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Counting Humps in Motzkin paths

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Abstract. In this paper we study the number of humps (peaks) in Dyck, Motzkin and Schröder paths. Recently A. Regev noticed that the number of peaks in all Dyck paths of order n is one half of the number of super Dyck paths of order n. He also computed the number of humps in Motzkin paths and found a similar relation, and asked for bijective proofs. We give a bijection and prove these results. Using this bijection we also give a new proof that the number of Dyck paths of order n with k peaks is the Narayana number. By double counting super Schröder paths, we also get an identity involving products of binomial coefficients.

Keywords: Dyck paths, Motzkin paths, Schröder paths, humps, peaks, Narayana number.

AMS Classification: 05A15.

1 Introduction

A *Dyck path* of order (semilength) n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to $(2n,0)$, using up-steps $(1,1)$ (denoted by U) and down-steps $(1,-1)$ (denoted by D) and never going below the x-axis. We use $\mathcal{D}_n$ to denote the set of Dyck paths of order n. It is well known that $\mathcal{D}_n$ is counted by the n-th *Catalan number* (A000108 in [8])

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A *peak* in a Dyck path is two consecutive steps UD. It is also well known (see, for example, [1, 4, 11]) that the number of Dyck paths of order n with k peaks is the *Narayana number* (A001263):

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

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Counting Dyck paths with restriction on peaks has been studied by many authors, see for example [2, 3, 5]. Here we are interested in counting peaks in all Dyck paths of order n. By summing over the above formula over k we immediately get the following result: the total number of peaks in all Dyck paths of order n is

$$pd_n = \sum_{k=1}^{n} kN(n,k) = \binom{2n-1}{n}.$$

If we allow a Dyck path to go bellow the x-axis, we get a *super Dyck path.* Let $\mathcal{SD}_n$ denote the set of super Dyck paths of order n. By standard arguments we have

$$sd_n = \#\mathcal{SD}_n = \binom{2n}{n} = 2\binom{2n-1}{n} = 2pd_n, \tag{1.1}$$

That is, the number of super Dyck paths of order n is twice the number of peaks in all Dyck paths of order n. This curious relation was first noticed by Regev [7], who also noticed that similar relation holds for Motzkin paths, which we will explain next.

A *Motzkin path of order* n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to $(n,0)$, using up-steps $(1,1)$, down-steps $(1,-1)$ and flat-steps $(1,0)$ (denoted by F) that never goes below the x-axis. Let $\mathcal{M}_n$ denote all the Motzkin paths of order n. The cardinality of $\mathcal{M}_n$ is the n-th *Motzkin number* m_n (A001006), which satisfies the following recurrence relation

$$m_0 = 1, \quad m_1 = 1, \quad m_n = m_{n-1} + \sum_{i=2}^{n} m_{i-2}m_{n-i}, \quad for \quad n \geq 2,$$

and have generating function

$$\sum_{n\geq 0} m_n x^n = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2}.$$

A *hump* in a Motzkin path is an up step followed by zero or more flat steps followed by a down step. We use hm_n to denote the total number of humps in all Motzkin paths of order n. We can similarly define *super Motzkin paths* to be Motzkin paths that are allowed to go below the x-axis, and use $\mathcal{SM}_n$ to denote the set of super Motzkin paths of order n. Using a recurrence relation and the WZ method [6, 12], Regev ([7]) proved that

$$sm_n = \#\mathcal{SM}_n = \sum_{j\geq 0} \binom{n}{j}\binom{n-j}{j} = 2hm_n + 1 \tag{1.2}$$

and asked for a bijective proof of (1.1) and (1.2). The main result of this paper is such a bijective proof.

Let $\mathcal{SM}_n^{UU}(k)$ $(\mathcal{SM}_n^{UD}(k))$ denote the set of paths in $\mathcal{SM}_n$ with k peaks and the first non-flat step is U, and the last non-flat step is U (D). Let $\mathcal{SM}_n^{U*}$ denote all paths in $\mathcal{SM}_n$ whose first non-flat step is U, and define

$$\mathcal{HM}_n = \{(M,P) | M \in \mathcal{M}_n, P \text{ is a hump of } M\}.$$

The main result of this paper is the following:

Theorem 1.1 *There is a bijection* $\Phi : \mathcal{HM}_n \to \mathcal{SM}_n^{U*}$ *such that if* $(M, P) \in \mathcal{HM}_n$ *and* $L = \Phi(M, P)$*, then there are* k *humps in* M *if and only if* $L \in \mathcal{SM}_n^{UU}(k-1) \cup \mathcal{SM}_n^{UD}(k)$*.*

The outline of the paper is as follows. In Section 2 we define the bijection Φ and prove Theorem 1.1. In section 3 we apply Φ to Dyck paths and give a new proof of the Narayana numbers. In section 4 we apply Φ to Schröder paths and get an identity involving products of binomial coefficients by double counting super Schröder paths whose F steps are m-colored.

2 The bijection $\Phi : \mathcal{HM}_n \leftrightarrow \mathcal{SM}_n^{U*}$

Note that a Motzkin path M of order n can also be considered as a sequence $M = M_1M_2 \cdots M_n$, with $M_i \in \{U, F, D\}$, and the number of U's is not less than the number of D's in every subsequence $M_1M_2 \cdots M_k$ of M. Hence a hump in M is a subsequence $P = M_iM_{i+1} \cdots M_{i+k+1}, k \geq 0$, such that $M_i = U$, $M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F$ and $M_{i+k+1} = D$. We call the end point of step M_i a *hump point*, and will also denoted as P. Similarly, if there exists i such that $M_i = D$, $M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F, k \geq 0$, $M_{i+k+1} = U$, then we call the subsequence $M_iM_{i+1} \cdots M_{i+k+1}$ a *valley* of M, and the end point of M_{i+k} is called a *valley point*. The end point $(n, 0)$ of M is also considered as a valley point.

Suppose L is a path in $\mathbb{Z} \times \mathbb{Z}$ from $O(0, 0)$ to $N(n, 0)$, and A a lattice point on M, we use x_A and y_A to denote the x-coordinate and y-coordinate of A, respectively. The sub-path of L from point A to point B is denoted by L_{AB}. We use $\bar{L}$ to denote the lattice path obtained from L by interchanging all the up-steps and down-steps in L, and keep the flat-steps unchanged.

Now we are ready to define the map Φ and prove Theorem 1.1.

Proof of Theorem 1.1:

(1) The map $\Phi : \mathcal{HM}_n \to \mathcal{SM}_n^{U*}$.

For any $(M, P) \in \mathcal{HM}_n$, we define $L = \Phi(M, P)$ by the following rules:

- Let C be the leftmost valley point in M such that $x_C > x_P$;
- Let B be the rightmost point in M such that $x_B < x_P, y_B = y_C$;
- Let A be the rightmost point in M such that $y_A = 0, x_A \leq x_B$;
- Set $L_0 = M_{OA}$, $L_1 = M_{AB}$, $L_2 = M_{BC}$, $L_3 = M_{CN}$ (Note that L_0, L_1 and L_3 may be empty);
- Define $L = \Phi(M, P) = L_0L_2\overline{L_3L_1}$.

Now we will prove that $L \in \mathcal{SM}_n^{U*}$. According to the above definition, L_0 and L_2 are both Motzkin paths, therefore $\#U = \#D$ in L_0 and L_2. And for L_1, we have $\#U - \#D = y_B - y_A = y_B = y_C$, for L_3, $\#U - \#D = -y_C$. Therefore the total number of U's is as much as that of D's in L. Thus L is a super Motzkin path of order n. Moreover, the first non-flat step in L must be in L_0 (when L_0 is not empty) or in L_2 (when L_0 is empty), and L_0, L_2 are both Motzkin paths, hence the first step leaving the x-axis must be a U. Therefore we proved that $L = \Phi(M, P) \in \mathcal{SM}_n^{U*}$.

(2) The inverse of Φ.

For any $L \in \mathcal{SM}_n^{U*}$, we define Ψ by the following rules:

- Let B be the leftmost point such that $y_B = 0$, and L goes below the x-axis after B. (If such a point does not exist, then set $B = N$);
- Let A be the rightmost point in L such that $x_A < x_B, y_A = 0$;
- Let C be the rightmost point in L such that $x_C \geq x_B$, and $\forall G$, $x_G \geq x_B$ implies that $y_C \geq y_G$;
- Let P be the rightmost hump point in L such that $x_P < x_B$;
- Set $L_0 = L_{OA}$, $L_1 = L_{AB}$, $L_2 = L_{BC}$, $L_3 = L_{CN}$ (Note that L_0, L_2 and L_3 may be empty);
- Set $M = L_0\overline{L_3}L_1\overline{L_2}$, and $\Psi(L) = (M, P)$.

Now we prove that $\Psi = \Phi^{-1}$. Since C is the highest point in L_3, and $\overline{L_3}$ and L_3 are symmetric with respect to the line $y = y_C$, C is mapped to the lowest point in $\overline{L_3}$. Moreover, L_0 and L_1 are both Motzkin paths, then $L_0\overline{L_3}L_1$ does not go below the x-axis, and the y-coordinate of the end point of $L_0\overline{L_3}L_1$ is y_C. In $\overline{L_2}$, the end point is the lowest point, and the start point of $\overline{L_2}$ is y_C higher than the end point. So $M = L_0\overline{L_3}L_1\overline{L_2}$ ends on the x-axis and never goes below it, i.e., $M \in \mathcal{M}_n$. Thus $\Psi(L) \in \mathcal{HM}_n$, and it is not hard to see that $\Psi = \Phi^{-1}$.

(3) There are k humps in M if and only if $\Phi(M, P) \in \mathcal{SM}_n^{UD}(k) \cup \mathcal{SM}_n^{UU}(k-1)$.

Since $\Phi(M) = L_0L_2\overline{L_3L_1} = L$, the number of humps changes only in sub-paths $\overline{L_3}$ and $\overline{L_1}$ when M is converted to L. If the last step of L_1 is U, then the last step in $\overline{L_1}$ becomes D. The number of humps in L_1 is the same as the number of humps in $\overline{L_1}$, and the number of humps in $\overline{L_3}$ is 1 less than the number of humps in L_3. The last step in $\overline{L_3}$ is U step, so concatenating $\overline{L_1}$ with $\overline{L_3}$ yields a new hump. Therefore the total number of humps in L is the same as in M. Thus we have $\Phi(M, P) \in \mathcal{SM}_n^{UD}(k)$.

If the last step in L_1 is D, then the last step in $\overline{L_1}$ is U. The number of humps in $\overline{L_1}$ is 1 less than the number of humps in L_1, and the humps in $\overline{L_3}$ is 1 less than the number of humps in L_3. Moreover, the last step in $\overline{L_3}$ is U, so concatenating $\overline{L_1}$ with $\overline{L_3}$ yields a new hump. Therefore the total number of humps in L is 1 less than the number humps in M. Thus we have $\Phi(M, P) \in \mathcal{SM}_n^{UU}(k-1)$.

∎

As an example, Figure 1 shows a Motzkin path $M \in \mathcal{M}_{41}$ with a circled hump point P, and Figure 2 shows a super Motzkin path $L \in \mathcal{SM}_{41}^{U*} = \Phi(M, P)$.

From Theorem 1.1 we can easily get the following result.

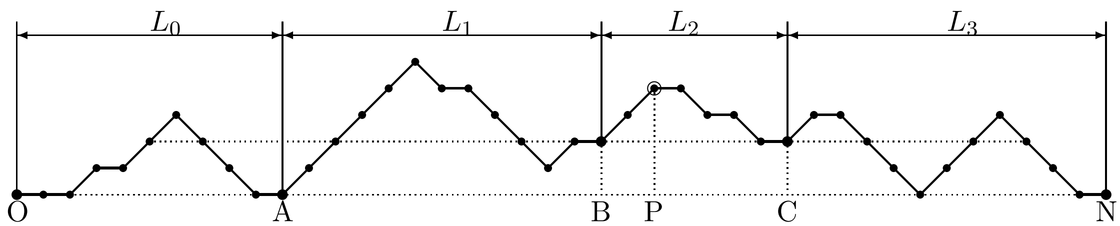

Figure 1: A Motzkin path $M \in \mathcal{M}_{41}$ with a circled hump point P.

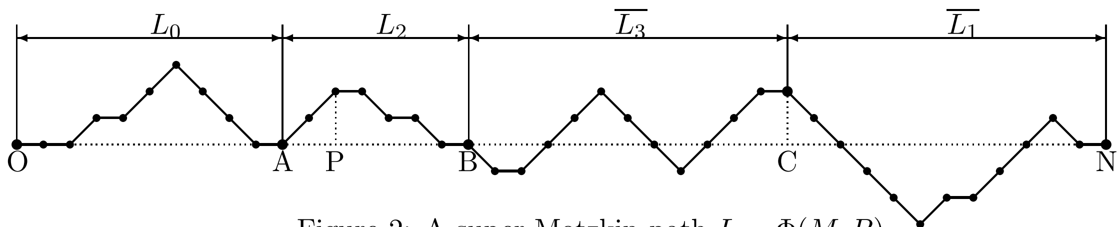

Figure 2: A super Motzkin path $L = \Phi(M, P)$.

Corollary 2.2 *For all $n \geq 0$, we have*

$$sm_n = 2hm_n + 1, \tag{2.1}$$

and

$$hm_n = \frac{1}{2}\left(\sum_{j\geq 0}\binom{n}{j}\binom{n-j}{j} - 1\right). \tag{2.2}$$

Proof. Equation (2.1) follows immediately from Theorem 1.1. To prove (2.2) we count super Motzkin paths with j U steps. We can first choose the j U steps among the total n steps, then choose j steps as D steps among the remaining $n-j$ steps. Thus we have

$$sm_n = \sum_{j\geq 0}\binom{n}{j}\binom{n-j}{j}.$$

Combine with equation (2.1) we get equation (2.2). ∎

3 Counting peaks in Dyck paths and the Narayana numbers

Note that when restricted to Dyck paths, Φ is a bijection between super Dyck paths and peaks in Dyck paths. Therefore we have the following result.

Corollary 3.3 *For all $n \geq 0$, we have*

$$sd_n = 2pd_n,$$

and

$$pd_n = \binom{2n-1}{n}.$$

Moreover, from the bijection Φ we can easily get a new proof for the Narayana numbers. To this end we need the following lemma.

Lemma 3.4 *Let $\mathcal{SD}_n^{UD}(k)$ ($\mathcal{SD}_n^{UU}(k)$) denote the set of super Dyck paths of order n with k peaks whose first step is U and last step is D (U), then we have*

$$\#\mathcal{SD}_n^{UD}(k) = \binom{n-1}{k-1}^2, \tag{3.1}$$

$$\#\mathcal{SD}_n^{UU}(k) = \binom{n-1}{k-1}\binom{n-1}{k}, \tag{3.2}$$

and the number of super Dyck paths with k peaks of order n is $\binom{n}{k}^2$.

Proof. Each $L \in \mathcal{SD}_n^{UD}(k)$ can be written uniquely as a word $L = U^{x_1}D^{y_1}U^{x_2}D^{y_2}\cdots U^{x_k}D^{y_k}$, such that

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n, & x_1, x_2, \cdots, x_k \geq 1, \\ y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \cdots, y_k \geq 1. \end{cases}$$

The number of solutions for the x_i's and for the y_i's both equal to $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$. Hence equation (3.1) is proved.

Each $L' \in \mathcal{SD}_n^{UU}(k)$ can be written uniquely as a word $L' = U^{x_1}D^{y_1}U^{x_2}D^{y_2}\cdots U^{x_k}D^{y_k}U^{x_{k+1}}$, such that

$$\begin{cases} x_1 + x_2 + \cdots + x_k + x_{k+1} = n, & x_1, x_2, \cdots, x_{k+1} \geq 1 \\ y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \cdots, y_k \geq 1 \end{cases}$$

There are $\binom{n-k+k+1-1}{k} = \binom{n}{k}$ solutions for the x_i's and $\binom{n-1}{k-1}$ solutions for the y_i's. Hence equation (3.2) is proved.

From (3.1) and (3.2) we have that the number of super Dyck paths with k peaks of order n is

$$\binom{n-1}{k-1}^2 + \binom{n-1}{k}^2 + 2\binom{n-1}{k-1}\binom{n-1}{k} = \binom{n}{k}^2.$$

∎

Corollary 3.5 *The number of Dyck paths of order n with k peaks is:*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Proof. From theorem 1.1 we know that each Dyck path of order n with k peaks is mapped to k super Dyck paths, and each of the k super Dyck paths is either in $\mathcal{SD}_n^{UU}(k-1)$ or in $\mathcal{SD}_n^{UD}(k)$. Therefore we have $kN(n,k) = \#\mathcal{SD}_n^{UU}(k-1) + \#\mathcal{SD}_n^{UD}(k)$. From Proposition 3.4 we can conclude that

$$N(n,k) = \frac{1}{k}\left(\binom{n-1}{k-1}^2 + \binom{n-1}{k-2}\binom{n-1}{k-1}\right) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

∎

Bijective proof of this result can also be found in [11, Exercise 6.36(a)].

4 Humps in Schröder paths

In this section we count the number of humps in a third kind of lattice paths: Schröder paths. A *Schröder path* of order n is a lattice path in $\mathbb{Z}\times\mathbb{Z}$, from $(0,0)$ to (n,n), using up-steps $(0,1)$, down-steps $(1,0)$ and flat-steps $(1,1)$ (denoted by U, D, F, respectively) and never going below the line $y=x$. Note that Schröder paths are different from rotating Motzkin paths 45 degrees counterclockwise, since the F steps in these two kinds of paths are different. However, the bijection Φ still works when counting humps in Schröder paths. Let ss_n denote the number of super Schröder paths of order n, and hs_n denote the number of humps in all Schröder paths of order n. We have the following result.

Corollary 4.6 *For all $n \geq 0$, we have*

$$ss_n = 2hs_n + 1, \tag{4.1}$$

and

$$hs_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k} - 1\right). \tag{4.2}$$

Proof. Apply the bijection Φ to Schröder paths we immediately get (4.1). Next we will count ss_n. Let L be a super Schröder path of order n with k humps, then there are k U steps, k D steps, and $n-k$ F steps in L. We can first choose a super Dyck path of order k and then "insert" $n-k$ F steps to get L. There are $\binom{2k}{k}$ ways to choose a super Dyck paths, and $\binom{n-k+2k+1-1}{2k} = \binom{n+k}{2k}$ ways for the insertion. Therefore we have

$$ss_n = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}.$$

From the above formula and (4.1) we get (4.2). ∎

The above proof inspired us to get the following identity, which is listed as an exercise in [9, Exercise 3(g) of Chapter 1].

Corollary 4.7 *For all $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}^2 (m+1)^k = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k} m^{n-k}. \tag{4.3}$$

Proof. We will first prove (4.3) $m=1$. From the proof of Corollary 4.6 we know that the right hand side of (4.3) is the number of super Schröder paths of order n when $m = 1$. Now we count ss_n with a different method to obtain the left hand. Let L be a super Dyck path of order n with k peaks, for each peak of L, we can either keep it invariant or change it into a F step to we get two super Schröder paths. Hence each L is mapped to 2^k super Schröder paths, thus the left hand side of (4.3) when $m=1$ also equals ss_n. Therefore we proved (4.3) for $m=1$.

For general m we count the number of super Schröder paths in which the F steps are m-colored. Now every super Dyck path with k peaks is mapped to $(m+1)^k$ colored super Schröder paths. So the total number of such path is $\sum_{k=0}^{n} \binom{n}{k}^2 (m+1)^k$. On the other hand, from the proof of Theorem 4.6 we know that the right hand side of (4.3) also counts the number of such paths, hence we proved (4.3). ∎

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References

[1] E. Deutsch, *Dyck path enumeration*, Discrete Math. 204 (1999), no. 1-3, 16–202.

[2] T. Mansour, *Counting Peaks at Height k in a Dyck Path*, Journal of Integer Sequences, Vol. 5 (2002), Article 02.1.1.

[3] T. Mansour, *Statistics on Dyck Paths*, Journal of Integer Sequences, Vol. 9:1 (2006), Article 06.1.5.

[4] T.V. Narayana, *A partial order and its applications to probability*, Sankhya 21 (1959) 91–98.

[5] P. Peart and W. Woan, *Dyck Paths With No Peaks At Height k*, Journal of Integer Sequences, Vol. 4 (2001), Article 01.1.3.

[6] M. Petkovesk, H. S. Wilf and D. Zeilberger, em A=B, AK Peters Ltd. (1996)

[7] A. Regev, *Humps for Dyck and for Motzkin paths*, arXiv: 1002. 4504 v1 [math. CO] 24 Feb 2010.

[8] N. J. A. Sloane, *Online Encyclopedia of Integer Sequence*, published electronically at http: // oeis. org

[9] R. P. Stanley, Enumerative Combinatorics (Volume 1 second edition), http://www-math.mit.edu/∼rstan/ec/ec1.pdf, 2011.

[10] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.

[11] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.

[12] D. Zeilberger,*The method of creative telescoping*, J. Symbolic Computation, 11 (1991) 195–204.