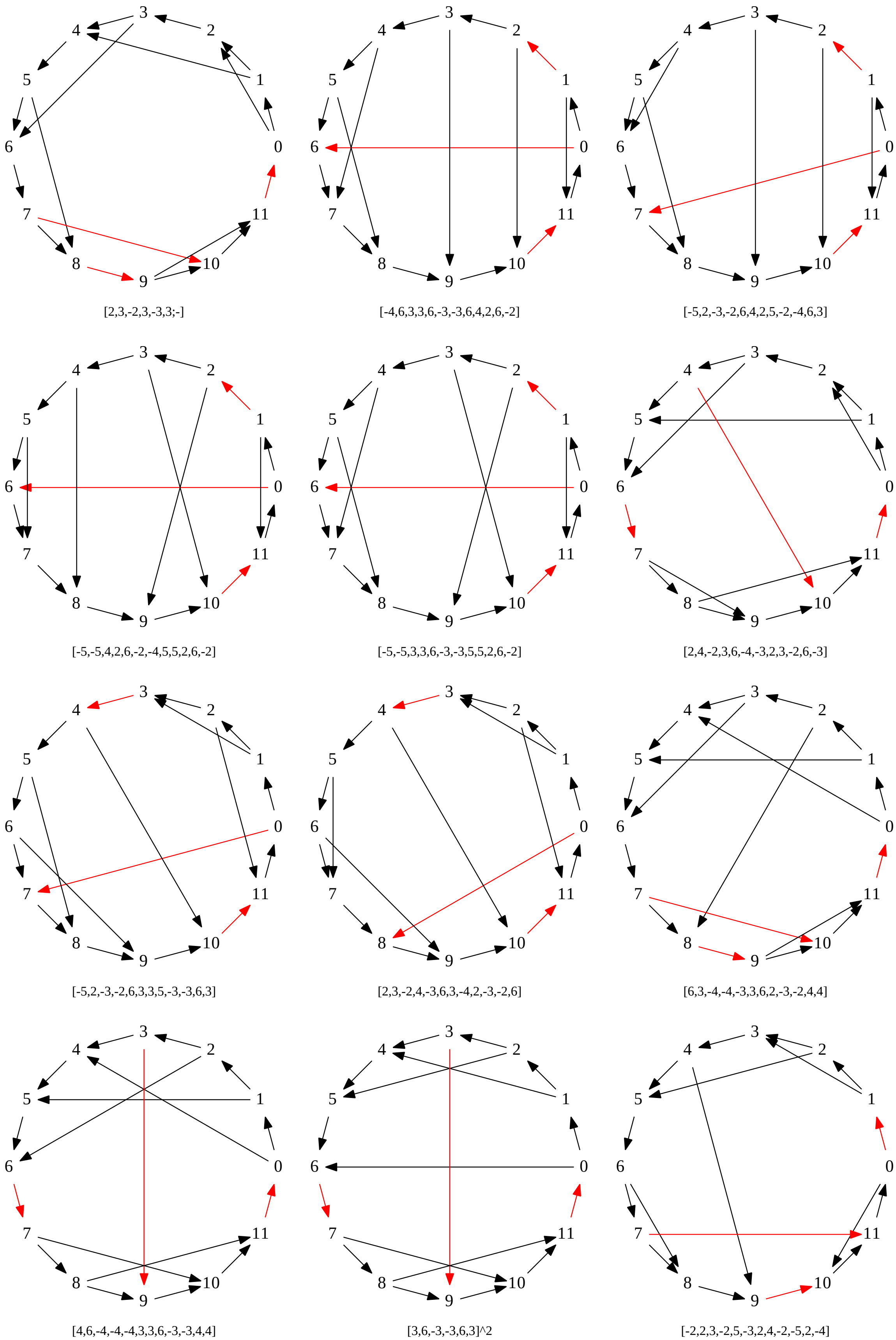


FIG. 10. 3-connected graphs on $n = 12$ vertices (start).

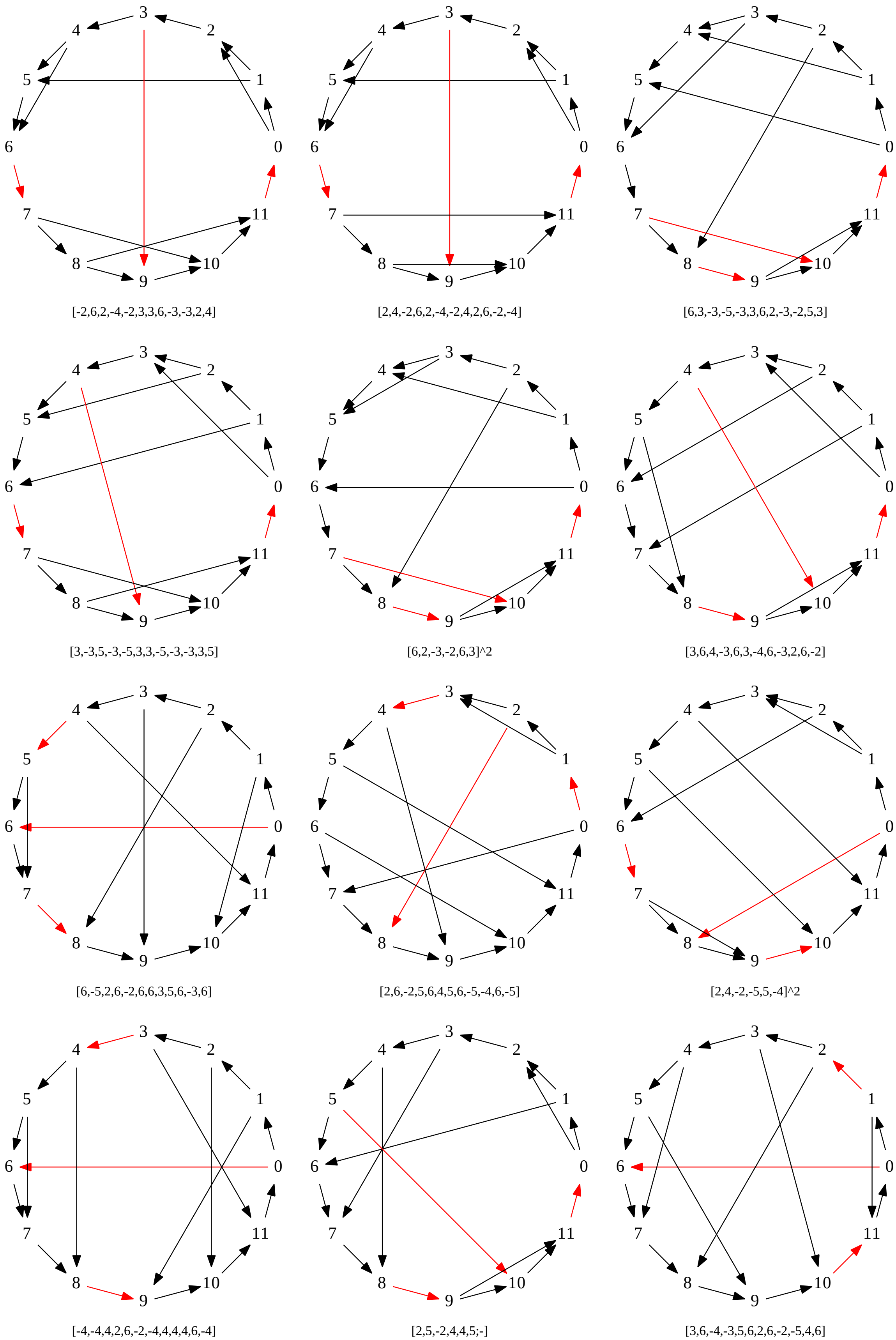


FIG. 11. 3-connected graphs on $n = 12$ vertices (continued).

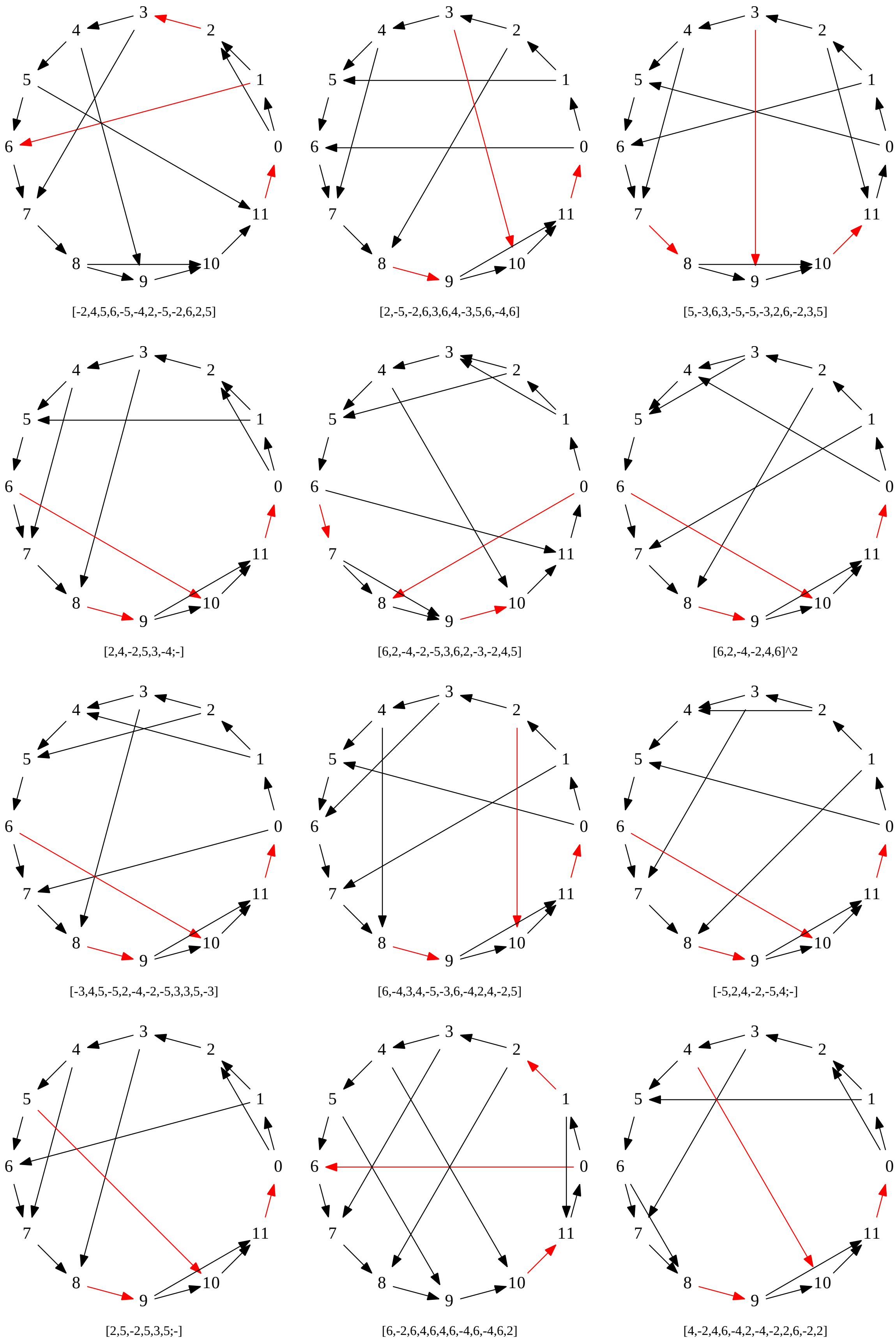


FIG. 12. 3-connected graphs on $n = 12$ vertices (continued).

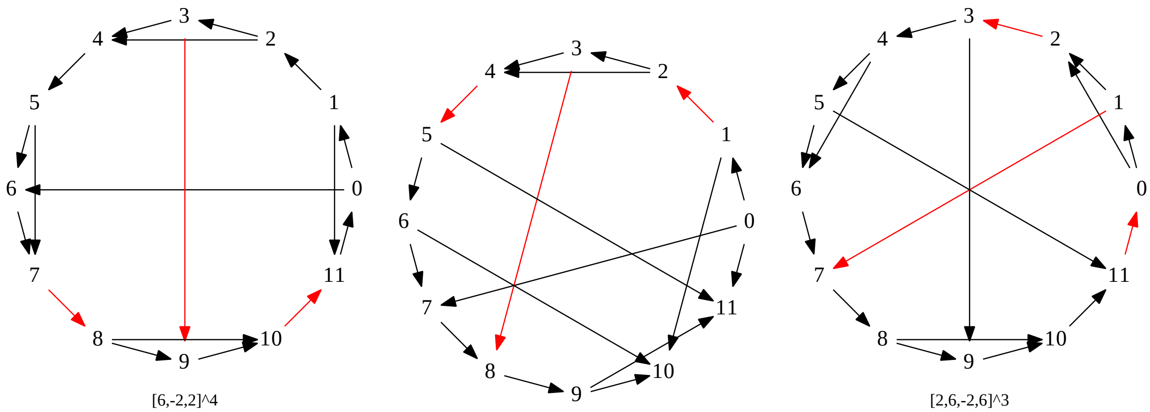


FIG. 13. 3-connected graphs on $n = 12$ vertices (end). Tietze's graph (`W129 d3 g3 EE41.70908`) does not have a Hamiltonian cycle.

D. Irreducible

The 18 graphs on $n = 12$ vertices, which are cyclically 4-connected or 5-connected and define classes of $18j$-symbols, follow in Figures 14–15. The translation to the enumeration by 18 capital letters in the reference work [45, App. 3] is:

- A `[6]^12`
- B `[-3,3]^6`
- C `[-5,5]^6`
- D `[4,-4,6]^4`. This representation is found by walking j_1, s_1, j_2, j_2' s_1', j_1', j_4', s_2, j_3', j_3, s_2, j_4 in [45, Fig. A 3.4].
- E `[3,5,5,-3,5,5;-]` This connection is found by walking j_3, l_2, j_3', k_1' s_2, k_1, s_1, k_2', j_4', l_1, j_4, k_2 in [45, Fig. A 3.5].
- F `[4,-5,4,-5,-4,4;-]` [45, Fig. A 3.6].
- G `[6,-5,5]^4` [45, Fig. A 3.7].
- H `[6,-5,-4,4,-5,4,6,-4,5,-4,4,5]` [45, Fig. A 3.8].
- I `[6,-4,5,-5,4,6,6,-5,-4,4,5,6]` [45, Fig. A 3.9].
- K `[-4,4,4,6,6,-4]^2` [45, Fig. A 3.10].
- L `[6,-3,6,6,3,6]^2` [45, Fig. A 3.12].
- M `[6,4,6,6,6,-4]^2` [45, Fig. A 3.13].
- N `[4,-3,4,5,-4,4;-]` [45, Fig. A 3.15].
- P `[6,-3,5,6,-5,3,6,-5,-3,6,3,5]` [45, Fig. A 3.16].
- R `[3,4,5,-3,5,-4;-]` [45, Fig. A 3.17].
- S `[-3,5,-3,4,4,5;-]` [45, Fig. A 3.18].
- T `[-4,6,3,6,6,-3,5,6,4,6,6,-5]` walking for example r, l, s, u, n, p, j, r', l', m', p', j' in [45, Fig. A 3.11].
- V `[6,-4,6,-4,3,5,6,-3,6,4,-5,4]` [45, Fig. A 3.14].

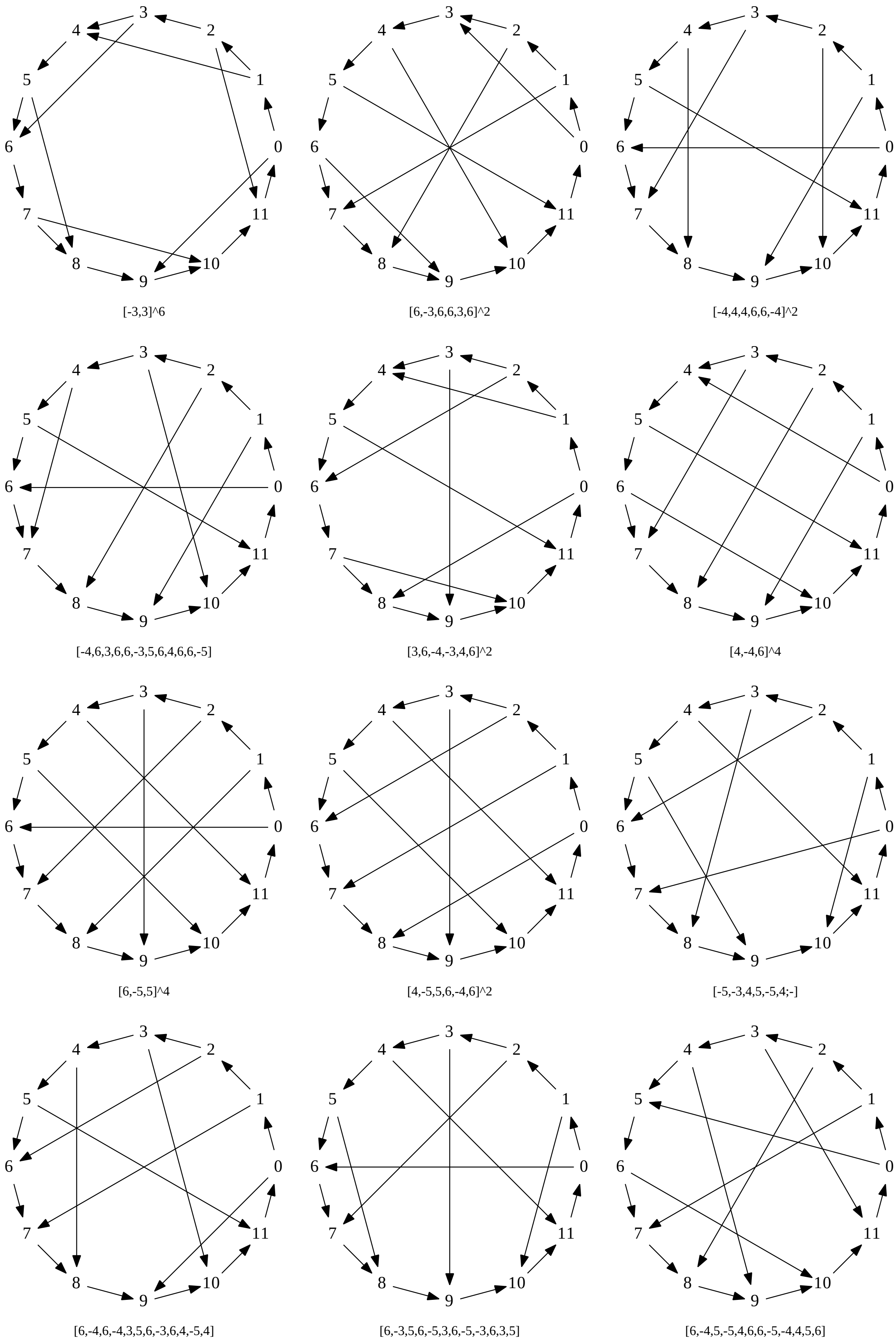


FIG. 14. 12 of the 18 graphs on $n = 12$ vertices which are irreducible.

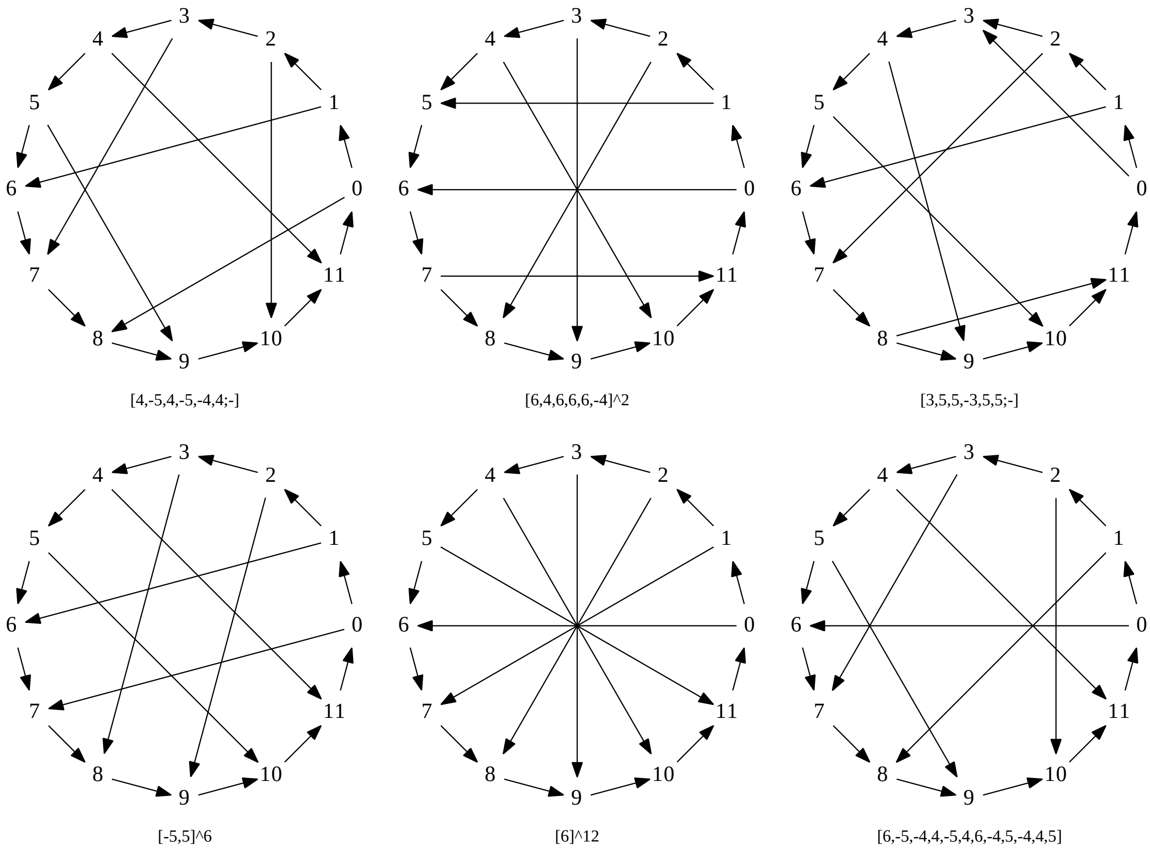


FIG. 15. The remaining 6 of the 18 graphs on $n = 12$ vertices which are irreducible. `[-5,5]^6` is the Franklin graph; `[6]^12` is the 6-prism graph.

VI. 14 VERTICES

The total number of graphs on 14 vertices is 509 [8, 9, 33, 41] [38, A002851].

Only the 84 of the diagrams which are irreducible are finally shown in Figures 16–22, each representing a $21j$-symbol. The 84 graphs can be characterized by the following Hamiltonian cycles:

Figure 16:

```
   LCF [3,-3,4,-3,5,3,-4;-]  = [-6,6,4,-5,7,5,-4]^2 =
[-5,6,4,-5,7,3,-4,-6,-3,5,3,7,5,-3] = [-5,7,-3,3,-6,6,-3,3,7,5,-3,-6,6,3] W209
d4 g4 EE48.48328
   LCF [3,7,7,-3,7,7,7]^2  = [7,7,7,-5,7,5,7]^2 = [3,-3,5,-3,5,3,5;-] =
[7,3,-5,7,-3,3,7,7,-3,3,7,5,-3,7] = [7,7,-3,-5,5,5,5,7,7,-5,-5,-5,5,3] =
[5,3,-5,5,-3,-5,7,3,-5,3,-3,5,-3,7] = [-5,5,5,-5,7,3,-5,-5,-3,5,3,7,5,-3] W209
d4 g4 EE48.36236
   LCF [7,3,-4,7,-3,-6,3,7,3,-3,7,-3,4,6]  = [-3,-6,3,7,4,-3,7,5,-4,6,7,3,-5,7]
= [-3,5,5,7,4,-6,-5,-5,-4,3,7,3,-3,6] = [-5,7,-4,3,-5,6,-3,3,7,5,-3,-6,4,5] =
[7,-3,6,-6,-5,5,3,7,-6,-3,-5,6,3,5] = [4,6,-4,3,-4,4,-3,-6,3,-4,3,-3,4,-3] =
[5,-4,4,5,-5,-5,-4,3,-5,3,-3,4,-3,5] = [4,-5,4,6,-4,-6,-4,5,3,-6,5,-3,-5,6] =
[-5,5,-4,-6,6,3,-5,6,-3,5,-6,6,4,-6] W200 d4 g4 EE47.76064
   LCF [-3,7,4,6,-5,7,-4,3,7,-6,-3,3,7,5]  =
[-3,6,4,7,4,-6,-4,-6,-4,3,7,3,-3,6] = [-5,5,5,-6,4,7,-5,-5,-4,5,3,6,7,-3] =
[5,-3,6,-6,5,-5,7,3,-6,-5,-3,6,3,7] = [-6,-4,3,7,-6,-3,3,6,6,-3,7,4,6,-6] =
[6,-4,3,5,-5,-3,-6,3,-5,3,-3,4,-3,5] = [-3,-5,4,6,-5,3,-4,5,-3,-6,5,3,-5,5] =
[5,-5,4,6,-5,-5,-4,5,3,-6,5,-3,-5,5] = [6,3,-6,4,-3,6,-6,-4,5,3,6,-6,-3,-5] =
[-3,-6,3,5,6,-3,6,6,-5,6,-6,3,-6,-6] = [-5,3,-6,5,-3,6,6,6,-5,5,6,-6,-6,-6]
W197 d4 g4 EE47.67915
   LCF [7,3,-3,7,-3,7,3]^2  = [5,-5,7,5,7,-5,7]^2 = [-3,5,-3,5,5,5,-5;-] =
[5,-3,7,5,7,-5,7,3,-5,7,-3,7,3,7] = [3,7,5,-3,5,7,5,-5,7,-5,3,-5,7,-3] =
[-5,-3,3,7,3,-3,5,-3,5,5,7,-5,3,-5] = [3,7,-5,-3,-5,5,3,5,7,-3,-5,5,-5,5] W205
d4 g4 EE47.93708
   LCF [6,6,-3,7,5,7,-6,-6,3,-5,7,-3,7,3]  =
[4,-3,5,7,-4,-6,3,-5,3,-3,7,-3,3,6] = [6,-3,5,7,5,-6,-6,-5,3,-5,7,-3,3,6] =
[-6,-6,4,-5,7,5,-4,6,6,6,-5,7,5,-6] = [-3,-5,5,3,-6,4,-3,-5,5,-4,5,3,6,-5] =
[6,6,6,-6,-6,4,-6,-6,-6,-4,3,6,6,-3] W201 d4 g4 EE47.97647
   LCF [4,-3,6,4,-4,6,4;-]  = [7,-4,3,7,4,-3,7]^2 = [4,-3,6,3,-4,6,-3;-] =
[3,-6,5,-3,-6,3,5;-] = [-5,-3,6,4,-5,6,4;-] = [7,-3,6,7,7,-6,3,7,-6,-3,7,7,3,6]
= [7,-3,6,-6,6,-6,3,7,-6,-3,-6,6,3,6] = [3,-6,-5,-3,4,-6,6,3,-4,6,-3,5,-6,6]
W197 d4 g4 EE47.72988
   LCF [-3,3,6,7,-3,-6,3,5,-6,-3,7,3,-5,6]  =
[6,-5,6,4,7,-6,-6,-4,-6,3,5,7,-3,6] = [-4,-6,3,-4,4,-3,5,5,-4,6,4,-5,-5,4] =
[-6,3,-4,6,-3,5,5,6,6,-6,-5,-5,4,-6] = [-6,5,-3,6,-6,6,-5,3,6,-6,-3,-6,6,3]
W195 d3 g4 EE47.64838
   LCF [-3,6,-3,6,6,3,6;-]  = [7,7,-6,6,-5,7,5]^2 = [-3,5,-3,6,6,3,-5;-] =
[6,-4,-4,5,5,5,-6;-] = [-5,-5,4,4,7,7,-4,-4,5,5,5,7,7,-5] =
[-5,-3,4,4,7,5,-4,-4,5,5,-5,7,3,-5] = [-6,3,-6,6,-3,7,5,6,6,-6,6,-5,7,-6] =
[5,3,6,-5,-3,-5,3,4,-6,-3,3,-4,5,-3] = [-6,3,-5,-5,-3,4,4,6,6,-4,-4,5,5,-6]
W199 d4 g4 EE48.03267
   LCF [-3,-6,3,7,3,-3,6,-3,5,6,7,3,-6,-5]  =
[-4,-6,5,3,7,-6,-3,-5,4,6,4,7,-4,6] = [6,-4,6,7,3,-6,-6,-3,-6,3,7,4,-3,6] =
[5,-6,6,4,7,-5,6,-4,-6,6,3,7,-6,-3] = [6,6,7,-5,6,-6,-6,-6,3,7,-6,-3,5,6] =
[-6,3,-4,3,-3,5,-3,4,6,4,-5,-4,4,-4] = [5,3,-6,5,-3,-5,5,6,-5,3,6,-5,-3,-6] =
[6,6,-3,-6,4,5,-6,-6,-4,3,-5,6,-3,3] W197 d4 g4 EE47.75062
   LCF [-4,-6,4,4,7,7,-4,-4,5,6,4,7,7,-5]  = [5,-6,6,7,7,-5,6,6,-6,6,7,7,-6,-6]
= [4,-3,3,7,-4,-3,3,4,5,-3,7,-4,3,-5] = [-6,-6,3,4,7,-3,6,-4,6,6,3,7,-6,-3] =
[-6,5,5,-6,6,7,-5,-5,6,4,-6,6,7,-4] = [4,4,-4,5,-4,-4,3,4,-5,-3,3,-4,4,-3] =
[4,4,6,-5,-4,-4,3,4,-6,-3,3,-4,5,-3] = [3,4,6,-3,-6,-4,3,4,-6,-3,3,-4,6,-3] =
[-6,3,4,-6,-3,5,-4,6,6,3,-5,6,-3,-6] = [-5,4,4,-6,6,-4,-4,6,4,5,-6,6,-4,-6] =
[4,-6,-6,5,-4,-6,5,5,-5,6,6,-5,-5,6] W197 d4 g4 EE47.83708
```

LCF [3,-6,6,-3,7,-6,6]^2 = [-6,6,4,-6,6,7,-4]^2 = [4,7,-3,6,-4,7,5,3,7,-6,-3,-5,7,3] = [-3,3,4,5,-3,-6,-4,3,-5,3,-3,3,-3,6] = [6,4,5,-6,6,-4,-6,-5,5,3,-6,6,-3,-5] W202 d4 g4 EE48.20639

Figure 17:

LCF [3,-3,5,-3,4,4,5;-] = [-6,6,7,3,7,7,-3]^2 = [3,-3,4,-3,4,4,-4;-] = [6,4,-5,6,6,-4,-6;-] = [6,6,7,7,7,7,-6,-6,5,7,7,7,7,-5] = [-5,5,6,4,7,7,-5,-4,-6,5,3,7,7,-3] = [4,-5,3,6,-4,-3,7,5,3,-6,5,-3,-5,7] W205 d4 g4 EE48.21641
LCF [-3,5,3,7,4,-3,-5,6,-4,3,7,3,-3,-6] = [5,-3,-6,6,4,-5,7,4,-4,-6,6,-4,3,7] W204 d4 g4 EE48.18647
LCF [6,7,7,-4,7,7,-6,3,7,7,-3,7,7,4] = [6,7,5,7,7,7,-6,-5,7,4,7,7,7,-4] = [7,4,-5,7,4,-4,7,7,-4,3,7,5,-3,7] = [7,7,-6,-5,-5,6,3,7,7,-3,6,-6,5,5] = [4,5,-3,7,-4,4,-5,5,3,-4,7,-3,-5,3] = [4,4,-4,7,-4,-4,3,6,3,-3,7,-3,4,-6] = [-3,3,4,7,-3,4,-4,6,4,-4,7,3,-4,-6] = [5,-3,-6,5,5,-5,7,4,-5,-5,6,-4,3,7] = [4,-3,-6,5,-4,-6,3,4,-5,-3,6,-4,3,6] = [-5,6,-4,3,4,6,-3,-6,-4,5,3,-6,4,-3] = [3,-3,6,-3,5,-6,4,4,-6,-5,-4,-4,3,6] W200 d4 g4 EE47.84785
LCF [3,-4,7,-3,4,-6,6]^2 W199 d4 g4 EE47.86854
LCF [-4,7,4,4,7,7,-4]^2 = [7,-6,6,7,7,-6,6]^2 = [4,-3,3,4,-4,-3,4;-] = [-4,7,4,4,-6,6,-4]^2 W197 d4 g4 EE48.08295
LCF [7,7,7,-6,6,7,7]^2 = [-3,3,-3,4,-3,3,4;-] = [6,-3,-5,6,6,3,-6;-] = [-4,7,5,3,7,7,-3,-5,7,4,4,7,7,-4] = [-5,6,6,3,7,7,-3,-6,-6,5,3,7,7,-3] = [4,4,-6,6,-4,-4,7,5,3,-6,6,-3,-5,7] W207 d4 g4 EE48.66179
LCF [6,6,7,7,7,-6,-6,-6,4,7,7,7,-4,6] = [7,7,-6,-4,-6,6,3,7,7,-3,6,-6,6,4] = [-6,3,-4,3,-3,6,-3,3,6,4,-3,-6,4,-4] = [5,6,-3,-5,6,-5,3,-6,3,-3,-6,-3,5,3] = [-6,6,-4,3,6,6,-3,-6,6,4,-6,-6,4,-4] W201 d4 g4 EE48.11933
LCF [7,7,-6,6,-6,6,7]^2 = [3,4,-5,-3,3,-4,3;-] = [6,-4,-4,6,6,3,-6;-] = [-6,-6,4,4,7,7,-4,-4,6,6,3,7,7,-3] W201 d4 g4 EE48.21995
LCF [7]^14 = [-3,3]^7 = [-5,7,5,3,7,7,-3]^2 W217 d4 g4 EE49.25357
LCF [3,6,-5,-3,5,3,6;-] = [5,-6,6,-6,6,-5,7]^2 = [3,-6,6,-3,7,5,7,5,-6,6,-5,7,-5,7] W197 d3 g4 EE47.87399
LCF [4,-3,5,5,-4,5,5;-] = [6,3,-5,5,-3,5,-6;-] = [-3,7,3,7,-6,-3,3,6,7,-3,7,3,6,-6] = [-6,5,-5,7,3,7,-5,-3,6,4,7,5,7,-4] = [3,-6,4,-3,7,5,-4,5,5,6,-5,7,-5,-5] = [-6,3,-6,4,-3,7,5,-4,6,4,6,-5,7,-4] W197 d4 g4 EE47.58692
LCF [-5,6,4,7,7,7,-4,-6,5,5,7,7,7,-5] = [6,6,7,5,7,7,-6,-6,-5,7,3,7,7,-3] = [-3,7,-3,3,-6,4,-3,3,7,-4,-3,3,6,3] = [-3,7,5,-4,-6,4,4,-5,7,-4,-4,3,6,4] = [-5,6,4,-5,7,5,-4,-6,5,5,-5,7,5,-5] = [-3,5,3,-5,4,-3,-5,3,-4,4,-3,3,5,-4] = [-4,-4,3,6,4,-3,6,6,-4,-6,4,4,-6,-6] W203 d4 g4 EE48.11421

Figure 18:

LCF [7,-3,-6,6,-5,3,5]^2 = [7,3,-4,6,-3,5,7,7,3,-6,-5,-3,4,7] = [7,-5,3,6,-6,-3,7,7,3,-6,5,-3,6,7] = [-4,5,-3,6,3,7,-5,-3,3,-6,4,-3,7,3] = [-5,-5,3,5,7,-3,6,6,-5,5,5,7,-6,-6] = [5,-4,-6,5,-5,-5,3,5,-5,-3,6,4,-5,5] = [-6,3,-4,6,-3,6,4,6,6,-6,-4,-6,4,-6] W197 d4 g4 EE47.60125
LCF [7,-5,7,-4,7,3,6,7,-3,7,5,7,-6,4] = [7,7,-6,-4,5,7,5,7,7,-5,6,-5,7,4] = [7,-4,3,7,-5,-3,3,7,4,-3,7,4,-4,5] = [-5,5,-6,4,7,7,-5,-4,5,5,6,7,7,-5] = [-3,6,4,7,-5,4,-4,-6,4,-4,7,3,-4,5] = [-5,4,5,-5,7,-4,4,-5,5,5,-4,7,5,-5] = [7,3,6,-6,-3,-6,4,7,-6,3,-4,6,-3,6] = [7,-6,-6,3,-5,6,-3,7,4,6,6,-6,-4,5] = [-5,5,-3,4,-6,5,-5,-4,3,5,-5,-3,6,3] = [-5,-4,3,5,6,-3,6,6,-5,5,-6,4,-6,-6] W195 d4 g4 EE47.32016
LCF [7,-4,6,7,-6,-6,3,7,-6,-3,7,4,6,6] = [4,-4,-6,5,-4,4,7,5,-5,-4,6,4,-5,7] = [-5,-4,3,7,4,-3,6,6,-4,5,7,4,-6,-6] W194 d4 g4 EE47.44879
LCF [-5,5,5,7,7,7,-5]^2 = [3,5,-5,-3,5,3,-5;-] = [-5,5,5,-5,7,5,-5]^2 = [7,7,-3,7,5,7,5,7,7,-5,7,-5,7,3] = [7,3,-3,7,-3,3,5,7,-3,3,7,-5,-3,3] W205 d4 g4 EE47.91606

LCF [7,-5,3,5,7,-3,7]^2 = [-5,-3,5,5,-5,5,5;-] =
[-3,3,5,7,-3,7,3,-5,5,-3,7,3,7,-5] = [7,-5,3,-5,5,-3,7,7,3,-5,5,-3,5,7] =
[-5,5,-5,7,3,7,-5,-3,5,5,7,5,7,-5] W201 d4 g4 EE47.49035
LCF [7,4,7,-5,6,-4,7,7,3,7,-6,-3,5,7] = [-5,-4,4,7,3,7,-4,-3,5,5,7,4,7,-5]
= [4,7,-3,7,-4,6,3,5,7,-3,7,-6,-5,3] = [7,4,-5,5,6,-4,7,7,-5,3,-6,5,-3,7] =
[-5,7,3,7,-6,-3,5,6,7,5,7,-5,6,-6] = [-3,-6,4,7,3,-6,-4,-3,4,6,7,3,-4,6] =
[-4,-6,4,4,7,-6,-4,-4,4,6,4,7,-4,6] = [-6,-4,4,-4,3,5,-4,-3,6,3,-5,4,-3,4] =
[-5,-4,4,-5,3,5,-4,-3,5,5,-5,4,5,-5] = [-6,-4,4,-5,3,5,-4,-3,6,4,-5,4,5,-4] =
[4,6,4,-6,-4,5,-4,-6,4,4,-5,6,-4,-4] W197 d4 g4 EE47.42171
LCF [3,-3,6,-3,3,6,3;-] = [-6,6,4,7,7,7,-4]^2 = [6,-3,3,6,6,-3,-6;-] =
[-6,-6,3,7,7,-3,6,6,6,6,7,7,-6,-6] = [-3,-5,3,6,4,-3,6,6,-4,-6,5,3,-6,-6] W203
d4 g4 EE48.46324
LCF [7,-4,3,-4,4,-3,4]^2 = [-5,5,-3,4,7,5,-5,-4,4,5,-5,7,-4,3] W197 d4 g4
EE47.32024
LCF [5,-3,5,6,6,-5,5;-] = [4,6,-6,-6,-4,3,6;-] =
[7,-4,3,5,-5,-3,4,7,-5,3,-4,4,-3,5] = [6,-6,-3,4,7,5,-6,-4,4,6,-5,7,-4,3] =
[7,5,-6,5,-5,6,-5,7,-5,3,6,-6,-3,5] W193 d3 g4 EE47.18167
LCF [-5,7,3,-5,7,-3,4,6,7,5,-4,7,5,-6] = [3,7,5,-3,6,7,5,-5,7,4,-6,-5,7,-4]
= [5,-6,-5,7,4,-5,7,5,-4,6,7,5,-5,7] = [-3,6,4,7,-6,3,-4,-6,-3,4,7,3,6,-4] =
[-4,4,-4,5,3,-4,5,-3,-5,4,4,-5,4,-4] = [-3,5,-3,5,-6,4,-5,3,-5,-4,-3,3,6,3] =
[-4,-4,5,3,-6,4,-3,-5,5,-4,4,4,6,-5] = [-4,-3,3,6,4,-3,5,6,-4,-6,4,-5,3,-6] =
[-3,6,-6,3,-6,3,-3,-6,-3,4,6,3,6,-4] W199 d4 g4 EE47.76644
LCF [4,6,-3,7,-4,7,3,-6,3,-3,7,-3,7,3] = [6,-5,7,5,7,-6,-6,5,-5,7,5,7,-5,6]
= [-4,-4,5,3,5,7,-3,-5,5,-5,4,4,7,-5] = [5,-3,6,7,-5,-5,3,4,-6,-3,7,-4,3,5] =
[-5,7,-4,3,6,6,-3,6,7,5,-6,-6,4,-6] = [6,6,-3,7,5,6,-6,-6,4,-5,7,-6,-4,3] =
[4,-4,6,-4,-4,5,3,5,-6,-3,-5,4,-5,4] = [-5,3,-5,-5,-3,3,4,6,-3,5,-4,5,5,-6] =
[-6,3,-5,3,-3,5,-3,5,6,4,-5,5,-5,-4] = [5,5,5,-5,6,-5,-5,-5,3,4,-6,-3,5,-4] =
[6,-5,6,6,-6,-6,-6,4,-6,-6,5,-4,6,6] W198 d4 g4 EE47.70929
LCF [3,-6,5,-3,7,5,7,-5,4,6,-5,7,-4,7] =
[-5,4,-4,7,3,-4,5,-3,5,5,7,-5,4,-5] = [6,-6,-3,7,3,6,-6,-3,4,6,7,-6,-4,3] =
[5,-6,5,-6,6,-5,7,-5,4,6,-6,6,-4,7] = [-5,3,-4,5,-3,5,5,6,-5,5,-5,-5,4,-6] =
[6,-6,-3,-6,3,5,-6,-3,4,6,-5,6,-4,3] W193 d3 g4 EE47.23870

Figure 19:

LCF [-3,6,4,7,4,7,-4,-6,-4,4,7,3,7,-4] =
[-5,-4,4,4,6,7,-4,-4,5,5,-6,4,7,-5] = [3,-6,5,-3,7,5,6,-5,5,6,-5,7,-6,-5] =
[3,4,-6,-3,4,-4,5,6,-4,3,6,-5,-3,-6] = [4,4,-6,5,-4,-4,5,5,-5,4,6,-5,-5,-4]
W194 d4 g4 EE47.39737
LCF [6,-4,7,7,3,-6,-6,-3,4,7,7,4,-4,6] = [7,3,-6,6,-3,6,7,7,4,-6,6,-6,-4,7]
= [-3,7,3,-5,3,-3,4,-3,7,4,-4,3,5,-4] = [5,5,-4,7,3,-5,-5,-3,3,4,7,-3,4,-4] =
[5,-6,5,3,7,-5,-3,-5,4,6,3,7,-4,-3] = [-4,5,5,5,7,-6,-5,-5,-5,3,4,7,-3,6] =
[6,6,7,-6,6,-6,-6,-6,4,7,-6,6,-4,6] = [5,-3,-6,5,-5,-5,3,4,-5,-3,6,-4,3,5] =
[-4,-3,5,3,5,6,-3,-5,5,-5,4,-6,3,-5] = [-6,3,-6,3,-3,6,-3,5,6,4,6,-6,-5,-4] =
[-5,6,-4,3,6,6,-3,-6,5,5,-6,-6,4,-5] = [5,-6,6,-6,-5,-5,6,3,-6,6,-3,6,-6,5]
W197 d4 g4 EE47.78069
LCF [-3,6,-3,5,5,5,6;-] = [3,-4,3,-3,4,-3,7]^2 =
[3,5,5,-3,-6,4,-5,-5,3,-4,3,-3,6,-3] = [3,-6,6,-3,6,-6,6,4,-6,6,-6,-4,-6,6]
W201 d4 g4 EE48.06221
LCF [6,7,5,-6,5,7,-6,-5,7,-5,3,6,7,-3] =
[3,4,6,-3,7,-4,4,6,-6,3,-4,7,-3,-6] = [7,-3,6,-6,-5,4,4,7,-6,-4,-4,6,3,5] =
[7,-5,-5,6,-5,3,6,7,-3,-6,5,5,-6,5] = [-4,4,-3,5,-6,-4,3,4,-5,-3,4,-4,6,3] =
[-5,4,-4,3,6,-4,-3,4,5,5,-6,-4,4,-5] = [-6,4,-4,3,6,-4,-3,4,6,4,-6,-4,4,-4] =
[4,4,-6,5,-4,-4,5,6,-5,3,6,-5,-3,-6] = [-5,-4,3,4,6,-3,6,-4,5,5,-6,4,-6,-5] =
[6,-3,6,-6,-5,4,-6,3,-6,-4,-3,6,3,5] = [6,-3,5,-6,5,5,-6,-5,5,-5,-5,6,3,-5]
W194 d4 g4 EE47.41178
LCF [4,7,-6,6,-4,7,7]^2 = [4,-4,-4,4,-4,3,4;-] = [-4,-3,5,-4,4,4,5;-] =
[-3,7,4,4,6,7,-4,-4,7,4,-6,3,7,-4] = [6,-4,7,7,4,-6,-6,5,-4,7,7,4,-5,6] =
[-6,5,7,-6,6,7,-5,6,6,7,-6,6,7,-6] = [-3,7,4,4,-6,5,-4,-4,7,4,-5,3,6,-4] =

[5,-3,-6,-4,4,-5,3,4,-4,-3,6,-4,3,4] W195 d4 g4 EE47.65770
LCF [3,6,-5,-3,4,4,6;-] = [3,5,-5,-3,4,4,-5;-] = [5,5,-5,6,6,-5,-5;-] = [-5,5,5,-6,6,7,-5]^2 = [-6,6,-5,6,6,-6,6;-] = [6,-3,7,-4,3,5,-6,-3,3,7,-5,-3,3,4] W195 d3 g4 EE47.62118
LCF [7,7,7,-4,7,7,4]^2 = [5,-4,7,7,4,-5,7]^2 = [4,-3,-5,4,-4,3,4;-] = [-4,-3,6,-4,3,6,3;-] = [7,-6,3,7,7,-3,6,7,5,6,7,7,-6,-5] W199 d4 g4 EE47.92512
LCF [7,-5,7,5,7,-6,6]^2 = [-4,-3,5,-4,5,3,5;-] = [-5,5,-4,-4,7,3,-5,3,-3,5,-3,7,4,4] = [4,4,6,-6,-4,-4,7,3,-6,3,-3,6,-3,7] = [7,5,6,-6,6,-6,-5,7,-6,3,-6,6,-3,6] W199 d4 g4 EE47.81083
LCF [-5,-4,3,7,4,-3,7,5,-4,5,7,4,-5,7] = [7,-6,3,5,7,-3,6,7,-5,6,3,7,-6,-3] = [7,7,-6,-4,-6,4,5,7,7,-4,6,-5,6,4] = [6,-6,5,7,7,-6,-6,-5,4,6,7,7,-4,6] = [-4,7,-3,3,-6,4,-3,4,7,-4,4,-4,6,3] = [-6,5,-4,7,3,6,-5,-3,6,4,7,-6,4,-4] = [-3,-6,3,-4,4,-3,4,5,-4,6,-4,3,-5,4] = [4,-4,6,6,-4,-6,4,5,-6,-6,-4,4,-5,6] W195 d4 g4 EE47.39167
LCF [-4,5,7,4,7,7,-5]^2 = [4,-3,4,5,-4,5,-4;-] = [7,-3,7,-6,4,-6,4,7,-4,7,-4,6,3,6] = [6,7,-4,7,3,6,-6,-3,7,4,7,-6,4,-4] = [5,-6,5,7,7,-5,6,-5,5,6,7,7,-6,-5] = [5,-4,7,-4,3,-5,4,-3,4,7,-4,4,-4,4] = [5,-5,6,4,7,-5,6,-4,-6,4,5,7,-6,-4] W193 d4 g4 EE47.05310
LCF [4,-5,5,-5,-4,3,5;-] = [6,-5,6,-5,5,6,-6;-] = [4,-4,5,7,-4,7,3,-5,5,-3,7,4,7,-5] = [-3,4,7,-5,4,-4,4,6,-4,7,-4,3,5,-6] = [-4,7,3,-5,5,-3,5,6,7,-5,4,-5,5,-6] = [7,-5,-4,5,5,-6,5,7,-5,-5,5,-5,4,6] = [-3,4,6,-5,5,-4,4,6,-6,-5,-4,3,5,-6] = [6,-4,5,-5,5,5,-6,-5,5,-5,-5,4,5,-5] W191 d3 g4 EE46.91447
LCF [6,7,-6,6,-5,7,-6,4,7,-6,6,-4,7,5] = [5,-4,4,7,3,-5,-4,-3,4,4,7,4,-4,-4] = [6,-4,-3,7,3,4,-6,-3,4,-4,7,4,-4,3] = [-4,5,6,4,7,-6,-5,-4,-6,3,4,7,-3,6] = [5,-3,6,-4,5,-5,4,4,-6,-5,-4,-4,3,4] W193 d3 g4 EE47.55605

Figure 20:

LCF [-4,3,5,-4,-3,3,5;-] = [7,-6,6,-6,6,-6,6]^2 W195 d3 g4 EE47.79409
LCF [6,-3,6,6,6,6,-6;-] = [7,3,-6,6,-3,7,7]^2 = [6,-3,-5,5,5,5,-6;-] = [6,6,-3,7,7,3,-6,-6,-3,3,7,7,-3,3] = [3,-6,3,-3,6,-3,7,5,3,6,-6,-3,-5,7] = [6,-5,-5,5,3,-6,-6,-3,-5,3,5,5,-3,6] W203 d4 g4 EE48.34463
LCF [6,-5,5,-4,7,3,-6,-5,-3,3,5,7,-3,4] = [5,-6,-3,7,3,-5,5,-3,4,6,7,-5,-4,3] = [4,-3,5,7,-4,6,3,-5,5,-3,7,-6,3,-5] = [5,-6,-3,7,4,-5,5,5,-4,6,7,-5,-5,3] = [5,-5,5,-6,4,-5,7,-5,-4,3,5,6,-3,7] = [-3,-6,3,-5,4,-3,4,6,-4,6,-4,3,5,-6] = [-5,4,-3,5,6,-4,6,4,-5,5,-6,-4,-6,3] = [-5,5,-4,4,6,6,-5,-4,5,5,-6,-6,4,-5] = [-5,5,-6,-5,3,6,-5,-3,5,5,6,-6,5,-5] = [-5,5,-6,4,-6,6,-5,-4,5,5,6,-6,6,-5] = [-5,4,6,-6,6,-4,6,6,-6,5,-6,6,-6,-6] W194 d4 g4 EE47.25303
LCF [5,-5,7,4,7,-5,7,-4,4,7,5,7,-4,7] = [7,-4,4,7,-5,3,-4,7,-3,3,7,4,-3,5] = [-5,4,-5,7,4,-4,7,5,-4,5,7,5,-5,7] = [-5,3,-4,7,-3,3,5,6,-3,5,7,-5,4,-6] = [7,-5,-3,6,-6,3,5,7,-3,-6,5,-5,6,3] = [6,6,-5,7,-5,4,-6,-6,4,-4,7,5,-4,5] = [7,-4,6,6,-6,-6,4,7,-6,-6,-4,4,6,6] = [3,-4,6,-3,6,-6,3,5,-6,-3,-6,4,-5,6] W195 d4 g4 EE47.32025
LCF [3,5,6,-3,5,6,-5;-] = [-4,6,6,-4,5,6,6;-] = [-4,6,3,-4,5,-3,6;-] = [-6,4,7,-5,7,-4,4,6,6,7,-4,7,5,-6] W194 d4 g4 EE47.31023
LCF [7,-5,3,5,7,-3,6,7,-5,4,5,7,-6,-4] = [5,-5,7,4,7,-5,6,-4,5,7,5,7,-6,-5] = [7,-5,-4,6,-5,3,5,7,-3,-6,5,-5,4,5] = [6,7,-5,4,-6,4,-6,-4,7,-4,3,5,6,-3] = [4,7,-4,6,-4,6,4,6,7,-6,-4,-6,4,-6] W193 d4 g4 EE46.99597
LCF [4,-4,4,7,-4,4,-4,5,5,-4,7,4,-5,-5] = [7,-6,3,-4,6,-3,5,7,4,6,-6,-5,-4,4] = [4,7,-4,6,-4,5,5,6,7,-6,-5,-5,4,-6] = [-6,-4,4,7,-6,4,-4,6,6,-4,7,4,6,-6] = [-6,4,7,-5,6,-4,5,6,6,7,-6,-5,5,-6] = [5,-5,6,-6,3,-5,6,-3,-6,4,5,6,-6,-4] W191 d3 g4 EE47.04310
LCF [-3,7,4,-5,3,5,-4]^2 = [4,7,-6,6,-4,6,7,5,7,-6,6,-6,-5,7] = [-4,4,6,3,7,-4,-3,5,-6,4,4,7,-5,-4] = [6,7,-5,-4,-6,4,-6,3,7,-4,-3,5,6,4] = [3,-6,-4,-3,4,-6,4,4,-4,6,-4,-4,4,6] W196 d4 g4 EE47.51335
LCF [-3,5,7,-5,3,5,-5]^2 = [5,-3,5,7,5,-5,5,-5,5,-5,7,-5,3,-5] W197 d4 g4

EE47.08518
LCF [5,-3,6,7,5,-5,5,6,-6,-5,7,-5,3,-6] =
[-4,-4,4,6,-5,3,-4,5,-3,-6,4,4,-5,5] = [4,-5,-3,5,-4,6,3,5,-5,-3,5,-6,-5,3] =
[6,3,-6,4,-3,6,-6,-4,4,4,6,-6,-4,-4] = [3,-6,-4,-3,4,6,4,6,-4,6,-4,-6,4,-6]
W196 d4 g4 EE47.35467
LCF [-5,-4,4,7,-5,3,-4,5,-3,5,7,4,-5,5] =
[7,-5,3,6,-5,-3,5,7,4,-6,5,-5,-4,5] = [-5,5,6,-4,7,5,-5,5,-6,5,-5,7,-5,4] =
[6,-4,5,-4,5,5,-6,-5,4,-5,-5,4,-4,4] = [-5,5,-5,-4,6,3,-5,5,-3,5,-6,5,-5,4] =
[-4,5,-4,6,3,6,-5,-3,5,-6,4,-6,4,-5] = [5,5,6,-5,6,-5,-5,4,-6,4,-6,-4,5,-4] =
[-5,4,6,-4,6,-4,6,4,-6,5,-6,-4,-6,4] = [-5,4,-5,5,6,-4,6,6,-5,5,-6,5,-6,-6] =
[6,-6,-6,4,-5,6,-6,-4,4,6,6,-6,-4,5] W192 d4 g4 EE46.91455
LCF [6,-6,5,-4,7,5,-6,-5,4,6,-5,7,-4,4] =
[-5,-4,4,-4,6,3,-4,5,-3,5,-6,4,-5,4] = [6,-4,5,-4,6,4,-6,-5,4,-4,-6,4,-4,4] =
[-4,6,3,6,-6,-3,5,-6,5,-6,4,-5,6,-5] W191 d3 g4 EE46.97169

Figure 21:

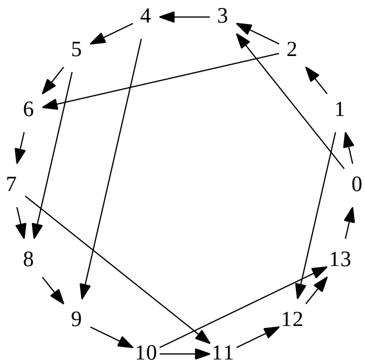
LCF [3,4,5,-3,5,-4,5;-] = [4,7,-3,7,-4,7,3]^2 = [4,4,-5,4,-4,-4,4;-] =
[-4,-4,4,4,5,7,-4,-4,5,-5,4,4,7,-5] = [-4,4,6,3,7,-4,-3,6,-6,3,4,7,-3,-6] =
[-6,5,-4,7,5,6,-5,6,6,-5,7,-6,4,-6] = [7,-6,-6,5,-5,6,6,7,-5,6,6,-6,-6,5] =
[4,-4,6,-4,-4,4,4,5,-6,-4,-4,4,-5,4] = [4,-5,3,6,-4,-3,6,4,5,-6,5,-4,-6,-5]
W195 d4 g4 EE47.49901
LCF [-4,3,-5,4,-3,7,3,-4,5,-3,4,5,7,-5] =
[7,3,-6,5,-3,-6,5,7,-5,3,6,-5,-3,6] = [6,-3,-6,4,-5,4,-6,-4,3,-4,6,-3,3,5] =
[-3,6,4,6,-6,5,-4,-6,5,-6,-5,3,6,-5] = [-4,6,4,6,-6,6,-4,-6,5,-6,4,-6,6,-5]
W193 d3 g4 EE47.32454
LCF [4,6,-3,6,-4,7,4,-6,3,-6,-4,-3,7,3] =
[6,6,-4,7,5,6,-6,-6,5,-5,7,-6,4,-5] = [-6,5,7,-5,6,6,-5,6,6,7,-6,-6,5,-6] =
[5,-3,6,-4,6,-5,3,4,-6,-3,-6,-4,3,4] = [6,4,-6,5,-5,-4,-6,4,-5,3,6,-4,-3,5]
W194 d4 g4 EE47.48901
LCF [7,-6,4,-6,6,-6,-4,7,4,6,-6,6,-4,6] =
[-4,6,-4,3,5,6,-3,-6,5,-5,4,-6,4,-5] W191 d3 g4 EE47.05742
LCF [3,6,6,-3,5,6,6;-] = [3,6,3,-3,5,-3,6;-] = [5,3,-5,4,-3,-5,4;-] =
[6,6,-4,7,5,-6,-6,-6,3,-5,7,-3,4,6] = [-3,6,6,7,-6,-6,3,-6,-6,-3,7,3,6,6] W200
d4 g4 EE48.02357
LCF [5,-6,6,-4,7,-5,6,3,-6,6,-3,7,-6,4] W193 d3 g4 EE47.48376
LCF [7,4,-4,6,-5,-4,4,7,3,-6,-4,-3,4,5] =
[7,-6,-5,5,6,-6,6,7,-5,6,-6,5,-6,6] = [-5,-4,4,-4,3,5,-4,-3,4,5,-5,4,-4,4] W191
d3 g4 EE47.14460
LCF [5,-3,4,6,6,-5,-4;-] = [-6,-3,5,6,6,-6,5;-] W193 d3 g4 EE47.30020
LCF [5,-6,6,-4,7,-5,4]^2 = [-6,5,-3,7,-6,4,-5,4,6,-4,7,-4,6,3] =
[-4,7,3,6,-6,-3,5,6,7,-6,4,-5,6,-6] = [6,-5,-4,4,-5,4,-6,-4,3,-4,5,-3,4,5] =
[-4,6,4,-5,5,6,-4,-6,5,-5,4,-6,5,-5] W191 d3 g4 EE47.04309
LCF [7,-4,3,7,4,-3,6,7,-4,4,7,4,-6,-4] = [5,-6,-4,7,4,-5,7,4,-4,6,7,-4,4,7]
= [6,4,-4,7,4,-4,-6,4,-4,4,7,-4,4,-4] = [7,-3,6,-6,5,-6,4,7,-6,-5,-4,6,3,6] =
[-4,6,-5,3,-6,4,-3,-6,5,-4,4,5,6,-5] W191 d3 g4 EE47.11027
LCF [5,6,-5,6,6,-5,6;-] = [5,-4,-4,4,5,-5,4;-] = [-4,6,-5,-4,4,4,6;-] =
[5,6,-6,-6,5,-5,6;-] W191 d3 g4 EE47.06319
LCF [5,-3,-5,4,5,-5,4;-] = [-4,6,-5,-4,5,3,6;-] =
[-5,5,6,-5,7,5,-5,6,-6,5,-5,7,5,-6] W195 d4 g4 EE47.34090

Figure 22:

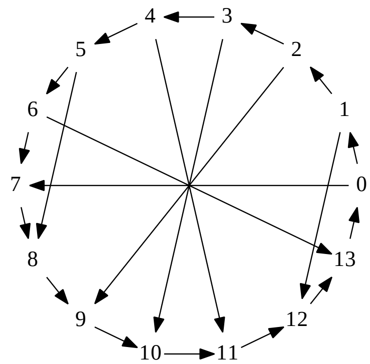
LCF [-3,4,-3,4,5,-4,4;-] = [-6,5,7,4,7,7,-5,-4,6,7,3,7,7,-3] =
[7,7,-6,-5,5,6,7,7,7,-5,6,-6,5,7] = [7,3,-3,7,-3,4,5,7,4,-4,7,-5,-4,3] W199 d4
g4 EE47.50380
LCF [5,3,-6,6,-3,-5,7]^2 W217 d4 g4 EE49.25588
LCF [7,-6,-4,7,4,6,7,7,-4,6,7,-6,4,7] = [5,-4,7,-4,4,-5,6,3,-4,7,-3,4,-6,4]
W193 d4 g4 EE47.19176

```
   LCF [4,7,-4,7,-4,4,7]^2  = [7,-4,4,7,-5,4,-4,7,4,-4,7,4,-4,5] =
[7,-6,-5,-4,4,5,6,7,-4,6,-5,5,-6,4] W189 d3 g5 EE46.75706
   LCF [-5,5,-4,7,4,-6,-5,4,-4,5,7,-4,4,6]  =
[5,-5,5,-6,4,-5,6,-5,-4,4,5,6,-6,-4] = [6,-6,5,-6,-5,5,-6,-5,4,6,-5,6,-4,5]
W189 d3 g5 EE46.62280
   LCF [-4,5,6,-4,5,6,-5;-]  = [-6,-4,5,7,-5,4,6,-5,6,-4,7,4,-6,5] W190 d4 g5
EE46.60848
   LCF [5,-6,5,-4,7,-5,4,-5,4,6,-4,7,-4,4]  =
[-5,5,-4,7,4,6,-5,6,-4,5,7,-6,4,-6] = [7,-5,6,-6,-5,4,6,7,-6,-4,5,6,-6,5] =
[5,-6,-5,-4,4,-5,4,5,-4,6,-4,5,-5,4] W189 d3 g5 EE46.70425
   LCF [-5,4,-4,7,4,-4,5,6,-4,5,7,-5,4,-6]  W189 d3 g5 EE46.67561
   LCF [5,-4,7,-4,4,-5,4]^2  = [4,7,-4,7,-4,4,5,6,7,-4,7,-5,4,-6] =
[7,-4,6,7,-5,4,6,7,-6,-4,7,4,-6,5] W190 d4 g5 EE46.77138
   LCF [4,6,-5,5,-4,5,6;-]  = [4,5,-5,5,-4,5,-5;-] = [-5,6,-5,5,-5,5,6;-] W189
d3 g5 EE46.50858
   LCF [-4,4,5,-4,5,-4,5;-]  = [7,-5,-4,5,-5,4,5]^2 =
[7,-6,-5,5,-5,5,6,7,-5,6,-5,5,-6,5] W189 d3 g5 EE46.64714
   LCF [-5,5]^7  W189 d3 g6 EE46.27353
```

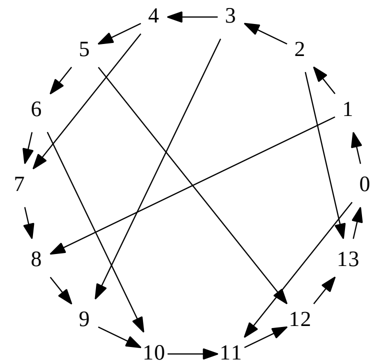
Some of these have appeared in the nuclear physics literature [15, 36]. Ponzano's figures (1)–(8) are number 58, 32, 68, 33, 71, 57, 59 and 82 in this list of 84.



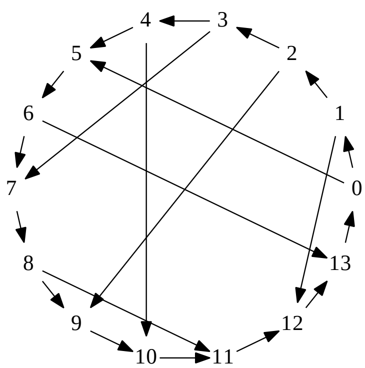
[3,-3,4,-3,5,3,-4;-]



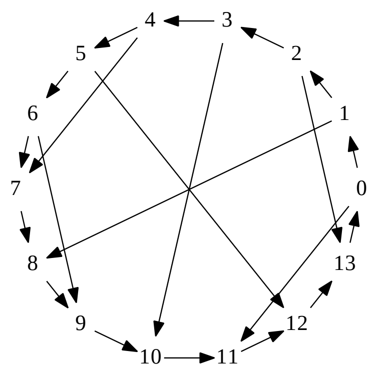
[3,7,7,-3,7,7,7]^2



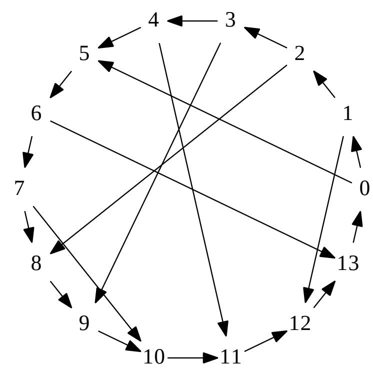
[7,3,-4,7,-3,-6,3,7,3,-3,7,-3,4,6]



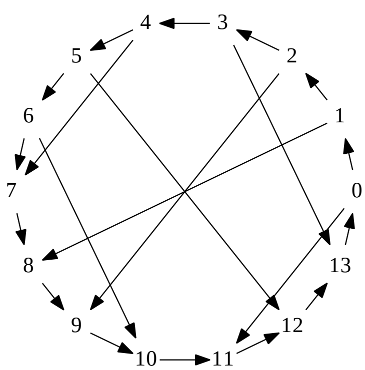
[-3,7,4,6,-5,7,-4,3,7,-6,-3,3,7,5]



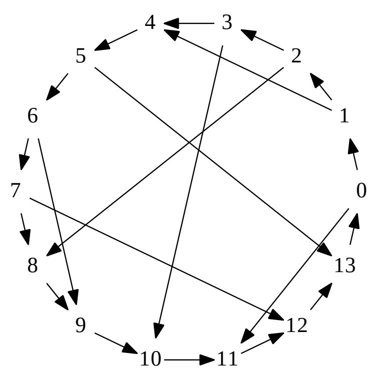
[7,3,-3,7,-3,7,3]^2



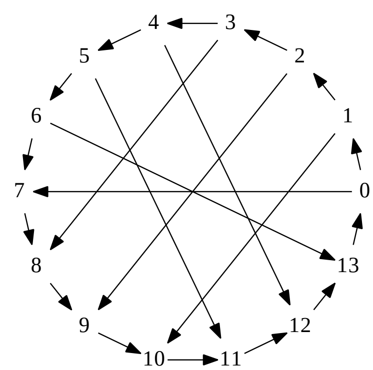
[6,6,-3,7,5,7,-6,-6,3,-5,7,-3,7,3]



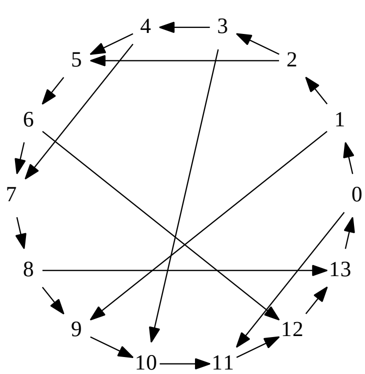
[7,-4,3,7,4,-3,7]^2



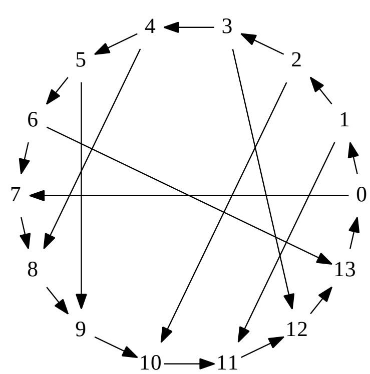
[-3,3,6,7,-3,-6,3,5,-6,-3,7,3,-5,6]



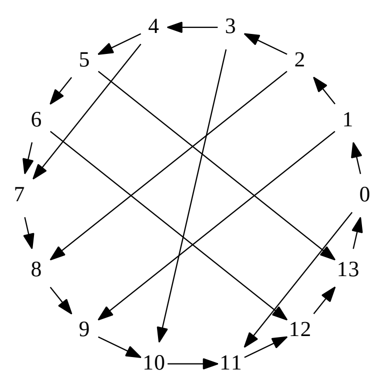
[7,7,-6,6,-5,7,5]^2



[-3,-6,3,7,3,-3,6,-3,5,6,7,3,-6,-5]



[-4,-6,4,4,7,7,-4,-4,5,6,4,7,7,-5]



[3,-6,6,-3,7,-6,6]^2

FIG. 16. Graphs on $n = 14$ vertices which are irreducible (start).

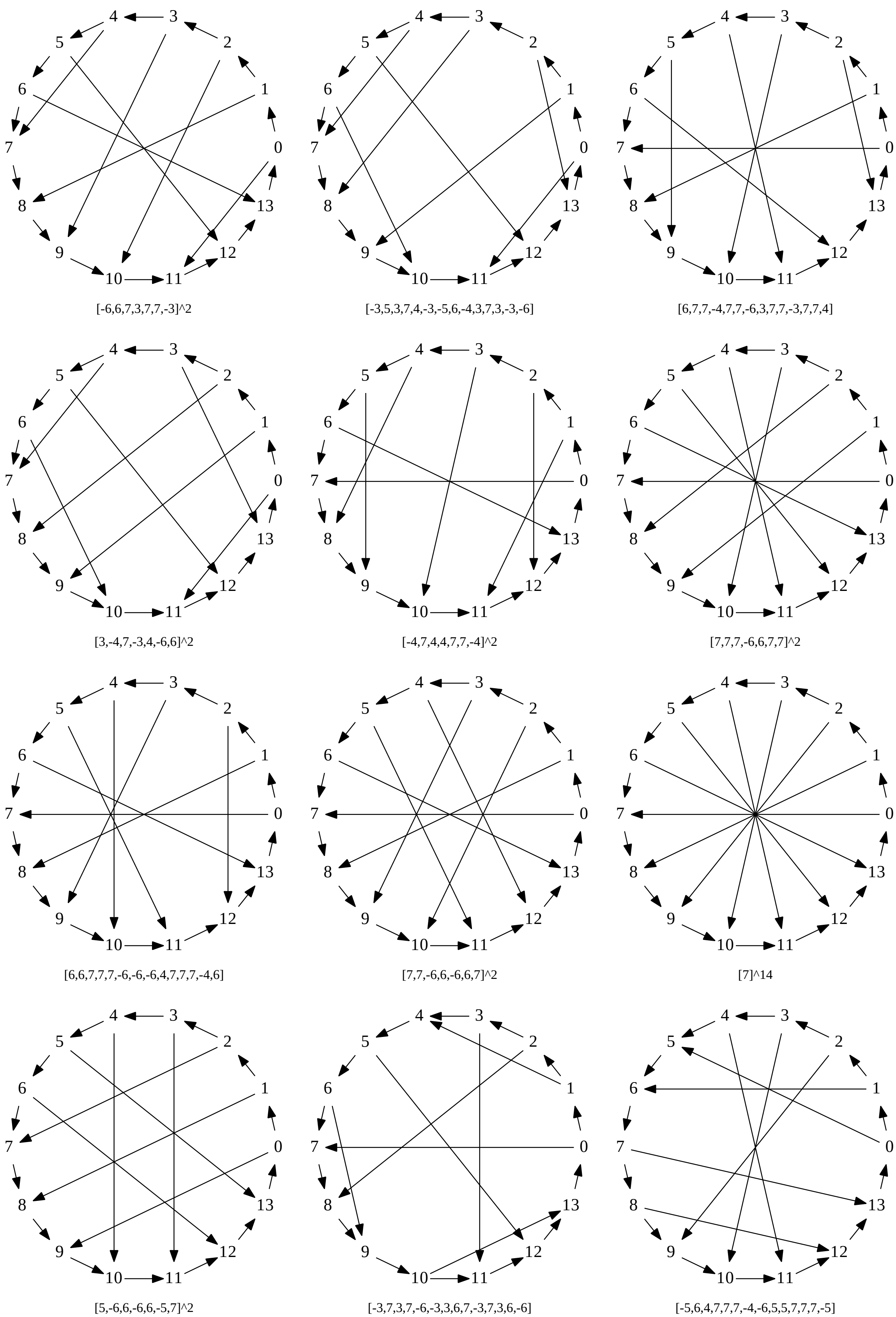


FIG. 17. Graphs on $n = 14$ vertices which are irreducible (continued).

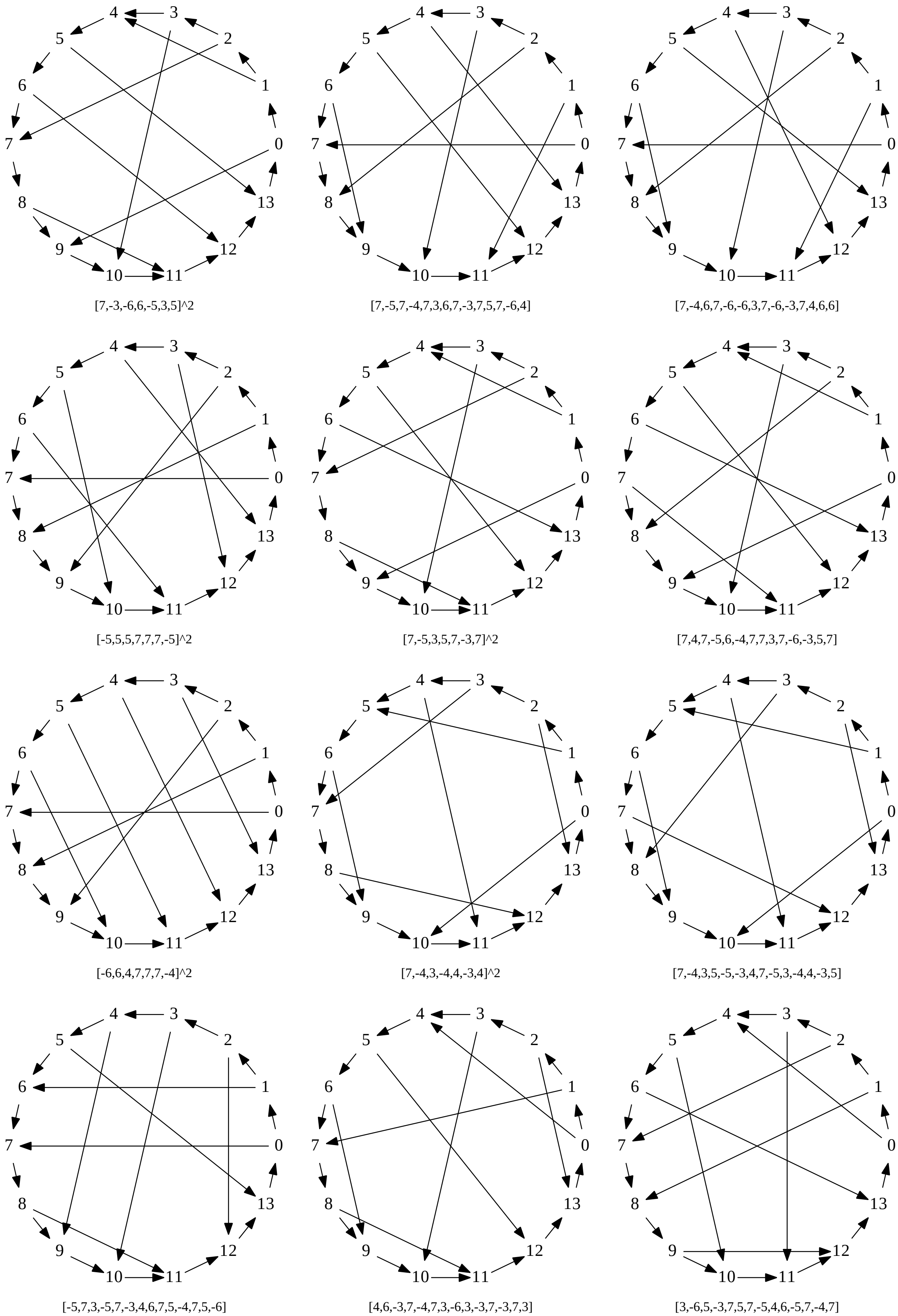


FIG. 18. Graphs on $n = 14$ vertices which are irreducible (continued).

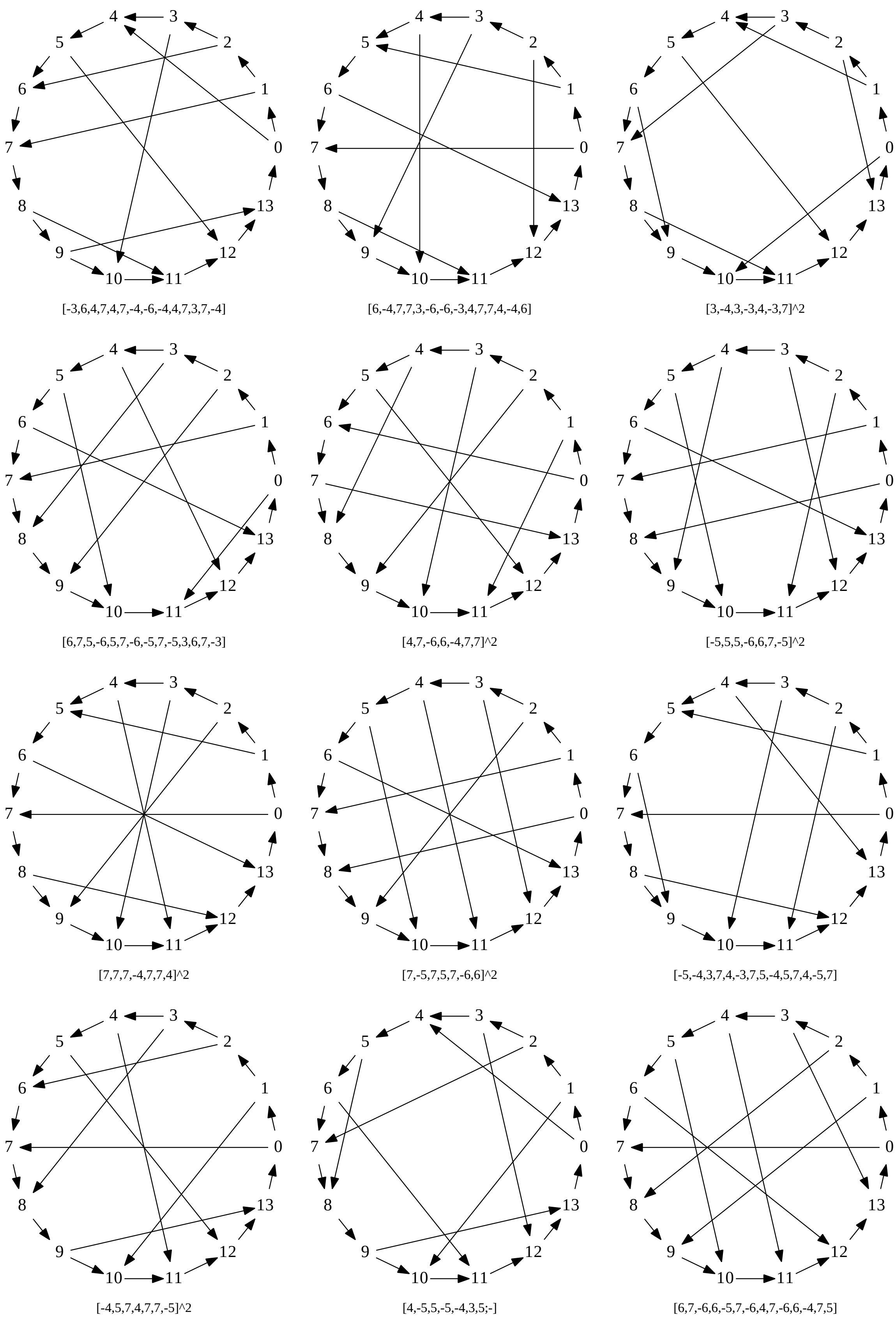
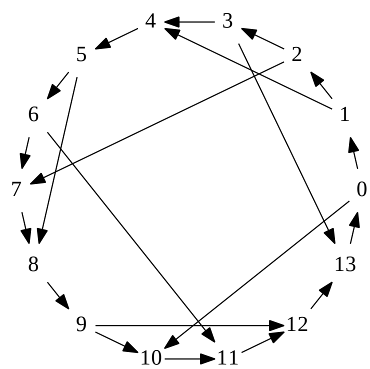
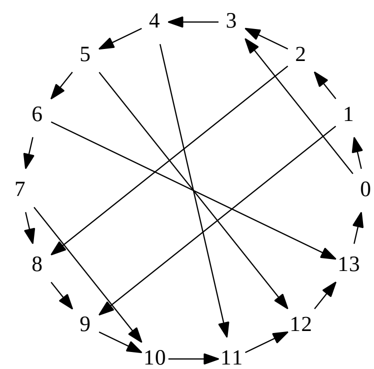


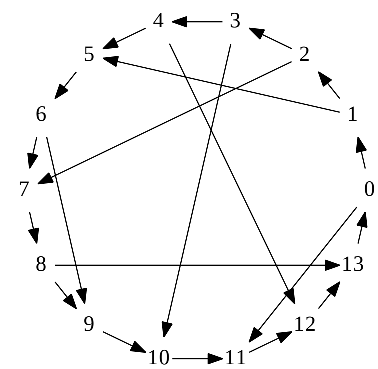
FIG. 19. Graphs on $n = 14$ vertices which are irreducible (continued).



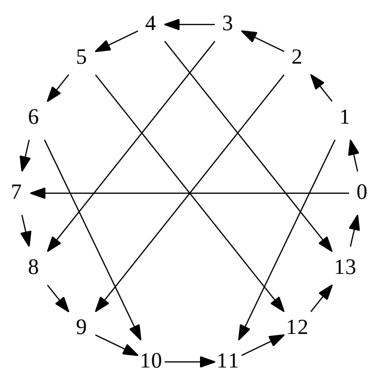
[-4,3,5,-4,-3,3,5;-]



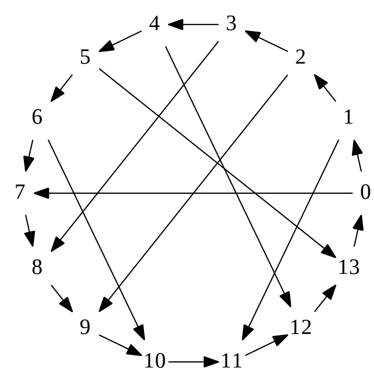
[7,3,-6,6,-3,7,7]^2



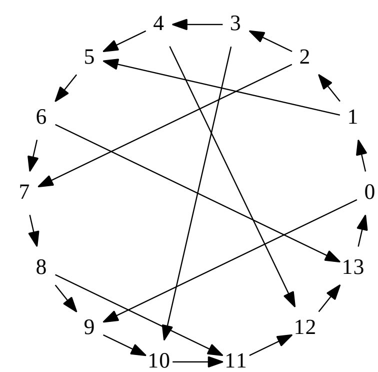
[-3,4,5,7,-6,-4,3,-5,5,-3,7,3,6,-5]



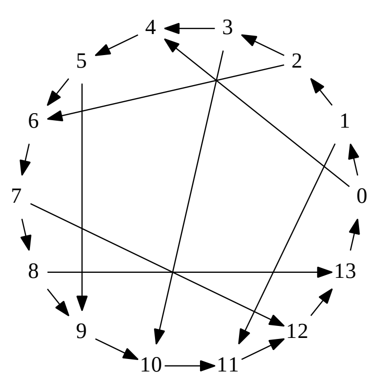
[5,-5,7,4,7,-5,7,-4,4,7,5,7,-4,7]



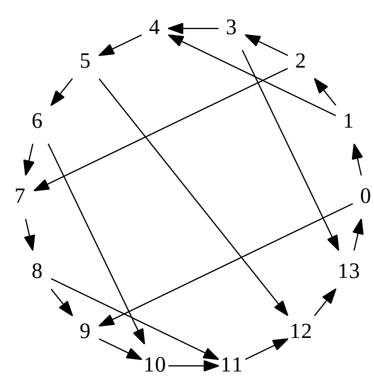
[-6,4,7,-5,7,-4,4,6,6,7,-4,7,5,-6]



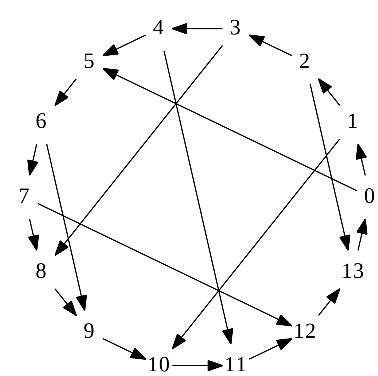
[7,-5,3,5,7,-3,6,7,-5,4,5,7,-6,-4]



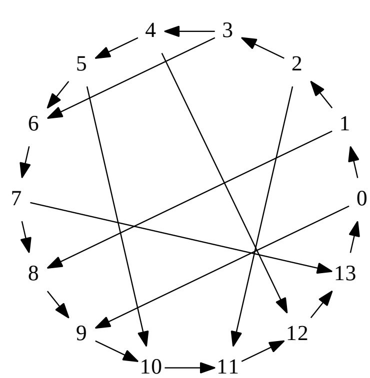
[4,-4,4,7,-4,4,-4,5,5,-4,7,4,-5,-5]



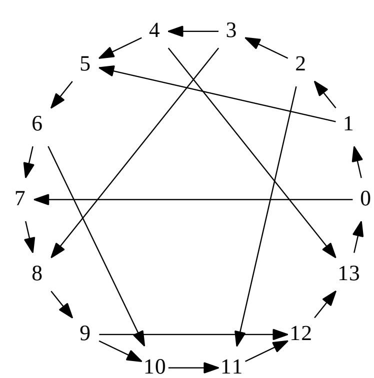
[-3,7,4,-5,3,5,-4]^2



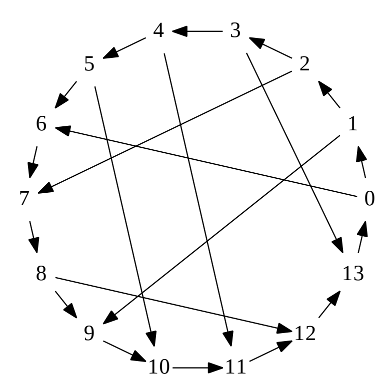
[-3,5,7,-5,3,5,-5]^2



[5,-3,6,7,5,-5,5,6,-6,-5,7,-5,3,-6]



[-5,-4,4,7,-5,3,-4,5,-3,5,7,4,-5,5]



[6,-6,5,-4,7,5,-6,-5,4,6,-5,7,-4,4]

FIG. 20. Graphs on $n = 14$ vertices which are irreducible (continued).

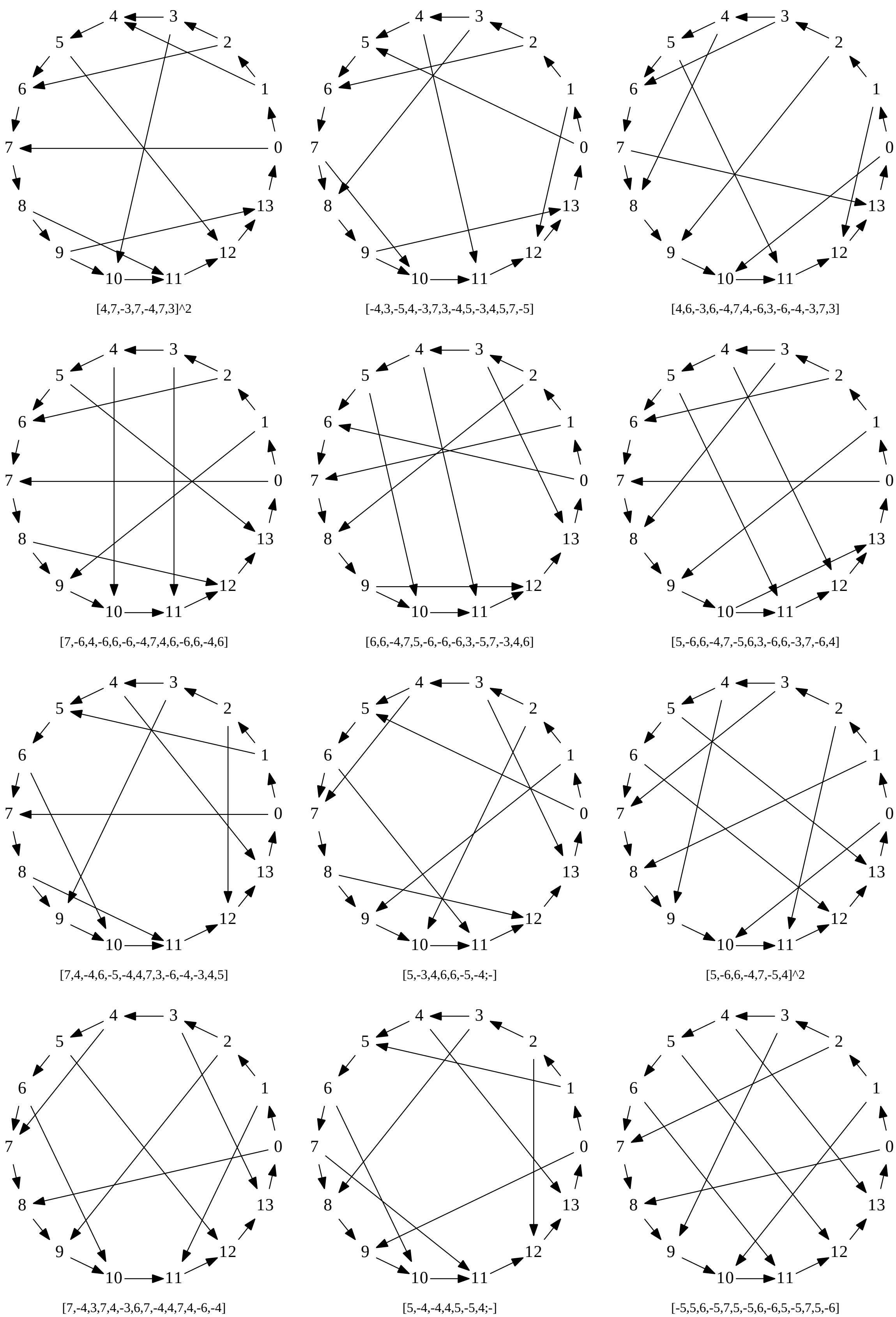
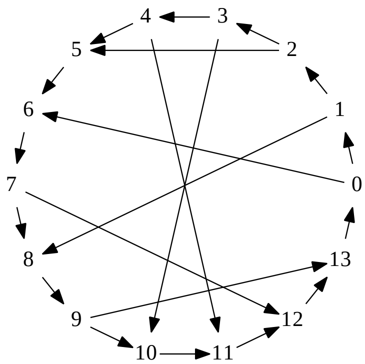
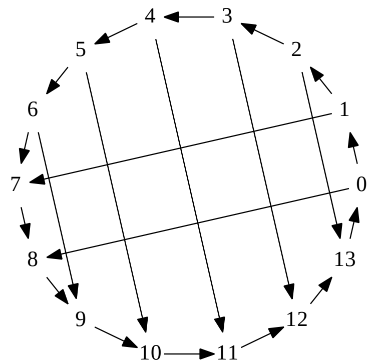


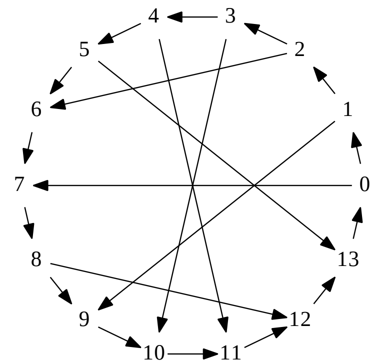
FIG. 21. Graphs on $n = 14$ vertices which are irreducible (continued).



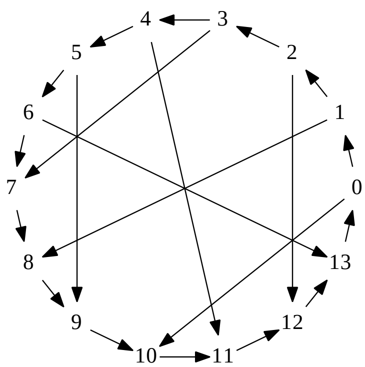
[-6,5,7,4,7,7,-5,-4,6,7,3,7,7,-3]



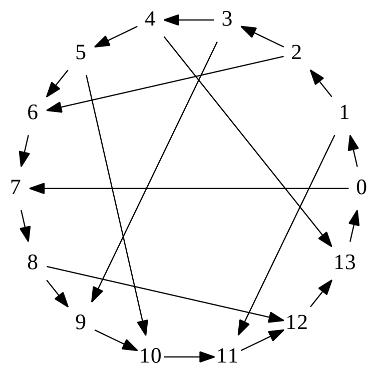
[5,3,-6,6,-3,-5,7]^2



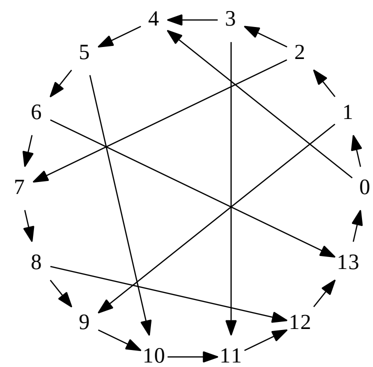
[7,-6,-4,7,4,6,7,7,-4,6,7,-6,4,7]



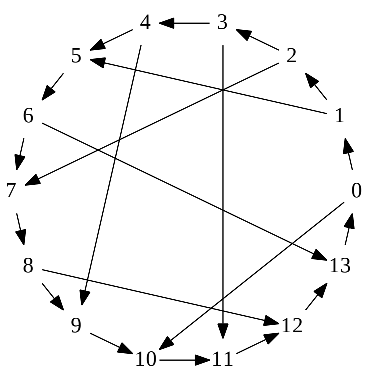
[4,7,-4,7,-4,4,7]^2



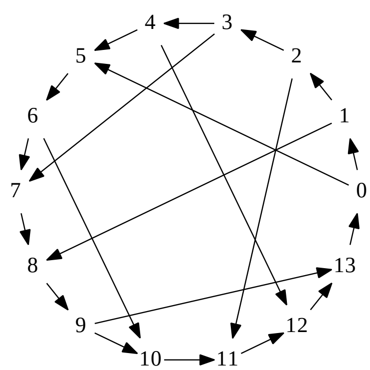
[-5,5,-4,7,4,-6,-5,4,-4,5,7,-4,4,6]



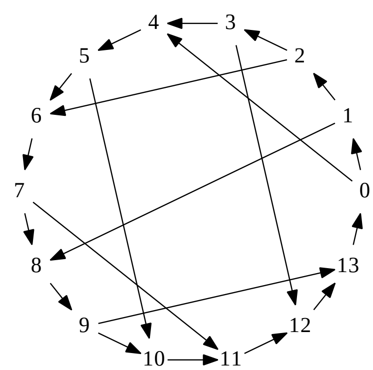
[-6,-4,5,7,-5,4,6,-5,6,-4,7,4,-6,5]



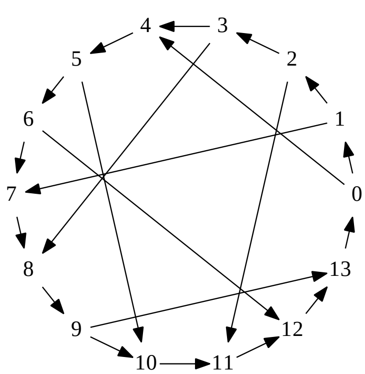
[5,-6,5,-4,7,-5,4,-5,4,6,-4,7,-4,4]



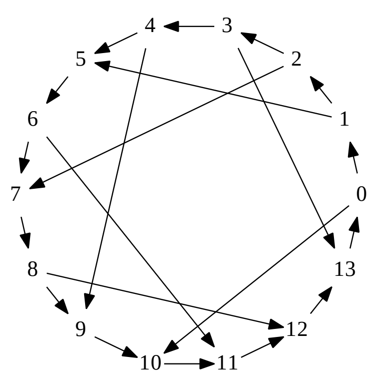
[-5,4,-4,7,4,-4,5,6,-4,5,7,-5,4,-6]



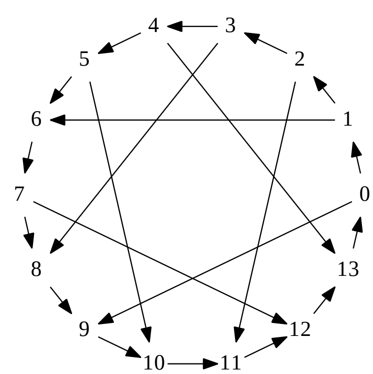
[5,-4,7,-4,4,-5,4]^2



[4,6,-5,5,-4,5,6;-]



[-4,4,5,-4,5,-4,5;-]



[-5,5]^7

FIG. 22. Graphs on $n = 14$ vertices which are irreducible (end).

VII. SUMMARY

We have plotted the non-isomorphic simple cubic graphs up to 12 vertices ($18j$-symbols) plus the subset on 14 vertices that defines classes of $21j$-symbols. Hamiltonian cycles have been identified. The associated LCF notation introduces a convenient ordering representation which combats the bewildering variety of planar graphical representations as the number of edges becomes large.

ACKNOWLEDGMENTS

The graphs were generated with Meringer's program `genreg` [33] and have been plotted with the `neato` program of the `graphviz` package.

[1] Aldred, R. E. L., D. Van Dyck, G. Brinkmann, V. Fack, and B. D. McKay, 2009, Discrete Math. **157**(2), 377.
[2] Crediting Truncated: Ališauskas, S., 2002, J. Math. Phys. **43**(3), 1547.
[3] Anderson, R. W., V. Aquilanti, and A. Marzuoli, 2009, J. Phys. Chem. A **113**(52), 15106.
[4] Aquilanti, V., A. C. P. Bitencourt, C. da S. Ferreira, A. Marzuoli, and M. Ragni, 2009, Theor. Chem. Acc. **123**(3–4), 237.
[5] Balcar, E., and S. W. Lovesey, 2009, *Introduction to the graphical theory of angular momentum*, volume 234 of *Springer Tracts in Modern Physics* (Springer).
[6] Bar-Shalom, A., and M. Klapisch, 1988, Comp. Phys. Commun. **50**(3), 375.
[7] Bau, S., 1990, Australas. J. Comb. **2**, 57.
[8] Brinkmann, G., 1996, J. Graph Theory **23**(2), 139.
[9] Brinkmann, G., J. Goedgebeur, and B. D. McKay, 2011, Disc. Math. Theor. Comp. Sci. **13**(2), 69.
[10] Brouder, C., and G. Brinkmann, 1997, J. Electron. Spectrosc. **86**(1–3), 127.
[11] Clark, L., and R. Entringer, 1983, Period. Math. Hung. **14**(1), 57.
[12] Coxeter, H. S. M., R. Frucht, and D. L. Powers, 1981, *Zero-symmetry graphs: trivalent graphical regular representations of groups* (Academic Press), ISBN 0-12-19458-04.
[13] Dalby, A., J. G. Nourse, W. D. Hounshell, A. K. I. Gushurst, D. L. Grier, B. A. Leland, and J. Laufer, 1992, J. Chem. Inf. Comput. Sci. **32**(3), 244.
[14] Danos, M., and U. Fano, 1998, Phys. Rep. **304**(4), 155.
[15] Di Leva, A., and G. Ponzano, 1967, Nuovo Cimento **51A**(4), 1107.
[16] Dürr, H. P., and F. Wagner, 1968, Nuovo Cimento **53A**(1), 255.
[17] Edmonds, A. R., 1957, *Angular momentum in quantum mechanics* (Princeton University Press), E: the factor $\sqrt{2j_2+1}$ in the denominator of (3.7.3) ought read $\sqrt{2j_3+1}$.
[18] El-Batanoni, F., M. El-Nadi, and G. L. Vysotsky, 1966, Nucl. Phys. **82**(2), 407.
[19] Fack, V., S. N. Pitre, and J. Van der Jeugt, 1997, Comput. Phys. Commun. **101**(1–2), 155.
[20] Frucht, R., 1976, J. Graph Theory **1**(1), 45.
[21] Granovskii, Y. I., and A. S. Zhednov, 1993, J. Phys. A.: Math. Gen. **26**(17), 4339.
[22] Graovac, A., and T. Pisanski, 1991, J. Math. Chem. **8**(1), 53.
[23] Gutman, I., and A. Graovac, 2007, Chem. Phys. Lett. **436**(1–4), 294.
[24] Imrich, W., 1971, Aequationes mathematicae **6**(1), 6.
[25] Jahn, H. A., and J. Hope, 1954, Phys. Rev. **93**(2), 318.
[26] Kent, R. D., and M. Schlesinger, 1989, Phys. Rev. A **40**(2), 536.
[27] Lievens, S., and J. Van der Jeugt, 2002, J. Math. Phys. **43**(7), 3824.
[28] Lima, P. M., 1991, Comput. Phys. Commun. **66**(1), 89.
[29] Louck, J. D., 2008, *Unitary symmetry and combinatorics* (World Scientific, Singapore), ISBN 981-281-472-8.
[30] Martello, S., 1983, ACM Trans. Math. Software **9**(1), 131.
[31] Massot, J.-N., E. El-Baz, and J. Lafoucriére, 1967, Rev. Mod. Phys. **39**(2), 288.
[32] Mattis, M. P., and E. Braaten, 1989, Phys. Rev. D. **39**(9), 2737.
[33] Meringer, M., 1999, J. Graph Theory **30**(2), 137.
[34] Newman, D. J., and J. Wallis, 1976, J. Phys. A: Math. Gen. **9**(12), 2021.
[35] Ord-Smith, R. J., 1954, Phys. Rev. **94**(5), 1227.
[36] Ponzano, G., 1965, Nuovo Cimento **36**(2), 385.
[37] Schwenk, A. J., 1989, J. Comb. Theory B **47**, 53.
[38] Sloane, N. J. A., 2003, Notices Am. Math. Soc. **50**(8), 912, http://oeis.org/, URL `http://oeis.org/`.
[39] Stone, A. P., 1956, Math. Proc. Cambr. Phil. Soc. **52**, 424.
[40] Stone, A. P., 1957, Proc. Phys. Soc. A **70**(12), 908.
[41] van Dyck, D., G. Brinkmann, V. Fack, and B. D. McKay, 2005, Comput. Phys. Commun. **173**(1–2), 61.
[42] van Dyck, D., and V. Fack, 2007, Disc. Math. **307**, 1506.
[43] Weininger, D., 1988, J. Chem. Inf. Comput. Sci. **28**(1), 31.
[44] Wormer, P. E. S., and J. Paldus, 2006, Adv. Quant. Chem. **51**, 59.
[45] Yutsis, A. P., I. B. Levinson, V. V. Vanagas, and A. Sen, 1962, *Mathematical Apparatus of the Theory of Angular Momentum* (Israel program for scientific translations), E: Eq. (20.6) in [18], Fig. 17.1 in [32].