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On Projections to the Pure Spinor Space

P.A. Grassi* and S. Guttenberg†,‡

*DISTA, Università del Piemonte Orientale,
Via Teresa Michel 11, Alessandria, 15121, Italy,
and INFN - Sezione di Torino.

†Dipartimento di Fisica Teorica, Università di Torino,
Via P. Giuria 1, I-10125 Torino, Italy
and INFN - Sezione di Torino.

‡CAMGSD, Departamento de Matemática,
Instituto Superior Técnico,
Av. Rovisco Pais 1,
1049–001 Lisboa, Portugal.

pietro.grassi@mfn.unipmn.it, sgutten@math.ist.utl.pt

Abstract

A family of covariant non-linear projections from the space of SO(10) Weyl spinors onto the space of pure SO(10) Weyl spinors is presented. The Jacobian matrices of these projections are related to a linear projector which was previously discussed in pure spinor string literature and which maps the antighost to its gauge invariant part. Only one representative of the family leads to a Hermitian Jacobian matrix and can itself be derived from a scalar potential. Comments on the SO(1,9) case are given as well as on the non-covariant version of the projection map. The insight is applied to the ghost action of pure spinor string theory, where the constraints on the fields can be removed using the projection, while introducing new gauge symmetries. This opens the possibility of choosing different gauges which might help to clarify the origin of the pure spinor ghosts. Also the measure of the pure spinor space is discussed from the projection point of view. The appendix contains the discussion of a toy model which served as a guideline for the pure spinor case.

Contents

1 Introduction

In recent years, pure spinors in 10 dimensions have become very important for string theory. This is mainly due to the invention by Berkovits [1] of a string theory sigma-model where a pure spinor ghost field λ^{α} plays a prominent role. But already before, pure spinors where used to describe 10-dimensional supersymmetric quantum field theories [2, 3, 4].

In spite of its remarkable success, the pure spinor sigma model is still conceptually not completely understood. Due to the presence of the quadratic pure spinor constraint on the ghost fields, it is highly challenging to derive the model from first principles as a gauge-fixing of a classical theory. In particular the diffeomorphism b-ghost appears only as a non-trivial composite field.

There have been many efforts to remove the ghost-constraints by either adding more ghosts like in [5] or [6] (with finitely many ghosts) or like in [7] or [8] (with infinitely many ghosts). Other efforts were relating the pure spinor string to the superembedding-formalism [9] or on the operator level directly to the Green Schwarz formalism [10, 12]. In [11] a connection to the pure spinor string was obtained by working out the BFT conversion of second class constraints (of the Green Schwarz action) into first class constraints. In addition in [13] as well as recently in [14] the pure spinor string was derived from classical ghost-free actions, and an attempt to derive the b-ghost from first principles by coupling to worldsheet-gravity was given in [15]. Although all these descriptions gave important insight and might eventually give a complete understanding, the picture so far remains a bit unsatisfactory.

At classical level, one natural way to remove constraints is to use a projection. To be more precise, replace in the action the constrained variables by projections of unconstrained ones and consider the result to be the new action of the unconstrained variables. A priori it is not clear whether this can solve some of the problems, since the free action would become non-free and the constraint would be replaced by another gauge symmetry. However, the latter might open the way to switch to different formulations by choosing different gauges. This was one of the motivations to look for a projection.

A linear projector has already appeared in the pure spinor string literature, in particular within the so-called Y-formalism in [16] and recently also in a covariant version in [17]. This projector is extracting the gauge invariant part of the antighost field. The transpose of this projector maps generic spinor variations to variations which are consistent with the pure spinor constraint. Therefore a projection of a general spinor to a pure spinor corresponds to the integration of this linearized projection. In this article we will present the full non-linear projection.

The article is organized as follows. In section 2 we introduce in equation (2.3) a family of nonlinear projection maps from the space of SO(10) Weyl spinors to SO(10) pure Weyl spinors and study some of its properties. Also the viewpoint of the projection being part of a variable transformation $\rho^\alpha \mapsto (\lambda^\alpha, \zeta^\alpha)$ is presented on page 7 and the non-covariant version of the projection using a reference spinor is discussed on page 9. In section 3 on page 10 we calculate the Jacobian matrix of the projection which provides the push-forward map for the vectors of the tangent spaces of the previously mentioned spaces. In particular, the variations of spinors are mapped with this linearized map. It is shown that on the constraint surface it reduces to a linear projector whose transpose is known to extract the gauge invariant part from the antighost of pure spinor string theory. In section 4 on page 13 we collect some properties of this and a few other projection matrices on the constraint surface, mainly to have them available as a toolbox for the subsequent section. In section 5 on page 16 a case is discussed where the Jacobian matrix is Hermitian which leads to several appealing properties. In particular it turns out that the projection then can be derived from a potential. In section 6 on page 19 we discuss the non-covariant version of the projection map for a particular reference spinor within the U(5)-covariant parametrization of SO(10) spinors.

In section 7 on page 21 we finally apply the mathematical insight to the pure spinor string. In subsection 7.1 we review the non-minimal pure spinor string and in particular the projection of the antighost field $\omega_{z\alpha}$ to its gauge invariant part. In addition we also introduce gauge invariant projections for the non-minimal fields $\bar{\omega}_z^\alpha$ and $\boldsymbol{s}_z^\alpha$ which had not yet been presented in the literature to our knowledge. In subsection 7.2 on page 26 we replace the pure spinor λ^α by the projection image of an unconstrained spinor ρ^α. The resulting constraint-free action is not very appealing by itself, but it is conceptionally interesting as it comes with an additional gauge symmetry which allows to look for different gauges than the pure spinor constraint. And in subsection 7.3 on page 31 we quickly review the form of the action in the U(5) formalism in order to see how the resulting gauge invariant antighost-combination corresponds to the previously discussed projection.

In section 8 on page 32 finally we regard the pure spinor space as being embedded in $\mathbb{C}^{16}$ and calculate the transformation of the holomorphic volume form of this ambient space under the aforementioned variable transformation $\rho^\alpha \mapsto (\lambda^\alpha, \zeta^a)$. We discuss the relation of the result to the pure spinor holomorphic volume form known from the literature.

The appendix A on page 35 contains the discussion of a toy model that served as a guide line to derive our projection map. Appendices B.1-B.3 starting from page 51 contain the detailed proofs of three propositions of the main part.

2 Nonlinear projection to the pure spinor space

An SO(10) or SO(1,9) Weyl spinor with complex components λ^α and ${\scriptstyle \alpha\in\{1,\ldots,16\}}$ is called a pure spinor[1] if it obeys the quadratic constraints

$$\lambda^\alpha \gamma^c_{\alpha\beta}\lambda^\beta = 0 \tag{2.1}$$

where the Latin index ${\scriptstyle c\in\{1,\ldots,10\}}$ for SO(10) or ${\scriptstyle c\in\{0,1\ldots,9\}}$ for $SO(1,9)$. The matrices $\gamma^c_{\alpha\beta}$ are the off-diagonal chiral blocks of the SO(10) or SO(1,9) Dirac gamma matrices in the standard Weyl representation[2]. The space of pure spinors is known to be a $\mathbb{C}^*$-fibration of $SO(10)/U(5)$. Thus pure spinors are a non-linear representation of SO(10). Therefore any projection to this space will

[1] In fact, in general dimensions d a pure spinor in even dimensions is defined to be annihilated by a maximally isotropic subspace of the Clifford vector space spanned by the Dirac gamma matrices. So roughly speaking it is annihilated by "half of the gamma matrices", meaning by $d/2$ linear combinations of gamma matrices. If one thinks of pure spinors as being a Clifford vacuum, then these $d/2$ generators are just the annihilators while the remaining ones will be the generators in the Clifford representation of spinors. In other words pure spinors provide possible vacua for a Clifford representation. It is well known that in 10 dimensions the above definition of pure spinors is equivalent to the quadratic constraint (2.1). $\diamond$

[2] In the standard Weyl-representation of SO(10) spinors one has

$$\Gamma^{a\,\underline{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix} 0 & \gamma^{a\,\alpha\beta} \\ \gamma^a_{\alpha\beta} & 0 \end{pmatrix}, \quad \underline{\alpha}\in\{1,\ldots,32\}, \quad \alpha\in\{1,\ldots,16\}, \quad a\in\{1,\ldots,10\}$$

with numerically $\gamma^a_{\alpha\beta} = \gamma^{a\,\alpha\beta}$ real and symmetric for $a\in\{1,\ldots,9\}$ and $\gamma^{10\,\alpha\beta} = -\gamma^{10}_{\alpha\beta} = i\delta_{\alpha\beta}$ symmetric and imaginary. The index a is pulled with the SO(10) metric δ_{ab}. If instead we think of SO(1,9) spinors, we have to use

$$\Gamma^0 \equiv -i\Gamma^{10} = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}$$

Then the index is pulled with $\eta_{ab} = \mathrm{diag}\,(-1,1,\ldots,1)$. In any case (so for either all $a\in\{1,\ldots,10\}$ or $a\in\{0,\ldots,9\}$) we thus have

$$(\gamma^a_{\alpha\beta})^* = \gamma_a^{\alpha\beta} = \gamma_a^{\beta\alpha}$$

So the notation is such that the $\gamma^a_{\alpha\beta}$ behave in contractions as if they were all real (for both, SO(1,9) and SO(10)). In particular we have

$$(\rho\gamma^a\rho)^* = (\bar\rho\gamma_a\bar\rho) \quad \diamond$$

necessarily be non-linear. In the following proposition we will observe that

$$P^\alpha(\rho,\bar\rho) \equiv \rho^\alpha - \tfrac{1}{2}\frac{(\rho\gamma^a\rho)(\bar\rho\gamma_a)^\alpha}{(\rho\bar\rho)+\sqrt{(\rho\bar\rho)^2-\frac{1}{2}(\rho\gamma^b\rho)(\bar\rho\gamma_b\bar\rho)}} \tag{2.2}$$

projects a general Weyl spinor to a pure spinor in an SO(10) covariant way. This fact will certainly not change if the projection is multiplied by some scalar function f which is 1 on the constraint surface. As this function will play an important role later on for tuning the properties of the Jacobian matrices, we will immediately include it into the discussion and promote P^α to a family of projections $P^\alpha_{(f)}$. In addition we will introduce some auxiliary variables ξ and ζ^a in (2.4) which will simplify the expressions and save some space.

Proposition 1 (Covariant projection to the pure spinor space)**.** *Consider an SO(10) Weyl spinor ρ^α with 16 complex components and the family of maps $P_{(f)}$*

$$(\rho^\alpha,\bar\rho_\beta)\mapsto\Big(P^\alpha_{(f)}(\rho,\bar\rho),\bar P_{(f)\beta}(\rho,\bar\rho)\Big)$$

acting on Weyl spinors via[3]

$$\begin{aligned} P^\alpha_{(f)}(\rho,\bar\rho) &\equiv f(\xi)\left(\rho^\alpha-\tfrac{1}{2}\frac{\zeta^a(\bar\rho\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) && (2.3)\\ \textit{with } f(0) &= 1,\quad \zeta^a\equiv\frac{(\rho\gamma^a\rho)}{(\rho\bar\rho)},\quad \xi\equiv\tfrac{1}{2}\zeta^a\bar\zeta_a && (2.4)\end{aligned}$$

with f any complex-valued function defined on the interval $[0,1]$ and obeying $f(0)=1$ and $\gamma^a_{\alpha\beta}$ and $\gamma_a^{\alpha\beta}$ being the chiral blocks of the Dirac-Γ-matrices in the 10d standard Weyl-representation of footnote 2. Then the following statements hold:

1. *$\forall f$, $P_{(f)}$ is a **projection map** from the space of Weyl spinors to the space of pure Weyl spinors, i.e. it obeys the following two properties*

$$\begin{aligned} P^\alpha_{(f)}(\rho,\bar\rho)\gamma^a_{\alpha\beta}P^\beta_{(f)}(\rho,\bar\rho) &= 0\quad\forall\rho^\alpha && (2.5)\\ P^\alpha_{(f)}(\lambda,\bar\lambda) &= \lambda^\alpha\quad\forall\lambda^\alpha \textit{ with } \lambda^\alpha\gamma^a_{\alpha\beta}\lambda^\beta=0 && (2.6)\end{aligned}$$

   The first one just states that the image of a general Weyl spinor is a pure Weyl spinor, while the second property is the projection property. It implies idempotency of the map $P_{(f)}$ (i.e. $P_{(f)}\circ P_{(f)}=P_{(f)}$), but it is slightly stronger as it also guarantees surjectivity onto the space of pure Weyl spinors.

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[3]About the generality of (2.3): The projection (2.3) is covariant and homogeneous of degree 1 for every function f. However, it is certainly not the most general covariant projection with this property. In a more general ansatz one could imagine terms like $\frac{(\rho\gamma^{[5]}\rho)(\bar\rho\gamma_{[5]})^\alpha}{(\rho\bar\rho)}$ (where $\gamma^{[5]}$ is shorthand for $\gamma^{a_1\cdots a_5}$) multiplied with a coefficient that depends not only on ξ, but also on other scale-invariant variables like $\frac{(\rho\gamma^{[5]}\rho)(\bar\rho\gamma_{[5]}\bar\rho)}{(\rho\bar\rho)^2}$ and probably many others. However, there is little motivation to make such a far more complicated ansatz. It might be interesting only if one wants to achieve that the projection behaves like the identity in quadratic contractions with $\gamma^{[5]}$, i.e. $(P(\rho,\bar\rho)\gamma^{[5]}P(\rho,\bar\rho))=(\rho\gamma^{[5]}\rho)$. ⋄

2. *$P_{(f)}$ is homogeneous of degree (1,0) for any f*

$$P^{\alpha}_{(f)}(c\rho,\bar{c}\bar{\rho}) = cP^{\alpha}_{(f)}(\rho,\bar{\rho}) \quad \forall c \in \mathbb{C} \tag{2.7}$$

and its modulus square is given by

$$P^{\alpha}_{(f)}(\rho,\bar{\rho})\bar{P}_{(f)\alpha}(\rho,\bar{\rho}) = 2(\rho\bar{\rho}) \mid f(\xi) \mid^2 \left(\frac{1-\xi}{1+\sqrt{1-\xi}}\right) \tag{2.8}$$

Although ξ and ζ^a are not well-defined at the origin $\mid \rho \mid = 0$, the projection map is still well-defined there in the sense that the limit exists if f is continuous on $[0,1]$:

$$P^{\alpha}_{(f)}(0,0) \equiv \lim_{|\rho|\to 0} P^{\alpha}_{(f)}(\rho,\bar{\rho}) = 0 \quad (\textit{for cont. } f) \tag{2.9}$$

3. *The zero-locus of the projection is given by*

$$P^{-1}_{(f)}(0) = \{0\} \cup \left\{\rho^{\alpha} \middle|\, \rho^{\alpha} = \tfrac{1}{2}\zeta^b(\gamma_b\bar{\rho})^{\alpha}\right\} \cup \{\rho^{\alpha}|f(\xi) = 0\} \tag{2.10}$$

In particular real vectors $\bar{\rho}_{\alpha} = \rho^{\alpha}$ are in the zero-locus. As this is not an $SO(10)$ invariant statement, also all vectors obeying an $SO(10)$ rotated version of this reality condition lie in the zero-locus.

4. *The projection $P_{(f)}$ is continuous everywhere if f is continuous, and it is differentiable everywhere but at the zero-locus subset $\{0\} \cup \{\rho^{\alpha} | \, \xi = 1\}$, if f is differentiable. One can choose f such that it will even become differentiable at $\xi = 1$, namely for $f(\xi) = \tilde{f}(\xi)\,(1-\xi)^{1+r}$ with a differentiable $\tilde{f}$ with $\tilde{f}(0) = 1$ and $r \geq 0$.*

5. *The auxiliary variables ξ, ζ^a obey*

$$\xi \in [0,1] \tag{2.11}$$

$$\xi = 1 \iff 0 \neq \rho^{\alpha} = \alpha^b(\gamma_b\bar{\rho})^{\alpha}\ {\scriptstyle for\, some}\ \alpha^b \in \mathbb{C} \iff 0 \neq \rho^{\alpha} = \tfrac{1}{2}\zeta^b(\gamma_b\bar{\rho})^{\alpha} \tag{2.12}$$

$$\zeta^a\zeta_a = 0, \quad \zeta^a(\gamma_a\rho)_{\alpha} = 0 \tag{2.13}$$

$$\xi|_{\rho=\lambda} = 0 = \zeta^a|_{\rho=\lambda} \quad \forall_{pure}\ \lambda^{\alpha} \neq 0 \tag{2.14}$$

$$\xi|_{c\rho,\bar{c}\bar{\rho}} = \xi|_{\rho,\bar{\rho}} \quad , \quad \zeta^a|_{c\rho,\bar{c}\bar{\rho}} = \tfrac{c}{\bar{c}}\,\zeta^a|_{\rho,\bar{\rho}} \quad \forall c \in \mathbb{C} \tag{2.15}$$

The proof of this proposition is given in appendix B.1 on page 51. Most of it contains only straightforward calculations. Only the proof of $\xi \leq 1$ turns out to be quite tricky.

Projection as part of a reparametrization We can regard the projection as part of a variable transformation

$$(\rho^{\alpha},\bar{\rho}_{\alpha}) \mapsto (\lambda^{\alpha},\zeta^a,\bar{\lambda}_{\alpha},\bar{\zeta}_a) \tag{2.16}$$
$$\text{with } \lambda^{\alpha} \equiv P^{\alpha}_{(f)}(\rho,\bar{\rho}), \quad \zeta^a \equiv \tfrac{(\rho\gamma^a\rho)}{(\rho\bar{\rho})}$$

where the variables on the righthand side are constrained by[4]

$$(\lambda\gamma^a\lambda) = 0 = \zeta^a(\lambda\gamma_a)_{\alpha} = \zeta^a\zeta_a \quad , \quad \tfrac{1}{2}\zeta^a\bar{\zeta}_a \leq 1 \tag{2.17}$$

[4]Proof of $\zeta^a(\gamma_a\lambda)_{\alpha} = 0$ in (2.17):

$$\zeta^a(\gamma_a\lambda)_{\alpha} \equiv \zeta^a\left(\gamma_a P_{(f)}(\rho,\bar{\rho})\right)_{\alpha} \overset{(2.3)}{=} f(\xi)\left(\zeta^a(\gamma_a\rho)_{\alpha} - \frac{1}{2}\frac{\overbrace{\zeta^a\zeta^c}^{\zeta^a\zeta_a}(\gamma_a\gamma_c\bar{\rho})^{\alpha}}{1+\sqrt{1-\xi}}\right) \overset{(2.13)}{=} 0 \qquad \diamond$$

One can see that they have effectively the same number of degrees of freedom by observing that the above reparametrization is invertible

$$\rho^\alpha = \frac{1+\sqrt{1-\frac{1}{2}\zeta^a\bar{\zeta}_a}}{2f(\frac{1}{2}\zeta^a\bar{\zeta}_a)\sqrt{1-\frac{1}{2}\zeta^a\bar{\zeta}_a}}\lambda^\alpha + \frac{1}{4\bar{f}(\frac{1}{2}\zeta^a\bar{\zeta}_a)\sqrt{1-\frac{1}{2}\zeta^a\bar{\zeta}_a}}\zeta^a\left(\bar{\lambda}\gamma_a\right)^\alpha \tag{2.18}$$

We have explicitly replaced here ξ by $\frac{1}{2}\zeta^a\bar{\zeta}_a$ in order to stress that we have a function of only ζ^a and λ^α on the righthand side. Instead for the other direction in (2.16) λ^α and ζ^a have to be seen as functions of ρ only. In particular all the appearances of ξ and ζ^a in the projection $P^\alpha_{(f)}(\rho,\bar{\rho})$ are mere placeholders for the ρ-expressions given in (2.4). The validity of (2.18) can easily be checked by plugging the explicit expression for the projection $P^\alpha_{(f)}(\rho,\bar{\rho})$ for λ^α and the same for its complex conjugate.[5] Note that (2.18) is singular at $\xi = 1$ (if not cured by an appropriately chosen $f(\xi)$) and at the zeros of f.

So to summarize, every Weyl spinor ρ^α can be parametrized by a pure spinor λ^α and a constrained vector ζ^a. It turns out to be very useful in some calculations to use (2.18) and write ρ^α as a linear combination of $\lambda^\alpha = P^\alpha_{(f)}(\rho,\bar{\rho})$ and $\bar{\lambda}_\alpha$.

We can use (2.8) to quickly determine the inverse transformation of the modulus

$$(\rho\bar{\rho}) = \frac{\left(1+\sqrt{1-\xi}\right)}{2\mid f(\xi)\mid^2(1-\xi)}(\lambda\bar{\lambda}) \tag{2.19}$$

Let us also finally provide two more useful contractions

$$(\lambda\bar{\rho}) \equiv P^\alpha_{(f)}(\rho,\bar{\rho})\bar{\rho}_\alpha \overset{(2.3)}{=} (\rho\bar{\rho})f(\xi)\sqrt{1-\xi} \tag{2.20}$$

$$(\lambda\gamma^c\rho) \equiv (P_{(f)}(\rho,\bar{\rho})\gamma^c\rho) \underset{(2.13)}{\overset{(2.3)}{=}} (\rho\bar{\rho})f(\xi)\frac{\zeta^c\sqrt{1-\xi}}{1+\sqrt{1-\xi}} \tag{2.21}$$

Alternative reparametrization Rewriting (2.18) in the form $\rho^\alpha = \frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\left(\lambda^\alpha + \frac{1}{2}\frac{f(\xi)\zeta^a}{\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)}\left(\bar{\lambda}\gamma_a\right)^\alpha\right)$ shows that this inverse variable transformation becomes particularly simple if one chooses

$$\tilde{\zeta}^a \equiv \frac{f(\xi)\zeta^a}{\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)} = \frac{f(\xi)}{\bar{f}(\xi)}\frac{(\rho\gamma^a\rho)}{(\rho\bar{\rho})+\sqrt{(\rho\bar{\rho})^2-\frac{1}{2}(\rho\gamma^a\rho)(\bar{\rho}\gamma_a\bar{\rho})}} \tag{2.22}$$

as new variable, so that the inverse transformation up to an overall prefactor is of the form $\rho^\alpha \propto \lambda^\alpha + \frac{1}{2}\tilde{\zeta}^a\left(\bar{\lambda}\gamma_a\right)^\alpha$. Note that, of course, one can also absorb the

[5] In order to prove (2.18) we start from the righthand side:

$$\begin{aligned}
& \frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}P^\alpha(\rho,\bar{\rho}) + \frac{1}{4\bar{f}(\xi)\sqrt{1-\xi}}\zeta^a\left(\bar{P}(\rho,\bar{\rho})\gamma_a\right)^\alpha = \\
\overset{(2.3)}{=} \; & \frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}f(\xi)\left(\rho^\alpha - \frac{1}{2}\frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) + \frac{1}{4\bar{f}(\xi)\sqrt{1-\xi}}\zeta^a\bar{f}(\xi)\left((\bar{\rho}\gamma_a)^\alpha - \frac{1}{2}\frac{\bar{\zeta}^b(\rho\overbrace{\gamma_b\gamma_a}^{-\gamma_a\gamma_b+2\delta_{ab}})^\alpha}{1+\sqrt{1-\xi}}\right) = \\
= \; & \frac{1}{2\sqrt{1-\xi}}\left(1+\sqrt{1-\xi}-\frac{\xi}{1+\sqrt{1-\xi}}\right)\rho^\alpha = \\
= \; & \rho^\alpha \quad \diamond
\end{aligned}$$

factor $\frac{1}{2}$ into the definition of $\tilde{\zeta}^a$. However, the way we have defined it here will guarantee that $\tilde{\xi}$, which we are going to introduce now, lies like ξ in the interval $[0, 1]$. In order to determine also the overall prefactor after reparametrization, let us calculate a few more relations between old and new variables. The absolute value squares are related via

$$\tilde{\xi} \equiv \tfrac{1}{2}\tilde{\zeta}^a \bar{\tilde{\zeta}}_a \quad = \quad \frac{\xi}{\left(1+\sqrt{1-\xi}\right)^2} = \frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}} \tag{2.23}$$

which implies the relations $(1-\tilde{\xi}) = \frac{2\sqrt{1-\xi}}{1+\sqrt{1-\xi}}$, $(1+\tilde{\xi}) = \frac{2}{1+\sqrt{1-\xi}}$ and thus in particular

$$\sqrt{1-\xi} = \frac{1-\tilde{\xi}}{1+\tilde{\xi}} \quad , \quad 1-\sqrt{1-\xi} = \frac{2\tilde{\xi}}{1+\tilde{\xi}} \quad , \quad 1+\sqrt{1-\xi} = \frac{2}{1+\tilde{\xi}} \tag{2.24}$$

The last can be used to invert the relation (2.23) and obtain

$$\xi = \frac{4\tilde{\xi}}{(1+\tilde{\xi})^2} \tag{2.25}$$

Because of $0 \leq \xi \leq 1$ also the new variable lives in this interval

$$0 \leq \tilde{\xi} \leq 1 \tag{2.26}$$

and also $\tilde{\zeta}^a$ and its absolute value vanish on the constraint surface

$$(\rho\gamma^a\rho) = 0\,\forall a \quad \overset{\text{for } \rho^\alpha \neq (0,\ldots,0)}{\Longleftrightarrow} \quad \tilde{\zeta}^a = 0\,\forall a \quad (\tilde{\xi} = 0) \tag{2.27}$$

Finally we can use the above relations to also write down the inverse of (2.22)

$$\zeta^a = \frac{\bar{\tilde{f}}(\tilde{\xi})}{\tilde{f}(\tilde{\xi})}\frac{2\tilde{\zeta}^a}{(1+\tilde{\xi})} \tag{2.28}$$

with[6]

$$\tilde{f}(\tilde{\xi}) \equiv f(\xi) \tag{2.29}$$

The Weyl spinor ρ^α expressed in terms of the pure spinor λ^α and the variable ζ^a as in (2.18) can now be written as

$$\rho^\alpha = \frac{1}{\tilde{f}(\tilde{\xi})(1-\tilde{\xi})}\left(\lambda^\alpha + \tfrac{1}{2}\tilde{\zeta}^a\left(\bar{\lambda}\gamma_a\right)^\alpha\right) \tag{2.30}$$

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[6] For the derivatives this implies

$$f'(\xi) = \frac{\partial\tilde{\xi}}{\partial\xi}\tilde{f}'(\tilde{\xi}) = \frac{1}{\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2}\tilde{f}'(\tilde{\xi})$$

I.e.

$$\tilde{f}'(\tilde{\xi}) = \sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2 f'(\xi)$$

or

$$f'(\xi) = \frac{(1+\tilde{\xi})^3}{4(1-\tilde{\xi})}\tilde{f}'(\tilde{\xi}) \qquad \diamond$$

The absolute value squared (2.19) turns into

$$(\rho\bar{\rho}) = \frac{1+\tilde{\xi}}{\mid \tilde{f}(\tilde{\xi}) \mid^2 (1-\tilde{\xi})^2}(\lambda\bar{\lambda}) \tag{2.31}$$

Apparently the inverse transformation (2.30) becomes most simple for the special choice $f = h$ where

$$h(\xi) \equiv \tilde{h}(\tilde{\xi}) \equiv \tfrac{1}{(1-\tilde{\xi})} = \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}} \tag{2.32}$$

It is interesting that we will rediscover this function a bit later in the context of Hermiticity and thus gave it the name h here. So we simply have

$$\rho^\alpha = \lambda^\alpha + \tfrac{1}{2}\tilde{\zeta}^a \left(\bar{\lambda}\gamma_a\right)^\alpha \quad \text{for } f = h. \tag{2.33}$$

Reference spinor The projector (2.3) with (2.4) is globally well defined, but it is non-holomorphic in ρ. This can be changed, if one simply replaces $\bar{\rho}_\alpha$ by some reference spinor $\bar{\chi}_\alpha$ which is not related to ρ^α by complex conjugation or in any other way. Almost everything still works in the same way as before, but it is useful to change the notation slightly, in order not to get confused:

$$\zeta^a \equiv \frac{\rho\gamma^a\rho}{(\rho\bar{\chi})}, \quad \bar{\eta}_a \equiv \frac{\bar{\chi}\gamma_a\bar{\chi}}{(\rho\bar{\chi})}, \quad \xi \equiv \tfrac{1}{2}\zeta^a\bar{\eta}_a \tag{2.34}$$

$$P^\alpha_{(f,\bar{\chi})}(\rho) \equiv f(\xi)\left(\rho^\alpha - \tfrac{1}{2}\frac{\zeta^a(\bar{\chi}\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) \tag{2.35}$$

One immediate consequence is that ξ is not in general real any longer and we have no guarantee that $\mid \xi \mid \leq 1$. If its absolute value exceeds 1, however, the projection map becomes non-continuous at the branch-cut of the square root. So one should better restrict manually to $\mid \xi \mid \leq 1$ which means that one cannot use one global projection map, but needs different projection maps for different neighbourhoods.

With the reference spinor $\bar{\chi}_\alpha$ being independent of ρ^α , it can itself be chosen to be a pure spinor, which implies

$$\xi = \bar{\eta}_a = 0 \quad (\text{if } \bar{\chi}_\alpha \text{ is pure}) \tag{2.36}$$

The projection then becomes independent of the function f and reduces to

$$P^\alpha_{(\bar{\chi})}(\rho) \equiv P^\alpha_{(f,\bar{\chi})}(\rho) = \rho^\alpha - \tfrac{1}{4}\frac{(\rho\gamma^a\rho)(\bar{\chi}\gamma_a)^\alpha}{(\rho\bar{\chi})} \quad (\text{if } \bar{\chi}_\alpha \text{ is pure}) \tag{2.37}$$

In the subsequent section 3 we will discuss the variation of the general nonlinear projection map (2.3) which leads to projection matrices. For the above simplified case of a projection map with a reference *pure* spinor (2.37) it is already recognizable that this will yield the non-covariant projection-matrix defined by Oda and Tonin in [16, eq.(17)] and used by them in [18, eq.(6)] in order to extract the gauge-invariant part of the antighost. We will come back to this in the remark on page 12.

3 Linearized

As we will quite frequently use the image of the projection $P^\alpha_{(f)}$ as an argument of another function, let us denote for simplicity like previously

$$\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho}) \tag{3.1}$$

Variations of ρ live in the tangent space and are mapped via the push-forward map, i.e. via the Jacobian-matrix defined by the derivatives of $P^\alpha_{(f)}$:

Proposition 2. *The Jacobian matrices $\Pi_{(f)\perp}$ and $\pi_{(f)\perp}$, defined for differentiable*[7] *f via $\delta P^\alpha_{(f)}(\rho,\bar{\rho}) = \Pi^\alpha_{(f)\perp\beta}(\rho,\bar{\rho})\delta\rho^\beta + \pi^{\alpha\beta}_{(f)\perp}(\rho,\bar{\rho})\delta\bar{\rho}_\beta$ or equivalently*

$$\begin{pmatrix}\delta\lambda^\beta\\ \delta\bar{\lambda}_\beta\end{pmatrix} \equiv \begin{pmatrix}\delta P^\alpha_{(f)}(\rho,\bar{\rho})\\ \delta\bar{P}_{(f)\alpha}(\rho,\bar{\rho})\end{pmatrix} \equiv \begin{pmatrix}\Pi^\alpha_{(f)\perp\beta}(\rho,\bar{\rho}) & \pi^{\alpha\beta}_{(f)\perp}(\rho,\bar{\rho})\\ \bar{\pi}_{(f)\perp\alpha\beta}(\rho,\bar{\rho}) & \bar{\Pi}_{(f)\perp\alpha}{}^\beta(\rho,\bar{\rho})\end{pmatrix}\begin{pmatrix}\delta\rho^\beta\\ \delta\bar{\rho}_\beta\end{pmatrix} \tag{3.2}$$

1. *... are explicitly given by*

$$\Pi^\alpha_{(f)\perp\beta}(\rho,\bar{\rho}) = \partial_{\rho^\beta}P^\alpha_{(f)}(\rho,\bar{\rho}) = \tag{3.3}$$

$$\begin{aligned} = \quad & f(\xi)\,\delta^\alpha_\beta + f'(\xi)\,\frac{\rho^\alpha\bar{\zeta}_b(\rho\gamma^b)_\beta}{\rho\bar{\rho}} - 2\xi f'(\xi)\,\frac{\rho^\alpha\bar{\rho}_\beta}{\rho\bar{\rho}} + \\ & -\tfrac{1}{1+\sqrt{1-\xi}}\left(f(\xi)\,\delta^a_b + \left(\tfrac{f(\xi)}{4\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + \tfrac{1}{2}f'(\xi)\right)\zeta^a\bar{\zeta}_b\right)\frac{(\gamma_a\bar{\rho})^\alpha(\rho\gamma^b)_\beta}{\rho\bar{\rho}} + \\ & + \left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + \left(1-\sqrt{1-\xi}\right)f'(\xi)\right)\frac{\zeta^a(\gamma_a\bar{\rho})^\alpha\bar{\rho}_\beta}{\rho\bar{\rho}} \end{aligned} \tag{3.4}$$

$$\pi^{\alpha\beta}_{(f)\perp}(\rho,\bar{\rho}) = \partial_{\bar{\rho}_\beta}P^\alpha_{(f)}(\rho,\bar{\rho}) = \tag{3.5}$$

$$\begin{aligned} = \quad & -\tfrac{1}{2}\tfrac{f(\xi)}{1+\sqrt{1-\xi}}\zeta^a\gamma_a^{\alpha\beta} - 2\xi f'(\xi)\,\frac{\rho^\alpha\rho^\beta}{\rho\bar{\rho}} + f'(\xi)\,\frac{\rho^\alpha\zeta^b(\bar{\rho}\gamma_b)^\beta}{\rho\bar{\rho}} + \\ & + \left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + \left(1-\sqrt{1-\xi}\right)f'(\xi)\right)\frac{\zeta^a(\gamma_a\bar{\rho})^\alpha\rho^\beta}{\rho\bar{\rho}} + \\ & -\tfrac{1}{2\left(1+\sqrt{1-\xi}\right)}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + f'(\xi)\right)\frac{\zeta^a(\gamma_a\bar{\rho})^\alpha\zeta^b(\bar{\rho}\gamma_b)^\beta}{\rho\bar{\rho}} \end{aligned} \tag{3.6}$$

with still $\zeta^a \equiv \frac{(\rho\gamma^a\rho)}{(\rho\bar{\rho})}$, $\quad \xi \equiv \frac{1}{2}\zeta^a\bar{\zeta}_a$ from (2.4).

---

[7] Remember that f is defined on the closed interval $[0,1]$. With differentiability at 0 we thus mean the existence of only a limit from the right ($\xi > 0$)

$$f'(0) \equiv \lim_{\xi\to 0^+}\frac{f(\xi)-f(0)}{\xi}$$

Similarly differentiability at 1 is understood as the existence of only a limit from the left ($\xi < 1$)

$$f'(1) \equiv \lim_{\xi\to 1^-}\frac{f(\xi)-f(1)}{\xi - 1} \quad \diamond$$

2. ... *or equivalently in terms of* $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar\rho)$ *and* ζ^a, ξ *by*[8]

$$
\begin{aligned}
&\Pi^\alpha_{(f)\perp\beta}(\rho,\bar\rho) = \\
&\quad = f(\xi)\left(\delta^\alpha_\beta - \tfrac{1}{2}\frac{(\gamma_a\bar\lambda)^\alpha(\lambda\gamma^a)_\beta}{(\lambda\bar\lambda)}\right) + \\
&\qquad -\left(\tfrac{\bar f(\xi)}{2\left(1+\sqrt{1-\xi}\right)} - \tfrac{\bar f(\xi)f'(\xi)(1-\xi)}{f(\xi)}\right)\frac{\lambda^\alpha\bar\zeta_c\,(\lambda\gamma^c)_\beta}{(\lambda\bar\lambda)} + \\
&\qquad -2(1-\xi)\left(1-\sqrt{1-\xi}\right)f'(\xi)\,\frac{\lambda^\alpha\bar\lambda_\beta}{(\lambda\bar\lambda)} - \tfrac{f(\xi)}{8\left(1+\sqrt{1-\xi}\right)^2}\frac{\bar\zeta_c(\gamma^c\gamma_b\lambda)^\alpha\zeta^d(\bar\lambda\gamma^b\gamma_d)_\beta}{(\lambda\bar\lambda)}
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\pi^{\alpha\beta}_{(f)\perp}(\rho,\bar\rho) = \\
&\quad = -\tfrac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)}\left(\zeta^a\gamma_a^{\alpha\beta} - \frac{\zeta^a(\gamma_a\bar\lambda)^\alpha\lambda^\beta}{(\lambda\bar\lambda)}\right) + \\
&\qquad +\left(1-\sqrt{1-\xi}\right)\left(\tfrac{\bar f(\xi)}{1+\sqrt{1-\xi}} - \tfrac{2(1-\xi)\bar f(\xi)f'(\xi)}{f(\xi)}\right)\frac{\lambda^\alpha\lambda^\beta}{(\lambda\bar\lambda)} + \\
&\qquad +(1-\xi)f'(\xi)\,\frac{\lambda^\alpha\zeta^c\left(\bar\lambda\gamma_c\right)^\beta}{(\lambda\bar\lambda)}
\end{aligned}
\tag{3.8}
$$

3. ... *map general variations* $\delta\rho$ *to variations* $\delta\lambda \equiv \Pi_{(f)\perp}\delta\rho + \pi_{(f)\perp}\delta\bar\rho$ *which are* γ-***orthogonal*** *to the pure spinor* $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar\rho)$, *i.e.* $(\lambda\gamma^c\delta\lambda) = 0$, *or equivalently*

$$
\lambda^\alpha\gamma^c_{\alpha\gamma}\Pi^\gamma_{(f)\perp\beta}(\rho,\bar\rho) = 0 \tag{3.9}
$$

$$
\lambda^\alpha\gamma^c_{\alpha\gamma}\pi^{\gamma\beta}_{(f)\perp}(\rho,\bar\rho) = 0 \qquad \forall\rho \tag{3.10}
$$

4. ... *on the constraint surface* ($\rho^\alpha = \lambda^\alpha$, $\xi = \zeta^a = 0$) *reduce for all* f *which are differentiable at* $\xi = 0$ *to*

$$
\Pi^\alpha_{\perp\beta} \equiv \Pi^\alpha_{(f)\perp\beta}(\lambda,\bar\lambda) = \delta^\alpha_\beta - \tfrac{1}{2}\frac{(\gamma_a\bar\lambda)^\alpha(\lambda\gamma^a)_\beta}{(\lambda\bar\lambda)} \tag{3.11}
$$

$$
\pi^{\alpha\beta}_{(f)\perp}(\lambda,\bar\lambda) = 0 \tag{3.12}
$$

with

$$
\left(\Pi_\perp\right)^2 = \Pi_\perp \tag{3.13}
$$

[8] In terms of the alternative parametrization $\tilde\zeta^a$ from equation (2.22), equations (3.7) and (3.8) become

$$
\begin{aligned}
\Pi^\alpha_{(f)\perp\beta}(\rho,\bar\rho) &= \tilde f(\tilde\xi)\left(\delta^\alpha_\beta - \tfrac{(\gamma_a\bar\lambda)^\alpha(\lambda\gamma^a)_\beta}{2(\lambda\bar\lambda)}\right) - \tfrac{1}{2}\left(\tilde f(\tilde\xi) - (1-\tilde\xi)\tilde f'(\tilde\xi)\right)\tfrac{\lambda^\alpha\bar{\tilde\zeta}_c(\lambda\gamma^c)_\beta}{(\lambda\bar\lambda)} + \\
&\quad - \tfrac{\tilde\xi(1-\tilde\xi)\tilde f'(\tilde\xi)\lambda^\alpha\bar\lambda_\beta}{(\lambda\bar\lambda)} - \tfrac{\tilde f(\tilde\xi)\bar{\tilde\zeta}_c(\gamma^c\gamma_b\lambda)^\alpha\tilde\zeta^d(\bar\lambda\gamma^b\gamma_d)_\beta}{8(\lambda\bar\lambda)} \\
\pi^{\alpha\beta}_{(f)\perp}(\rho,\bar\rho) &= -\tfrac{\bar{\tilde f}(\tilde\xi)}{2}\left(\tilde\zeta^a\gamma_a^{\alpha\beta} - \tfrac{\tilde\zeta^a(\gamma_a\bar\lambda)^\alpha\lambda^\beta}{(\lambda\bar\lambda)}\right) + \tilde\xi\left(\bar{\tilde f}(\tilde\xi) - \tfrac{(1-\tilde\xi)\bar{\tilde f}(\tilde\xi)\tilde f'(\tilde\xi)}{\tilde f(\tilde\xi)}\right)\tfrac{\lambda^\alpha\lambda^\beta}{(\lambda\bar\lambda)} + \\
&\quad + \tfrac{(1-\tilde\xi)}{2}\tfrac{\bar{\tilde f}(\tilde\xi)\tilde f'(\tilde\xi)\lambda^\alpha\tilde\zeta^c(\bar\lambda\gamma_c)^\beta}{\tilde f(\tilde\xi)(\lambda\bar\lambda)}
\end{aligned}
$$

For example for $f = 1$ one obtains

$$
\begin{aligned}
\Pi^\alpha_{(1)\perp\beta}(\rho,\bar\rho) &= \left(\delta^\alpha_\beta - \tfrac{(\gamma_a\bar\lambda)^\alpha(\lambda\gamma^a)_\beta}{2(\lambda\bar\lambda)}\right) - \tfrac{\lambda^\alpha\bar{\tilde\zeta}_c(\lambda\gamma^c)_\beta}{2(\lambda\bar\lambda)} - \tfrac{\bar{\tilde\zeta}_c(\gamma^c\gamma_b\lambda)^\alpha\tilde\zeta^d(\bar\lambda\gamma^b\gamma_d)_\beta}{8(\lambda\bar\lambda)} \\
\pi^{\alpha\beta}_{(1)\perp}(\rho,\bar\rho) &= -\tfrac{1}{2}\left(\tilde\zeta^a\gamma_a^{\alpha\beta} - \tfrac{\tilde\zeta^a(\gamma_a\bar\lambda)^\alpha\lambda^\beta}{(\lambda\bar\lambda)}\right) + \tfrac{\tilde\xi\lambda^\alpha\lambda^\beta}{(\lambda\bar\lambda)} \qquad \diamond
\end{aligned}
$$

5. ... *obey projection properties in the sense*[9]

$$\begin{aligned}\Pi_{(f)\perp}(\lambda,\bar\lambda)\Pi_{(f)\perp}(\rho,\bar\rho) &= \Pi_{\perp(f)}(\rho,\bar\rho) && (3.14)\\ \Pi_{(f)\perp}(\lambda,\bar\lambda)\pi_{(f)\perp}(\rho,\bar\rho) &= \pi_{\perp(f)}(\rho,\bar\rho) && (3.15)\end{aligned}$$

6. ... *have trace*

$$tr\Pi_{(f)\perp}(\rho,\bar\rho) = \left(11 - \tfrac{4(1-\sqrt{1-\xi})}{1+\sqrt{1-\xi}}\right) f\,(\xi) - 2(1-\xi)\left(1-\sqrt{1-\xi}\right) f'\,(\xi) \qquad (3.16)$$

which in general reduces only on the constraint surface $\rho = \lambda$, $\xi = 0$ *to* $tr\Pi_\perp = 11$.

The proof of this proposition is given in appendix B.2 on page 55.

Remarks

- The projection matrix on the constraint surface (3.11) is the transpose of the projection matrix $(1-K)$ given in equations (2.11) and (2.9) of [17]. For the non-covariant projection (2.35) we would replace (3.11) by

$$\Pi^{\alpha}_{\perp\beta} \equiv \Pi_{(f,\bar\chi)\perp\beta}{}^{\alpha}(\lambda) = \delta^\alpha_\beta - \tfrac{1}{2}\frac{(\gamma_a\bar\chi)^\alpha(\lambda\gamma^a)_\beta}{(\lambda\bar\chi)} \qquad (3.17)$$

  which is the transpose of $(1-K)$ in equation (17) of [16] (see also (7) in [18], (14) in [19] or more recently (2.12) in [20]). The relation of this non-covariant $\Pi_{(f,\bar\chi)\perp\beta}{}^{\alpha}(\lambda)$ to (2.35) is via the derivative like in the covariant case. This is particularly obvious in the case where $\bar\chi_\alpha$ is pure and one can start from (2.37):

$$\begin{aligned}\partial_{\rho^\beta} P^\alpha_{(f,\bar\chi)}(\rho) &= \delta^\alpha_\beta - \tfrac{1}{2}\frac{(\gamma_a\bar\chi)^\alpha(\rho\gamma^a)_\beta}{(\rho\bar\chi)} + \tfrac{1}{4}\frac{(\rho\gamma^a\rho)(\bar\chi\gamma_a)^\alpha\bar\chi_\beta}{(\rho\bar\chi)^2} = && (3.18)\\ &\overset{\text{if }\rho=\lambda}{=} \delta^\alpha_\beta - \tfrac{1}{2}\frac{(\gamma_a\bar\chi)^\alpha(\lambda\gamma^a)_\beta}{(\lambda\bar\chi)} && (3.19)\end{aligned}$$

- Concerning the 6th statement of the proposal, there exists an f for which $\mathrm{tr}\,\Pi_{(f)\perp}(\rho,\bar\rho) = 11 \quad \forall\rho^\alpha$. This solution is derived in footnote 34 on page 62. It could therefore be that this solution obeys a strict linear projector property $\Pi_{(f)\perp}(\rho,\bar\rho)^2 = \Pi_{(f)\perp}(\rho,\bar\rho)$ as this is the case for $\Pi_{(1)\perp}(\rho,\bar\rho)$ within the toy-model in (A.58) on page 43.

[9]The equations (3.14) and (3.15) are not usual projection properties of a linear projection which would be of the form $\Pi^2 = \Pi$. Instead they are understood as being part of a projection $\mathcal{P}_{(f)}$ acting on the tangent bundle of the spinor space:

$$\begin{aligned}\mathcal{P}_{(f)}: \quad (\rho,\bar\rho) &\overset{P_{(f)}}{\mapsto} (\lambda,\bar\lambda)\\ \begin{pmatrix}\delta\rho\\ \delta\bar\rho\end{pmatrix} &\mapsto \begin{pmatrix}\Pi_{(f)\perp}(\rho,\bar\rho) & \pi_{(f)\perp}(\rho,\bar\rho)\\ \pi_{(f)\perp}(\rho,\bar\rho) & \bar\Pi_{(f)\perp}(\rho,\bar\rho)\end{pmatrix}\begin{pmatrix}\delta\rho\\ \delta\bar\rho\end{pmatrix}\end{aligned}$$

The projection property $\mathcal{P}_{(f)}\circ\mathcal{P}_{(f)} = \mathcal{P}_{(f)}$ for the tangent bundle map then is equivalent to the three equations (3.14), (3.15) and $P_{(f)}\circ P_{(f)} = P_{(f)}$ (2.6). ◇

- The projection matrix maps general variations $\delta\rho^\alpha$ to some $\delta\lambda^\alpha$ that are $\gamma-$orthogonal to $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar\rho)$, i.e. such that $\lambda\gamma^a\delta\lambda = 0$. As one can easily check, any pure spinor variable together with a variation with the just mentioned property $\lambda\gamma^a\delta\lambda = 0$ obeys

$$\delta\lambda^\alpha = (\Pi_\perp\delta\lambda)^\alpha \tag{3.20}$$

with $\Pi_\perp = \mathbb{1} - \frac{1}{2}\frac{(\gamma_a\bar\lambda)\otimes(\lambda\gamma^a)}{(\lambda\lambda)}$. In particular for $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar\rho)$ this follows from (3.2) together with (3.11) and (3.12) at $\rho = \lambda$. A variation of the constrained variable λ thus can be written as

$$\delta\lambda^\alpha\partial_{\lambda^\alpha} \quad = \quad (\Pi_\perp\delta\lambda)^\alpha\partial_{\lambda^\alpha} = \delta\lambda^\alpha(\Pi_\perp^T\partial_\lambda)_\alpha \tag{3.21}$$

The resulting **covariant derivative** $(\Pi_\perp^T\partial_\lambda)_\alpha$ contains the transposed projection matrix and it leaves the constraint invariant

$$(\Pi_\perp^T\partial_\lambda)_\alpha(\lambda\gamma^c\lambda) = 0 \tag{3.22}$$

In the sigma model the antighost $\omega_{z\alpha}$ plays the role of the partial derivative $\partial_{\lambda^\alpha}$. As was noted in the above cited references (e.g. [18] or [17]), the expression $(\Pi_\perp^T\omega_z)_\alpha$ is gauge invariant. Gauge invariance means that via Poisson bracket or commutator the constraint $\lambda\gamma^c\lambda$ (generating the gauge transformation) annihilates $(\Pi_\perp^T\omega_z)_\alpha$, or equivalently the other way round, $(\Pi_\perp^T\omega_z)_\alpha$ annihilates the constraint. The latter point of view makes it a covariant derivative.

4 Some natural projection matrices on the constraint surface

Let us give in the following an overview over several projection matrices acting on the tangent or cotangent space of the constraint surface and list some of their properties. They will appear frequently in the remaining discussion and therefore a summary at this point will be very convenient. We are restricting to the constraint surface, as we will later present a way to express the linear projectors away from the surface in terms of those at the surface.

Let us start with our familiar

$$\Pi_\perp = \mathbb{1} - \frac{(\gamma^a\bar\lambda)\otimes(\lambda\gamma_a)}{2(\bar\lambda\lambda)} \quad , \quad \Pi_\perp^\dagger = \Pi_\perp \tag{4.1}$$

It projects to an eleven dimensional subspace ($\mathrm{tr}\,\Pi_\perp = 11$) and further obeys

$$(\Pi_\perp)^2 \stackrel{(B.43)}{=} \Pi_\perp \tag{4.2}$$

$$\Pi_\perp\lambda = \lambda \quad , \quad \bar\lambda\Pi_\perp = \bar\lambda$$

$$(\lambda\gamma^c\Pi_\perp) = 0 = (\Pi_\perp\gamma^c\bar\lambda) \tag{4.3}$$

The last line shows again that it projects to tangent 'vectors' (spinors) that are γ-orthogonal to λ.

The unit matrix minus a projection matrix is always another projection matrix that maps to the complementary subspace. It is thus not surprising that

$$\Pi_{\|} \equiv \mathbb{1} - \Pi_{\perp} = \frac{(\gamma^a\bar{\lambda}) \otimes (\lambda\gamma_a)}{2(\bar{\lambda}\lambda)} \quad , \quad \Pi_{\|}^{\dagger} = \Pi_{\|} \tag{4.4}$$

maps to a 5-dimensional space ($\mathrm{tr}\,\Pi_{\|} = 5$) and obeys

$$\left(\Pi_{\|}\right)^2 = \Pi_{\|} \tag{4.5}$$

$$\Pi_{\|}\lambda = 0 = \bar{\lambda}\Pi_{\|} \tag{4.6}$$

$$(\lambda\gamma^c\Pi_{\|}) = (\lambda\gamma^c) \quad , \quad (\Pi_{\|}\gamma^c\bar{\lambda}) = (\gamma^c\bar{\lambda}) \tag{4.7}$$

From the last line we would say that $\Pi_{\|}$ maps to spinors which are γ-'parallel' to λ^{α}. The matrices $\Pi_{\perp}$ and $\Pi_{\|}$ are of course orthogonal to each other

$$\Pi_{\|}\Pi_{\perp} = \Pi_{\perp}\Pi_{\|} = 0 \tag{4.8}$$

Another projection matrix that will frequently appear is

$$\Pi_{\lambda} \equiv \frac{\lambda \otimes \bar{\lambda}}{(\bar{\lambda}\lambda)} \quad , \quad \Pi_{\lambda}^{\dagger} = \Pi_{\lambda} \tag{4.9}$$

It maps to the 1-dimensional subspace ($\mathrm{tr}\,\Pi_{\lambda} = 1$) spanned by λ itself. Apart from that it shares several properties with $\Pi_{\perp}$:

$$(\Pi_{\lambda})^2 = \Pi_{\lambda} \tag{4.10}$$

$$\Pi_{\lambda}\lambda = \lambda \quad , \quad \bar{\lambda}\Pi_{\lambda} = \bar{\lambda} \tag{4.11}$$

$$(\lambda\gamma^c\Pi_{\lambda}) = 0 = (\Pi_{\lambda}\gamma^c\bar{\lambda}) \tag{4.12}$$

And then there is of course $\mathbb{1} - \Pi_{\lambda}$ with trace 15 and sharing many properties with $\Pi_{\|}$:

$$(\mathbb{1} - \Pi_{\lambda})\lambda = 0 = \bar{\lambda}(\mathbb{1} - \Pi_{\lambda}) \tag{4.13}$$

$$(\lambda\gamma^c(\mathbb{1} - \Pi_{\lambda})) = (\lambda\gamma^c) \quad , \quad ((\mathbb{1} - \Pi_{\lambda})\gamma^c\bar{\lambda}) = (\gamma^c\bar{\lambda}) \tag{4.14}$$

Finally we could take the transpose or complex conjugate of each of the above projection matrices. As they are all Hermitian, the result is the same:

$$\bar{\Pi}_{\perp} = \Pi_{\perp}^T = \mathbb{1} - \frac{(\gamma^a\lambda) \otimes (\bar{\lambda}\gamma_a)}{2(\bar{\lambda}\lambda)} \tag{4.15}$$

$$\bar{\Pi}_{\|} = \Pi_{\|}^T = \frac{(\gamma^a\lambda) \otimes (\bar{\lambda}\gamma_a)}{2(\bar{\lambda}\lambda)} \tag{4.16}$$

$$\bar{\Pi}_{\lambda} = \Pi_{\lambda}^T = \frac{\bar{\lambda} \otimes \lambda}{(\bar{\lambda}\lambda)} \tag{4.17}$$

They have the same trace as the original one and all the other properties are obtained by interchanging λ and $\bar{\lambda}$.

It will be useful to see that one can express Π_λ in terms of $\bar{\Pi}_\|$ and vice versa[10]:

$$\begin{aligned}\Pi_\| &= \tfrac{1}{2}(\gamma^a\bar{\Pi}_\lambda\gamma_a) && (4.18)\\ \Pi_\lambda &= \tfrac{1}{4\xi}\zeta^b(\gamma_b\bar{\Pi}_\|\gamma^c)\bar{\zeta}_c && (4.19)\end{aligned}$$

As intermediate steps between the above equations, we also have

$$\begin{aligned}(\Pi_\|\gamma_a)\zeta^a &= \zeta^a(\gamma_a\bar{\Pi}_\lambda) \quad , \quad (\bar{\Pi}_\|\gamma^a)\bar{\zeta}_a = \bar{\zeta}_a(\gamma^a\Pi_\lambda) && (4.20)\\ \bar{\zeta}_a(\gamma^a\Pi_\|) &= (\bar{\Pi}_\lambda\gamma^a)\bar{\zeta}_a \quad , \quad \zeta^a(\gamma_a\bar{\Pi}_\|) = (\Pi_\lambda\gamma_a)\zeta^a && (4.21)\end{aligned}$$

Note that the same relations hold if one replaces ζ^a and $\bar{\zeta}_a$ by $(\gamma^a\lambda)_\alpha$ and $(\gamma_a\bar{\lambda})^\alpha$ where the proof works just like the one in footnote 10:

$$\begin{aligned}(\Pi_\|\gamma_a)(\gamma^a\lambda)_\alpha &= (\gamma^a\lambda)_\alpha(\gamma_a\bar{\Pi}_\lambda) \quad , \quad (\bar{\Pi}_\|\gamma^a)(\gamma_a\bar{\lambda})^\alpha = (\gamma_a\bar{\lambda})^\alpha(\gamma^a\Pi_\lambda) && (4.22)\\ (\gamma_a\bar{\lambda})^\alpha(\gamma^a\Pi_\|) &= (\bar{\Pi}_\lambda\gamma^a)(\gamma_a\bar{\lambda})^\alpha \quad , \quad (\gamma^a\lambda)_\alpha(\gamma_a\bar{\Pi}_\|) = (\Pi_\lambda\gamma_a)(\gamma^a\lambda)_\alpha && (4.23)\end{aligned}$$

In addition one can easily check from the definitions that we have

$$\begin{aligned}\Pi_\perp\Pi_\lambda &= \Pi_\lambda\Pi_\perp = \Pi_\lambda && (4.24)\\ \Pi_\perp\gamma^c\bar{\Pi}_\lambda &= \bar{\Pi}_\lambda\gamma^c\Pi_\perp = \Pi_\lambda\gamma^c\bar{\Pi}_\perp = \bar{\Pi}_\perp\gamma^c\Pi_\lambda = 0 && (4.25)\end{aligned}$$

We will later also need the variation of $\Pi_\perp = \mathbb{1} - \frac{1}{2}\frac{(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^a)}{(\lambda\bar{\lambda})}$ and some of the above formulas simplify this calculation:

$$\delta\Pi_{\perp\beta}^{\alpha} = -\tfrac{(\gamma_a\bar{\lambda})^\alpha}{2(\lambda\bar{\lambda})}\underbrace{\left(\gamma^a_{\beta\gamma} - \frac{(\lambda\gamma^a)_\beta\bar{\lambda}_\gamma}{(\lambda\bar{\lambda})}\right)}_{\gamma^a_{\beta\delta}\left(\mathbb{1}-\Pi_{(\lambda)}\right)^\delta{}_\gamma}\delta\lambda^\gamma - \delta\bar{\lambda}_\gamma\underbrace{\left(\gamma_a^{\gamma\alpha} - \lambda^\gamma\frac{(\gamma_a\bar{\lambda})^\alpha}{(\lambda\bar{\lambda})}\right)}_{\left(\mathbb{1}-\Pi_{(\lambda)}\right)^\gamma{}_\delta\gamma_a^{\delta\alpha}}\tfrac{(\lambda\gamma^a)_\beta}{2(\lambda\bar{\lambda})} \tag{4.26}$$

Using (4.22) and (4.23) we obtain

$$\delta\Pi_{\perp\beta}^{\alpha} = -\tfrac{(\gamma_a\bar{\lambda})^\alpha}{2(\lambda\bar{\lambda})}(\bar{\Pi}_\perp\gamma^a)_{\beta\gamma}\delta\lambda^\gamma - \delta\bar{\lambda}_\gamma(\gamma_a\bar{\Pi}_\perp)^{\gamma\alpha}\tfrac{(\lambda\gamma^a)_\beta}{2(\lambda\bar{\lambda})} \tag{4.27}$$

5 Hermitian projection matrix and the projection potential

The matrix $\Pi_\perp = \mathbb{1} - \frac{(\gamma^a\bar{\lambda})\otimes(\lambda\gamma_a)}{2(\lambda\bar{\lambda})}$ (4.1),(3.11) on the constraint surface is Hermitian. We will later in the field theory application see that this is a very

[10](4.18) is obvious from the definitions (4.4) and (4.9) of $\Pi_\|$ and Π_λ. Contracting now (4.18) from the right with $\gamma_b\zeta^b$, one obtains

$$(\Pi_\|\gamma_b)\zeta^b = \tfrac{1}{2}(\gamma^a\bar{\Pi}_\lambda\underbrace{\gamma_a\gamma_b}_{-\gamma_b\gamma_a+2\eta_{ab}})\zeta^b = -\tfrac{1}{2}(\gamma^a\underbrace{\bar{\Pi}_\lambda\gamma_b\gamma_a)\zeta^b}_{=0\quad(2.17)} + \zeta^a(\gamma_a\bar{\Pi}_\lambda)$$

Together with its complex conjugate version, this gives (4.20). Equation (4.21) is then simply obtained by Hermitian conjugation, using that the projection matrices and the gamma-matrices are all Hermitian. Finally, contracting the second equation of (4.20) from the left with $\frac{1}{4\xi}\zeta^b\gamma_b$, we obtain

$$\tfrac{1}{4\xi}\zeta^b(\gamma_b\bar{\Pi}_\|\gamma^a)\bar{\zeta}_a = \tfrac{1}{4\xi}\zeta^b\bar{\zeta}_a(\underbrace{\gamma_b\gamma^a}_{-\gamma^a\gamma_b+2\delta^a_b}\Pi_\lambda) = -\tfrac{1}{4\xi}\bar{\zeta}_a\underbrace{\zeta^b(\gamma^a\gamma_b\Pi_\lambda)}_{=0\quad(2.13)} + \underbrace{\tfrac{1}{2\xi}\zeta^a\bar{\zeta}_a}_{=1}\Pi_\lambda$$

which proves (4.19). $\diamond$

useful property. It is therefore natural to ask, whether this can be realized also off the constraint surface. Off the constraint surface we have a non-vanishing contribution $\pi_{(f)\perp}$ in (3.2) and therefore should consider the complete matrix $\begin{pmatrix} \Pi_{(f)\perp} & \pi_{(f)\perp} \\ \bar{\pi}_{(f)\perp} & \bar{\Pi}_{(f)\perp} \end{pmatrix}$ instead of only the block $\Pi_{(f)\perp}$.

Proposition 3 (Hermiticity)**.** *Remember the definition in equation (2.32) of the function h*

$$h(\xi) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}} \quad \text{for } \xi \in [0,1[\tag{5.1}$$

and assume that the function f which defines the projection $P^{\alpha}_{(f)}$ *is differentiable*[11] *in an interval* $I \subset [0, b[, \quad b \leq 1$ *(so in a neighbourhood of the constraint surface* $\xi = 0 \in I$*) with continuous* f' *at least at* 0*. Then the following statements hold:*

1. *The matrix* $\begin{pmatrix} \Pi^{\alpha}_{(f)\perp\beta}(\rho,\bar{\rho}) & \pi^{\alpha\beta}_{(f)\perp}(\rho,\bar{\rho}) \\ \bar{\pi}_{(f)\perp\alpha\beta}(\rho,\bar{\rho}) & \bar{\Pi}_{(f)\perp\alpha}{}^{\beta}(\rho,\bar{\rho}) \end{pmatrix}$ *is Hermitian for all* ρ *where* $\xi \equiv \frac{(\rho\gamma^a\rho)(\bar{\rho}\gamma_a\bar{\rho})}{2(\rho\bar{\rho})} \in I$ *if and only if* $f = h$ *in* I*.*
For the blocks of the matrix, this means that

$$\Pi^{\dagger}_{(h)\perp} = \Pi_{(h)\perp} \tag{5.2}$$

$$\pi^{T}_{(h)\perp} = \pi_{(h)\perp} \tag{5.3}$$

2. *There exists a potential*

$$\Phi(\rho,\bar{\rho}) \equiv \tfrac{(\rho\bar{\rho})}{2}(1+\sqrt{1-\xi}) \tag{5.4}$$

such that

$$P^{\alpha}_{(h)} = \partial_{\bar{\rho}_\alpha}\Phi \quad , \quad \bar{P}_{(h)\alpha} = \partial_{\rho^\alpha}\Phi \tag{5.5}$$

3. *The potential* Φ *can be written as*

$$\Phi(\rho,\bar{\rho}) = P^{\alpha}_{(h)}(\rho,\bar{\rho})\bar{P}_{(h)\alpha}(\rho,\bar{\rho}) \tag{5.6}$$

The proof of this proposition is given in appendix B.3 on page 62.

Remarks

- The Kähler potential on the pure spinor space is given by $(\lambda\bar{\lambda})$. The pontential (5.6) is therefore the pullback of the Kähler potential into the ambient space along the projection $P^{\alpha}_{(h)}$. That does not explain, however, why this is at the same time a potential for the projection itself.
- The function h defined in (5.1) obviously is divergent at $\xi = 1$. It thus does not really belong to the class of functions that we discussed in the previous propositions, as there we assumed f to be defined on the closed interval $[0, 1]$. Statements of these propositions that where about $\xi = 1$ thus need to be reconsidered. In particular the projection $P^{\alpha}_{(h)}$ is not defined for those Weyl spinors ρ^α for which $\xi = 1$. Thus $\xi = 1$ also drops

[11] At 0 differentiability is again understood in the sense of footnote 7 on page 10. $\diamond$

out of the zero-locus (2.10). In addition h does not have any zeroes, so the zero-locus is simply given by

$$P^{-1}_{(h)}(0) = \{0\} \tag{5.7}$$

where $P^{\alpha}_{(h)}(0,0)$ is defined just via the limit $\lim_{|\rho|\to 0} P^{\alpha}_{(h)}(\rho,\bar{\rho}) = 0$ like in proposition 1. Having no well-defined projection at $\xi = 1$ seems odd, but a priori it is most important that $P^{\alpha}_{(h)}$ is well-behaved close to the constraint surface where $\xi = 0$ and which is thus 'far away' from the troublesome points. In addition at least the absolute value of $P^{\alpha}_{(h)}(\rho,\bar{\rho})$ which is given according to (5.6) by the potential (5.4) is well behaved at $\xi = 1$ and converges to

$$\lim_{\xi\to 1}\left(P^{\alpha}_{(h)}(\rho,\bar{\rho})\bar{P}_{(h)\alpha}(\rho,\bar{\rho})\right) = \tfrac{1}{2}(\rho\bar{\rho}) \tag{5.8}$$

Let us rewrite some of the formulas for our projection in the specific case $f(\xi) = h(\xi) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ whose derivative is

$$h'(\xi) = \tfrac{1}{4}\tfrac{1}{\sqrt{1-\xi}^3} \tag{5.9}$$

The non-linear projection map (2.3) becomes[12]

$$P^{\alpha}_{(h)}(\rho,\bar{\rho}) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\rho^{\alpha} - \frac{\zeta^a(\bar{\rho}\gamma_a)^{\alpha}}{4\sqrt{1-\xi}} \tag{5.10}$$

with still (2.4)

$$\zeta^a \equiv \frac{(\rho\gamma^a\rho)}{(\rho\bar{\rho})}, \quad \bar{\zeta}_a \equiv \frac{(\bar{\rho}\gamma_a\bar{\rho})}{(\rho\bar{\rho})}, \quad \xi \equiv \tfrac{1}{2}\zeta^a\bar{\zeta}_a \tag{5.11}$$

The modulus squared (2.8) is now according to (5.6) in the proposition given by Φ in (5.4).

The linearized tangent space projection matrices (3.4) and (3.6) become

$$\begin{aligned}
&\Pi^{\alpha}_{(h)\perp\beta}(\rho,\bar{\rho}) = \\
&= \tfrac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\delta^{\alpha}_{\beta} - \tfrac{1}{2\sqrt{1-\xi}}\frac{(\gamma_a\bar{\rho})^{\alpha}(\gamma^a\rho)_{\beta}}{(\rho\bar{\rho})} - \tfrac{1}{8}\tfrac{1}{\sqrt{1-\xi}^3}\frac{\zeta^a(\gamma_a\bar{\rho})^{\alpha}\bar{\zeta}_b(\gamma^b\rho)_{\beta}}{(\rho\bar{\rho})} + \\
&\quad -\tfrac{1}{2}\tfrac{\xi}{\sqrt{1-\xi}^3}\frac{\rho^{\alpha}\bar{\rho}_{\beta}}{(\rho\bar{\rho})} + \tfrac{1}{4}\tfrac{1}{\sqrt{1-\xi}^3}\frac{\rho^{\alpha}\bar{\zeta}_b(\gamma^b\rho)_{\beta}}{(\rho\bar{\rho})} + \tfrac{1}{4}\tfrac{1}{\sqrt{1-\xi}^3}\frac{\zeta^a(\gamma_a\bar{\rho})^{\alpha}\bar{\rho}_{\beta}}{(\rho\bar{\rho})}
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&\pi^{\alpha\beta}_{(h)\perp}(\rho,\bar{\rho}) = \\
&= -\tfrac{1}{2}\tfrac{\xi}{\sqrt{1-\xi}^3}\frac{\rho^{\alpha}\rho^{\beta}}{(\rho\bar{\rho})} + \tfrac{1}{4}\tfrac{1}{\sqrt{1-\xi}^3}\frac{\rho^{\alpha}\zeta^b(\gamma_b\bar{\rho})^{\beta}}{(\rho\bar{\rho})} + \tfrac{1}{4\sqrt{1-\xi}^3}\frac{\zeta^a(\gamma_a\bar{\rho})^{\alpha}\rho^{\beta}}{(\rho\bar{\rho})} + \\
&\quad -\tfrac{1}{8\sqrt{1-\xi}^3}\frac{\zeta^a\zeta^b(\gamma_a\bar{\rho})^{\alpha}(\gamma_b\bar{\rho})^{\beta}}{(\rho\bar{\rho})} - \tfrac{1}{4\sqrt{1-\xi}}\zeta^a\gamma_a^{\alpha\beta}
\end{aligned} \tag{5.13}$$

[12] Another property of the Hermitian projector (though probably quite meaningless) is that the difference from ρ^{α} to its projection $P^{\alpha}_{(h)}(\rho,\bar{\rho})$ can be nicely expressed in terms of $\bar{P}_{(h)\alpha}(\rho,\bar{\rho})$:

$$\rho^{\alpha} - P^{\alpha}_{(h)}(\rho,\bar{\rho}) = \tfrac{1}{2(1+\sqrt{1-\xi})}\zeta^a\gamma_a^{\alpha\beta}\bar{P}_{(h)\beta}(\rho,\bar{\rho}) \qquad \diamond$$

The equation (2.18) for expressing ρ^α in terms of $\lambda^\alpha \equiv P^\alpha(\rho,\bar\rho)$ and $\bar\lambda_\alpha \equiv \bar P_\alpha(\rho,\bar\rho)$ turns for $f(\xi)=\frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ into

$$\rho^\alpha = \lambda^\alpha + \frac{1}{2\left(1+\sqrt{1-\xi}\right)}\zeta^a\left(\bar\lambda\gamma_a\right)^\alpha \tag{5.14}$$

In the Hermitian case the rewriting of the projection matrices in terms of λ^α and its complex conjugate as was done in general in (3.7) and (3.8) becomes particularly useful, as the $\lambda\otimes(\lambda\gamma^c)$-term drops (in addition to the already missing $(\gamma^c\bar\lambda)\otimes\bar\lambda$-terms) and one is left with[13,14]:

$$\begin{aligned}
\Pi^\alpha_{(h)\perp\beta}(\rho,\bar\rho) &= \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\left(\delta^\alpha_\beta - \frac{1}{2}\frac{(\gamma^a\bar\lambda)^\alpha(\gamma_a\lambda)_\beta}{(\lambda\bar\lambda)}\right) - \frac{1-\sqrt{1-\xi}}{2\sqrt{1-\xi}}\frac{\lambda^\alpha\bar\lambda_\beta}{(\lambda\bar\lambda)} + \\
&\quad -\frac{1}{16\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\frac{\bar\zeta_c(\gamma^c\gamma_b\lambda)^\alpha\zeta^d(\bar\lambda\gamma^b\gamma_d)_\beta}{(\lambda\bar\lambda)} &(5.15)\\
\pi^{\alpha\beta}_{(h)\perp}(\rho,\bar\rho) &= -\frac{\zeta^a}{4\sqrt{1-\xi}}\left(\gamma_a^{\alpha\beta} - \tfrac{(\gamma_a\bar\lambda)^\alpha\lambda^\beta}{(\lambda\bar\lambda)} - \tfrac{\lambda^\alpha(\gamma_a\bar\lambda)^\beta}{(\lambda\bar\lambda)}\right) &(5.16)
\end{aligned}$$

It is further convenient in the Hermitian case, to express the projection matrices off the constraint surface in terms of the projection matrices $\Pi^\alpha_{\perp\beta}$ (4.1) and $\Pi^\alpha_{(\lambda)\beta}$ (4.9) defined on the surface:

$$\begin{aligned}
\Pi^\alpha_{(h)\perp\beta}(\rho,\bar\rho) &= \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\Pi^\alpha_{\perp\beta} - \frac{1-\sqrt{1-\xi}}{2\sqrt{1-\xi}}\Pi^\alpha_{(\lambda)\beta} + \\
&\quad -\frac{1}{8\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\frac{\bar\zeta_c(\gamma^c\bar\Pi_{\parallel}\gamma_d)_\beta\zeta^d}{(\lambda\bar\lambda)} &(5.17)\\
\pi^{\alpha\beta}_{(h)\perp}(\rho,\bar\rho) &= -\frac{1}{4\sqrt{1-\xi}}\left((\Pi_\perp\gamma_a)^{\alpha\beta}\zeta^a - (\Pi_{(\lambda)}\gamma_a)^{\alpha\beta}\zeta^a\right) = &(5.18)\\
&\overset{(4.20)}{\underset{(4.21)}{=}} -\frac{1}{4\sqrt{1-\xi}}\left(\zeta^a(\gamma_a\bar\Pi_\perp)^{\alpha\beta} - \zeta^a(\gamma_a\bar\Pi_\lambda)^{\alpha\beta}\right) &(5.19)
\end{aligned}$$

---

[13]The ρ-derivatives of ζ^a and ξ in (B.27)-(B.29), rewritten in terms of $\lambda\equiv P_{(f)}(\rho,\bar\rho)$, as it is done in the appendix on page 55 in footnote 32, now turn for $f(\xi)=h(\xi)\equiv\frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ into

$$\begin{aligned}
\partial_{\rho^\beta}\zeta^a &= \frac{1}{4(\lambda\bar\lambda)}\Big\{4\left(1+\sqrt{1-\xi}\right)(\lambda\gamma^a)_\beta - \zeta^a\bar\zeta_b\left(\lambda\gamma^b\right)_\beta + \\
&\quad -2\zeta^b\left(\gamma_b\gamma^a\bar\lambda\right)_\beta + 2\left(1-\sqrt{1-\xi}\right)\zeta^a\bar\lambda_\beta\Big\}\\
\partial_{\bar\rho_\beta}\zeta^a &= -\frac{\zeta^a}{4(\lambda\bar\lambda)}\Big\{2\left(1+\sqrt{1-\xi}\right)\lambda^\beta + \zeta^b\left(\bar\lambda\gamma_b\right)^\beta\Big\}\\
\partial_{\rho^\beta}\xi &= \frac{\sqrt{1-\xi}}{2(\lambda\bar\lambda)}\Big\{\left(1+\sqrt{1-\xi}\right)\bar\zeta_c\left(\gamma^c\lambda\right)_\beta - 2\xi\bar\lambda_\beta\Big\} \qquad \diamond
\end{aligned}$$

[14]In terms of the alternative parametrization $\tilde\zeta^a$ from equation (2.22), equations (5.15) and (5.16) turn into

$$\begin{aligned}
\Pi^\alpha_{(h)\perp\beta}(\rho,\bar\rho) &= \tfrac{1}{1-\tilde\xi}\left(\delta^\alpha_\beta - \tfrac{(\gamma_a\bar\lambda)^\alpha(\lambda\gamma^a)_\beta}{2(\lambda\bar\lambda)}\right) - \tfrac{\tilde\xi}{(1-\tilde\xi)}\tfrac{\lambda^\alpha\bar\lambda_\beta}{(\lambda\bar\lambda)} - \tfrac{1}{8(1-\tilde\xi)}\tfrac{\bar{\tilde\zeta}_c(\gamma^c\gamma_b\lambda)^\alpha\tilde\zeta^d(\bar\lambda\gamma^b\gamma_d)_\beta}{(\lambda\bar\lambda)}\\
\pi^{\alpha\beta}_{(h)\perp}(\rho,\bar\rho) &= -\tfrac{\tilde\zeta^a}{2(1-\tilde\xi)}\left(\gamma_a^{\alpha\beta} - \tfrac{(\gamma_a\bar\lambda)^\alpha\lambda^\beta}{(\lambda\bar\lambda)} - \tfrac{\lambda^\alpha(\bar\lambda\gamma_a)^\beta}{(\lambda\bar\lambda)}\right) \qquad \diamond
\end{aligned}$$

The same result is obtained when using $\tilde h(\tilde\xi)\equiv h(\xi)=\frac{1}{1-\tilde\xi}$ (2.32) and $\tilde h'(\tilde\xi)=\frac{1}{(1-\tilde\xi)^2}$ in the equations of footnote 8 on page 11. $\diamond$

For the field theory application in section 7, one of the most important properties of the Hermitian projection-matrices will be the fact that the order of the matrix multiplication in the projection property (3.14) will not matter any longer as these matrices will commute. Starting from the original projection property

$$\begin{pmatrix} \Pi^{\alpha}_{\perp\gamma} & 0 \\ 0 & \bar{\Pi}_{\perp\alpha}{}^{\gamma} \end{pmatrix}\begin{pmatrix} \Pi^{\gamma}_{(h)\perp\beta}(\rho,\bar{\rho}) & \pi^{\gamma\beta}_{(h)\perp}(\rho,\bar{\rho}) \\ \bar{\pi}_{(h)\perp\gamma\beta}(\rho,\bar{\rho}) & \bar{\Pi}_{(h)\perp\gamma}{}^{\beta}(\rho,\bar{\rho}) \end{pmatrix} = \begin{pmatrix} \Pi^{\alpha}_{(h)\perp\beta}(\rho,\bar{\rho}) & \pi^{\alpha\beta}_{(h)\perp}(\rho,\bar{\rho}) \\ \bar{\pi}_{(h)\perp\alpha\beta}(\rho,\bar{\rho}) & \bar{\Pi}_{(h)\perp\alpha}{}^{\beta}(\rho,\bar{\rho}) \end{pmatrix} \tag{5.20}$$

and taking the Hermitian conjugate on both sides, we arrive (because of Hermiticity) at

$$\begin{pmatrix} \Pi^{\alpha}_{(h)\perp\gamma}(\rho,\bar{\rho}) & \pi^{\alpha\gamma}_{(h)\perp}(\rho,\bar{\rho}) \\ \bar{\pi}_{(h)\perp\alpha\gamma}(\rho,\bar{\rho}) & \bar{\Pi}_{(h)\perp\alpha}{}^{\gamma}(\rho,\bar{\rho}) \end{pmatrix}\begin{pmatrix} \Pi^{\gamma}_{\perp\beta} & 0 \\ 0 & \bar{\Pi}_{\perp\gamma}{}^{\beta} \end{pmatrix} = \begin{pmatrix} \Pi^{\alpha}_{(h)\perp\beta}(\rho,\bar{\rho}) & \pi^{\alpha\beta}_{(h)\perp}(\rho,\bar{\rho}) \\ \bar{\pi}_{(h)\perp\alpha\beta}(\rho,\bar{\rho}) & \bar{\Pi}_{(h)\perp\alpha}{}^{\beta}(\rho,\bar{\rho}) \end{pmatrix} \tag{5.21}$$

which is equivalent to the following two equations and their complex conjugates respectively

$$\Pi^{\alpha}_{(h)\perp\gamma}(\rho,\bar{\rho})\Pi^{\gamma}_{\perp\beta} = \Pi^{\alpha}_{\perp\gamma}\Pi^{\gamma}_{(h)\perp\beta}(\rho,\bar{\rho}) \quad \left(= \Pi^{\alpha}_{(h)\perp\gamma}(\rho,\bar{\rho})\right) \tag{5.22}$$

$$\pi^{\alpha\gamma}_{(h)\perp}\bar{\Pi}_{\perp\gamma}{}^{\beta} = \Pi^{\alpha}_{\perp\gamma}\pi^{\gamma\beta}_{(h)\perp} \quad \left(= \pi^{\alpha\beta}_{(h)\perp}\right) \tag{5.23}$$

Using (4.24),(4.21) and (4.8) it is also easy to check these equations explicitly.

6 Natural projection in the U(5) formalism

The 10 Dirac gamma matrices Γ^a can be used to define 5 pairs of creation and annihilation matrices

$$b^{\mathfrak{a}} \equiv \tfrac{1}{2}\left(\Gamma^{2\mathfrak{a}-1} - i\Gamma^{2\mathfrak{a}}\right) \tag{6.1}$$

$$b^{\dagger}_{\mathfrak{b}} \equiv \tfrac{1}{2}\left(\Gamma_{2\mathfrak{b}-1} + i\Gamma_{2\mathfrak{b}}\right), \quad {\scriptstyle \mathfrak{a},\mathfrak{b}\in\{1,\ldots,5\}} \tag{6.2}$$

$$\{b^{\mathfrak{a}}, b^{\dagger}_{\mathfrak{b}}\} = \delta^{\mathfrak{a}}_{\mathfrak{b}} \tag{6.3}$$

For $SO(1,9)$ one simply replaces Γ^{10} by $i\Gamma^0$ or Γ_{10} by $-i\Gamma_0$. The creation matrices can be used to build a Fock space representation of spinors by acting on a vacuum spinor $|\Omega\rangle$ which is annihilated by all annihilation matrices (see e.g. Appendix B.1 of [21, p.430] and in particular in the pure spinor context appendix D of [22] for a more detailed discussion of this parametrization):

$$|_{\mathfrak{a}}\rangle \equiv b^{\dagger}_{\mathfrak{a}}|\Omega\rangle, \qquad \binom{5}{1} = 5 \text{ states} \tag{6.4}$$

$$|_{\mathfrak{a}_1\mathfrak{a}_2}\rangle \equiv b^{\dagger}_{\mathfrak{a}_1}b^{\dagger}_{\mathfrak{a}_2}|\Omega\rangle, \qquad \binom{5}{2} = 10 \text{ states} \tag{6.5}$$

$$\ddots$$

$$|_{\mathfrak{a}_1\ldots\mathfrak{a}_5}\rangle \equiv b^{\dagger}_{\mathfrak{a}_1}\cdots b^{\dagger}_{\mathfrak{a}_5}|\Omega\rangle, \qquad \binom{5}{5} = 1 \text{ state} \tag{6.6}$$

Together with the vacuum $|\Omega\rangle$, these are precisely $\sum_{k=0}^{5}\binom{5}{k} = 2^5 = 32$ states. An arbitrary Dirac spinor $|\Psi\rangle$ can therefore be expanded in this basis. It is a well known fact that chirality in this picture corresponds to an even number of creators:

$$|\Psi\rangle \equiv \underbrace{\Psi^+|\Omega\rangle + \tfrac{1}{2}\Psi^{\mathfrak{a}_1\mathfrak{a}_2}|_{\mathfrak{a}_1\mathfrak{a}_2}\rangle + \tfrac{1}{4!}\overbrace{\Psi_{\mathfrak{a}}\epsilon^{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}}^{\Psi^{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}}|_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rangle}_{\text{chiral}} + \tag{6.7}$$

$$+\underbrace{\tfrac{1}{5!}\overbrace{\Psi_+\epsilon^{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5}}^{\Psi^{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5}}|_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5}\rangle + \tfrac{1}{3!}\overbrace{\tfrac{1}{2}\Psi_{\mathfrak{a}_1\mathfrak{a}_2}\epsilon^{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3}}^{\Psi^{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}}|_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3}\rangle + \Psi^{\mathfrak{a}}|_{\mathfrak{a}}\rangle}_{\text{antichiral}}$$

Chiral SO(10) Weyl spinors ρ^α can therefore be U(5)-covariantly parametrized by a U(5)-singlet ρ^+, a U(5)-bivector $\rho^{\mathfrak{a}_1\mathfrak{a}_2}$ (antisymmetric with $\mathfrak{a}_1,\mathfrak{a}_2\in\{1,\ldots,5\}$, i.e. 10 components) and a U(5) covector $\rho_{\mathfrak{a}}$ (with 5 components). The pure spinor constraint $(\lambda\gamma^a\lambda) = 0$ turns into $\langle\lambda|Cb^{\mathfrak{a}}|\lambda\rangle = 0$ and $\langle\lambda|Cb^\dagger_{\mathfrak{a}}|\lambda\rangle = 0$ where C is the charge conjugation matrix. The first one turns out to be a consequence of the second, while the second can be calculated to be of the form (see up to a conventional sign [1] or again appendix D of [22])

$$\lambda^+\lambda_{\mathfrak{a}} = \tfrac{1}{8}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\lambda^{\mathfrak{b}_1\mathfrak{b}_2}\lambda^{\mathfrak{b}_3\mathfrak{b}_4} \tag{6.8}$$

For $\lambda^+ \neq 0$ one can obviously solve for $\lambda_{\mathfrak{a}}$. A natural projection from the space of Weyl spinors ρ^α to pure Weyl spinors λ^α is then simply to replace the general component $\rho_{\mathfrak{a}}$ by the solution to the above equation, i.e. $(\rho^+,\rho^{\mathfrak{a}_1\mathfrak{a}_2},\rho_{\mathfrak{a}}) \mapsto (\rho^+,\rho^{\mathfrak{a}_1\mathfrak{a}_2},\frac{1}{8\rho^+}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rho^{\mathfrak{b}_1\mathfrak{b}_2}\rho^{\mathfrak{b}_3\mathfrak{b}_4})$. The claim is that there is some reference spinor $\bar\chi$, such the previously discussed $P_{(\bar\chi)}$ of equation (2.35) is precisely this projection:

$$(\rho^+,\rho^{\mathfrak{a}_1\mathfrak{a}_2},\rho_{\mathfrak{a}}) \overset{P_{(\bar\chi)}}{\mapsto} P_{(\bar\chi)}(\rho) \overset{!}{=} (\rho^+,\rho^{\mathfrak{a}_1\mathfrak{a}_2},\tfrac{1}{8\rho^+}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rho^{\mathfrak{b}_1\mathfrak{b}_2}\rho^{\mathfrak{b}_3\mathfrak{b}_4}) \equiv (\lambda^+,\lambda^{\mathfrak{a}_1\mathfrak{a}_2},\lambda_{\mathfrak{a}}) \tag{6.9}$$

This should determine the reference spinor $\bar\chi_\alpha$. As there appear no square roots in the above terms, it is reasonable to assume that $\bar\chi_\alpha$ is a pure spinor and the general form (2.35) of the projection reduces to (2.37), i.e.

$$P^\alpha_{(\bar\chi)}(\rho) = \rho^\alpha - \tfrac{1}{4}\frac{(\rho\gamma^a\rho)(\gamma_a\bar\chi)^\alpha}{(\rho\bar\chi)} \quad (\text{pure } \bar\chi) \tag{6.10}$$

The antichiral spinor $\bar\chi_\alpha$ has in general only a non-vanishing second line in the expansion (6.7). Note that the state (6.6) which appears in this expansion is an alternative vacuum if one interchanges the role of annihilators and creators, as it is annihilated by all $b^\dagger_{\mathfrak{a}}$'s. Being a vacuum it is automatically a pure spinor. So instead of making a general ansatz for $\bar\chi_\alpha$, let us simply try this alternative vacuum as reference spinor

$$(\bar\chi_+,\bar\chi_{\mathfrak{a}\mathfrak{a}},\bar\chi^{\mathfrak{a}}) = (1,0,0) \quad (\text{ansatz}) \tag{6.11}$$

The translation of (6.10) into U(5) language is then rather simple. First we note that due to the definitions (6.1),(6.2) we have $\gamma^a\otimes\gamma_a \to 2b^{\mathfrak{a}}\otimes b^\dagger_{\mathfrak{a}} + 2b^\dagger_{\mathfrak{a}}\otimes b^{\mathfrak{a}}$. Applied to (6.10), only the second term survives with

$$b^{\mathfrak{a}}|\bar\chi\rangle = \tfrac{1}{5!}\epsilon^{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5}b^{\mathfrak{a}}|_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5}\rangle = \tfrac{1}{4!}\epsilon^{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}|_{\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rangle \tag{6.12}$$

$$\langle\rho|Cb^\dagger_{\mathfrak{a}}|\rho\rangle = 2\rho^+\rho_{\mathfrak{a}} - \tfrac{1}{4}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rho^{\mathfrak{b}_1\mathfrak{b}_2}\rho^{\mathfrak{b}_3\mathfrak{b}_4} \tag{6.13}$$

So the only non-vanishing component of $b^{\mathfrak{a}}|\bar{\chi}\rangle$ is $(b^{\mathfrak{a}}\bar{\chi})_{\mathfrak{c}} = \delta^{\mathfrak{a}}_{\mathfrak{c}}$ while $(b^{\mathfrak{a}}\bar{\chi})^{+} = (b^{\mathfrak{a}}\bar{\chi})^{\mathfrak{cd}} = 0$. The projection (6.10) thus becomes

$$
\begin{aligned}
P^{+}_{(\bar{\chi})}(\rho) &= \rho^{+} && (6.14)\\
P^{\mathfrak{cd}}_{(\bar{\chi})}(\rho) &= \rho^{\mathfrak{cd}} && (6.15)\\
P_{(\bar{\chi})\mathfrak{c}}(\rho) &= \rho_{\mathfrak{c}} - \tfrac{1}{2}\frac{\langle\rho|Cb^{\dagger}_{\mathfrak{a}}|\rho\rangle(b^{\mathfrak{a}}\bar{\chi})_{\mathfrak{c}}}{\rho^{+}} = && (6.16)\\
&= \rho_{\mathfrak{c}} - \tfrac{1}{2}\frac{\left(2\rho^{+}\rho_{\mathfrak{a}} - \frac{1}{4}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rho^{\mathfrak{b}_1\mathfrak{b}_2}\rho^{\mathfrak{b}_3\mathfrak{b}_4}\right)\delta^{\mathfrak{a}}_{\mathfrak{c}}}{\rho^{+}} = && (6.17)\\
&= \tfrac{1}{8\rho^{+}}\epsilon_{\mathfrak{c}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\rho^{\mathfrak{b}_1\mathfrak{b}_2}\rho^{\mathfrak{b}_3\mathfrak{b}_4} && (6.18)
\end{aligned}
$$

This is indeed the projection that we suggested in (6.9).

Let us define for double indices the partial derivative such that the variation comes with a factor $\frac{1}{2}$ which takes into account the antisymmetry:

$$
\delta = \delta\rho^{+}\partial_{\rho^{+}} + \tfrac{1}{2}\delta\rho^{\mathfrak{ab}}\partial_{\rho^{\mathfrak{ab}}} + \delta\rho_{\mathfrak{a}}\partial_{\rho_{\mathfrak{a}}} \tag{6.19}
$$

In other words $\partial_{\rho^{\mathfrak{ab}}}$ is twice the naive partial derivative. Let us think of the spinor index α (in the U(5) basis) to be a collective index $^{\alpha} \in \{^{+}, {}^{\mathfrak{a}_1\mathfrak{a}_2}, {}_{\mathfrak{a}}\}$. The projection matrix $\Pi_{\perp}$ then becomes

$$
\Pi_{\perp(\bar{\chi})}{}^{\alpha}{}_{\beta}(\rho,\bar{\rho}) = \frac{\partial\left(P^{+}_{(\bar{\chi})}(\rho), P^{\mathfrak{a}_1\mathfrak{a}_2}_{(\bar{\chi})}(\rho), P_{(\bar{\chi})\mathfrak{a}}(\rho)\right)}{\partial(\rho^{+}, \rho^{\mathfrak{b}_1\mathfrak{b}_2}, \rho_{\mathfrak{b}})} = \tag{6.20}
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^{\mathfrak{a}_1}_{\mathfrak{b}_1}\delta^{\mathfrak{a}_2}_{\mathfrak{b}_2} - \delta^{\mathfrak{a}_1}_{\mathfrak{b}_2}\delta^{\mathfrak{a}_2}_{\mathfrak{b}_1} & 0 \\ -\frac{1}{8(\rho^{+})^2}\epsilon_{\mathfrak{a}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3\mathfrak{c}_4}\rho^{\mathfrak{c}_1\mathfrak{c}_2}\rho^{\mathfrak{c}_3\mathfrak{c}_4} & \frac{1}{2\rho^{+}}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{c}_1\mathfrak{c}_2}\rho^{\mathfrak{c}_1\mathfrak{c}_2} & 0 \end{pmatrix} \tag{6.21}
$$

We will come back to this U(5) form of the projection matrix in subsection 7.3 on page 31.

7 Ghost action of the pure spinor string

We are now ready to apply some of the mathematical insight to the pure spinor string. Remember that in the introduction the transpose of the linearized projection was claimed to project the antighost field of the pure spinor string to its gauge invariant part. In the following first subsection we will quickly recall the ghost action of the so-called non-minimal formalism as a functional of constrained fields and discuss the constrained variation and the corresponding gauge transformations. The known projector to a gauge invariant part of the antighost will be extended to the so-called non-minimal fields. After that, in subsection 7.2, we will replace the constrained variables by projections of unconstrained variables and discuss the variation and the gauge symmetries of the resulting constraint-free ghost action. And in subsection 7.3, we will recall the minimal ghost action in the U(5) parametrization and quickly review how in this formulation the antighosts automatically combine to gauge invariant combinations. We will then provide the explicit reference spinor $\bar{\chi}$ for which the non-covariant projector $\bar{\Pi}_{\perp(\bar{\chi})}$ yields precisely these expressions.

7.1 With constrained variables

The ghost action (left-moving sector) in the non-minimal formalism [23] of Berkovits' pure spinor string theory is given by

$$S_{\mathrm{gh}}[\lambda, \omega_z, \bar{\lambda}, \bar{\omega}_z, \boldsymbol{r}, \boldsymbol{s}_z] \quad = \quad \int d^2 z \left(\bar{\partial}\lambda^\alpha \omega_{z\alpha} + \bar{\partial}\bar{\lambda}_\alpha \bar{\omega}_z^\alpha + \bar{\partial}\boldsymbol{r}_\alpha \boldsymbol{s}_z^\alpha \right) \tag{7.1}$$

together with the constraints

$$(\lambda\gamma^a\lambda) = (\bar{\lambda}\gamma^a\bar{\lambda}) = (\bar{\lambda}\gamma^a\boldsymbol{r}) = 0 \tag{7.2}$$

The variations consistent with these constraints accordingly have to obey

$$(\delta\lambda\gamma^a\lambda) = (\delta\bar{\lambda}\gamma^a\bar{\lambda}) = (\delta\bar{\lambda}\gamma^a\boldsymbol{r}) + (\bar{\lambda}\gamma^a\delta\boldsymbol{r}) = 0 \tag{7.3}$$

The constraints (7.2) generate via the Poisson bracket gauge transformations of the form

$$\delta_{(\mu)}\omega_{z\alpha} \quad \equiv \quad \mu_{za}(\gamma^a\lambda)_\alpha \tag{7.4}$$

$$\delta_{(\bar{\mu},\sigma)}\bar{\omega}_z^\alpha \quad \equiv \quad \bar{\mu}_z^a(\gamma_a\bar{\lambda})^\alpha - \boldsymbol{\sigma}_z^a(\gamma_a\boldsymbol{r})^\alpha \tag{7.5}$$

$$\delta_{(\sigma)}\boldsymbol{s}_z^\alpha \quad \equiv \quad \boldsymbol{\sigma}_z^a(\gamma_a\bar{\lambda})^\alpha \tag{7.6}$$

where $\mu_{za}, \bar{\mu}_z^a$ are some even and $\boldsymbol{\sigma}_z^a$ are some odd gauge parameters. In equations (6) and (7) of [18] within the minimal formalism (in the absence of $\bar{\lambda}_\alpha, \bar{\omega}_z^\alpha, \boldsymbol{r}_\alpha$ and $\boldsymbol{s}_z^\alpha$) the authors presented a linear projection of the antighost $\omega_{z\alpha}$ to its gauge invariant part $\tilde{\omega}_{z\alpha}$, using some reference spinor (see also (14) and (18) in [19] or more recently (2.12) and (2.15) in [20]). Its covariantized version (where $\bar{\lambda}_\alpha$ plays the role of the reference spinor) is given by

$$\tilde{\omega}_{z\alpha} \equiv (\bar{\Pi}_\perp\omega_z)_\alpha = \omega_{z\alpha} - \tfrac{1}{2}\frac{(\gamma^a\lambda)_\alpha(\bar{\lambda}\gamma_a\omega_z)}{(\lambda\bar{\lambda})} \tag{7.7}$$

and was presented in equations (2.9) and (2.11) of [17]. The gauge transformation of $\boldsymbol{s}_z^\alpha$ in (7.6) is of the same type as the one of $\omega_{z\alpha}$, just that the role of λ^α and $\bar{\lambda}_\alpha$ is interchanged. It is therefore easy to guess also a projection to a gauge invariant part of $\boldsymbol{s}_z^\alpha$:

$$\tilde{\boldsymbol{s}}_z^\alpha \equiv (\Pi_\perp\boldsymbol{s}_z)^\alpha = \boldsymbol{s}_z^\alpha - \tfrac{1}{2}\frac{(\gamma_a\bar{\lambda})^\alpha(\lambda\gamma^a\boldsymbol{s}_z)}{(\lambda\bar{\lambda})} \tag{7.8}$$

For $\bar{\omega}_z^\alpha$ at least the $\delta_{(\sigma)}$-part of (7.5) is of a slightly different form, so that the same naive guess as above does not work. However, having in mind the BRST transformation $\mathbf{s}\boldsymbol{s}_z^\alpha = \bar{\omega}_z^\alpha$ and the fact that the BRST differential forms together with the gauge transformations a closed algebra[15], a natural guess for the gauge

[15] We have not presented the complete action of the pure spinor string and neither the BRST transformations of all its fields, but those of the non-minimal sector are given by [23]

$$\mathbf{s}\boldsymbol{s}_z^\alpha \quad = \quad \bar{\omega}_z^\alpha, \quad \mathbf{s}\bar{\omega}_z^\alpha = 0, \qquad \mathbf{s}\bar{\lambda}_\alpha = \boldsymbol{r}_\alpha, \quad \mathbf{s}\boldsymbol{r}_\alpha = 0$$

These BRST transformations commute with the $\delta_{(\sigma)}$ gauge transformations in (7.5)-(7.6).

$$[\mathbf{s}, \delta_{(\sigma)}] = 0$$

The commutator with $\delta_{(\bar{\mu})}$ instead is non-vanishing but produces another $\delta_{(\sigma)}$-transformation:

$$[\mathbf{s}, \delta_{(\bar{\mu})}]\bar{\omega}_z^\alpha \quad = \quad \bar{\mu}_z^a(\gamma_a\boldsymbol{r})^\alpha$$

$$[\mathbf{s}, \delta_{(\bar{\mu})}]\boldsymbol{s}_z^\alpha \quad = \quad -\bar{\mu}_z^a(\gamma_a\bar{\lambda})^\alpha$$

So $[\delta_{(\bar{\mu})}, \mathbf{s}] = \boldsymbol{\delta}_{(\tilde{\sigma})}$ with $\tilde{\sigma}_z^a = \bar{\mu}_z^a$. The parameter has changed parity, but that does not change invariance properties at linearized level. $\diamond$

invariant part of $\bar{\omega}_{z\alpha}$ is simply the BRST transformation of $\tilde{\boldsymbol{s}}_z^\alpha$

$$\begin{aligned}
\tilde{\bar{\omega}}_z^\alpha &\equiv \mathbf{s}\tilde{\boldsymbol{s}}_z^\alpha = && (7.9)\\
&= (\Pi_\perp\bar{\omega}_z)^\alpha + \left(-(\gamma_a\boldsymbol{r})^\alpha + (\lambda\boldsymbol{r})\frac{(\gamma_a\bar{\lambda})^\alpha}{(\lambda\bar{\lambda})}\right)\frac{(\lambda\gamma^a\boldsymbol{s}_z)}{2(\lambda\bar{\lambda})} = && (7.10)\\
&= (\Pi_\perp\bar{\omega}_z)^\alpha + \left(-(\gamma_a\boldsymbol{r})^\alpha + (\gamma_a\bar{\Pi}_\lambda\boldsymbol{r})^\alpha\right)\frac{(\lambda\gamma^a\boldsymbol{s}_z)}{2(\lambda\bar{\lambda})} && (7.11)
\end{aligned}$$

Using the identity $(\gamma_a\bar{\Pi}_\lambda)^{\alpha\beta}(\lambda\gamma^a)_\gamma = (\Pi_\parallel\gamma_a)^{\alpha\beta}(\lambda\gamma^a)_\gamma$ of equation (4.22), we obtain the linear projection[16]

$$\tilde{\bar{\omega}}_z^\alpha = (\Pi_\perp\bar{\omega}_z)^\alpha - (\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\frac{(\lambda\gamma^a\boldsymbol{s}_z)}{2(\lambda\bar{\lambda})} \qquad (7.12)$$

Gauge invariance of this expression now follows from gauge invariance of $\tilde{\boldsymbol{s}}_z^\alpha$ and the fact that the BRST differential builds a closed algebra with the gauge transformations according to footnote 15:

$$\begin{aligned}
\delta_{(\sigma)}\tilde{\bar{\omega}}_z^\alpha &= \delta_{(\sigma)}\mathbf{s}\tilde{\boldsymbol{s}}_z^\alpha \overset{\text{fn (15)}}{=} \mathbf{s}\underbrace{\delta_{(\sigma)}\tilde{\boldsymbol{s}}_z^\alpha}_{=0} = 0 && (7.13)\\
\delta_{(\bar{\mu})}\tilde{\bar{\omega}}_z^\alpha &= \delta_{(\bar{\mu})}\mathbf{s}\tilde{\boldsymbol{s}}_z^\alpha \overset{\text{fn (15)}}{=} \mathbf{s}\underbrace{\delta_{(\bar{\mu})}\tilde{\boldsymbol{s}}_z^\alpha}_{=0} + \boldsymbol{\delta}_{(\tilde{\sigma})}\tilde{\boldsymbol{s}}_z^\alpha = 0 && (7.14)
\end{aligned}$$

It can also easily been checked by direct calculation[17]. To our knowledge the gauge invariant projections $\tilde{\mathbf{s}}_z^\alpha$ and $\tilde{\bar{\omega}}_z^\alpha$ for the non-minimal variables had not yet been mentioned in the literature. Note that $\tilde{\omega}_{z\alpha}$, $\tilde{\boldsymbol{s}}_z^\alpha$ and $\tilde{\bar{\omega}}_z^\alpha$ are gauge invariant only up to the constraints (7.2). We will remove this restriction a bit later in this

[16] The expression for $\tilde{\bar{\omega}}_z$ in (7.12) is not everywhere linear in $\bar{\omega}_z^\alpha$, but instead some terms are linear in $\mathbf{s}_z$. It should thus be understood as a linear projection from the variables $(\boldsymbol{s}_z^\alpha, \bar{\omega}_z^\alpha)$ to $(\tilde{\boldsymbol{s}}_z^\alpha, \tilde{\bar{\omega}}_z^\alpha)$, which indeed obeys the projection-property in addition to being gauge-invariant. The projection property on the constraint surface is inherited from $\Pi_\perp$ (Because of $\Pi_\perp^2 = \Pi_\perp$ we have $(\mathbf{s}\Pi_\perp)\Pi_\perp + \Pi_\perp(\mathbf{s}\Pi_\perp) = \mathbf{s}\Pi_\perp$):

$$\begin{aligned}
\tilde{\bar{\omega}}_z &\overset{(7.9)}{=} \Pi_\perp\bar{\omega}_z + (\mathbf{s}\Pi_\perp)\boldsymbol{s}_z\\
\tilde{\tilde{\bar{\omega}}}_z &= \Pi\tilde{\bar{\omega}}_z + (\mathbf{s}\Pi)\tilde{\boldsymbol{s}}_z =\\
&= \Pi\left(\Pi\bar{\omega}_z + (\mathbf{s}\Pi)\boldsymbol{s}_z\right) + (\mathbf{s}\Pi)\Pi\boldsymbol{s}_z =\\
&\underset{(\mathbf{s}\Pi)\Pi+\Pi(\mathbf{s}\Pi)=\mathbf{s}\Pi}{\overset{\Pi^2=\Pi}{=}} \bar{\omega}_z + (\mathbf{s}\Pi)\boldsymbol{s}_z = \tilde{\bar{\omega}}_z \qquad \diamond
\end{aligned}$$

[17] Let us check the gauge invariance of (7.12) by direct calculation

$$\begin{aligned}
\delta_{(\bar{\mu},\sigma)}\tilde{\bar{\omega}}_z^\alpha &= \bar{\mu}_z^a\underbrace{(\Pi_\perp\gamma_a\bar{\lambda})^\alpha}_{=0} - \boldsymbol{\sigma}_z^\alpha(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha - (\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\frac{\boldsymbol{\sigma}_z^b(\lambda\overbrace{\gamma^a\gamma_b}^{-\gamma_b\gamma^a+2\delta_b^a}\bar{\lambda})}{2(\lambda\bar{\lambda})} =\\
&= (\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\frac{\boldsymbol{\sigma}_z^b(\lambda\gamma_b\gamma^a\bar{\lambda})}{2(\lambda\bar{\lambda})}
\end{aligned}$$

Using the Fierz identity $(\gamma_a\boldsymbol{r})^\alpha(\gamma^a\bar{\lambda})^\beta = -(\bar{\lambda}\gamma_a\boldsymbol{r})\gamma^{a\,\alpha\beta} - (\gamma_a\bar{\lambda})^\alpha(\gamma^a\boldsymbol{r})^\beta$ and remembering the constraint $(\bar{\lambda}\gamma_a\boldsymbol{r}) = 0$, the above expression becomes

$$\delta_{(\bar{\mu},\sigma)}\tilde{\bar{\omega}}_z^\alpha = \underbrace{(\Pi_\perp\gamma_a\bar{\lambda})^\alpha}_{=0}\frac{\boldsymbol{\sigma}_z^b(\lambda\gamma_b\gamma^a\boldsymbol{r})}{2(\lambda\bar{\lambda})} = 0 \qquad \diamond$$

section. Let us now elaborate a bit further on the variation of the constrained variables and recover the gauge invariant expressions in the equations of motion. To this end, let us first note that by using our projection maps, the constraints (7.2) can equivalently be written as

$$\lambda^\alpha = P^\alpha_{(f)}(\lambda,\bar\lambda), \quad \bar\lambda_\alpha = P^\alpha_{(f)}(\lambda,\bar\lambda), \quad \boldsymbol{r}_\alpha = \bar\Pi_{\perp\alpha}{}^\beta \boldsymbol{r}_\beta \tag{7.15}$$

Varying these constraints on both sides leads for the ghost λ^α to $\delta\lambda^\alpha = (\Pi_\perp \delta\lambda)^\alpha$ which was already given in equation (3.20). For $\bar\lambda_\alpha$ one obtains the complex conjugate relation, while for the $\boldsymbol{r}_\alpha$-variation of the last constraint in (7.2) we need the variation of $\Pi_{\perp\alpha}{}^\beta$ given in (4.27) (or actually its complex conjugate). Using $(\bar\lambda\gamma_a\boldsymbol{r}) = 0$ this yields

$$\delta\boldsymbol{r}_\alpha = \bar\Pi_{\perp\alpha}{}^\beta\delta\boldsymbol{r}_\beta - \tfrac{(\gamma^a\lambda)_\alpha}{2(\lambda\bar\lambda)}(\underbrace{\boldsymbol{r}\Pi_\perp}_{\boldsymbol{r}}\gamma_a)^\gamma \underbrace{\delta\bar\lambda_\gamma}_{(\bar\Pi_\perp\delta\bar\lambda)_\gamma} \tag{7.16}$$

So altogether we end up with the following relations for the variations

$$\delta\lambda^\alpha = \Pi^\alpha_{\perp\beta}\delta\lambda^\beta, \quad \delta\bar\lambda_\alpha = \bar\Pi_{\perp\alpha}{}^\beta\delta\bar\lambda_\beta, \quad \delta\boldsymbol{r}_\alpha = \bar\Pi_{\perp\alpha}{}^\beta\delta\boldsymbol{r}_\beta - \tfrac{(\gamma^a\lambda)_\alpha}{2(\lambda\bar\lambda)}(\boldsymbol{r}\gamma_a\bar\Pi_\perp)^\gamma\delta\bar\lambda_\gamma \tag{7.17}$$

These relations are on the one side just equivalent to the constraints in (7.3) but on the other hand they enable to rewrite the variations in terms of projections that extract the independent degrees of freedom of the variation. In particular the variation of a function $f(\lambda,\bar\lambda,\boldsymbol{r})$ (instead of a functional) can be rewritten as follows

$$\delta f(\lambda,\bar\lambda,\boldsymbol{r}) = \left(\delta\lambda^\alpha\partial_{\lambda^\alpha} + \delta\bar\lambda_\alpha\partial_{\bar\lambda_\alpha} + \delta\boldsymbol{r}_\alpha\partial_{\boldsymbol{r}_\alpha}\right) f(\lambda,\bar\lambda,\boldsymbol{r}) = \tag{7.18}$$

$$\stackrel{(7.17)}{=} \Big\{\delta\lambda^\alpha\bar\Pi_{\perp\alpha}{}^\beta\partial_{\lambda^\beta} + \delta\bar\lambda_\alpha\Big(\Pi^\alpha_{\perp\beta}\partial_{\bar\lambda_\beta} - (\underbrace{\boldsymbol{r}\gamma_a\bar\Pi_\perp}_{\Pi_\perp\gamma_a\boldsymbol{r}})^\alpha\tfrac{(\lambda\gamma^a\partial_{\boldsymbol{r}})}{2(\lambda\bar\lambda)}\Big) +$$

$$+\delta\boldsymbol{r}_\alpha\Pi^\alpha_{\perp\beta}\partial_{\boldsymbol{r}_\beta}\Big\} f(\lambda,\bar\lambda,\boldsymbol{r}) \tag{7.19}$$

and then naturally defines **covariant derivatives** in the sense in which it was already discussed on page 13:

$$D_{\lambda^\alpha} \equiv \bar\Pi_{\perp\alpha}{}^\beta\partial_{\lambda^\beta}\ , \quad D_{\bar\lambda_\alpha} \equiv \Pi^\alpha_{\perp\beta}\partial_{\bar\lambda_\beta} - (\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\tfrac{(\lambda\gamma^a\partial_{\boldsymbol{r}})}{2(\lambda\bar\lambda)}\ , \quad D_{\boldsymbol{r}_\beta} \equiv \Pi^\alpha_{\perp\beta}\partial_{\boldsymbol{r}_\beta} \tag{7.20}$$

They annihilate the constraints (7.2) and remarkably reproduce the projector that we have proposed for $\bar\omega^\alpha_z$ in (7.12). The same insertions of projection matrices using (7.17) should be done in the constrained variation of functionals, in particular of the action (7.1):

$$
\begin{aligned}
&\delta S_{\mathrm{gh}}[\lambda,\omega_z,\bar\lambda,\bar\omega_z,\boldsymbol{r},\boldsymbol{s}_z] = \\
&= \int d^2z\,\Big[\bar\partial\lambda^\alpha\delta\omega_{z\alpha}+\bar\partial\bar\lambda_\alpha\delta\bar\omega_z^\alpha+\bar\partial\boldsymbol{r}_\alpha\delta\boldsymbol{s}_z^\alpha+ \\
&\quad -\delta\lambda^\alpha\bar\partial\omega_{z\alpha}-\delta\bar\lambda_\alpha\bar\partial\bar\omega_z^\alpha-\delta\boldsymbol{r}_\alpha\bar\partial\boldsymbol{s}_z^\alpha+ \\
&\quad +\bar\partial\left(\delta\lambda^\alpha\omega_{z\alpha}+\delta\bar\lambda_\alpha\bar\omega_z^\alpha+\delta\boldsymbol{r}_\alpha\boldsymbol{s}_z^\alpha\right)\Big] = \qquad (7.21)\\
&\overset{(7.17)}{=} \int d^2z\,\Big[\bar\partial\lambda^\alpha\delta\omega_{z\alpha}+\bar\partial\bar\lambda_\alpha\delta\bar\omega_z^\alpha+\bar\partial\boldsymbol{r}_\alpha\delta\boldsymbol{s}_z^\alpha+ \\
&\quad -\delta\lambda^\alpha\bar\Pi_{\perp\alpha}{}^\beta\bar\partial\omega_{z\beta}-\delta\bar\lambda_\alpha\Big(\Pi^\alpha_{\perp\beta}\bar\partial\bar\omega_z^\beta-(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\tfrac{(\lambda\gamma^a\bar\partial\boldsymbol{s}_z)}{2(\lambda\bar\lambda)}\Big)-\delta\boldsymbol{r}_\alpha\Pi^\alpha_{\perp\beta}\bar\partial\boldsymbol{s}_z^\beta+ \\
&\quad +\bar\partial\Big(\delta\lambda^\alpha\bar\Pi_{\perp\alpha}{}^\beta\omega_{z\beta}+\delta\bar\lambda_\alpha\Big(\Pi^\alpha_{\perp\beta}\bar\omega_z^\beta-(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\tfrac{(\lambda\gamma^a\boldsymbol{s}_z)}{2(\lambda\bar\lambda)}\Big)+\delta\boldsymbol{r}_\alpha\Pi^\alpha_{\perp\beta}\boldsymbol{s}_z^\beta\Big)\Big] \qquad (7.22)
\end{aligned}
$$

The equations of motion from the constrained variation are thus (we suppress the worldsheet arguments of the functional derivative)

$$
0 \overset{\mathrm{eom}}{=} \begin{pmatrix} \frac{\delta}{\delta\omega_{z\alpha}}S_{\mathrm{gh}} \\ \frac{\delta}{\delta\bar\omega_z^\alpha}S_{\mathrm{gh}} \\ \frac{\delta}{\delta\boldsymbol{s}_z^\alpha}S_{\mathrm{gh}} \end{pmatrix} = \begin{pmatrix} \bar\partial\lambda^\alpha \\ \bar\partial\bar\lambda_\alpha \\ -\bar\partial\boldsymbol{r}_\alpha \end{pmatrix} \qquad (7.23)
$$

$$
0 \overset{\mathrm{eom}}{=} \begin{pmatrix} \bar\Pi_{\perp\alpha}{}^\beta & & \\ & \Pi^\alpha_{\perp\beta} & -(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\frac{(\lambda\gamma^a)_\beta}{2(\lambda\bar\lambda)} \\ & & \Pi^\alpha_{\perp\beta} \end{pmatrix} \begin{pmatrix} \frac{\delta}{\delta\lambda^\beta}S_{\mathrm{gh}} \\ \frac{\delta}{\delta\bar\lambda_\beta}S_{\mathrm{gh}} \\ \frac{\delta}{\delta\boldsymbol{r}_\beta}S_{\mathrm{gh}} \end{pmatrix} = \qquad (7.24)
$$

$$
= \; -\begin{pmatrix} \bar\Pi_{\perp\alpha}{}^\beta\bar\partial\omega_{z\beta} \\ \Pi^\alpha_{\perp\beta}\bar\partial\bar\omega_z^\beta-(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\frac{(\lambda\gamma^a\bar\partial\boldsymbol{s}_z)}{2(\lambda\bar\lambda)} \\ \Pi^\alpha_{\perp\beta}\bar\partial\boldsymbol{s}_z^\beta \end{pmatrix} \qquad (7.25)
$$

In the same way the boundary conditions from the variational principle (in the absence of an additional boundary action) become[18]

$$
\bar\Pi_{\perp\alpha}{}^\beta\omega_{z\beta}\big|_{\partial\Sigma} \overset{\mathrm{bc}}{=} \left(\Pi^\alpha_{\perp\beta}\bar\omega_z^\beta-(\Pi_\perp\gamma_a\boldsymbol{r})^\alpha\tfrac{(\lambda\gamma^a\boldsymbol{s}_z)}{2(\lambda\bar\lambda)}\right)\Big|_{\partial\Sigma} \overset{\mathrm{bc}}{=} \Pi^\alpha_{\perp\beta}\boldsymbol{s}_z^\beta\big|_{\partial\Sigma} \overset{\mathrm{bc}}{=} 0 \qquad (7.26)
$$

The expressions in the equations of motion for the conjugate momenta (7.25) already resemble the gauge invariant variables (7.7),(7.8) and (7.12), apart from the $\bar\partial$-derivative. But as the matrix $\Pi_\perp$ depends only on λ and $\bar\lambda$, we have on-shell (7.23) $\bar\partial\Pi_\perp=0$ which allows to pull in (7.25) the $\bar\partial$-derivative to the front. The equations of motion then simply become

$$
\bar\partial\tilde\omega_{z\alpha} \overset{\mathrm{eom}}{=} \bar\partial\tilde{\bar\omega}_z^\alpha \overset{\mathrm{eom}}{=} \bar\partial\tilde{\boldsymbol{s}}_z^\alpha \overset{\mathrm{eom}}{=} 0 \qquad (7.27)
$$

Note finally that (not surprisingly) also the action itself can be rewritten in terms of the gauge invariant variables. As the relations (7.17) are for arbitrary variations, they hold in particular for the worldsheet derivatives $\bar\partial$. Using this

[18] In the presence of right-movers, the expressions would not be zero, but coincide on the boundary with the corresponding projections for the right-movers. $\diamond$

fact in the action (7.1), we directly obtain the gauge invariant variables:

$$S_{\mathrm{gh}}[\lambda,\omega_z,\bar{\lambda},\bar{\omega}_z,\boldsymbol{r},\boldsymbol{s}_z] = \stackrel{(7.17)}{=} \int d^2z\, \Big[\bar{\partial}\lambda^\alpha \underbrace{\bar{\Pi}_{\perp\alpha}{}^\beta\omega_{z\alpha}}_{\tilde{\omega}_{z\alpha}} + \bar{\partial}\bar{\lambda}_\alpha \underbrace{\left(\Pi^\alpha_{\perp\beta}\bar{\omega}^\beta_z - (\Pi_\perp\gamma_a\boldsymbol{r})^\alpha \tfrac{(\lambda\gamma^a\boldsymbol{s})}{2(\lambda\bar{\lambda})}\right)}_{\tilde{\bar{\omega}}^\alpha_z} + \\ + \bar{\partial}\boldsymbol{r}_\alpha \underbrace{\Pi^\alpha_{\perp\beta}\boldsymbol{s}^\beta_z}_{\tilde{\boldsymbol{s}}^\alpha_z}\Big] \tag{7.28}$$

7.2 With unconstrained variables

So far we have not really used our projection $P_{(f)}$, but only its Jacobian matrix at the constraint surface which was known already previously. However, having a projection at hand, we can easily get rid of the constraint. The resulting action is certainly not free and not very pleasant, but conceptionally it is of some interest. In particular it will have additional gauge symmetries that can be fixed to different constraints than those we started with. So the idea to remove the pure spinor constraint is simply to replace λ^α by $P^\alpha_{(f)}(\rho,\bar{\rho})$ of (2.3) with an unconstrained spinor ρ^α.

$$\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho}) \tag{7.29}$$

We have a priori two natural options in order to remove also the constraint $(\bar{\lambda}\gamma^a\boldsymbol{r}) = 0$ on $\boldsymbol{r}_\alpha$. Either replace it by

$$\boldsymbol{r}_\alpha \stackrel{?}{\equiv} \bar{\Pi}_{(f)\perp}(\rho,\bar{\rho})_\alpha{}^\beta\boldsymbol{t}_\beta \tag{7.30}$$

with some unconstrained $\boldsymbol{t}_\beta$ or by

$$\boldsymbol{r}_\alpha \equiv (\bar{\Pi}_\perp\boldsymbol{t})_\alpha \equiv \bar{\Pi}_{(f)\perp}\left(\lambda,\bar{\lambda}\right)_\alpha{}^\beta\boldsymbol{t}_\beta \tag{7.31}$$

with of course still $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$. For both options, the result will be γ-orthogonal to $\bar{\lambda}$. Only the latter choice is a projection by itself (with $\bar{\Pi}^2_\perp = \bar{\Pi}_\perp$), but also the first is (part of) a projection, if not regarded independently, but together with $\rho^\alpha \mapsto \lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$, as we have discussed previously. One might want to prefer the first one in order to get formally exactly the same equations of motion for $\boldsymbol{t}_\alpha$ as for example for $\bar{\rho}_\alpha$. However, the variation of $\bar{\Pi}_{(f)\perp}(\rho,\bar{\rho})$ can be quite complicated to calculate, while the variation of $\bar{\Pi}_\perp\left(\lambda,\bar{\lambda}\right) = \mathbb{1} - \frac{(\gamma^a\lambda)\otimes(\gamma_a\bar{\lambda})}{2(\lambda\bar{\lambda})}$ with $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$ is relatively easy to perform and was provided in equation (4.27). We thus choose the second option and define a family of action functionals of unconstrained variables via

$$S_{\mathrm{gh}(f)}[\rho,\omega_z,\bar{\rho},\bar{\omega}_z,\boldsymbol{t},\boldsymbol{s}_z] \equiv S_{\mathrm{gh}}[\underbrace{P_{(f)}(\rho,\bar{\rho})}_{\lambda},\omega_z,\underbrace{\bar{P}_{(f)}(\rho,\bar{\rho})}_{\bar{\lambda}},\bar{\omega}_z,\underbrace{\bar{\Pi}_\perp\boldsymbol{t}}_{\boldsymbol{r}},\boldsymbol{s}_z] \tag{7.32}$$

Obviously the antighost gauge symmetries of the original action (7.4)-(7.6) will still be **gauge transformations** $\delta_{(\mu,\bar{\mu},\sigma)}$ of the new action $S_{\mathrm{gh(f)}}$. However, we expect new gauge symmetries, namely all transformations of $\rho,\bar{\rho}$ and $\boldsymbol{t}$ that leave $\lambda,\bar{\lambda}$ and $\boldsymbol{r}$ unchanged. So let us express the variation of the latter in terms of the former. For $\delta \boldsymbol{r}$ we need the variation of the projection matrix $\delta\Pi_\perp$ given in (4.27) (or better its complex conjugate) together with (7.31). This yields[19]

$$\delta \boldsymbol{r}_\alpha \equiv (\bar{\Pi}_\perp \delta \boldsymbol{t})_\alpha - \frac{(\boldsymbol{t}\gamma_a\bar{\lambda})}{2(\lambda\bar{\lambda})}(\bar{\Pi}_\perp\gamma^a\delta\lambda)_\alpha - \frac{(\gamma^a\lambda)_\alpha}{2(\lambda\bar{\lambda})}(\underbrace{\boldsymbol{t}\Pi_\perp}_{\boldsymbol{r}}\gamma_a\delta\bar{\lambda}) \tag{7.33}$$

Using $\delta\lambda = \Pi_{(f)\perp}(\rho,\bar{\rho})\delta\rho + \pi_{(f)\perp}(\rho,\bar{\rho})\delta\bar{\rho}$ (3.2), the constrained variation of $\delta\lambda$ can be further expressed in terms of the free variation $\delta\rho$. So altogether this yields in matrix notation

$$\begin{pmatrix}\delta\lambda\\ \delta\bar{\lambda}\\ \delta\boldsymbol{r}\end{pmatrix} = \begin{pmatrix} \Pi_{(f)\perp}(\rho,\bar{\rho}) & \pi_{(f)\perp}(\rho,\bar{\rho}) & 0\\ \bar{\pi}_{(f)\perp}(\rho,\bar{\rho}) & \bar{\Pi}_{(f)\perp}(\rho,\bar{\rho}) & 0\\ \left\{\begin{matrix}-(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\bar{\Pi}_\perp\gamma^a\Pi_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})}+\\ -\frac{(\gamma^a\lambda)\otimes(\boldsymbol{t}\Pi_\perp\gamma_a\bar{\pi}_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})}\end{matrix}\right\} & \left\{\begin{matrix}-(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\bar{\Pi}_\perp\gamma^a\pi_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})}+\\ -\frac{(\gamma^a\lambda)\otimes(\boldsymbol{t}\Pi_\perp\gamma_a\bar{\Pi}_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})}\end{matrix}\right\} & \bar{\Pi}_\perp \end{pmatrix}\begin{pmatrix}\delta\rho\\ \delta\bar{\rho}\\ \delta\boldsymbol{t}\end{pmatrix} \tag{7.34}$$

where the terms in the curly brackets where broken into two lines just because of lack of space and thus have to be understood as being summands in the same row of the block-matrix. Also for place reasons λ was used like many times before as a place holder for $P_{(f)}(\rho,\bar{\rho})$ and also the implicit dependence of $\Pi_\perp \equiv \Pi_{(f)\perp}(\lambda,\bar{\lambda})$ on λ and thus ρ is not indicated above.

Equation (7.34) is valid for general variations, but as mentioned above we are looking for **new gauge transformations** $\delta_{(\nu,\bar{\nu},\tau)}$ of $\rho,\bar{\rho}$ and $\boldsymbol{t}$ which lead to a vanishing variation of $\lambda,\bar{\lambda}$ and $\boldsymbol{r}$ on the left-hand side of (7.34). On the constraint surface $\rho=\lambda$ the matrix-action on $\delta\rho$ in (7.34) reduces to $\delta\lambda = \Pi_\perp\delta\rho$. Remember that on the constraint surface we have proper projection properties of the form $\Pi_\perp^2 = \Pi_\perp$ (4.5) implying $\Pi_\perp\Pi_\parallel = 0$ (4.8). This suggests that the new gauge symmetry transformations for ρ is (at the constraint surface) of the form $\delta_{(\mathrm{sym})}\rho \propto \Pi_\parallel$. However, off the constraint surface we do not have $\Pi_\perp^2 = \Pi_\perp$ but something of the form $\Pi_{(f)\perp}(\lambda,\bar{\lambda})\Pi_{(f)\perp}(\rho,\bar{\rho}) = \Pi_{(f)\perp}(\rho,\bar{\rho})$ ((3.14) and (3.15)) implying $\Pi_{(f)\parallel}(\lambda,\bar{\lambda})\Pi_{(f)\perp}(\rho,\bar{\rho}) = 0$. Still choosing $\delta_{(\mathrm{sym})}\rho \propto \Pi_{(f)\parallel}(\lambda,\bar{\lambda})$ would lead in (7.34) to $\Pi_{(f)\perp}(\rho,\bar{\rho})\delta_{(\mathrm{sym})}\rho \propto \Pi_{(f)\perp}(\rho,\bar{\rho})\Pi_{(f)\parallel}(\lambda,\bar{\lambda})$ which is the wrong order of matrices, if we want to use the just mentioned $\Pi_{(f)\parallel}(\lambda,\bar{\lambda})\Pi_{(f)\perp}(\rho,\bar{\rho}) = 0$. This is the main reason for us to discuss later the new gauge symmetries only in the case $f(\xi) = h(\xi) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ where we get Hermitian projection matrices which have additional properties that will resolve this problem. In spite of this issue, there will certainly be also for general f a new gauge symmetry taking care of the artificial degrees of freedom that we introduced by using the projection $P_{(f)}$. These symmetries are given by the 0-eigenvectors of the matrix in (7.34). For a general function f it would thus require some work to determine this symmetry, while in the Hermitian case the above naive guess will already work.

We have now argued about the new gauge symmetries by just discussing the variation of the constrained variables in terms of the unconstrained variables. Let us now provide the general variation of the ghost action. Starting from the variation (7.21) of the ghost action and applying the relation in (7.34) not only

[19] If the variables on the righthand side of (7.33) are also on the constraint surface, i.e. $\boldsymbol{t}_\alpha = \boldsymbol{r}_\alpha$, $\delta\boldsymbol{t}_\alpha = \delta\boldsymbol{r}_\alpha$, $\rho^\alpha = \lambda^\alpha$, $\delta\rho^\alpha = \delta\lambda^\alpha$, it reduces to the constraint on $\delta\boldsymbol{r}$ given in (7.17). $\diamond$

for the general variation δ, but also for the partial derivative $\delta \to \bar{\partial}$, we obtain

$$
\begin{aligned}
&\delta S_{\mathrm{gh}(f)}[\rho, \omega_z, \bar{\rho}, \bar{\omega}_z, \boldsymbol{t}, \boldsymbol{s}_z] = \\
&= \quad -\int_\Sigma d^2 z\ \left(\delta\rho^T, \delta\bar{\rho}^T, \delta \boldsymbol{t}^T\right) \times \\
&\quad \times \begin{pmatrix} \Pi^T_{(f)\perp}(\rho,\bar{\rho}) & \bar{\pi}^T_{(f)\perp}(\rho,\bar{\rho}) & \left\{ \begin{array}{c} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\Pi^T_{(f)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})} + \\ -\frac{(\bar{\pi}^T_{(f)\perp}(\rho,\bar{\rho})\gamma_a\Pi^T_\perp \boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{array} \right\} \\ \pi^T_{(f)\perp}(\rho,\bar{\rho}) & \bar{\Pi}^T_{(f)\perp}(\rho,\bar{\rho}) & \left\{ \begin{array}{c} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\pi^T_{(f)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})} + \\ -\frac{(\bar{\Pi}^T_{(f)\perp}(\rho,\bar{\rho})\gamma_a\Pi^T_\perp \boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{array} \right\} \\ 0 & 0 & \bar{\Pi}^T_\perp \end{pmatrix} \begin{pmatrix} \bar{\partial}\omega_z \\ \bar{\partial}\bar{\omega}_z \\ \bar{\partial}\boldsymbol{s}_z \end{pmatrix} \\
&\quad + \int_\Sigma d^2 z\ \left(\delta\omega^T, \delta\bar{\omega}^T, \delta \boldsymbol{s}_z^T\right) \times \\
&\quad \times \begin{pmatrix} \Pi_{(f)\perp}(\rho,\bar{\rho}) & \pi_{(f)\perp}(\rho,\bar{\rho}) & 0 \\ \bar{\pi}_{(f)\perp}(\rho,\bar{\rho}) & \bar{\Pi}_{(f)\perp}(\rho,\bar{\rho}) & 0 \\ \left\{ \begin{array}{c} (\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\Pi^T_\perp\gamma^a\Pi_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})} + \\ +\frac{(\gamma^a\lambda)\otimes(\boldsymbol{t}\Pi_\perp\gamma_a\bar{\pi}_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})} \end{array} \right\} & \left\{ \begin{array}{c} (\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\Pi^T_\perp\gamma^a\pi_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})} + \\ +\frac{(\gamma^a\lambda)\otimes(\boldsymbol{t}\Pi_\perp\gamma_a\bar{\Pi}_{(f)\perp}(\rho,\bar{\rho}))}{2(\lambda\bar{\lambda})} \end{array} \right\} & -\bar{\Pi}_\perp \end{pmatrix} \begin{pmatrix} \bar{\partial}\rho \\ \bar{\partial}\bar{\rho} \\ \bar{\partial}\boldsymbol{t} \end{pmatrix}
\end{aligned}
\tag{7.35}
$$

Note that because of the projection properties (3.14) and (3.15), the fact that the matrices have argument $(\rho, \bar{\rho})$ is not in contradiction with (7.22), where they are evaluated at $(\lambda, \bar{\lambda})$. The equations of motion for the conjugate momenta $\omega_z, \bar{\omega}_z$ and $\boldsymbol{s}_z$ that one reads off from (7.35) look formally different from (7.25) but are in fact equivalent. The new gauge symmetries that we have mentioned above will always allow to fix $\rho^\alpha = \lambda^\alpha$ and $\boldsymbol{t}^\alpha = \boldsymbol{r}^\alpha$ and thus return to the original action. For this gauge the equations of motion from (7.35) precisely reduce to the ones in (7.25).

Note that from (7.35) we can also re-derive the antighost gauge transformations (7.4)-(7.6) in terms of the projector matrices. If we vary only $\omega_z, \bar{\omega}_z$ and $\boldsymbol{s}_z$ and require the variation (7.35) to vanish for any $\bar{\partial}\lambda, \bar{\partial}\bar{\lambda}$ and $\bar{\partial}\boldsymbol{t}$, we obtain

$$
\begin{pmatrix} \Pi^T_{(f)\perp}(\rho,\bar{\rho}) & \bar{\pi}^T_{(f)\perp}(\rho,\bar{\rho}) & \left\{ \begin{array}{c} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\Pi^T_{(f)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})} + \\ -\frac{(\bar{\pi}^T_{(f)\perp}(\rho,\bar{\rho})\gamma_a\Pi^T_\perp \boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{array} \right\} \\ \pi^T_{(f)\perp}(\rho,\bar{\rho}) & \bar{\Pi}^T_{(f)\perp}(\rho,\bar{\rho}) & \left\{ \begin{array}{c} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\pi^T_{(f)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})} + \\ -\frac{(\bar{\Pi}^T_{(f)\perp}(\rho,\bar{\rho})\gamma_a\Pi^T_\perp \boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{array} \right\} \\ 0 & 0 & \bar{\Pi}^T_\perp \end{pmatrix} \begin{pmatrix} \delta_{(\mathrm{sym})}\omega_z \\ \delta_{(\mathrm{sym})}\bar{\omega}_z \\ \delta_{(\mathrm{sym})}\boldsymbol{s}_z \end{pmatrix} \stackrel{!}{=} 0
\tag{7.36}
$$

Again on the constraint surface it is obvious that the symmetry transformation $\delta_{(\mathrm{sym})}\omega_{z\alpha} \equiv \delta_{(\mu)}\omega_{z\alpha} = (\Pi^T_\parallel \mu_z)_\alpha$ with some spinorial gauge parameter $\mu_{z\alpha}$ obeys this condition. And for the antighosts this even holds off the constraint surface, because the order of the projection matrices is here due to the transposition the correct one, i.e. $\Pi^T_{(f)\perp}(\rho,\bar{\rho})\Pi^T_{(f)\parallel}(\lambda,\bar{\lambda}) = 0 \quad \forall f$ according to (3.14). For the variable $\boldsymbol{s}_z$ a first guess would be $\delta_{(\boldsymbol{\sigma})}\boldsymbol{s}_z^\alpha = (\Pi_\parallel \boldsymbol{\sigma}_z)^\alpha$, which turns out to need a compensating transformation of $\bar{\omega}_z$. One thus ends up with the following gauge transformations for the antighosts

$$\delta_{(\mu)}\omega_{z\alpha} = (\bar{\Pi}_{\|}\mu_z)_\alpha \tag{7.37}$$

$$\delta_{(\bar{\mu},\boldsymbol{\sigma})}\bar{\omega}_z^\alpha = (\Pi_{\|}\bar{\mu}_z)^\alpha - (\lambda\gamma^a\Pi_{\|}\boldsymbol{\sigma}_z)\frac{(\gamma_a\overbrace{\bar{\Pi}_\perp\boldsymbol{t}}^{\boldsymbol{r}})^\alpha}{2(\lambda\bar{\lambda})} \tag{7.38}$$

$$\delta_{(\boldsymbol{\sigma})}\boldsymbol{s}_z^\alpha = (\Pi_{\|}\boldsymbol{\sigma}_z)^\alpha \tag{7.39}$$

with as usual $\Pi_{\|} \equiv \Pi_{(f)\|}(\lambda,\bar{\lambda})$ where $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$ and spinorial gauge parameters $\mu_{z\alpha}, \bar{\mu}_z^\alpha$ and $\boldsymbol{\sigma}_z^\alpha$. Having in mind that $\Pi_{\|} = \frac{(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})}$ and comparing with the form of the gauge transformations in (7.4)-(7.6), we can deduce the relation between the spinorial gauge parameters here and the vectorial ones there:[20]

$$\mu_{za} = \frac{(\bar{\lambda}\gamma_a)^\alpha\mu_{z\alpha}}{2(\lambda\bar{\lambda})}, \quad \bar{\mu}_z^a = \frac{(\lambda\gamma^a)_\alpha\bar{\mu}_z^\alpha}{2(\lambda\bar{\lambda})}, \quad \boldsymbol{\sigma}_z^a = \frac{(\lambda\gamma^a)_\alpha\boldsymbol{\sigma}_z^\alpha}{2(\lambda\bar{\lambda})} \tag{7.40}$$

Both parametrizations, either with $\mu_{za}, \bar{\mu}_z^a$ and $\boldsymbol{\sigma}_z^a$ or with $\mu_{z\alpha}, \bar{\mu}_z^\alpha$ and $\boldsymbol{\sigma}_z^\alpha$ are reducible. Remembering that $\operatorname{tr}\Pi_{\|} = 5$, it is apparent from (7.37)-(7.39) that for each of these spinors only 5 components effectively enter the gauge transformation.

Gauge transformations in the Hermitian case

The antighost gauge transformations (7.37)-(7.39) formally look the same for any particular choice of f in our projection $P_{(f)}$. The only way f enters there is via the dependence of $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$ on it. The situation is different for the new gauge transformations of the ghosts $\rho^\alpha, \bar{\rho}_\alpha$ and $\boldsymbol{t}_\alpha$, because for the particular choice $f(\xi) = h(\xi) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ the matrix $\Pi_{\|} \equiv \Pi_{(h)\|}(\lambda,\bar{\lambda})$ on the constraint surface commutes with $\Pi_{(h)\|}(\rho,\bar{\rho})$ off the constraint surface (see (5.22) and (5.23)).

So looking again at (7.34) and having in mind (5.22) and (5.23), it is obvious that

$$\delta_{(\boldsymbol{\tau})}\boldsymbol{t}_\alpha \equiv (\bar{\Pi}_{\|}\boldsymbol{\tau})_\alpha \tag{7.41}$$

for some spinorial gauge parameter $\boldsymbol{\tau}_\alpha$ will not induce a variation of $\lambda^\alpha, \bar{\lambda}_\alpha$ or $\boldsymbol{r}_\alpha$ on the left-hand side of (7.34) and thus be a gauge transformation. Similarly one can try

$$\delta_{(\nu)}\rho^\alpha = (\Pi_{\|}\nu)^\alpha \tag{7.42}$$

$$\delta_{(\bar{\nu})}\bar{\rho}_\alpha = (\bar{\Pi}_{\|}\bar{\nu})_\alpha \tag{7.43}$$

---

[20] While in (7.40) the expressions for μ_{za} and $\boldsymbol{\sigma}_z^a$ can directly be read off from $\delta_{(\mu)}\omega_{z\alpha}$ and $\delta_{(\boldsymbol{\sigma})}\boldsymbol{s}_z^\alpha$, the one for $\bar{\mu}_z^a$ requires some Fierzing:

$$\begin{aligned}\delta_{(\bar{\mu},\boldsymbol{\sigma})}\bar{\omega}_z^\alpha &\overset{(7.38)}{=} \frac{(\gamma_a\bar{\lambda})^\alpha(\lambda\gamma^a\bar{\mu}_z)}{2(\lambda\bar{\lambda})} - \frac{(\lambda\overbrace{\gamma^a\gamma_b}^{-\gamma_b\gamma^a+2\delta_b^a}\bar{\lambda})(\lambda\gamma^b\boldsymbol{\sigma}_z)}{4(\lambda\bar{\lambda})^2}(\gamma_a\boldsymbol{r})^\alpha = \\ &\overset{\text{Fierz}}{=} \underbrace{\frac{(\lambda\gamma^a\bar{\mu}_z)}{2(\lambda\bar{\lambda})}}_{\bar{\mu}_z^a}(\gamma_a\bar{\lambda})^\alpha - \underbrace{\frac{(\lambda\gamma^a\boldsymbol{\sigma}_z)}{2(\lambda\bar{\lambda})}}_{\boldsymbol{\sigma}_z^a}(\gamma_a\boldsymbol{r})^\alpha \qquad \diamond\end{aligned}$$

with spinorial gauge parameters ν^α and $\bar{\nu}_\alpha$. As in (7.34) $\delta\rho$ and $\delta\bar{\rho}$ enter not only in the variation of $\delta\lambda$ and $\delta\bar{\lambda}$, one might suspect that one needs also some compensating transformation of $\delta_{(\nu,\bar{\nu})}\boldsymbol{t}$. But one can easily check that the transformations as they are written above are already a symmetry and do not need such a compensation.

Gauge fixing One interesting aspect of this new artificial gauge symmetry is, that we can try to find different gauge fixings that bring us from the pure spinor constraint to some other constraint.

In particular the role of ρ^α and $\omega_{z\alpha}$ can now be interchanged by fixing $\omega_{z\alpha}$ (with a constraint linear in $\omega_{z\alpha}$) and leaving ρ unconstrained. Choosing a linear constraint for $\omega_{z\alpha}$ might give a clue how to obtain such a ghost system from an underlying gauge freedom. A natural gauge fixing constraint might be

$$(\omega_z\gamma^a\bar{P}_{(h)}(\rho,\bar{\rho})) \stackrel{!}{=} 0 \quad \text{(gauge 1)} \tag{7.44}$$

which is equivalent to

$$\tilde{\omega}_{z\alpha} \equiv (\bar{\Pi}_\perp\omega_z)_\alpha \stackrel{!}{=} \omega_{z\alpha} \quad \text{(gauge 1)} \tag{7.45}$$

with $\bar{\Pi}_\perp \equiv \bar{\Pi}_{(h)\perp}(P_{(h)}(\rho,\bar{\rho}),\bar{P}_{(h)}(\rho,\bar{\rho}))$. It fixes only the gauge parameter μ_z^a. Now the antighost $\omega_{z\alpha}$ would obey a constraint, while the ghost ρ^α would be unconstrained, but not gauge invariant. $\lambda^\alpha \equiv P_{(h)}(\rho,\bar{\rho})$ would still be the gauge invariant combination. However, according to (7.28) (with λ replaced by $P_{(h)}(\rho,\bar{\rho})$), this gauge would not lead to a free action $S_{\mathrm{gh}(h)}$.

Looking at (7.34) it is clear that absorbing the transpose of the matrix there into the antighosts would lead to another free action (we use Hermiticity of $\Pi_{(h)\perp}$ and symmetry of $\pi_{(h)\perp}$):

$$\begin{pmatrix} \bar{\Pi}_{(h)\perp}(\rho,\bar{\rho}) & \bar{\pi}_{(h)\perp}(\rho,\bar{\rho}) & \left\{\begin{matrix} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\bar{\Pi}_{(h)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})}+ \\ -\frac{(\bar{\pi}_{(h)\perp}(\rho,\bar{\rho})\gamma_a\bar{\Pi}_\perp\boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{matrix}\right\} \\ \pi_{(h)\perp}(\rho,\bar{\rho}) & \Pi_{(h)\perp}(\rho,\bar{\rho}) & \left\{\begin{matrix} -(\boldsymbol{t}\gamma_a\bar{\lambda})\frac{(\pi_{(h)\perp}(\rho,\bar{\rho})\gamma^a\Pi_\perp)}{2(\lambda\bar{\lambda})}+ \\ -\frac{(\Pi_{(h)\perp}(\rho,\bar{\rho})\gamma_a\bar{\Pi}_\perp\boldsymbol{t})\otimes(\lambda\gamma^a)}{2(\lambda\bar{\lambda})} \end{matrix}\right\} \\ 0 & 0 & \Pi_\perp \end{pmatrix}\begin{pmatrix}\omega_z \\ \bar{\omega}_z \\ \boldsymbol{s}_z\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}\omega_z \\ \bar{\omega}_z \\ \boldsymbol{s}_z\end{pmatrix}$$

$$\text{(gauge 2)} \tag{7.46}$$

The above (somewhat ugly) constraint would therefore replace in this gauge the original constraints $(\lambda\gamma^a\lambda) = (\bar{\lambda}\gamma_a\bar{\lambda}) = (\bar{\lambda}\gamma_a\boldsymbol{r}) = 0$ while keeping a free action.

Conceptionally more interesting would be those gauges which are linear in each spinorial variable, as $(\bar{\rho}\gamma^a\omega_z) = 0$. This sort of constraint on the ρ-ghosts could be easily translated into an equivalent constraint on the odd gauge parameters of some underlying gauge symmetry. In contrast the pure spinor constraint $(\lambda\gamma^a\lambda) = 0$ would be trivial for anticommuting parameters. However, it seems that the gauge $(\bar{\rho}\gamma^a\omega_z) = 0$ is too strong and would fix 10 instead of five degrees of freedom.

Another aim for a possible gauge fixing would be that ρ^α and $\bar{\rho}_\alpha$ as well as $\omega_{z\alpha}$ and $\bar{\omega}_z^\alpha$ are treated as complex conjugates in all transformations. This is so far not the case in the BRST transformations, as well as in the gauge transformations. Having in mind also the matter fields of the pure spinor string, with

BRST transformation $\mathbf{s}\boldsymbol{\theta}^\alpha = \lambda^\alpha$, this would require to identify some variable with $\bar{\boldsymbol{\theta}}^\alpha$ (perhaps after turning the gauge parameter $\bar{\nu}_\alpha$ into an anticommuting ghost). It is not yet clear if this would lead anywhere.

7.3 Ghost action in the U(5) formalism

We will not translate the complete previous discussion into the U(5) formalism. Instead we just want to relate some known fact from the literature to our presentation. For simplicity, we even restrict to the minimal formalism, i.e. neglect the variables $\bar{\lambda}_\alpha, \bar{\omega}_z^\alpha, \boldsymbol{r}_\alpha, \boldsymbol{s}_z^\alpha$ from (7.1). In the minimal formalism the ghost action in U(5) coordinates thus reads

$$S[\lambda, \omega_z] = \int d^2z \quad \bar{\partial}\lambda^+ \omega_{z+} + \tfrac{1}{2}\bar{\partial}\lambda^{\mathfrak{a}_1\mathfrak{a}_2}\omega_{z\mathfrak{a}_1\mathfrak{a}_2+} + \bar{\partial}\lambda_{\mathfrak{a}}\omega_z^{\mathfrak{a}} \tag{7.47}$$

Using the constraint (6.8) we can replace (on the patch $\lambda^+ \neq 0$) $\lambda^{\mathfrak{a}}$ by

$$\lambda_{\mathfrak{a}} = \tfrac{1}{8\lambda^+}\epsilon_{\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\lambda^{\mathfrak{b}_1\mathfrak{b}_2}\lambda^{\mathfrak{b}_3\mathfrak{b}_4} \tag{7.48}$$

which leads to

$$\begin{aligned} S[\lambda, \omega_z] \quad = \quad & \int d^2z \quad \bar{\partial}\lambda^+ \left(\omega_{z+} - \tfrac{1}{8(\lambda^+)^2}\epsilon_{\mathfrak{c}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4}\lambda^{\mathfrak{b}_1\mathfrak{b}_2}\lambda^{\mathfrak{b}_3\mathfrak{b}_4}\omega_z^{\mathfrak{c}}\right) + \\ & + \tfrac{1}{2}\bar{\partial}\lambda^{\mathfrak{a}_1\mathfrak{a}_2}\left(\omega_{z\mathfrak{a}_1\mathfrak{a}_2} + \tfrac{1}{2\lambda^+}\epsilon_{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{b}_1\mathfrak{b}_2\mathfrak{c}}\lambda^{\mathfrak{b}_1\mathfrak{b}_2}\omega_z^{\mathfrak{c}}\right) \end{aligned} \tag{7.49}$$

Now let us compare the antighost combinations in the brackets with the gauge invariant antighost $\tilde{\omega}_z \equiv (\Pi^T\omega_z)$ defined in (7.7), but with its non-covariant version (call it ω'_z), where $\bar{\lambda}$ is replaced by the reference spinor $\bar{\chi} = (1, 0, 0)$ of equation (6.11) so that we can use the projection matrix (6.21) evaluated at $\rho = \lambda$ which we denote by $\Pi_{(\bar{\chi})} \equiv \Pi_{(\bar{\chi})}(\lambda, \bar{\lambda})$:

$$\omega'_{z\alpha} \quad \equiv \quad (\Pi^T_{(\bar{\chi})})_\alpha{}^\beta \omega_{z\beta} = \tag{7.50}$$

$$\overset{(6.21)}{=} \begin{pmatrix} 1 & 0 & -\frac{1}{8(\lambda^+)^2}\epsilon_{\mathfrak{b}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3\mathfrak{c}_4}\lambda^{\mathfrak{c}_1\mathfrak{c}_2}\lambda^{\mathfrak{c}_3\mathfrak{c}_4} \\ 0 & \delta^{\mathfrak{b}_1}_{\mathfrak{a}_1}\delta^{\mathfrak{b}_2}_{\mathfrak{a}_2} - \delta^{\mathfrak{b}_1}_{\mathfrak{a}_2}\delta^{\mathfrak{b}_2}_{\mathfrak{a}_1} & \frac{1}{2\lambda^+}\epsilon_{\mathfrak{b}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{c}_1\mathfrak{c}_2}\lambda^{\mathfrak{c}_1\mathfrak{c}_2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_{z+} \\ \omega_{z\mathfrak{b}_1\mathfrak{b}_2} \\ \omega_z^{\mathfrak{b}} \end{pmatrix} \tag{7.51}$$

In order to be consistent with (6.19) the matrix multiplication has to be understood as coming with a factor $\frac{1}{2}$ when summing over the double-indices $\mathfrak{b}_1\mathfrak{b}_2$. We thus arrive at

$$\omega'_{z\alpha} = \begin{pmatrix} \omega_{z+} - \frac{1}{8(\lambda^+)^2}\epsilon_{\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3\mathfrak{c}_4\mathfrak{b}}\lambda^{\mathfrak{c}_1\mathfrak{c}_2}\lambda^{\mathfrak{c}_3\mathfrak{c}_4}\omega_z^{\mathfrak{b}} \\ \omega_{z\mathfrak{a}_1\mathfrak{a}_2} + \frac{1}{2\lambda^+}\epsilon_{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{c}_1\mathfrak{c}_2\mathfrak{b}}\lambda^{\mathfrak{c}_1\mathfrak{c}_2}\omega_z^{\mathfrak{b}} \\ 0 \end{pmatrix} \tag{7.52}$$

These are precisely the terms that appeared naturally in the action above, so we can write

$$S[\lambda, \omega_z] \quad = \quad \int d^2z \quad \bar{\partial}\lambda^+\omega'_{z+} + \tfrac{1}{2}\bar{\partial}\lambda^{\mathfrak{a}_1\mathfrak{a}_2}\omega'_{z\mathfrak{a}_1\mathfrak{a}_2} \tag{7.53}$$

This rewriting of the action was first presented in equations (26) and (27) of [16] and our ω'_z agrees up to a conventional sign of the components $\omega_z^{\mathfrak{a}}$ (and $\lambda_{\mathfrak{a}}$). The upshot of the discussion in this subsection, however, is that these gauge invariant combinations ω'_z fit into our more general projection-picture if one chooses the reference spinor $\bar{\chi} = (1, 0, 0)$ of equation (6.11).

8 Volume form of the pure spinor space

The holomorphic volume form of the pure spinor space is already well known. Originally it was given in [23] p.13,(4.4), or even in [24] p.14, (3.6). It was further elaborated on it in [25] p. 28,29 or p.33, (2.85)-(2.87) and [26]p.5, (2.5)-(2.10) where it takes the form

$$[d^{11}\lambda] = \tfrac{1}{11!5!(\lambda\bar{\lambda})^3}(\bar{\lambda}\gamma_a)^{\alpha_1}(\bar{\lambda}\gamma_b)^{\alpha_2}(\bar{\lambda}\gamma_c)^{\alpha_3}(\gamma^{abc})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathbf{d}\lambda^{\beta_1}\cdots\mathbf{d}\lambda^{\beta_{11}} \tag{8.1}$$

See also [27] p.16 and [28]. At first sight the 11-form (8.1) does not look holomorphic in λ^α. However, going to U(5)-coordinates, one can show that all $\bar{\lambda}$-dependence disappears.

Wedging the holomorphic volume form with its complex conjugate yields according to the above cited references

$$[d^{11}\lambda]\wedge[d^{11}\bar{\lambda}] = \tfrac{1}{11!(\lambda\bar{\lambda})^3}(\mathbf{d}\lambda^\alpha\mathbf{d}\bar{\lambda}_\alpha)^{11} \tag{8.2}$$

Let us think of the pure spinor space as being embedded in the ambient space $\mathbb{C}^{16}$ at $\xi = 0$. Can we obtain the above holomorphic volume form from the holomorphic volume form of $\mathbb{C}^{16}$ by the variable transformations $\rho^\alpha \mapsto (\lambda^\alpha, \zeta^a)$ or $\rho^\alpha \mapsto (\lambda^\alpha, \tilde{\zeta}^a)$ discussed on page 6 and page 7, or by yet another similar one? The canonical holomorphic volume form of the ambient space $\mathbb{C}^{16}$ is given by

$$[d^{16}\rho] \equiv \tfrac{1}{16!}\epsilon_{\alpha_1\ldots\alpha_{16}}\mathbf{d}\rho^{\alpha_1}\cdots\mathbf{d}\rho^{\alpha_{16}} \tag{8.3}$$

The corresponding measure of the complete space is therefore

$$[d^{16}\rho]\wedge[d^{16}\bar{\rho}] = \tfrac{1}{(16!)^2}\underbrace{\epsilon_{\alpha_1\ldots\alpha_{16}}\epsilon^{\beta_1\ldots\beta_{16}}}_{16!\delta^{\beta_1\ldots\beta_{16}}_{\alpha_1\ldots\alpha_{16}}}\mathbf{d}\rho^{\alpha_1}\cdots\mathbf{d}\rho^{\alpha_{16}}\mathbf{d}\bar{\rho}_{\beta_1}\cdots\mathbf{d}\bar{\rho}_{\beta_{16}} = \tag{8.4}$$

$$= -\tfrac{1}{16!}(\mathbf{d}\rho^\alpha\mathbf{d}\bar{\rho}_\alpha)^{16} \tag{8.5}$$

In the corresponding discussion within the toy-model in the appendix around page 49 it became clear that in order to obtain the expected holomorphic volume form of the constrained space after the variable transformation it was necessary to redefine ζ or $\tilde{\zeta}$ once more in (A.110). We will try the same here and define

$$\check{\zeta}^a \equiv (\lambda\bar{\lambda})\tilde{\zeta}^a \quad , \quad \check{\xi} \equiv \tfrac{1}{2}\check{\zeta}^a\bar{\check{\zeta}}_a = (\lambda\bar{\lambda})^2\tilde{\xi} \tag{8.6}$$

where λ^α will be understood as being $P^\alpha_{(h)}(\rho,\bar{\rho})$. We have expressed $\check{\zeta}^a$ in terms of $\tilde{\zeta}^a$, as we will need some equations which where particularly simple for $\tilde{\zeta}^a$. However, a very good motivation for this reparametrization (apart from the results of the toy-model) is that $\check{\zeta}^a$ as a function of ρ^α becomes holomorphic and very simple. In order to see this, we have to use explicit form of $\tilde{\zeta}^a$ in (2.22) and the fact that according to (5.6) and (5.4) we have $(\lambda\bar{\lambda}) = \frac{1}{2}(1+\sqrt{1+\xi})(\rho\bar{\rho})$. Then altogether, we have the following variable transformation[21]:

$$\begin{aligned} (\rho^\alpha, \bar{\rho}_\alpha) &\mapsto (\lambda^\alpha, \bar{\lambda}_\alpha, \check{\zeta}^a, \bar{\check{\zeta}}_a) & (8.7)\\ \text{with } \lambda^\alpha &\equiv P^\alpha_{(h)}(\rho,\bar{\rho}) & (8.8)\\ \check{\zeta}^a &= \tfrac{1}{2}(\rho\gamma^a\rho) & (8.9)\end{aligned}$$

[21] With $(\rho\gamma^a\rho)$ being the most obvious choice to parametrize the distance of ρ^α from being a pure spinor, it is a fair question why we have not used $\check{\zeta}^a$ as it is given in (8.9) from the beginning. There are two aspects. One is that the equivalence between (8.9) and (8.6) holds only in the case $f = h$. For general f the definition $\check{\zeta}^a \equiv (\lambda\bar{\lambda})\tilde{\zeta}^a$ would be equivalent to

Its inverse becomes (based on (2.33) with $f = h$ and using (8.6))

$$\rho^\alpha = \lambda^\alpha + \tfrac{1}{2}\frac{\check{\zeta}^a(\bar{\lambda}\gamma_a)^\alpha}{(\lambda\bar{\lambda})} \tag{8.10}$$

whose differential reads

$$\begin{aligned}\mathbf{d}\rho^\alpha &= \mathbf{d}\lambda^\beta\Big(\delta^\alpha_\beta - \tfrac{1}{2}\frac{\check{\zeta}^a(\bar{\lambda}\gamma_a)^\alpha\bar{\lambda}_\beta}{(\lambda\bar{\lambda})^2}\Big) + \tfrac{1}{2}\frac{\mathbf{d}\check{\zeta}^a(\bar{\lambda}\gamma_a)^\alpha}{\lambda\bar{\lambda}} + \\ &\quad + \tfrac{1}{2}\frac{\mathbf{d}\bar{\lambda}_\beta\check{\zeta}^a}{\lambda\bar{\lambda}}\Big(\gamma_a^{\alpha\beta} - \frac{(\bar{\lambda}\gamma_a)^\alpha\lambda^\beta}{(\lambda\bar{\lambda})}\Big)\end{aligned} \tag{8.11}$$

It is still clear that the inverse transformation (8.10) is not holomorphic so that it does not seem like a good idea to try deriving a holomorphic volume form with it. But note that at least on the constraint surface $\check{\zeta}^a = 0$ it becomes holomorphic. The same is true for the original transformation (8.8) because of $\frac{\partial\lambda^\alpha}{\partial\bar{\rho}_\beta}\Big|_{\check{\zeta}^a=0} = \pi^{\alpha\beta}_{(h)\perp}\Big|_{\check{\zeta}^a=0} = 0$. Instead the one-forms in (8.11) transform even on the constraint surface non-holomorphically. But at least all $\mathbf{d}\bar{\lambda}_\alpha$-appearance drops:

$$\mathbf{d}\rho^\alpha|_{\check{\zeta}^a=0} = \mathbf{d}\lambda^\alpha + \tfrac{1}{2}\frac{\mathbf{d}\check{\zeta}^a\left(\bar{\lambda}\gamma_a\right)^\alpha}{\lambda\bar{\lambda}} \tag{8.12}$$

This suggests that the variable transformation might lead to the holomorphic λ-volume form at least on the surface $\check{\zeta}^a = 0$. So the hope is, that after the transformation we have a split of $[d^{16}\rho]$ into the holomorphic volume form $[d^{11}\lambda]$ of (8.1) and some volume form for the $\check{\zeta}^a$-space.[22] Using (8.12), the holomorphic volume form (8.3) transforms on the constraint surface as

$$[d^{16}\rho]\big|_{\check{\zeta}^a=0} = \tfrac{\epsilon_{\alpha_1\ldots\alpha_{16}}}{16!}\big(\mathbf{d}\lambda^{\alpha_1} + \tfrac{\mathbf{d}\check{\zeta}^a(\bar{\lambda}\gamma_a)^{\alpha_1}}{2(\lambda\bar{\lambda})}\big)\cdots\big(\mathbf{d}\lambda^{\alpha_{16}} + \tfrac{\mathbf{d}\check{\zeta}^a(\bar{\lambda}\gamma_a)^{\alpha_{16}}}{2(\lambda\bar{\lambda})}\big) \tag{8.13}$$

$\check{\zeta}^a = 2(\rho\bar{\rho})f(\xi)^2\frac{(1-\xi)\zeta^a}{(1+\sqrt{1-\xi})^2}$. In turn, defining $\check{\zeta}^a \equiv \frac{1}{2}(\rho\gamma^a\rho)$ would not be equivalent to (8.6) any more and thus will not have the simple inverse transformation (8.10). The other aspect is that although also $\zeta^a = \frac{(\rho\gamma^a\rho)}{(\rho\bar{\rho})}$ has a non-simple inverse (2.18), that variable appeared very naturally in the original projection (2.3). It was introduced as a placeholder to make equations shorter significantly without loosing the feeling for the ρ^α-dependence within the projection. If we had worked with $\check{\zeta}^a$, a lot of factors $(\rho\bar{\rho})$ would be floating around everywhere. The variable $\tilde{\zeta}^a$ on the other hand was introduced in (2.22) to get the most simple form of the inverse transformation, while $\check{\zeta}^a$ as a function of ρ^α and $\bar{\rho}_\alpha$ is in turn more complicated. ◇

[22] We could think about a suitable covariant holomorphic volume form for the $\check{\zeta}^a$-space without going through the transformation of the ambient space volume form. It has to contain five $\mathbf{d}\check{\zeta}^a$. The five indices have to be saturated and the only invariant tensors are the 10d epsilon tensor $\epsilon_{a_1\ldots a_{10}}$, and the metric. Otherwise we could contract (if we do not allow λ-dependence) only with $\check{\zeta}^a$ or its complex conjugate. Contracting with the ϵ-tensor is of no help, because it leads again to five free indices and saturating them with $\check{\zeta}^a$ or $\bar{\check{\zeta}}_a$ always leads to vanishing results. It is therefore impossible to construct a covariant holomorphic volume form for the $\check{\zeta}^a$-space which is λ^α-independent. If we instead allow λ-dependence, a natural candidate would be $(\lambda\gamma_{a_1\ldots a_5}\lambda)$. However, because of the constraint $\check{\zeta}_a(\gamma^a\lambda)_\alpha = 0$ this would again vanish. So we need to use also the complex conjugate $\bar{\lambda}_\alpha$ and hope that this non-holomorphic $\bar{\lambda}$-dependence might drop when dividing by appropriate powers of $(\lambda\bar{\lambda})$ as it is the case in (8.1). So the only ansatz which has a chance to be a holomorphic volume form on the $\check{\zeta}$-space is

$$[d^5\check{\zeta}] \overset{?}{=} \frac{1}{(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{a_1\ldots a_5}\bar{\lambda})\mathbf{d}\check{\zeta}^{a_1}\cdots\mathbf{d}\check{\zeta}^{a_5} \qquad \diamond$$

As λ^α and $\check{\zeta}^a$ are constrained variables, they effectively have 11 and 5 components respectively and we have the identities

$$\mathbf{d}\lambda^{\alpha_1}\cdots\mathbf{d}\lambda^{\alpha_{12}} = 0 \tag{8.14}$$
$$\mathbf{d}\check{\zeta}^{a_1}\cdots\mathbf{d}\check{\zeta}^{a_6} = 0 \tag{8.15}$$

Using these, the holomorphic volume form of the ambient space becomes on the constraint surface

$$[d^{16}\rho]\big|_{\check{\zeta}^a=0} = \tfrac{1}{32\cdot 5!11!(\lambda\bar{\lambda})^5}(\bar{\lambda}\gamma_{a_1})^{\alpha_1}\cdots(\bar{\lambda}\gamma_{a_5})^{\alpha_5}\epsilon_{\alpha_1\ldots\alpha_{16}}\mathbf{d}\lambda^{\alpha_6}\cdots\mathbf{d}\lambda^{\alpha_{16}}\mathbf{d}\check{\zeta}^{a_1}\cdots\mathbf{d}\check{\zeta}^{a_5} \tag{8.16}$$

Any matrix with two upper indices can be expanded in $\gamma_a^{\alpha\beta}, \gamma_{abc}^{\alpha\beta}$ and $\gamma_{abcde}^{\alpha\beta}$ where only the middle one is antisymmetric. This means that any antisymmetric matrix $A^{[\alpha\beta]}$ can be written as (see e.g. (D.142) of [29])

$$A^{[\alpha\beta]} = \tfrac{1}{16\cdot 3!}\gamma_{abc}^{\alpha\beta}\gamma_{\gamma\delta}^{abc}A^{\gamma\delta} \tag{8.17}$$

So we can in particular replace

$$\tfrac{1}{(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{a_1})^{[\alpha_1}(\bar{\lambda}\gamma_{a_2})^{\alpha_2]} = \tfrac{1}{16\cdot 3!(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{a_1}\gamma^{bcd}\gamma_{a_2}\bar{\lambda})\gamma_{bcd}^{\alpha_1\alpha_2} \tag{8.18}$$

to obtain

$$\begin{aligned}[d^{16}\rho]\big|_{\check{\zeta}^a=0} = {} & \tfrac{1}{5!11!3!32\cdot 16}\tfrac{1}{(\lambda\bar{\lambda})^3}\gamma_{bcd}^{\alpha_1\alpha_2}(\bar{\lambda}\gamma_{a_3})^{\alpha_3}\cdots(\bar{\lambda}\gamma_{a_5})^{\alpha_5}\epsilon_{\alpha_1\ldots\alpha_{16}}\mathbf{d}\lambda^{\alpha_6}\cdots\mathbf{d}\lambda^{\alpha_{16}}\times \\ & \times\tfrac{1}{(\lambda\bar{\lambda})^2}(\bar{\lambda}\gamma_{a_1}{}^{bcd}{}_{a_2}\bar{\lambda})\mathbf{d}\check{\zeta}^{a_1}\cdots\mathbf{d}\check{\zeta}^{a_5}\end{aligned} \tag{8.19}$$

This is already reasonably close to a split into (8.1) and a volume form for the $\check{\zeta}^a$-space, in particular if one has in mind footnote 22 in which we argued that the $\check{\zeta}^a$-volume form is expected to be λ-dependent. There just remains the problem that the contraction of the indices bcd is between the two "factors" so that factorization is not yet obvious. We believe, however, that because of the constraints on λ^α **and** on $\check{\zeta}^a$ there will be identities similar to (A.108) for the toy-model which will rearrange the contractions and thus solve this issue. We leave this for further study.

9 Conclusions

In this article we have introduced a family $P_{(f)}^\alpha$ (parametrized by a function f) of covariant non-linear projections to the pure spinor space whose linearization on the constraint surface reduced for all of them to the transpose of the known projector to the gauge invariant part of the antighost $\omega_{z\alpha}$. In addition we introduced similar linear projectors to gauge invariant parts for the non-minimal variables $\bar{\omega}_z^\alpha$ and $\boldsymbol{s}_z^\alpha$.

A priori the simple choice $f = 1$ seems natural for the non-linear projection, but does not lead to any particular properties. Instead we presented a non-trivial function h for which the projection can be derived from a potential Φ and has Hermitian Jacobian matrices. Hermiticity was essential to derive the explicit form for the additional gauge transformations that one obtains if one replaces the pure spinors in the string action by projections of an unconstrained spinor. We have discussed possible gauge fixings but it remains to see whether one can transform in this way to a formalism which has some advantage to the original.

Regarding the pure spinor space (parametrized by λ^α) as being embedded into an ambient space $\mathbb{C}^{16}$ (parametrized by ρ^α), the projection becomes part of a variable transformation: $\rho^\alpha \mapsto (\lambda^\alpha, \zeta^a)$. Depending on the application the latter variable was also redefined to either $\tilde{\zeta}^a$ or $\check{\zeta}^a$. All of them have in common that they are constrained. We have derived the transformation of the holomorphic volume form under of $\mathbb{C}^{16}$ under this variable transformation, hoping that it would factorize into the holomorphic volume form of the pure spinor space and a volume form of the rest. In the toy-model such a factorization worked perfectly. In the pure spinor case the result is promising, but contains contractions between the two "factors" which hopefully can be resolved via some identities. If this could be completed, the corresponding reparametrization might by a way to derive path-integral measures for higher genus. We leave this for further study.

Having a projection to the pure spinor space allows to obtain solutions to the pure spinor constraint without switching to the U(5) formalism. The latter is based on a Fock space representation whose vacuum has to be a pure spinor itself. Having concrete pure spinors in the standard representation should thus allow to derive concrete intertwiners between the standard and the Fock space representation.

One further application of the present formalism might be the projection of unphysical contributions in the computation of the force between D-branes. Indeed, this check in the context of the pure spinor formalism has never been done. Similarly, the computation of the partition function [30] can be probably expressed in closed form using this non-linear projections. We hope to report on these applications in the future.

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A Toy model

Many of the properties of pure spinor space can also be observed for a simple toy model which helped us to find the projection that we presented in the main part. In this appendix we will be less rigorous in some statements and put more emphasis to present the logic that we followed.

Let us consider complex variables $\lambda^I \in \mathbb{C}$, $I \in \{1, \ldots, \mathcal{N}\}$ obeying the quadratic constraint

$$\lambda^2 \equiv \lambda^I \delta_{IJ} \lambda^J = 0 \tag{A.1}$$

We will denote the complex conjugate by $\bar{\lambda}_I$ and use the metric δ_{IJ} to pull indices up or down, i.e. $\lambda_I = \delta_{IJ}\lambda^J$. In principle there is thus no need to distinguish between upper and lower indices, but we will still do so, because we understand the Einstein summation convention to be applied only when an index is repeated with opposite vertical position.

If one wants to explicitly solve the constraint, it is convenient to introduce 'lightcone-coordinates'

$$\lambda^{\pm} \equiv \tfrac{1}{\sqrt{2}}\left(\lambda^{\mathcal{N}-1} \pm i\lambda^{\mathcal{N}}\right) \tag{A.2}$$

in which the metric becomes off-diagonal for these two entries, i.e. $\lambda^+ = \lambda_-$, $\lambda^- = \lambda_+$ while it remains diagonal for the remaining $\mathcal{N}-2$ indices i, i.e. $\lambda^i = \lambda_i$:

$$\lambda^2 = 2\lambda^+\lambda^- + \lambda^i\lambda_i = 0 \tag{A.3}$$

For $\mathcal{N} = 3$ this is the model discussed for example in section 3.1.3. of [32]. In a patch where $\lambda^+ \neq 0$, one can thus solve the constraint $\lambda^2 = 0$ for λ^- as a function of λ^+ and the λ^i's. Let us give this function the name λ^-_{sol}, as we will later refer to it:

$$\lambda^-_{\mathrm{sol}}(\lambda^+, \lambda^i) \equiv -\tfrac{1}{2\lambda^+}\lambda^i\lambda_i \tag{A.4}$$

These coordinates correspond in the pure spinor case to the U(5)-covariant coordinates.

A covariant derivative[23] which respects the constraint (A.1) can be constructed using some reference vector $\bar{\chi}_I$:

$$D_{\lambda^I} \quad\equiv\quad \partial_{\lambda^I} - \tfrac{1}{(\lambda\bar{\chi})}\lambda_I\bar{\chi}^J\partial_{\lambda^J} \equiv \Pi^T_{(\bar{\chi})\perp}{}_I{}^J\partial_{\lambda^J} \tag{A.5}$$

This defines the projection matrix $\Pi^T_{(\bar{\chi})\perp}$ whose transpose

$$\Pi_{(\bar{\chi})\perp} = \mathbb{1} - \tfrac{\bar{\chi}\otimes\lambda}{(\bar{\chi}\lambda)} \tag{A.6}$$

projects a general vector to one which is orthogonal (therefore the subscript $\perp$) to λ^K or equivalently

$$\lambda^K\delta_{KI}\Pi_{(\bar{\chi})\perp}{}^I{}_J \quad=\quad 0 \tag{A.7}$$

and in addition is idempotent, as one can easily check:

$$(\Pi_{(\bar{\chi})\perp})^2 = \Pi_{(\bar{\chi})\perp} \tag{A.8}$$

Note that in particular every variation of λ^I consistent with (A.1) is orthogonal to λ^I (i.e. $\lambda^I\delta_{IJ}\delta\lambda^J = 0$). $\Pi_{(\bar{\chi})\perp}$ thus acts on $\delta\lambda$ like the identity:

$$\Pi_{(\bar{\chi})\perp}{}^I{}_J\delta\lambda^J = \delta\lambda^J \tag{A.9}$$

A general variation of a function of λ^I is thus of the form

$$\delta\lambda^T\partial_\lambda = (\Pi_{(\bar{\chi})\perp}\delta\lambda)^T\partial_\lambda = \delta\lambda^T(\Pi^T_{(\bar{\chi})\perp}\partial_\lambda) = \delta\lambda^T D_\lambda \tag{A.10}$$

and thus reproduces the covariant derivative from which we started.

[23]The covariant derivative D_{λ^I} corresponds in the pure spinor sigma model to the gauge invariant part of the antighost ω_z. The projection to this gauge invariant part as presented in [16] and [17] was the starting point for this article. ◇

We will not restrict $\bar{\chi}$ to be only a constant reference vector, but allow e.g. also $\bar{\chi}_I = \bar{\lambda}_I$. As this will be the most important case, we denote

$$\Pi_\perp \equiv \Pi_{(\bar{\lambda})\perp} = \mathbb{1} - \tfrac{\bar{\lambda}\otimes\lambda}{(\bar{\lambda}\lambda)} \tag{A.11}$$

Particular properties of this projection matrix are its Hermiticity and the fact that (seen as a function of λ and $\bar{\lambda}$) it is homogeneous of degree (0,0), i.e.

$$\Pi_\perp(c\lambda, \bar{c}\bar{\lambda}) = \Pi_\perp(\lambda, \bar{\lambda}) \quad \forall c \in \mathbb{C} \tag{A.12}$$

A natural question is now, whether one can integrate (A.9). Or in other words, whether we can find a projection

$$P^I_{(\bar{\chi})}: \quad \rho^I \mapsto \lambda^I \equiv P^I_{(\bar{\chi})}(\rho) \tag{A.13}$$

such that its variation just produces the linear projector $\Pi_\perp$. I.e. if we define $\Pi_{(\bar{\chi})\perp}(\rho)$ via $\delta P^I_{(\bar{\chi})}(\rho) \equiv (\Pi_{(\bar{\chi})\perp}(\rho)\delta\rho)^I$ or equivalently

$$\Pi_{(\bar{\chi})\perp}(\rho)^I{}_J \equiv \partial_{\rho^J} P^I_{(\bar{\chi})}(\rho) \tag{A.14}$$

then this matrix should reduce on the constraint surface $\rho = \lambda$ to $\Pi_{(\bar{\chi})\perp}(\lambda)$. So the condition is

$$\partial_{\rho^J} P^I_{(\bar{\chi})}(\rho)\Big|_{\rho=\lambda} \stackrel{!}{=} \delta^I_J - \tfrac{\bar{\chi}^I\lambda_J}{(\bar{\chi}\lambda)} \equiv \Pi_{(\bar{\chi})\perp}{}^I{}_J \tag{A.15}$$

The naive extension of the righthand side off the constraint surface to $\partial_{\rho^J} P^I_{(\bar{\chi})}(\rho) \stackrel{!}{=} \delta^I_J - \frac{\bar{\chi}^I\rho_J}{(\bar{\chi}\rho)}$ is not so easily integrable, but already the terms one obtains by integrating it while treating the ρ-dependent denominator as a constant leads for certain $\bar{\chi}$ to a solution of (A.15)

$$P^I_{(\bar{\chi})}(\rho) \equiv \rho^I - \frac{1}{2(\bar{\chi}\rho)}\rho^2\bar{\chi}^I \quad (\text{if } \bar{\chi}^I\bar{\chi}_I = 0) \tag{A.16}$$

However, the extra condition $\bar{\chi}^I\bar{\chi}_I = 0$ is necessary for $\lambda^I \equiv P^I_{(\bar{\chi})}(\rho)$ to obey the constraint $\lambda^2 = 0$ for all ρ. It is clear that we need additional terms in the case where $\bar{\chi}$ is not a constrained vector, e.g. for the covariant choice $\bar{\chi} = \bar{\rho}$. An idea would be to replace $\bar{\chi}$ in turn by a projection that uses ρ as reference spinor $\bar{\chi}_I \to \bar{\chi}_I - \frac{1}{2(\bar{\chi}\rho)}(\bar{\chi})^2\rho_I$ and then again replace ρ_I by the projection (A.16) and then continue iteratively. The main insight from this procedure is that we will obtain something of the form

$$P^I_{(\bar{\chi})}(\rho) \equiv f(\xi)\rho^I - g(\xi)\frac{\rho^2}{(\rho\bar{\chi})}\bar{\chi}^I \tag{A.17}$$

$$\text{with } \xi \equiv \frac{\rho^2\bar{\chi}^2}{(\rho\bar{\chi})^2} \tag{A.18}$$

with some functions f, g to be determined. We will understand this as an ansatz from now on and can forget about the iterative motivation[24]. The variable ξ

[24]Coming from the iteration idea of the projection (A.16), our original approach was via a power series ansatz

$$P^I_{(\bar{\chi})}(\rho) \equiv \rho^I\big(\textstyle\sum_{k=0}^\infty \alpha_k\xi^k\big) - \bar{\chi}^I\frac{\rho^2}{(\rho\bar{\chi})}\big(\sum_{k=0}^\infty \beta_k\xi^k\big), \quad \alpha_k, \beta_k \in \mathbb{C}$$

In order to be a projection, one needs that for constrained λ^I (i.e. with $\lambda^2 = 0$) it should is homogeneous of degree 0 w.r.t. ρ and also w.r.t. $\bar{\chi}$. The above ansatz for $P^I_{(\bar{\chi})}(\rho)$ is therefore homogeneous of degree 1 in ρ and 0 in $\bar{\chi}$.

For a vector λ^I that lies already on the constraint-surface $\lambda^2 \equiv \lambda^I \delta_{IJ} \lambda^J = 0$, the map $P^I_{(\bar{\chi})}$ (being a projection) should be the identity:

$$\lambda^I \stackrel{!}{=} P^I_{(\bar{\chi})}(\lambda) = \lambda^I f(0) \tag{A.19}$$

This determines

$$f(0) = 1 \tag{A.20}$$

The other requirement for $P^I_{(\bar{\chi})}$ is that the image $P^I_{(\bar{\chi})}(\rho)$ lies on the constraint surface for every ρ^I, i.e.

$$0 \stackrel{!}{=} P^I_{(\bar{\chi})}(\rho) P_{(\bar{\chi})I}(\rho) = \tag{A.21}$$

$$= \rho^2 \left(f(\xi)^2 - 2 f(\xi) g(\xi) + \xi g(\xi)^2 \right) \tag{A.22}$$

So at least away from the constraint surface (i.e. for $\rho^2 \neq 0$) we need the bracket to vanish and thus obtain a priori two solutions for f in terms of g or vice versa

$$f(\xi) = g(\xi) \left(1 \pm \sqrt{1 - \xi} \right) \tag{A.23}$$

The condition $f(0) = 1$ fixes the sign to be the upper one. This fixes g uniquely in terms of f:

$$g(\xi) = \frac{f(\xi)}{1 + \sqrt{1 - \xi}}, \quad f(0) = 1 \tag{A.24}$$

Plugging this back into the ansatz (A.17), we obtain a **family of projections** that still depends on the choice of the reference spinor $\bar{\chi}$ and the function f:

$$P^I_{(f,\bar{\chi})}(\rho) \equiv f(\xi) \left(\rho^I - \frac{1}{1 + \sqrt{1 - \xi}} \frac{\rho^2}{(\rho\bar{\chi})} \bar{\chi}^I \right) \quad \text{with } \xi \equiv \frac{\rho^2 \bar{\chi}^2}{(\rho\bar{\chi})^2} \tag{A.25}$$

For $\bar{\chi} = \bar{\rho}$, it is convenient to define

$$\zeta \equiv \frac{\rho^2}{(\rho\bar{\rho})} \tag{A.26}$$

which implies

$$\xi = \zeta\bar{\zeta} \tag{A.27}$$

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reduce to the identity map $P^I_{(\bar{\chi})}(\lambda) \stackrel{!}{=} \lambda^I$, which fixes $\alpha_0 = 1$. From the condition that $(P_{(\bar{\chi})}(\rho))^2 = 0$ one obtains $\beta_0 = \frac{1}{2}$ and in addition (using already $\alpha_0 = 1$,$\beta_0 = \frac{1}{2}$) the recursion relation $(\alpha_n - 2\beta_n) = -\sum_{k=1}^{n-1} \alpha_k (\alpha_{n-k} - 2\beta_{n-k}) - \sum_{k=0}^{n-1} \beta_k \beta_{n-1-k}$.

If one fixes by hand $\alpha_k = 0 \quad \forall k \geq 1$ and makes a reparametrization $\beta_n \equiv \frac{\gamma_n}{2^{2n+1}}$, it leads to the new series $\gamma_{n+1} = \sum_{k=0}^{n} \gamma_k \gamma_{n-k}$ ($\gamma_0 = 1$), whose first entries (starting with γ_0) are $\{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...\}$. According to oeis.org [31] (thanks to Nicolas Orantin and Stavros Papadakis for information about this web-site!), these are the Catalan numbers or Segner numbers $\gamma_n = \frac{(2n)!}{n!(n+1)!}$. For them it is well known that $\sum_{n=0}^{\infty} \gamma_n \xi^n = \frac{2}{1+\sqrt{1-4\xi}}$. Coming back to $\beta_n = \frac{\gamma_n}{2^{2n+1}}$, we obtain $\sum_{n=0}^{\infty} \beta_n \xi^n = \frac{1}{1+\sqrt{1-\xi}}$. Therefore the power series ansatz for our projector with the manual choice $\alpha_k = 0 \quad \forall k \geq 1$ turns into

$$P^I_{(\bar{\chi})}(\rho) = \rho^I - \bar{\chi}^I \frac{\rho^2}{(\rho\bar{\chi})} \frac{1}{1+\sqrt{1-\xi}} \qquad \diamond$$

and the projector becomes[25,26]

$$\lambda^I \equiv P^I_{(f)}(\rho,\bar\rho) \equiv f(\xi)\left(\rho^I - \frac{\zeta}{1+\sqrt{1-\xi}}\bar\rho^I\right) \tag{A.28}$$

As long as $\bar\rho$ is treated independently from ρ and as long as we are not talking about reality properties, there is no loss of generality when using this second form. We can simply see $\bar\rho$ as an independent vector and are back at $P_{(f,\bar\chi)}$. Then $\bar\zeta$ is not the complex conjugate any more but it is still formally defined in the same way as before, i.e. as $\bar\zeta \equiv \frac{\bar\chi^2}{(\rho\bar\chi)}$. The notation will thus reflect only which point of view we take. We will stress it, as soon as it makes a fundamental difference. The absolute value square of the image of the projection is

$$(\lambda\bar\lambda) = 2(\rho\bar\rho) \mid f(\xi)\mid^2 \frac{1-\xi}{1+\sqrt{1-\xi}} \tag{A.29}$$

The absolute value (A.29) shows that if $f(1)$ is non-singular, the projection (A.28) maps to the origin for $\xi = 1$.

$$P^I_{(f)}(\rho,\bar\rho)\Big|_{\xi=1} = 0 \tag{A.30}$$

In particular when f has no zero's (like the valid choice $f=1$), $\xi = 1$ is the only case where the projection vanishes (apart from $(\rho\bar\rho) = 0$ for which ξ is

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[25] When $\bar\rho_I$ is really the complex conjugate of ρ^I, then we have $\xi \le 1$

$$\begin{aligned}(\rho\bar\rho)^2 &= \left(\textstyle\sum_I \mid \rho^I\mid^2\right)^2 = \left(\textstyle\sum_I \mid (\rho^I)^2\mid\right)^2 \ge \mid \textstyle\sum_I(\rho^I)^2\mid^2 = \mid \rho^2\mid^2 = \rho^2\bar\rho^2 \\ \Rightarrow (\rho\bar\rho)^2 &\ge \rho^2\bar\rho^2 \quad \Rightarrow \xi \ge 1\end{aligned}$$

The fact $\xi \le 1$ makes the projection (A.28) particularly well-behaved. But note that this statement is true only if the reference spinor $\bar\chi = \bar\rho$ is really the complex conjugate of ρ. $\diamond$

[26] Let us discuss two special cases.

- For $\mathcal{N}=1$ the constraint $0 = \lambda^2 = (\Re(\lambda) + i\Im(\lambda))^2 = \Re(\lambda)^2 - \Im(\lambda)^2 + 2i\Re(\lambda)\Im(\lambda)$ implies $\Re(\lambda) = \Im(\lambda) = 0 \Rightarrow \lambda = 0$. Consistent with this, the projection maps every complex number ρ to 0. Note that at quantum level, or even in the ring of supernumbers, $\lambda^2 = 0$ has also nontrivial solutions.
- In dimension $\mathcal{N}=2$, the solution space to $0 = \lambda^2 = 2\lambda^+\lambda^-$ is the union of $\lambda^+ = 0$ and $\lambda^- = 0$. The projection (A.28) reads for $f=1$

$$\begin{aligned}P^+_{(f=1)}(\rho,\bar\rho) &= \rho^+ - \bar\rho^+\frac{2\rho^+\rho^-}{\rho^+\bar\rho_+ + \rho^-\bar\rho_- + \sqrt{(\rho^+\bar\rho_+ - \rho^-\bar\rho_-)^2}} \\ &= \rho^+ - \bar\rho_-\frac{2\rho^+\rho^-}{\rho^+\bar\rho_+ + \rho^-\bar\rho_- + \mid \underbrace{\rho^+\bar\rho_+ - \rho^-\bar\rho_-}_{\text{real}}\mid} = \\ &= \begin{cases}\rho^+ - \frac{\bar\rho_-\rho^-}{\bar\rho_+} & \text{for} \quad \mid\rho^+\mid > \mid\rho^-\mid \\ 0 & \text{for} \quad \mid\rho^+\mid \le \mid\rho^-\mid\end{cases} \\ P^-_{(f=1)}(\rho,\bar\rho) &= \begin{cases}0 & \text{for} \quad \mid\rho^+\mid \ge \mid\rho^-\mid \\ \rho^- - \frac{\bar\rho_+\rho^+}{\bar\rho_-} & \text{for} \quad \mid\rho^+\mid < \mid\rho^-\mid\end{cases}\end{aligned}$$

On the constraint surface this implies:

$$\begin{pmatrix} P^+_{(f=1)}(\lambda^+,0) \\ P^-_{(f=1)}(\lambda^+,0)\end{pmatrix} = \begin{pmatrix}\lambda^+ \\ 0\end{pmatrix} \quad , \quad \begin{pmatrix} P^+_{(f=1)}(0,\lambda^-) \\ P^-_{(f=1)}(0,\lambda^-)\end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^-\end{pmatrix} \quad \diamond$$

not well-defined). Using this insight and plugging $\xi = 1$ and $\lambda^I = 0$ back into (A.28), we obtain $0 = \rho^I - \zeta\bar{\rho}^I$ and therefore

$$\xi = 1 \quad \Longleftrightarrow \quad \rho^I = \zeta\bar{\rho}^I \quad (\text{or } \bar{\rho}_I = \bar{\zeta}\rho_I) \tag{A.31}$$

Projection as part of a variable transformation One can partially invert the projection and express ρ^I as a linear combination of λ^I and $\bar{\lambda}_I$ with ζ-dependent coefficients (having in mind that $\xi = \zeta\bar{\zeta}$):

$$\rho^I(\lambda, \bar{\lambda}, \zeta, \bar{\zeta}) = \frac{1 + \sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^I + \frac{\zeta}{2\bar{f}(\xi)\sqrt{1-\xi}}\bar{\lambda}^I \tag{A.32}$$

The unconstrained vector ρ^I can thus be seen as a non-holomorphic function of λ^I and ζ and one can regard the projection as part of a variable transformation $\rho^I \mapsto (\lambda^I, \zeta)$ and the above equation as the inverse variable transformation. This means that in (A.32) ζ is regarded as an independent variable, while in (A.28) it was just a placeholder for $\frac{\rho^2}{(\rho\bar{\rho})}$ and λ^I was regarded as a function of ρ^I only.

The inverse transformation of the absolute value squared is obviously (looking at (A.29)) given by

$$(\rho\bar{\rho}) = \frac{(\lambda\bar{\lambda})\left(1 + \sqrt{1-\xi}\right)}{2 \mid f(\xi) \mid^2 (1-\xi)} \tag{A.33}$$

Alternative reparametrization with simple inverse If one rewrites the righthand side of equation (A.32) as $\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\left(\lambda^I + \frac{f(\xi)}{\bar{f}(\xi)}\frac{\zeta}{1+\sqrt{1-\xi}}\bar{\lambda}^I\right)$, it becomes obvious that the inverse variable transformation would become particularly simple upon choosing

$$\tilde{\zeta} \equiv \frac{f(\xi)}{\bar{f}(\xi)}\frac{\zeta}{1+\sqrt{1-\xi}} = \frac{f(\xi)}{\bar{f}(\xi)}\frac{\rho^2}{(\rho\bar{\rho}) + \sqrt{(\rho\bar{\rho})^2 - \rho^2\bar{\rho}^2}} \tag{A.34}$$

as a new variable. The absolute value square of this variable is

$$\tilde{\xi} \equiv \tilde{\zeta}\bar{\tilde{\zeta}} = \frac{\xi}{(1+\sqrt{1-\xi})^2} = \frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}} \tag{A.35}$$

Because of $0 \leq \xi \leq 1$ also $\tilde{\xi}$ obeys

$$0 \leq \tilde{\xi} \leq 1 \tag{A.36}$$

In order to invert the relation between $\tilde{\zeta}$ and ζ in (A.34), it is useful to invert first the above relation (A.35) of their absolute values. Multiplying both sides of (A.35) with the denominator $\left(2 - \xi + 2\sqrt{1-\xi}\right)$ and putting the term that contains the square root on the left-hand side and the rest on the righthand side, one obtains $2\tilde{\xi}\sqrt{1-\xi} = \xi(1+\tilde{\xi}) - 2\tilde{\xi}$. Squaring and putting everything to the left yields $\xi\left(\xi(1+\tilde{\xi})^2 - 4\tilde{\xi}\right) = 0$. At least for $\xi \neq 0$, the inverse transformation of the absolute value square is thus given by

$$\xi = \frac{4\tilde{\xi}}{(1+\tilde{\xi})^2} \tag{A.37}$$

Together with $0 \leq \tilde{\xi} \leq 1$ this implies

$$\sqrt{1-\xi} = \frac{1-\tilde{\xi}}{1+\tilde{\xi}} \tag{A.38}$$

Plugging this into (A.34) and solving for ζ, we arrive at

$$\zeta \quad = \quad \frac{\bar{\tilde{f}}(\tilde{\xi})}{\tilde{f}(\tilde{\xi})} \frac{2\tilde{\zeta}}{(1+\tilde{\xi})} \tag{A.39}$$

where

$$\tilde{f}(\tilde{\xi}) \equiv f(\xi(\tilde{\xi})) \tag{A.40}$$

We introduced the new variable $\tilde{\zeta}$ in order to obtain a simple inverse transformation of $\rho^I \mapsto (\lambda^I, \tilde{\zeta})$, and indeed, after plugging the above expressions into (A.32) this inverse transformation becomes

$$\rho^I(\lambda, \bar{\lambda}, \tilde{\zeta}, \bar{\tilde{\zeta}}) \quad = \quad \frac{1}{\tilde{f}(\tilde{\xi})(1-\tilde{\xi})} \left(\lambda^I + \tilde{\zeta}\bar{\lambda}^I\right) \tag{A.41}$$

The absolute value squared is given by

$$(\rho\bar{\rho}) \quad = \quad \frac{(1+\tilde{\xi})}{\mid \tilde{f}(\tilde{\xi}) \mid^2 (1-\tilde{\xi})^2} (\lambda\bar{\lambda}) \tag{A.42}$$

Apparently the inverse transformation (A.41) simplifies even further if the function $f(\xi)$ (which determines the precise form of the projection (A.28) is chosen to coincide with $\frac{1}{(1-\tilde{\xi})}$ in the new variables. We will later rediscover this function in the context of Hermiticity, so we give it the name h:

$$h(\xi) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}} = \frac{1}{(1-\tilde{\xi})} \tag{A.43}$$

In this case the inverse transformation (A.41) is simply

$$\rho^I(\lambda, \bar{\lambda}, \tilde{\zeta}, \bar{\tilde{\zeta}}) = \lambda^I + \tilde{\zeta}\bar{\lambda}^I \quad \text{if } f = h \tag{A.44}$$

with $(\rho\bar{\rho}) = (\lambda\bar{\lambda})\left(1+\tilde{\xi}\right)$.

It is interesting to see that from the simple inverse transformations (A.41) (for general f) or (A.44) (for $f = h$) one can indeed reconstruct the original (in terms of ρ^I quite complicated) form of the projection (A.28), just exploiting the constraint $\lambda^2 = 0$. As this is basically inverting the inverse transformation, it is clear that it has to work. The reader might perhaps convince himself for the special case $f = h$ that plugging (A.44) into (A.34) (with $f = h$ written in terms of $\rho, \bar{\rho}$) indeed gives $\tilde{\zeta}$ and that plugging (A.44) into (A.28) indeed gives λ^I.

Jacobian matrices The projection P^I in the manifold is naturally accompanied by push-forward maps on vectors in the tangent space, which are given by the Jacobian matrix of the projection:

$$\begin{aligned} \delta\lambda^I &\equiv \delta P^I_{(f)}(\rho, \bar{\rho}) \equiv \Pi^I_{(f)\perp K}\delta\rho^K + \pi^{IK}_{(f)\perp}\delta\bar{\rho}_K && \text{(A.45)} \\ \Pi^I_{(f)\perp K}(\rho, \bar{\rho}) &\equiv \partial_{\rho^K} P^I_{(f)}(\rho, \bar{\rho}) && \text{(A.46)} \\ \pi^{IK}_{(f)\perp}(\rho, \bar{\rho}) &\equiv \partial_{\bar{\rho}_K} P^I_{(f)}(\rho, \bar{\rho}) && \text{(A.47)} \end{aligned}$$

The subscript '$\perp$' indicates the fact that the matrices $\Pi_{(f)\perp}$ and $\pi_{(f)\perp}$ are mapping to subspaces which are orthogonal to λ

$$\lambda_I \delta\lambda^I = 0\,, \quad P_I(\rho,\bar\rho)\Pi^I_{(f)\perp K}(\rho,\bar\rho) = P_I(\rho,\bar\rho)\pi^{IK}_{(f)\perp}(\rho,\bar\rho) = 0 \quad \forall\rho \qquad \text{(A.48)}$$

In order to calculate $\Pi_{(f)\perp}$ and $\pi_{(f)\perp}$ we repeatedly need the following partial derivatives:[27]

$$\frac{\partial}{\partial\rho^K}\zeta = 2\frac{\rho_K}{(\rho\bar\rho)} - \zeta\frac{\bar\rho_K}{(\rho\bar\rho)} \qquad \text{(A.49)}$$

$$\frac{\partial}{\partial\bar\rho_K}\zeta = -\zeta\frac{\rho^K}{(\rho\bar\rho)} \qquad \text{(A.50)}$$

$$\frac{\partial}{\partial\rho^K}\xi = 2\frac{\bar\zeta\rho_K}{(\rho\bar\rho)} - 2\xi\frac{\bar\rho_K}{(\rho\bar\rho)} \qquad \text{(A.51)}$$

Using these, we obtain

$$\begin{aligned}\Pi^I_{(f)\perp K}(\rho,\bar\rho) &= \\ = \; & f(\xi)\,\delta^I_K + 2\bar\zeta f'(\xi)\tfrac{\rho^I\rho_K}{(\rho\bar\rho)} - 2\xi f'(\xi)\tfrac{\rho^I\bar\rho_K}{(\rho\bar\rho)} - \left(\frac{f(\xi)}{\sqrt{1-\xi}} + \frac{2\xi f'(\xi)}{1+\sqrt{1-\xi}}\right)\tfrac{\bar\rho^I\rho_K}{(\rho\bar\rho)} + \\ & +2\zeta\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + \left(1-\sqrt{1-\xi}\right)f'(\xi)\right)\tfrac{\bar\rho^I\bar\rho_K}{(\rho\bar\rho)} \end{aligned} \qquad \text{(A.52)}$$

with trace

$$\operatorname{tr}\Pi_{(f)\perp}(\rho,\bar\rho) = f(\xi)\,(N-1) - 2(1-\xi)\left(1-\sqrt{1-\xi}\right)f'(\xi) \qquad \text{(A.53)}$$

and

$$\begin{aligned}\pi^{IK}_{(f)\perp}(\rho,\bar\rho) &= \\ = \; & -\frac{f(\xi)}{\left(1+\sqrt{1-\xi}\right)}\zeta\delta^{IK} - 2\xi f'(\xi)\tfrac{\rho^I\rho^K}{(\rho\bar\rho)} + 2\zeta f'(\xi)\tfrac{\rho^I\bar\rho^K}{(\rho\bar\rho)} + \\ & +2\zeta\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + \left(1-\sqrt{1-\xi}\right)f'(\xi)\right)\tfrac{\bar\rho^I\rho^K}{(\rho\bar\rho)} + \\ & -\frac{2\zeta^2}{1+\sqrt{1-\xi}}\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + f'(\xi)\right)\tfrac{\bar\rho^I\bar\rho^K}{(\rho\bar\rho)}\end{aligned} \qquad \text{(A.54)}$$

Neither the matrix $\Pi_{(f)\perp}(\rho,\bar\rho)$ nor the more appropriate full matrix

$$\begin{pmatrix} \Pi_{(f)\perp}(\rho,\bar\rho) & \pi_{(f)\perp}(\rho,\bar\rho) \\ \bar\pi_{(f)\perp}(\rho,\bar\rho) & \bar\Pi_{(f)\perp}(\rho,\bar\rho) \end{pmatrix} \qquad \text{(A.55)}$$

[27] Using (A.32) and (A.33), the derivatives (A.49) to (A.51) can be rewritten as

$$\frac{\partial}{\partial\rho^K}\zeta = \bar f(\xi)\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)\frac{\lambda_K}{(\lambda\bar\lambda)} + f(\xi)\frac{\zeta\sqrt{1-\xi}\left(1-\sqrt{1-\xi}\right)}{1+\sqrt{1-\xi}}\frac{\bar\lambda_K}{(\lambda\bar\lambda)}$$

$$\frac{\partial}{\partial\bar\rho_K}\zeta = -\zeta\sqrt{1-\xi}\bar f(\xi)\frac{\lambda^K}{(\lambda\bar\lambda)} - \zeta\frac{f(\xi)\sqrt{1-\xi}}{\left(1+\sqrt{1-\xi}\right)}\frac{\zeta\bar\lambda^K}{(\lambda\bar\lambda)}$$

$$\frac{\partial}{\partial\rho^K}\xi = 2\bar f(\xi)\,(1-\xi)\frac{\bar\zeta\lambda_K}{(\lambda\bar\lambda)} - 2f(\xi)\frac{\xi\,(1-\xi)}{1+\sqrt{1-\xi}}\frac{\bar\lambda_K}{(\lambda\bar\lambda)} \qquad \diamond$$

are in general projection matrices by themselves in the sense that they are idempotent for all ρ. So in general $\Pi^2_{(f)\perp}(\rho,\bar\rho) \neq \Pi_{(f)\perp}(\rho,\bar\rho)$ and the square of the matrix (A.55)

$$\begin{pmatrix} \Pi^2_{(f)\perp} + \pi_{(f)\perp}\bar\pi_{(f)\perp} & \Pi_{(f)\perp}\pi_{(f)\perp} + \pi_{(f)\perp}\bar\Pi_{(f)\perp} \\ \bar\pi_{(f)\perp}\Pi_{(f)\perp} + \bar\Pi_{(f)\perp}\bar\pi_{(f)\perp} & \bar\Pi^2_{(f)\perp} + \bar\pi_{(f)\perp}\pi_{(f)\perp} \end{pmatrix} \tag{A.56}$$

(for notational simplicity, we have suppressed here the argument $(\rho,\bar\rho)$) is in general not equal to (A.55). Matrices obeying this strict kind of projection property would have an integer trace, namely the dimension of the projected subspace. Looking at the general trace in (A.53), $f = 1$ is an obvious choice where at least the trace becomes integer

$$\mathrm{tr}\,\Pi_{(1)\perp}(\rho,\bar\rho) \quad = \quad N-1 \tag{A.57}$$

and it turns out that indeed $\Pi_{(1)\perp}(\rho,\bar\rho)$ is a proper projection matrix

$$\Pi^2_{(1)\perp}(\rho,\bar\rho) = \Pi_{(1)\perp}(\rho,\bar\rho) \tag{A.58}$$

It is therefore worth to spell out this special case, in which the matrices also become particularly simple:

$$\Pi^I_{(1)\perp K}(\rho,\bar\rho) \quad = \quad \delta^I_K - \frac{1}{\sqrt{1-\xi}}\frac{\bar\rho^I\rho_K}{(\rho\bar\rho)} + \frac{\zeta}{\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\frac{\bar\rho^I\bar\rho_K}{(\rho\bar\rho)} \tag{A.59}$$

$$\pi^{IK}_{(1)\perp}(\rho,\bar\rho) \quad = \quad -\frac{\zeta}{1+\sqrt{1-\xi}}\Pi^{IK}_{(1)\perp}(\rho,\bar\rho) \tag{A.60}$$

$\Pi^I_{(1)\perp K}(\rho,\bar\rho)$ can be even further simplified by noticing that some of the terms combine to the projection $\lambda_K \equiv P_{(1)K}(\rho,\bar\rho)$:

$$\Pi^I_{(1)\perp K}(\rho,\bar\rho) = \delta^I_K - \frac{1}{\sqrt{1-\xi}}\frac{\bar\rho^I\lambda_K}{(\rho\bar\rho)} \tag{A.61}$$

Note that $f = 1$ for the pure spinor case does not lead to $\Pi^2_{(1)\perp}(\rho,\bar\rho) = \Pi_{(1)\perp}(\rho,\bar\rho)$ nor to an integer trace as one can see in (3.16) on page 12 (See also footnote 34 on page 62). If $f \neq 1$, the matrices can only be seen as part of a tangent bundle projection, namely

$$\mathcal{P}_{(f)}: \quad (\rho^I,\bar\rho_I) \quad \mapsto \quad (\lambda^I,\bar\lambda_I) \equiv (P^I_{(f)}(\rho,\bar\rho), \bar P_{(f)I}(\rho,\bar\rho)) \tag{A.62}$$

$$\begin{pmatrix} \delta\rho^I \\ \delta\bar\rho_I \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \delta\lambda^I \\ \delta\bar\lambda_I \end{pmatrix} = \begin{pmatrix} \Pi_{(f)\perp}(\rho,\bar\rho) & \pi_{(f)\perp}(\rho,\bar\rho) \\ \bar\pi_{(f)\perp}(\rho,\bar\rho) & \bar\Pi_{(f)\perp}(\rho,\bar\rho) \end{pmatrix}\begin{pmatrix} \delta\rho^I \\ \delta\bar\rho_I \end{pmatrix} \tag{A.63}$$

Its projection property

$$\mathcal{P}_{(f)} \circ \mathcal{P}_{(f)} = \mathcal{P}_{(f)} \tag{A.64}$$

implies for the matrices

$$\Pi^I_{(f)\perp K}(\lambda,\bar\lambda)\Pi^K_{(f)\perp J}(\rho,\bar\rho) + \underbrace{\pi^{IK}_{(f)\perp}(\lambda,\bar\lambda)}_{=0}\bar\pi_{(f)\perp KJ}(\rho,\bar\rho) \quad = \quad \Pi^I_{(f)\perp J}(\rho,\bar\rho) \tag{A.65}$$

$$\underbrace{\pi^{IK}_{(f)\perp}(\lambda,\bar\lambda)}_{=0}\bar\Pi^{\perp J}_K(\rho,\bar\rho) + \Pi^I_{\perp K}(\lambda,\bar\lambda)\pi^{KJ}_{(f)\perp}(\rho,\bar\rho) \quad = \quad \pi^{IJ}_{(f)\perp}(\rho,\bar\rho) \tag{A.66}$$

Comparing the left side of these two equations with (A.56), the difference is only that here the first factor of each term is evaluated at $\lambda^I \equiv P^I_{(f)}(\rho, \bar{\rho})$ and only the second at ρ^I while in (A.56) both have argument ρ as mentioned in the line below (A.56).

The Jacobian matrices are mapping to a subspace of vectors that is perpendicular to $\lambda^I \equiv P^I_{(f)}(\rho, \bar{\rho})$:

$$\lambda_I \Pi^I_{(f)\perp K}(\rho, \bar{\rho}) = 0 \tag{A.67}$$

$$\lambda_I \pi^{IK}_{(f)\perp}(\rho, \bar{\rho}) = 0 \tag{A.68}$$

On the constraint surface where $\rho = \lambda$ with $\lambda^2 = 0$ (and thus $\zeta = \xi = 0$, $f(0) = 1$), the matrix $\Pi^I_{(f)\perp K}(\rho, \bar{\rho})$ reduces for all f to the projection matrix we started with (as it was asked for in (A.15), just now with $\bar{\chi} = \bar{\rho}$) while $\pi^{IK}_{(f)\perp}(\rho, \bar{\rho})$ vanishes there:

$$\Pi^I_{\perp K} \equiv \Pi^I_{(f)\perp K}(\lambda, \bar{\lambda}) = \delta^I_K - \tfrac{\bar{\lambda}^I \lambda_K}{(\lambda \bar{\lambda})} \tag{A.69}$$

$$\pi^{IK}_{(f)\perp}(\lambda, \bar{\lambda}) = 0 \tag{A.70}$$

The matrix $\Pi_\perp \equiv \Pi_{(f)\perp}(\lambda, \bar{\lambda})$ depends on the function f only implicitly via the dependence of λ on f. Note that $\Pi_{(f)\perp}$ evaluated on the constraint surface, i.e. $\Pi_\perp \equiv \Pi_{(f)\perp}(\lambda, \bar{\lambda})$, is a proper projection matrix for any f

$$\Pi^2_\perp = \Pi_\perp \tag{A.71}$$

$$\operatorname{tr} \Pi_\perp = N - 1 \tag{A.72}$$

Furthermore this matrix is Hermitian

$$\Pi^\dagger_\perp = \Pi_\perp \tag{A.73}$$

It is thus a natural question whether this Hermiticity can be extended from $\Pi_\perp = \Pi_{(f)\perp}(\lambda, \bar{\lambda})$ off the constraint surface to $\Pi_{(f)\perp}(\rho, \bar{\rho})$.

Hermitian Jacobian matrices The answer to the above question is that $\Pi_{(f)\perp}(\rho, \bar{\rho})$ is Hermitian for all ρ if $f = h$, where

$$h(\xi) \equiv \frac{1 + \sqrt{1-\xi}}{2\sqrt{1-\xi}}, \quad (h(0) = 1) \tag{A.74}$$

In fact for this choice the matrices become

$$\begin{aligned}\Pi^I_{(h)\perp K}(\rho, \bar{\rho}) = {}& \frac{1}{2\sqrt{1-\xi}}\left(1 + \sqrt{1-\xi}\right)\delta^I_K - \frac{2-\xi}{2\sqrt{1-\xi}^3}\tfrac{\bar{\rho}^I \rho_K}{(\rho\bar{\rho})} + \\ & + \frac{\bar{\zeta}}{2\sqrt{1-\xi}^3}\tfrac{\rho^I \rho_K}{(\rho\bar{\rho})} - \frac{\xi}{2\sqrt{1-\xi}^3}\tfrac{\rho^I \bar{\rho}_K}{(\rho\bar{\rho})} + \frac{\zeta}{2\sqrt{1-\xi}^3}\tfrac{\bar{\rho}^I \bar{\rho}_K}{(\rho\bar{\rho})}\end{aligned} \tag{A.75}$$

$$\begin{aligned}\pi^{IK}_{(h)\perp}(\rho, \bar{\rho}) = {}& -\frac{\xi}{2\sqrt{1-\xi}^3}\tfrac{\rho^I \rho^K}{(\rho\bar{\rho})} + \frac{\zeta}{2\sqrt{1-\xi}^3}\tfrac{\rho^I \bar{\rho}^K}{(\rho\bar{\rho})} + \frac{\zeta}{2\sqrt{1-\xi}^3}\tfrac{\bar{\rho}^I \rho^K}{(\rho\bar{\rho})} + \\ & - \frac{\zeta^2}{2\sqrt{1-\xi}^3}\tfrac{\bar{\rho}^I \bar{\rho}^K}{(\rho\bar{\rho})} - \frac{\zeta}{2\sqrt{1-\xi}}\delta^{IK}\end{aligned} \tag{A.76}$$

This shows that not only $\Pi_{(h)\perp}(\rho,\bar\rho)$ is Hermitian, but in addition $\pi_{(h)\perp}$ is symmetric

$$\begin{aligned}\Pi^\dagger_{(h)\perp}(\rho,\bar\rho) &= \Pi_{(h)\perp}(\rho,\bar\rho) && \text{(A.77)}\\ \pi^T_{(h)\perp}(\rho,\bar\rho) &= \pi_{(h)\perp}(\rho,\bar\rho) && \text{(A.78)}\end{aligned}$$

These equations are in turn equivalent to the statement that the complete block matrix $\left(\begin{smallmatrix}\Pi_{(h)\perp}(\rho,\bar\rho) & \pi_{(h)\perp}(\rho,\bar\rho)\\ \bar\pi_{(h)\perp}(\rho,\bar\rho) & \bar\Pi_{(h)\perp}(\rho,\bar\rho)\end{smallmatrix}\right)$ is Hermitian.

The non-linear projection (A.28) itself becomes for $f=h$

$$\lambda^I \equiv P^I_{(h)}(\rho,\bar\rho) \equiv \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\rho^I - \frac{\zeta}{2\sqrt{1-\xi}}\bar\rho^I \qquad \text{(A.79)}$$

with 'inverse' transformation (A.32)

$$\rho^I = \lambda^I + \frac{\zeta}{1+\sqrt{1-\xi}}\bar\lambda^I \qquad \text{(A.80)}$$

In the alternative parametrization (A.34) this becomes according to (A.44) simply $\rho^I = \lambda^I + \tilde\zeta\bar\lambda^I$.

In particular in the Hermitian case it turns out to be convenient for calculations to use the above inverse transformation, in order to express the Jacobian matrices in terms of λ^I, ζ and their complex conjugates:

$$\begin{aligned}\Pi^I_{(h)\perp K}(\rho,\bar\rho) &= \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\left(\delta^I_K - \frac{\bar\lambda^I\lambda_K}{(\lambda\bar\lambda)} - \frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}\frac{\lambda^I\bar\lambda_K}{(\lambda\bar\lambda)}\right) && \text{(A.81)}\\ &\overset{\xi=0}{=} \delta^I_K - \frac{\bar\lambda^I\lambda_K}{(\lambda\bar\lambda)} \quad (\equiv \Pi^I_{\perp K}) && \text{(A.82)}\\ \pi^{IK}_{(h)\perp}(\rho,\bar\rho) &= -\frac{\zeta}{2\sqrt{1-\xi}}\Bigg(\underbrace{\delta^{IK} - \frac{\bar\lambda^I\lambda^K}{(\lambda\bar\lambda)}}_{\Pi_\perp} - \underbrace{\frac{\lambda^I\bar\lambda^K}{(\lambda\bar\lambda)}}_{\equiv\Pi^T_\parallel}\Bigg) && \text{(A.83)}\\ &\overset{\zeta=\xi=0}{=} 0 && \text{(A.84)}\end{aligned}$$

The ρ-dependence of these matrices is now only implicitly, with ζ,ξ and $\lambda^I \equiv P^I_{(h)}(\rho,\bar\rho)$ being functions of ρ. The trace of the first matrix is given by

$$\operatorname{tr}\Pi_{(h)\perp}(\rho,\bar\rho) = \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}N - \frac{1}{\sqrt{1-\xi}} \qquad \text{(A.85)}$$

It is useful to note that in the Hermitian case $\bar\pi_{(h)\perp}$ is proportional to $\pi_{(h)\perp}$

$$\bar\pi_{(h)\perp IK}(\rho,\bar\rho) = \frac{\bar\zeta}{\zeta}\pi_{(h)\perp IK}(\rho,\bar\rho) \qquad \text{(A.86)}$$

If we denote just for a moment $\partial_I \equiv \partial_{\rho^I}$ and $\bar\partial^I \equiv \partial_{\bar\rho_I}$, then we have due to Hermiticity $\quad \partial_{[I|}\bar P_{(h)|K]} = \bar\partial^{[I}P^{K]}_{(h)} = \partial_I P^K_{(h)} - \bar\partial^K\bar P_{(h)I} = 0 \qquad$ (A.87)

This in turn implies that $P^I\mathbf{d}\bar\rho_I + \bar P_I\mathbf{d}\rho^I$ is a closed 1-form and thus locally exact. I.e. there exists a scalar function $\Phi(\rho,\bar\rho)$ such that

$$\lambda^I \equiv P^I(\rho,\bar\rho) = \bar\partial^I\Phi(\rho,\bar\rho) \qquad \text{(A.88)}$$

Indeed an explicit solution for the "potential" Φ is given by

$$\Phi(\rho,\bar{\rho}) = \frac{(\rho\bar{\rho})}{2}\left(1+\sqrt{1-\xi}\right) \tag{A.89}$$

One can further check by direct calculation that

$$(\lambda\bar{\lambda}) \equiv P^K(\rho,\bar{\rho})\bar{P}_K(\rho,\bar{\rho}) = \Phi(\rho,\bar{\rho}) \tag{A.90}$$

Holomorphic volume form and measure

Ambient space The canonical holomorphic volume form in the unconstrained ambient space $\mathbb{C}^{\mathcal{N}}$ is given by

$$[d^{\mathcal{N}}\rho] \quad = \quad \mathbf{d}\rho^1 \wedge \ldots \wedge \mathbf{d}\rho^{\mathcal{N}} = \tfrac{1}{\mathcal{N}!}\epsilon_{K_1\ldots K_{\mathcal{N}}}\mathbf{d}\rho^{K_1} \wedge \ldots \wedge \mathbf{d}\rho^{K_{\mathcal{N}}} \tag{A.91}$$

Wedged with its complex conjugate and using $\epsilon_{K_1\ldots K_{\mathcal{N}}}\epsilon^{L_1\ldots L_{\mathcal{N}}} = \mathcal{N}!\delta^{L_1\ldots L_{\mathcal{N}}}_{K_1\ldots K_{\mathcal{N}}}$ and for the signs $\sum_{k=0}^{\mathcal{N}-1} k = \frac{\mathcal{N}(\mathcal{N}-1)}{2}$, one obtains the canonical integration measure of the ambient space

$$[d^{\mathcal{N}}\rho] \wedge [d^{\mathcal{N}}\bar{\rho}] = (-)^{\frac{\mathcal{N}(\mathcal{N}-1)}{2}}\tfrac{1}{\mathcal{N}!}(\mathbf{d}\rho^I\mathbf{d}\bar{\rho}_I)^{\mathcal{N}} \tag{A.92}$$

where on the righthand side we have omitted the wedge symbol $\wedge$ and will omit it also in the following.

λ-space In the λ-space (which we think of being embedded in the ambient ρ-space at $\zeta = 0$) things complicate a bit, because of the constraint $\lambda^2 = 0$, in particular when trying to build a covariant holomorphic volume form. So let us first collect a few identities which we will need in the following. They are all related to the fact that λ^K and $\mathbf{d}\lambda^K$ effectively have only $\mathcal{N}-1$ independent components and thus the antisymmetrization of $\mathcal{N}$ vectors vanishes.

$$\begin{aligned}
\mathbf{d}\lambda^{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}}} &= 0 & \text{(A.93)}\\
\mathbf{d}\lambda^{[I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-1}}\lambda^{I_{\mathcal{N}}]} &= 0 & \text{(A.94)}\\
\mathcal{N}\mathbf{d}\lambda^{[1}\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v^{-]} &= \tfrac{1}{\lambda^+}\mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v_I\lambda^I & \text{(A.95)}
\end{aligned}$$

We will prove these identities in a footnote[28]on the next page. This will require at least for the last identity (A.95) the explicit solution (A.4) of the constraint together with its derivatives, which obey

$$\partial_+\lambda^-_{\mathrm{sol}} = -\frac{\lambda^-_{\mathrm{sol}}}{\lambda^+} \quad , \quad \partial_i\lambda^-_{\mathrm{sol}} = -\frac{\lambda_i}{\lambda^+} \tag{A.96}$$

The holomorphic volume form that we are looking for has to contain $\mathcal{N}-1$ powers of $\mathbf{d}\lambda^I$. Their $\mathcal{N}-1$ indices have to be contracted covariantly with $\epsilon_{I_1\ldots I_{\mathcal{N}}}$, δ_{IJ} or the only available vectors, namely λ^I and $\bar{\lambda}_I$. It is clear that we cannot do without the ϵ-tensor. All contractions avoiding it would vanish. But after contracting the $\mathcal{N}-1$ one-forms with the covariant ϵ-tensor of the ambient space $\epsilon_{I_1\ldots I_{\mathcal{N}-1}I_{\mathcal{N}}}\mathbf{d}\lambda^{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-1}}$, there is one index $I_{\mathcal{N}}$ remaining, which needs to be saturated. So we have to contract it with another vector, so either with λ^I or with $\bar{\lambda}^I$. Contracting with λ^I yields a vanishing result due to (A.94).

Contracting it instead with $\bar{\lambda}^I$ seems to be nonsense a priori, as we want a holomorphic form. However, from (A.95) it is clear that this $\bar{\lambda}^I$ will actually be contracted with a λ^I so that this non-holomorphic factor can be divided out. So the following covariant expression is indeed holomorphic:

$$[d\lambda^{\mathcal{N}-1}] \quad \equiv \quad \frac{1}{(\mathcal{N}-1)!(\lambda\bar{\lambda})}\epsilon_{I_1\ldots I_{\mathcal{N}}}\mathbf{d}\lambda^{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-1}}\bar{\lambda}^{I_{\mathcal{N}}} \tag{A.97}$$

The corresponding measure of the total space is therefore

$$\begin{aligned}&[d\lambda^{\mathcal{N}-1}]\wedge[d\bar{\lambda}^{\mathcal{N}-1}] = \\ &= \frac{\bar{\lambda}^K\lambda_L}{(\mathcal{N}-1)!^2(\lambda\bar{\lambda})^2}\underbrace{\epsilon_{I_1\ldots I_{\mathcal{N}-1}K}\epsilon^{J_1\ldots J_{\mathcal{N}-1}L}}_{\mathcal{N}!\delta^{J_1\ldots J_{\mathcal{N}-1}L}_{I_1\ldots I_{\mathcal{N}-1}K}}\mathbf{d}\lambda^{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-1}}\mathbf{d}\bar{\lambda}_{J_1}\cdots\mathbf{d}\bar{\lambda}_{J_{\mathcal{N}-1}}\end{aligned} \tag{A.98}$$

---

[28] Proof of (A.93)-(A.95): Remember that the constraint on λ^I is quadratic, i.e. $\lambda^I\lambda_I = 2\lambda^+\lambda^- + \lambda^i\lambda_i = 0$, while the one on $\mathbf{d}\lambda^I$ is linear in the $\mathbf{d}\lambda$'s, i.e. $\lambda_I\mathbf{d}\lambda^I = 0$. Solving the latter equation for one of the $\mathbf{d}\lambda$'s, e.g. for $\mathbf{d}\lambda^{\mathcal{N}}$ yields an expression linear in the other $\mathbf{d}\lambda$'s, so that (A.93) is rather obvious. Instead, in order to show (A.94) or (A.95), it is essential to observe that the constraint-function $\lambda^I\lambda_I$ is homogeneous of degree 2 (or in other words has definite "ghost number" 2). Solving for any of the λ's, e.g. for λ^-, yields some function of the other variables

$$\lambda^- = \lambda^-_{\mathrm{sol}}(\lambda^1,\ldots,\lambda^{\mathcal{N}-2},\lambda^+) \quad (*)$$

At least for showing (A.94), it is actually not essential here to switch to "lightcone coordinates" with λ^+,λ^-. One could also solve for $\lambda^{\mathcal{N}}_{\mathrm{sol}}(\lambda^1,\ldots,\lambda^{\mathcal{N}-1})$, although the explicit form of the solution $\lambda^{\mathcal{N}}_{\mathrm{sol}} = \pm\sqrt{-(\lambda^1)^2-\ldots-(\lambda^{\mathcal{N}-1})^2}$ is not unique and is also not so nice as the one in lightcone coordinates $\lambda^-_{\mathrm{sol}} = -\frac{1}{2\lambda^+}\lambda^i\lambda_i$. However, what matters for the argument is only homogeneity: Because of the homogeneity of the constraint function also the functions $\lambda^{\mathcal{N}}_{\mathrm{sol}}$ or λ^-_{sol} are homogeneous, but now of degree 1 (They have ghost number 1), i.e.

$$\left(\lambda^i\partial_i + \lambda^+\partial_+\right)\lambda^-_{\mathrm{sol}} = \lambda^-_{\mathrm{sol}} \quad (\#)$$

In order to show (A.94) and (A.95), first take the differential of (*):

$$\mathbf{d}\lambda^-_{\mathrm{sol}} = \left(\mathbf{d}\lambda^i\partial_i + \mathbf{d}\lambda^+\partial_+\right)\lambda^-_{\mathrm{sol}} \quad (**)$$

Now we can start with the left-hand side of (A.95) (for a particular permutation of the indices), make the antisymmetrization of the index '$-$' explicit and then replace $\mathbf{d}\lambda^-$ using (**).

$$\begin{aligned}
&\mathcal{N}\underbrace{\mathbf{d}\lambda^{[1}\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+}_{\mathcal{N}-1}v^{-]} = \\
&= \quad \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v^- - \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^-v^+ + (\mathcal{N}-2)\mathbf{d}\lambda^+\mathbf{d}\lambda^-\mathbf{d}\lambda^{[1}\cdots\mathbf{d}\lambda^{\mathcal{N}-3}v^{\mathcal{N}-2]} = \\
&\overset{(**)}{=} \quad \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v^- - \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\left(\mathbf{d}\lambda^i\partial_i\lambda^-_{\mathrm{sol}} + \mathbf{d}\lambda^+\partial_+\lambda^-_{\mathrm{sol}}\right)v^+ + \\
&\qquad +(\mathcal{N}-2)\mathbf{d}\lambda^+\left(\mathbf{d}\lambda^i\partial_i\lambda^-_{\mathrm{sol}} + \mathbf{d}\lambda^+\partial_+\lambda^-_{\mathrm{sol}}\right)\mathbf{d}\lambda^{[1}\cdots\mathbf{d}\lambda^{\mathcal{N}-3}v^{\mathcal{N}-2]} = \\
&= \quad \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v^- - \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+\partial_+\lambda^-_{\mathrm{sol}}v^+ + \\
&\qquad +(-)^{\mathcal{N}-3}\mathbf{d}\lambda^+\mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}v^i\partial_i\lambda^-_{\mathrm{sol}} = \\
&= \quad \mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+\left(v^- - v^+\partial_+\lambda^-_{\mathrm{sol}} - v^i\partial_i\lambda^-_{\mathrm{sol}}\right)
\end{aligned}$$

If at this point we replace the general vector v^I by λ^I, then because of (#) the last line indeed vanishes, which proves (A.94) without making use of the explicit form of the constraint. In order to prove (A.95), however, we need the explicit form (A.4) or in particular its derivatives (A.96). Using these derivatives for the above equation with general v^I yields:

$$\begin{aligned}\mathcal{N}\mathbf{d}\lambda^{[1}\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v^{-]} &= \tfrac{1}{\lambda^+}\mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+\left(v^-\lambda^+ + v^+\lambda^- + v^i\lambda_i\right) = \\ &= \tfrac{1}{\lambda^+}\mathbf{d}\lambda^1\cdots\mathbf{d}\lambda^{\mathcal{N}-2}\mathbf{d}\lambda^+v_I\lambda^I \quad \diamond\end{aligned}$$

Using $\mathcal{N}\delta^{J_1\ldots J_{\mathcal{N}-1}L}_{I_1\ldots I_{\mathcal{N}-1}K}{=}\delta^{J_1\ldots J_{\mathcal{N}-1}}_{I_1\ldots I_{\mathcal{N}-1}}\delta^L_K{-}(\mathcal{N}{-}1)\delta^{J_1\ldots J_{\mathcal{N}-1}}_{[I_1\ldots I_{\mathcal{N}-2}|K}\delta^L_{|I_{\mathcal{N}-1}]}$ and $\lambda_L\mathbf{d}\lambda^L = 0$ and for the signs $\sum_{k=0}^{\mathcal{N}-2}k{=}\frac{(\mathcal{N}-1)(\mathcal{N}-2)}{2}$, this becomes

$$[d\lambda^{\mathcal{N}-1}]\wedge[d\bar{\lambda}^{\mathcal{N}-1}] = \frac{1}{(\mathcal{N}{-}1)!(\lambda\bar{\lambda})}(-)^{\frac{(\mathcal{N}-1)(\mathcal{N}-2)}{2}}(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1} \tag{A.99}$$

From the total space point of view, the prefactor $\frac{1}{(\lambda\bar{\lambda})}$ is not really necessary. It comes only from the requirement that we want it to factorize into holomorphic and anti-holomorphic volume form, so from compatibility with the complex structure.

From the ambient space to the λ-space Thinking of the constrained λ-space as being embedded at $\zeta = 0$ (or $\tilde{\zeta} = 0$ in the parametrization of (A.34)) into the ambient space, it is a natural question if the above volume form and measure can be derived from the ambient space. The idea is to make a variable transformation from ρ^I to (λ^I, ζ) or $(\lambda^I, \tilde{\zeta})$ and see if the transformed volume form factorizes. Let us consider the case $f = h$ where the parametrization with the variables $\tilde{\zeta}$ of (A.34) becomes according to (A.44) simply

$$\rho^I = \lambda^I + \tilde{\zeta}\bar{\lambda}^I \tag{A.100}$$

It is obvious that this transformation is not holomorphic which might spoil the idea to obtain a holomorphic λ-volume form after the transformation. However, on the constraint surface $\tilde{\zeta} = 0$, the transformation is holomorphic. On the other hand, the transformation of one-forms, given by

$$\mathbf{d}\rho^I = \mathbf{d}\lambda^I + \tilde{\zeta}\mathbf{d}\bar{\lambda}^I + \mathbf{d}\tilde{\zeta}\bar{\lambda}^I \tag{A.101}$$

is not holomorphic even at $\tilde{\zeta} = 0$. But this does not exclude the possibility that the volume form might still transform holomorphically at $\tilde{\zeta} = 0$ and this is what we are going to test in the following.

But before we discuss the transformation of the holomorphic volume form (A.91), let us first consider the transformation of the total measure (A.92) where the problem of holomorphicity does not yet appear. For the measure it remains to see, if we can factorize the result covariantly and whether one factor coincides with (A.99). So let us start by replacing in (A.92) all $\mathbf{d}\rho^I$ using (A.101):

$$\begin{aligned}(\mathbf{d}\rho^I\mathbf{d}\bar{\rho}_I)^{\mathcal{N}} &= \\ &= \Big(\Big(\mathbf{d}\lambda^I + \tilde{\zeta}\mathbf{d}\bar{\lambda}^I + \mathbf{d}\tilde{\zeta}\bar{\lambda}^I\Big)\Big(\mathbf{d}\bar{\lambda}_I + \bar{\tilde{\zeta}}\mathbf{d}\lambda_I + \mathbf{d}\bar{\tilde{\zeta}}\lambda_I\Big)\Big)^{\mathcal{N}} = \qquad (A.102)\\ &\overset{\mathbf{d}\lambda^I\lambda_I=0}{=} \Big\{\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I\Big(1-\tilde{\xi}\Big) + \tilde{\zeta}\lambda_I\mathbf{d}\bar{\lambda}^I\mathbf{d}\bar{\tilde{\zeta}} + \\ &\qquad -\bar{\tilde{\zeta}}\bar{\lambda}^I\mathbf{d}\lambda_I\mathbf{d}\tilde{\zeta} + (\lambda\bar{\lambda})\mathbf{d}\tilde{\zeta}\mathbf{d}\bar{\tilde{\zeta}}\Big\}^{\mathcal{N}} \qquad (A.103)\end{aligned}$$

Now we can use the identity (A.93) and its complex conjugate which imply that only terms survive which have $\mathcal{N} - 1$ factors of $\mathbf{d}\lambda^I$ and of $\mathbf{d}\bar{\lambda}_I$ respectively, as well as one factor of $\mathbf{d}\tilde{\zeta}$ and $\mathbf{d}\bar{\tilde{\zeta}}$. In addition we are using the constraints $\lambda_I\lambda^I = \lambda_I\mathbf{d}\lambda^I = \mathbf{d}\lambda_I\mathbf{d}\lambda^I = 0$ wherever possible:

$$\begin{aligned}(\mathbf{d}\rho^I\mathbf{d}\bar{\rho}_I)^{\mathcal{N}} \overset{(A.93)}{=} &\ \mathcal{N}\Big(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I(1-\tilde{\xi})\Big)^{\mathcal{N}-1}(\lambda\bar{\lambda})\mathbf{d}\tilde{\zeta}\mathbf{d}\bar{\tilde{\zeta}} + \qquad (A.104)\\ &+\mathcal{N}(\mathcal{N}-1)\Big(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I(1-\tilde{\xi})\Big)^{\mathcal{N}-2}\bar{\lambda}^J\mathbf{d}\lambda_J\lambda_K\mathbf{d}\bar{\lambda}^K\tilde{\xi}\mathbf{d}\tilde{\zeta}\mathbf{d}\bar{\tilde{\zeta}}\end{aligned}$$

In order to combine the two terms which have different contractions among the $\mathbf{d}\lambda$'s, we can use the identity (A.94) in the following way:

$$\begin{aligned}
0 &\overset{(A.94)}{=} \mathcal{N}\mathbf{d}\lambda^{[I_1}\mathbf{d}\bar{\lambda}_{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-2}}\mathbf{d}\bar{\lambda}_{I_{\mathcal{N}-2}}\mathbf{d}\lambda^{J}\mathbf{d}\bar{\lambda}_{K}\lambda^{K]}\bar{\lambda}_J = && \text{(A.105)}\\
&= \mathbf{d}\lambda^{I_1}\mathbf{d}\bar{\lambda}_{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-2}}\mathbf{d}\bar{\lambda}_{I_{\mathcal{N}-2}}\mathbf{d}\lambda^{J}\mathbf{d}\bar{\lambda}_{K}\lambda^{K}\bar{\lambda}_J + \\
&\quad -\mathbf{d}\lambda^{I_1}\mathbf{d}\bar{\lambda}_{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-2}}\mathbf{d}\bar{\lambda}_{I_{\mathcal{N}-2}}\mathbf{d}\lambda^{K}\mathbf{d}\bar{\lambda}_{K}\lambda^{J}\bar{\lambda}_J + \\
&\quad +(\mathcal{N}-2)\mathbf{d}\lambda^{I_1}\mathbf{d}\bar{\lambda}_{I_1}\cdots\mathbf{d}\lambda^{I_{\mathcal{N}-3}}\mathbf{d}\bar{\lambda}_{I_{\mathcal{N}-3}}\mathbf{d}\lambda^{J}\mathbf{d}\bar{\lambda}_{I_{\mathcal{N}-2}}\mathbf{d}\lambda^{K}\mathbf{d}\bar{\lambda}_{K}\lambda^{I_{\mathcal{N}-2}}\bar{\lambda}_J = && \text{(A.106)}\\
&= (\mathcal{N}-1)(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-2}\mathbf{d}\lambda^{J}\mathbf{d}\bar{\lambda}_{K}\lambda^{K}\bar{\lambda}_J - (\lambda\bar{\lambda})(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1} && \text{(A.107)}
\end{aligned}$$

This implies the identity

$$(\mathcal{N}-1)(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-2}\mathbf{d}\lambda^{J}\mathbf{d}\bar{\lambda}_{K}\lambda^{K}\bar{\lambda}_J = (\lambda\bar{\lambda})(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1} \qquad \text{(A.108)}$$

Plugging this back into (A.104) yields

$$(\mathbf{d}\rho^I\mathbf{d}\bar{\rho}_I)^{\mathcal{N}} \quad = \quad \mathcal{N}\cdot(\lambda\bar{\lambda})(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1}\times(1-\tilde{\xi})^{\mathcal{N}-2}\mathbf{d}\tilde{\zeta}\mathbf{d}\bar{\tilde{\zeta}} \qquad \text{(A.109)}$$

This, as indicated, factorizes into a λ- and a $\tilde{\zeta}$-volume form. However, the λ-volume form $\mathcal{N}\cdot(\lambda\bar{\lambda})(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1}$ does not come with the expected negative power of $(\lambda\bar{\lambda})$ as in (A.99).

This problem disappears if we redefine

$$\check{\zeta} \equiv (\lambda\bar{\lambda})\tilde{\zeta} \quad , \quad \check{\xi} \equiv \check{\zeta}\bar{\check{\zeta}} = (\lambda\bar{\lambda})^2\tilde{\xi} \qquad \text{(A.110)}$$

Note, however, that then the factorization breaks down in general (the $\check{\zeta}$-measure becomes λ-dependent) and we need to restrict to the constraint surface $\check{\xi}=0$, if we want to restore it:

$$\begin{aligned}
(\mathbf{d}\rho^I\mathbf{d}\bar{\rho}_I)^{\mathcal{N}} &= \tfrac{\mathcal{N}}{(\lambda\bar{\lambda})}\cdot(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1}\times(1-\tfrac{1}{(\lambda\bar{\lambda})^2}\check{\xi})^{\mathcal{N}-2}\mathbf{d}\check{\zeta}\mathbf{d}\bar{\check{\zeta}} = && \text{(A.111)}\\
&\overset{\check{\xi}=0}{=} \tfrac{\mathcal{N}}{(\lambda\bar{\lambda})}\cdot(\mathbf{d}\lambda^I\mathbf{d}\bar{\lambda}_I)^{\mathcal{N}-1}\times\mathbf{d}\check{\zeta}\mathbf{d}\bar{\check{\zeta}} && \text{(A.112)}
\end{aligned}$$

So using the variables (A.110) we have achieved to arrive at the measure (A.99) on the constraint surface. Note finally that the new variables $\check{\zeta}$ turn out to be very simple in terms of ρ, if one uses that according to (A.89), (A.90), (A.34) and (A.26) it is nothing else but

$$\check{\zeta} = \tfrac{1}{2}\rho^2 \qquad \text{(A.113)}$$

This means in particular that in contrast to ζ or $\tilde{\zeta}$ it depends holomorphically on ρ^{α}.

Transformation of the target space holomorphic volume form Now let us see if we can achieve the same for the holomorphic volume form. Using the redefined variables $\check{\zeta}$ of (A.110), the reparametrization of ρ^I and the corresponding transformation of $\mathbf{d}\rho^I$ read

$$\begin{aligned}
\rho^I &= \lambda^I + \check{\zeta}\frac{\bar{\lambda}^I}{(\lambda\bar{\lambda})} && \text{(A.114)}\\
\mathbf{d}\rho^I &= \Big(\delta^I_J - \check{\zeta}\frac{\bar{\lambda}^I\bar{\lambda}_J}{(\lambda\bar{\lambda})^2}\Big)\mathbf{d}\lambda^J + \frac{\bar{\lambda}^I}{(\lambda\bar{\lambda})}\mathbf{d}\check{\zeta} + \frac{\check{\zeta}}{(\lambda\bar{\lambda})}\Big(\delta^I_J - \frac{\bar{\lambda}^I\lambda_J}{(\lambda\bar{\lambda})}\Big)\mathbf{d}\bar{\lambda}^J && \text{(A.115)}
\end{aligned}$$

For the transformation of the holomorphic volume form (A.91) of the ambient space, we will now immediately restrict to the constraint surface $\check{\zeta} = 0$ where $\mathbf{d}\rho^I = \mathbf{d}\lambda^I + \frac{\bar{\lambda}^I}{(\lambda\bar{\lambda})}\mathbf{d}\check{\zeta}$. This yields

$$
\begin{aligned}
[d^{\mathcal{N}}\rho]\big|_{\check{\zeta}=0} &= \tfrac{1}{\mathcal{N}!}\epsilon_{K_1\ldots K_{\mathcal{N}}}\Big(\mathbf{d}\lambda^{K_1} + \frac{\bar{\lambda}^{K_1}}{(\lambda\bar{\lambda})}\mathbf{d}\check{\zeta}\Big)\cdots\Big(\mathbf{d}\lambda^{K_{\mathcal{N}}} + \frac{\bar{\lambda}^{K_{\mathcal{N}}}}{(\lambda\bar{\lambda})}\mathbf{d}\check{\zeta}\Big) = \qquad \text{(A.116)}\\
&\overset{\text{(A.93)}}{=} \tfrac{1}{(\lambda\bar{\lambda})(\mathcal{N}-1)!}\epsilon_{K_1\ldots K_{\mathcal{N}}}\mathbf{d}\lambda^{K_1}\cdots\mathbf{d}\lambda^{K_{\mathcal{N}-1}}\bar{\lambda}^{K_{\mathcal{N}}} \times \mathbf{d}\check{\zeta} \qquad \text{(A.117)}
\end{aligned}
$$

This indeed factorizes into the holomorphic λ-measure (A.97) and $\mathbf{d}\check{\zeta}$. So on the constraint surface the non-holomorphic transformation (A.114) seems to transform the volume form holomorphically.

From toy-model formulas to pure spinor formulas

The toy-model equations have been proven to be a valuable guide for the pure spinor case, although they do not make use of something like the Clifford algebra nor of Fierz identities. Nevertheless many equations look almost identical. A recipe to go from pure spinor equations back to toy-model equations is to make the following replacements

$$
\begin{aligned}
\rho^\alpha, \bar{\rho}_\alpha &\to \rho^I, \bar{\rho}_I\\
\zeta^a, \bar{\zeta}_a &\to \zeta, \bar{\zeta}\\
\gamma^a_{\alpha\beta} &\to \delta_{IJ}\\
\sum_a(\ldots) &\to 2\cdot(\ldots)\\
\xi &\to \xi
\end{aligned}
$$

There are a few deviations from this rule which is therefore not rigorous but rather serves as a guideline. For example the last term of (5.15) does not appear in the toy-model. It would be interesting to see if one can modify the recipe in a way which also correctly covers such terms.

B Proofs

B.1 Proof of proposition 1

Proof. Let us start from the end with the properties of the auxiliary variables:

5. The equations in (2.13), i.e. $\zeta^a\zeta_a = 0$ and $\zeta^a(\gamma_a\rho)_\alpha = 0$, follow directly from the Fierz identity.[29] That ξ and ζ^a vanish for pure spinors away from the origin (2.14) follows directly from their definitions in (2.4). Equations (2.15) are also obvious from the definitions.

If ρ and its complex conjugate are 'proportional' in the sense $\rho^\alpha \propto (\gamma_b\bar\rho)^\alpha$, or more explicitly $\rho^\alpha = \alpha^b(\rho,\bar\rho)(\gamma_b\bar\rho)^\alpha$ for some complex-valued $\alpha^b(\rho,\bar\rho)$, or equivalently $\bar\rho_\alpha = \bar\alpha^b(\gamma_b\rho)_\alpha$ and $(\rho\bar\rho) \neq 0$ (which implies with the preceding assumption that $\bar\alpha_b(\rho,\bar\rho)(\rho\gamma^b\rho) \neq 0$), then we have

$$\tfrac{1}{2}\zeta^a(\gamma_a\bar\rho)^\alpha = \tfrac{1}{2}\frac{\rho\gamma^a\rho}{\bar\alpha^c(\rho\gamma_c\rho)}(\underbrace{\gamma_a\gamma_b}_{-\gamma_b\gamma_a+2\delta_{ab}}\rho)^\alpha\bar\alpha^b = \tag{B.1}$$

$$\stackrel{\text{Fierz}}{=} \frac{(\rho\gamma_b\rho)\bar\alpha^b}{(\rho\gamma_c\rho)\bar\alpha^c}\rho^\alpha = \rho^\alpha \quad \surd \tag{B.2}$$

This shows that if ρ^α is 'proportional' to $\bar\rho_\alpha$ in the sense $\rho^\alpha = \alpha^b(\gamma_b\bar\rho)^\alpha$, then the expansion coefficients α^b can always be chosen to be

$$\alpha^b = \tfrac{1}{2}\zeta^b = \tfrac{\rho\gamma^b\rho}{2(\rho\bar\rho)} \tag{B.3}$$

and thus proves the second equivalence relation in (2.12).

In order to show the first equivalence relation in (2.12), we will show for the absolute value square of the difference $|\,\rho - \frac{1}{2}\zeta^a(\gamma_a\bar\rho)\,|^2 = 0 \iff \xi = 1$or $\rho\bar\rho = 0$:

$$|\,\rho - \tfrac{1}{2}\zeta^a(\gamma_a\bar\rho)\,|^2 = \left(\rho - \tfrac{1}{2}\zeta^a(\gamma_a\bar\rho)\right)\left(\bar\rho - \tfrac{1}{2}\bar\zeta^b(\gamma_b\rho)\right) = \tag{B.4}$$

$$= \rho\bar\rho - \tfrac{1}{2}\zeta^a(\bar\rho\gamma_a\bar\rho) - \tfrac{1}{2}\bar\zeta^a(\rho\gamma_a\rho) + \tfrac{1}{4}\zeta^a\bar\zeta^b(\bar\rho\underbrace{\gamma_a\gamma_b}_{-\gamma_b\gamma_a+2\delta_{ab}}\rho) = \tag{B.5}$$

$$= \rho\bar\rho\,(1-\xi) \tag{B.6}$$

This completes the proof of (2.12).

---

[29] The 10d Fierz identity that we will use most frequently is

$$\gamma_{a(\alpha\beta}\gamma^a_{\gamma)\delta} = 0 \quad \text{or } \gamma^{a(\alpha\beta}\gamma_a^{\gamma)\delta} = 0$$

So in particular

$$(\rho\gamma_a\rho)(\gamma^a\rho)_\alpha = 0$$

or, when we work with pure spinors λ:

$$(\gamma_a\lambda)_\alpha(\gamma^a\lambda)_\beta = \tfrac{1}{2}(\lambda\gamma_a\lambda)\gamma^a_{\alpha\beta} = 0$$

The less well known Fierz identity given in footnote 30 can be found in e.g. [17], equation (2.12). One can find how to derive this kind of chiral Fierz identities e.g. around page 179 of [29]. The first identity of above is there given in equation (D.160) (p.180), while a symmetrized version of the identity given in footnote 30 can there be found in equation (D.166), which would actually already be enough for our purposes here. Nevertheless, in order to get the full one of footnote 30, one can take there a linear combination of equations (D.154) on page 179 and (D.70) on page 174 (taking the chiral block with the index structure that matches the identity in footnote 30). ◇

In order to prove (2.11), we remember the definition $\xi = \frac{1}{2}\zeta^a\bar{\zeta}_a$ and thus

$$\mathbb{R} \ni \xi \geq 0 \quad \text{with } \xi = 0 \, \text{only if } \zeta^a = 0 \tag{B.7}$$

Now assume that $\xi > 1$ and consider the spinor $\frac{1}{2}\frac{\zeta^a(\gamma_a\bar{\rho})^\alpha}{1+i\sqrt{\xi-1}}$. The choice of this spinor is suggested by the appearance of such a term in the projector map (2.3) (with $\sqrt{-1} \equiv i$), but the following argument is independent from our motivation to look at this particular spinor. We claim that this spinor would be equal to ρ^α for every $\xi > 1$ by calculating the modulus squared of the difference:

$$\begin{aligned}
&\left|\rho^\alpha - \frac{1}{2}\frac{\zeta^a(\gamma_a\bar{\rho})^\alpha}{1+i\sqrt{\xi-1}}\right|^2 = \\
&\overset{\xi \geq 1}{=} \quad \left(\rho^\alpha - \frac{1}{2}\frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{1+i\sqrt{\xi-1}}\right)\left(\bar{\rho}_\alpha - \frac{1}{2}\frac{\bar{\zeta}_b(\gamma^b\rho)_\alpha}{1-i\sqrt{\xi-1}}\right) = && \text{(B.8)}\\
&= \quad (\rho\bar{\rho}) - \frac{1}{2}\frac{\bar{\zeta}_b(\rho\gamma^b\rho)}{1-i\sqrt{\xi-1}} - \frac{1}{2}\frac{\zeta^a(\bar{\rho}\gamma_a\bar{\rho})}{1+i\sqrt{\xi-1}} + \frac{1}{4}\frac{\zeta^a(\bar{\rho}\,\overbrace{\gamma_a\gamma^b}^{-\gamma^b\gamma_a+2\delta_a^b}\,\rho)\bar{\zeta}_b}{1+\xi-1} = && \text{(B.9)}\\
&\overset{(2.13)}{=} \quad (\rho\bar{\rho})\left(2 - \frac{\xi}{1-i\sqrt{\xi-1}} - \frac{\xi}{1+i\sqrt{\xi-1}}\right) = && \text{(B.10)}\\
&= \quad 0 && \text{(B.11)}
\end{aligned}$$

However, a few lines above we have seen that a linear combination of $(\gamma_a\bar{\rho})^\alpha$ can be equal to ρ if and only if $\xi = 1$ which disproves the assumption $\xi > 1$ and thus shows that[30]

$$\xi \leq 1 \tag{B.12}$$

The proof of the properties of the auxiliary variables is now complete. From now on let us stick to the order in the proposition and continue with its 1st statement. □

1. The second projection property (2.6) follows directly from (2.14) and the fact that we required $f(0) = 1$. What remains to show is that the spinor indeed

[30] This was a very indirect proof of $\xi \leq 1$. One might think that a more direct way is to use the following Fierz identity (see also footnote 29 on page 51):

$$4\gamma^a_{\delta\beta}\gamma_a^{\alpha\gamma} \quad = \quad 8\delta^\alpha_\delta\delta^\gamma_\beta + 2\delta^\alpha_\beta\delta^\gamma_\delta - \gamma^{ab\alpha}{}_\beta\gamma_{ba}{}^\gamma{}_\delta$$

Contracting it with two ρ's and two $\bar{\rho}$'s, we obtain

$$4(\rho\gamma^a\rho)(\bar{\rho}\gamma_a\bar{\rho}) \quad = \quad 10(\rho\bar{\rho})^2 - (\bar{\rho}\gamma^{ab}\rho)(\bar{\rho}\gamma_{ba}\rho) \leq 10(\rho\bar{\rho})^2$$

Strange enough, this shows only $\xi \leq \frac{5}{4}$ which is weaker than what we had before. The bound $\xi = \frac{5}{4}$ would be reached only for $(\bar{\rho}\gamma^{ab}\rho) = 0$. Together with the proven $\xi \leq 1$ this implies

$$(\bar{\rho}\gamma^{ab}\rho) = 0 \iff \rho\bar{\rho} = 0 \qquad \diamond$$

becomes pure after the mapping:

$$
\begin{aligned}
& P^{\alpha}_{(f)}(\rho,\bar\rho)\gamma^{c}_{\alpha\beta}P^{\beta}_{(f)}(\rho,\bar\rho) = \\
= \quad & f(\xi)^2\left(\rho^{\alpha}-\tfrac{1}{2}\frac{\zeta^{a}(\bar\rho\gamma_a)^{\alpha}}{1+\sqrt{1-\xi}}\right)\gamma^{c}_{\alpha\beta}\left(\rho^{\beta}-\tfrac{1}{2}\frac{\zeta^{b}(\bar\rho\gamma_b)^{\beta}}{1+\sqrt{1-\xi}}\right) = && \text{(B.13)}\\
= \quad & f(\xi)^2\left((\rho\gamma^c\rho)-\frac{\zeta^{a}(\rho\overbrace{\gamma^{c}\gamma_a}^{-\gamma_a\gamma^c+2\delta^c_a}\bar\rho)}{1+\sqrt{1-\xi}}+\tfrac{1}{4}\frac{\zeta^a\zeta^b(\bar\rho\gamma_a\overbrace{\gamma^c\gamma_b}^{-\gamma_b\gamma^c+2\delta^c_b}\bar\rho)}{\left(1+\sqrt{1-\xi}\right)^2}\right) = && \text{(B.14)}\\
\overset{(2.13)}{=} \quad & \frac{f(\xi)^2}{\rho\bar\rho}\left(\zeta^c-\frac{2\zeta^c}{1+\sqrt{1-\xi}}+\frac{\zeta^c\xi}{\left(1+\sqrt{1-\xi}\right)^2}\right)=0 \quad \surd && \text{(B.15)}
\end{aligned}
$$

This proves (2.6). □

2. The homogeneity (2.7) follows directly from the scaling behaviour (2.15) of the variables ξ and ζ^a. The modulus square of the projected spinor is

$$
\begin{aligned}
& P^{\alpha}_{(f)}(\rho,\bar\rho)\bar P_{(f)\alpha}(\rho,\bar\rho) = \\
= \quad & f(\xi)\,\bar f(\xi)\left(\rho^{\alpha}-\tfrac{1}{2}\frac{\zeta^{a}(\bar\rho\gamma_a)^{\alpha}}{1+\sqrt{1-\xi}}\right)\left(\bar\rho_{\alpha}-\tfrac{1}{2}\frac{\bar\zeta^{b}(\gamma_b\rho)_{\alpha}}{1+\sqrt{1-\xi}}\right) = && \text{(B.16)}\\
= \quad & \mid f(\xi)\mid^2\left((\rho\bar\rho)-\tfrac{1}{2}\tfrac{\bar\zeta^b(\rho\gamma_b\rho)}{1+\sqrt{1-\xi}}-\tfrac{1}{2}\tfrac{\zeta^a(\bar\rho\gamma_a\bar\rho)}{1+\sqrt{1-\xi}}+\tfrac{1}{4}\tfrac{\zeta^a(\bar\rho\overbrace{\gamma_a\gamma_b}^{-\gamma_b\gamma_a+2\delta_{ab}}\rho)\bar\zeta^b}{\left(1+\sqrt{1-\xi}\right)^2}\right) = && \text{(B.17)}\\
\overset{(2.13)}{=} \quad & \mid f(\xi)\mid^2(\rho\bar\rho)\left(1-\tfrac{1}{2}\tfrac{\bar\zeta^b\zeta_b}{1+\sqrt{1-\xi}}-\tfrac{1}{2}\tfrac{\zeta^a\bar\zeta_a}{1+\sqrt{1-\xi}}+\tfrac{1}{2}\tfrac{\zeta^a\bar\zeta_a}{\left(1+\sqrt{1-\xi}\right)^2}\right) = && \text{(B.18)}\\
= \quad & 2(\rho\bar\rho)\mid f(\xi)\mid^2\left(\frac{1-\xi}{1+\sqrt{1-\xi}}\right) && \text{(B.19)}
\end{aligned}
$$

This agrees with the claim in (2.8). At the origin $(\rho\bar\rho)=0$ the variable ξ is ill-defined, but when we assume $f(\xi)$ to be continuous on the whole interval $[0,1]$ then it will clearly be bounded and we have a well defined limit when approaching the origin:

$$
\begin{aligned}
& \lim_{|\rho|\to 0}P^{\alpha}_{(f)}(\rho,\bar\rho)\bar P_{(f)\alpha}(\rho,\bar\rho) = \\
= \quad & \lim_{|\rho|\to 0}2\mid\rho\mid^2\mid f(\xi)\mid^2\left(\frac{1-\xi}{1+\sqrt{1-\xi}}\right)\le && \text{(B.20)}\\
\overset{0\le\xi\le 1}{\le} \quad & \lim_{|\rho|\to 0}2\mid\rho\mid^2\mid f(\xi)\mid^2= && \text{(B.21)}\\
\overset{f(\xi)\text{bounded}}{=} \quad & 0 && \text{(B.22)}
\end{aligned}
$$

This proves (2.9). Note that the projection properties (2.5) and (2.6) also obviously hold in this limit. □

3. From the above result we see that the zero-vector is in the zero-locus of $P^{\alpha}_{(f)}$, at least if the latter is analytically continued to that point. Looking at the definition (2.3) of the projector in the remaining regime, it is obvious that it vanishes if and only if either $f(\xi)=0$ or the term in the bracket, i.e. $\rho^{\alpha}-\frac{1}{2}\frac{\zeta^a(\bar\rho\gamma_a)^{\alpha}}{1+\sqrt{1-\xi}}$ is zero. From (2.12) we know that this is the case if and only

if $\rho^\alpha = \frac{1}{2}\zeta^a(\bar\rho\gamma_a)^\alpha$ (or $\xi = 1$). This completes the proof of (2.10). Note that writing $(\rho\bar\rho)\rho^\alpha = \frac{1}{2}(\rho\gamma^a\rho)(\bar\rho\gamma_a)^\alpha$ instead of $\rho^\alpha = \frac{1}{2}\zeta^a(\bar\rho\gamma_a)^\alpha$ would include also the case $\rho^\alpha = 0$ into this set. Now let us assume that ρ^α is real, which is a non-covariant statement:

$$\underbrace{\bar\rho_\alpha}_{(\rho^\alpha)^*} = \rho^\alpha \quad \text{(assumption)} \tag{B.23}$$

Remember from footnote 2 on page 4 that we have $\gamma^{10\alpha\beta} = -\gamma^{10}_{\alpha\beta} = i\delta_{\alpha\beta}$ ($\gamma^0_{\alpha\beta} = -\gamma^{0\alpha\beta} = \gamma_0^{\alpha\beta} = -\delta_{\alpha\beta}$). Using the above assumption for the following expression, we thus obtain

$$\frac{\zeta^a}{2}(\gamma_a\bar\rho)^\alpha = \sum_{i=1}^{9}\frac{(\rho\gamma^i\rho)}{2}\underbrace{(\gamma_i\bar\rho)^\alpha}_{(\gamma_i\rho)_\alpha} + \frac{(\rho^\gamma\gamma^{10}_{\gamma\delta}\rho^\delta)}{2(\rho\bar\rho)}(\underbrace{\gamma^{\alpha\beta}_{10}}_{-\gamma^{10}_{\alpha\beta}}\underbrace{\bar\rho_\beta}_{\rho^\beta}) = \tag{B.24}$$

$$\stackrel{(B.23)}{=} \frac{(\rho\gamma^a\rho)}{2(\rho\rho)}(\gamma_a\rho)^\alpha - \frac{(\rho^\gamma\overbrace{\gamma^{10}_{\gamma\delta}}^{-i\delta_{\gamma\delta}}\rho^\delta)}{(\rho\rho)}(\underbrace{\gamma^{10}_{\alpha\beta}}_{-i\delta_{\alpha\beta}}\rho^\beta) = \tag{B.25}$$

$$= \rho^\alpha \tag{B.26}$$

From what we have proved before, this shows that $\xi = 1$ if ρ is real.[31] The complete zero-locus is SO(10) covariant. Any SO(10) rotation of a real spinor (the result of the rotation is in general not real any longer) thus also has to lie in the zero-locus. The corresponding (infinitesimally) rotated reality condition would be

$$(L_{ab}\gamma^{ab}{}_\alpha{}^\beta\bar\rho_\beta) = (L_{ab}\gamma^{ab\,\alpha}{}_\beta\rho^\beta)$$

with any antisymmetric parameter L_{ab}. □

4. Continuity is obvious. If f is differentiable, then there are only two possible problems for differentiability of the projector. One is at $\rho^\alpha = 0$, where ξ and ζ^a are not well defined. And the other is at $\xi = 1$, as the square root is not differentiable at 0. We will later study the variations (and thus the derivatives) of the projector and will see in footnote 33 that they have a pole for $|\rho| \to 0$ and also in general for $\xi = 1$, but that the latter can be avoided by choosing $f(\xi) = \tilde f(\xi)\,(1-\xi)^{1+r}$ with $\tilde f$ differentiable everywhere. This will complete the proof of proposition 1. □

[31] Note that the same calculation (B.24)-(B.26) goes through if ρ^α is only almost real in the sense $\bar\rho_\alpha = c\rho^\alpha$ for some $c \in \mathbb{C}$ (implying that ρ^α is a complex multiple of some real spinor $\tilde\rho^\alpha$, i.e. $\rho^\alpha = \tilde c\tilde\rho^\alpha, \quad \tilde c \in \mathbb{C}$). Also in this case $\xi = 1$. ◇

B.2 Proof of proposition 2

Proof of Proposition 2. Remember $\zeta^a = \frac{\rho\gamma^a\rho}{\rho\bar\rho}$, $\xi = \frac{1}{2}\zeta^a\bar\zeta_a$ of equation (2.4). Their derivatives read[32]

$$\partial_{\rho^\beta}\zeta^a = \frac{2(\gamma^a\rho)_\beta - \zeta^a\bar\rho_\beta}{\rho\bar\rho} \tag{B.27}$$

$$\partial_{\bar\rho_\beta}\zeta^a = -\frac{\zeta^a\rho^\beta}{\rho\bar\rho} \tag{B.28}$$

$$\partial_{\rho^\beta}\xi = \tfrac{1}{2}\partial_{\rho^\beta}\zeta^a\bar\zeta_a + \tfrac{1}{2}\zeta^a\partial_{\rho^\beta}\bar\zeta_a = \frac{\bar\zeta_a(\gamma^a\rho)_\beta - 2\xi\bar\rho_\beta}{\rho\bar\rho} \tag{B.29}$$

1. Now we can apply the ρ-derivative to the projector (2.3):

$$\begin{aligned}
&\partial_{\rho^\beta}P^\alpha_{(f)}(\rho,\bar\rho) = \\
&= \partial_{\rho^\beta}\xi\cdot f'(\xi)\left(\rho^\alpha - \tfrac{1}{2}\frac{\zeta^a(\bar\rho\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) + \\
&\quad + f(\xi)\left(\delta^\alpha_\beta - \tfrac{1}{2}\frac{\partial_{\rho^\beta}\zeta^a(\bar\rho\gamma_a)^\alpha}{1+\sqrt{1-\xi}} - \tfrac{1}{2}\frac{\partial_{\rho^\beta}\xi\cdot\zeta^a(\bar\rho\gamma_a)^\alpha}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2}\right) = \qquad \text{(B.30)} \\
&\overset{\text{(B.27)(B.29)}}{=} f(\xi)\,\delta^\alpha_\beta + f'(\xi)\,\frac{\bar\zeta_a\rho^\alpha(\gamma^a\rho)_\beta}{\rho\bar\rho} - 2f'(\xi)\,\xi\frac{\rho^\alpha\bar\rho_\beta}{\rho\bar\rho} + \\
&\quad -\left(\frac{f(\xi)}{1+\sqrt{1-\xi}}\delta^a_b + \frac{f(\xi)\,\zeta^a\bar\zeta_b}{4\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2} + \tfrac{1}{2}\frac{f'(\xi)\,\zeta^a\bar\zeta_b}{1+\sqrt{1-\xi}}\right)\frac{(\gamma_a\bar\rho)^\alpha(\rho\gamma^b)_\beta}{\rho\bar\rho} + \\
&\quad +\left(\frac{f'(\xi)\,\xi}{1+\sqrt{1-\xi}} + \frac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)} + \frac{f(\xi)\,\xi}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2}\right)\frac{\zeta^a(\bar\rho\gamma_a)^\alpha\bar\rho_\beta}{\rho\bar\rho} \qquad \text{(B.31)}
\end{aligned}$$

After a slight rewriting this agrees with (3.4). The derivative with respect to

[32] Using (2.18), the derivatives of the auxiliary variables can also be written in terms of $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar\rho)$:

$$\begin{aligned}
\partial_{\rho^\beta}\zeta^a &= \frac{\sqrt{1-\xi}}{(\lambda\bar\lambda)}\Big\{2\bar f(\xi)\,(\lambda\gamma^a)_\beta - \frac{\bar f(\xi)}{2\left(1+\sqrt{1-\xi}\right)}\zeta^a\bar\zeta_b\left(\lambda\gamma^b\right)_\beta + \\
&\quad - \frac{f(\xi)}{1+\sqrt{1-\xi}}\zeta^b\left(\gamma_b\gamma^a\bar\lambda\right)_\beta + \frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}\zeta^a f(\xi)\bar\lambda_\beta\Big\} \\
\partial_{\bar\rho_\beta}\zeta^a &= -\zeta^a\frac{\sqrt{1-\xi}}{(\lambda\bar\lambda)}\Big\{\bar f(\xi)\lambda^\beta + \frac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)}\zeta^b\left(\bar\lambda\gamma_b\right)^\beta\Big\} \\
\partial_{\rho^\beta}\xi &= \frac{(1-\xi)}{(\lambda\bar\lambda)}\Big\{\bar f(\xi)\bar\zeta_c\left(\gamma^c\lambda\right)_\beta - \frac{2\xi}{1+\sqrt{1-\xi}}f(\xi)\bar\lambda_\beta\Big\}
\end{aligned}$$

Written in this form it is particularly easy to see that on the subspace where $\xi = 1$ (basically the zero-locus of the projection), the ρ-derivative of ξ vanishes

$$\partial_{\rho^\beta}\xi\Big|_{\xi=1} = 0$$

while at the constraint surface (where $\xi = \zeta^a = \bar\zeta_a = 0$) we have even

$$\partial_{\rho^\beta}\xi\Big|_{\xi=\zeta^a=0} = \partial_{\bar\rho_\beta}\zeta^a\Big|_{\xi=\zeta^a=0} = 0, \quad \partial_{\rho^\beta}\zeta^a\Big|_{\xi=\zeta^a=0} = \frac{2\,(\lambda\gamma^a)_\beta}{(\lambda\bar\lambda)}$$

So in both subspaces the ρ-derivative of ξ vanishes which means that ξ is stationary (so e.g. extremal) at these values. This agrees with the observation that $0 \le \xi \le 1$. $\diamond$

$\bar{\rho}_\beta$ instead has the form

$$\begin{aligned}
&\partial_{\bar{\rho}_\beta} P^\alpha_{(f)}(\rho,\bar{\rho}) = \\
= \quad & \partial_{\bar{\rho}_\beta}\xi \cdot f'(\xi)\left(\rho^\alpha - \tfrac{1}{2}\frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) + \\
&+ f(\xi)\left(-\tfrac{1}{2}\frac{\partial_{\bar{\rho}_\beta}\zeta^a(\bar{\rho}\gamma_a)^\alpha}{1+\sqrt{1-\xi}} - \tfrac{1}{2}\frac{\zeta^a\gamma_a^{\alpha\beta}}{1+\sqrt{1-\xi}} - \tfrac{1}{2}\frac{\partial_{\bar{\rho}_\beta}\xi\zeta^a(\bar{\rho}\gamma_a)^\alpha}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2}\right) = \qquad \text{(B.32)} \\
\overset{\text{(B.28)(B.29)}}{=} \quad & \frac{\zeta^a(\gamma_a\bar{\rho})^\beta - 2\xi\rho^\beta}{\rho\bar{\rho}} \times \\
&\times \left(f'(\xi)\left(\rho^\alpha - \tfrac{1}{2}\frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{1+\sqrt{1-\xi}}\right) - \tfrac{1}{2}f(\xi)\frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)^2}\right) + \\
&+ f(\xi)\left(-\tfrac{1}{2}\frac{-\frac{\zeta^a\rho^\beta}{\rho\bar{\rho}}(\bar{\rho}\gamma_a)^\alpha}{1+\sqrt{1-\xi}} - \tfrac{1}{2}\frac{\zeta^a\gamma_a^{\alpha\beta}}{1+\sqrt{1-\xi}}\right) \qquad \text{(B.33)}
\end{aligned}$$

Collecting the terms, one arrives at (3.6)[33]. □

[33] Looking at the denominators of (3.4) and (3.6) it becomes clear that the only possible poles of $\Pi_{(f)\perp}$ and $\pi_{(f)\perp}$ are either at $(\rho\bar{\rho}) = 0$, at $\xi = 1$ or at some poles of f itself. This shows that $P^\alpha_{(f)}$ is differentiable everywhere but at $\{0\} \cup \{\rho^\alpha | \xi = 1\}$ if f is everywhere differentiable. This was one of the statements of the 4th point of proposition 1. Furthermore the singularities $\frac{1}{\sqrt{1-\xi}}$ come with $f(\xi)$ and could be removed by $f(\xi) \propto \sqrt{1-\xi}$ which, however, would introduce new singularities in other terms because of $f'(\xi) \propto \frac{1}{\sqrt{1-\xi}}$. A save choice instead would be

$$f(\xi) = \tilde{f}(\xi)(1-\xi)^{1+r}, \quad \tilde{f}(0) = 1, \quad r \geq 0 \qquad \text{(B.34)}$$

with $\tilde{f}$ differentiable everywhere. The exponent $(1+r)$ guarantees that the derivative

$$f'(\xi) = \tilde{f}'(\xi)(1-\xi)^{1+r} - (1+r)\tilde{f}(\xi)(1-\xi)^r \qquad \text{(B.35)}$$

does not have singularities, while at the same time the factor $(1-\xi)^{1+r}$ will cure the singularities of the form $\frac{1}{\sqrt{1-\xi}}$ which come along with f only. This completes the proof of point 4 of proposition 1. ◇

2. Using (2.18) and (2.19), we can rewrite the the linear projection matrix (3.4) in terms of λ^α and its complex conjugate:

$$
\begin{aligned}
&\Pi^\alpha_{(f)\perp\beta}(\rho,\bar\rho) = f\,(\xi)\,\delta^\alpha_\beta + \\
&+f'(\xi)\frac{\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\alpha+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^c\left(\bar\lambda\gamma_c\right)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\bar\zeta_b(\lambda\gamma^b)_\beta+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^d\bar\zeta_b(\bar\lambda\overbrace{\gamma_d\gamma^b}^{-\gamma^b\gamma_d+2\delta^b_d})_\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)}+ \\
&-2\xi f'(\xi)\frac{\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\alpha+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^c\left(\bar\lambda\gamma_c\right)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}\bar\lambda_\beta+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d\left(\lambda\gamma^d\right)_\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)}+ \\
&-\frac{1}{1+\sqrt{1-\xi}}\left(f\,(\xi)\,\delta^a_b+\left(\frac{f(\xi)}{4\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+\frac{1}{2}f'(\xi)\right)\zeta^a\bar\zeta_b\right)\times \\
&\times\frac{\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}(\gamma_a\bar\lambda)^\alpha+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_c(\overbrace{\gamma_a\gamma^c}^{-\gamma^c\gamma_a+2\delta^c_a}\lambda)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}(\lambda\gamma^b)_\beta+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^d(\bar\lambda\overbrace{\gamma_d\gamma^b}^{-\gamma^b\gamma_d+2\delta^b_d})_\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)}+ \\
&+\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+\left(1-\sqrt{1-\xi}\right)f'\,(\xi)\right)\times \\
&\times\frac{\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}\zeta^a(\gamma_a\bar\lambda)^\alpha+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_c\zeta^a(\overbrace{\gamma_a\gamma^c}^{-\gamma^c\gamma_a+2\delta^c_a}\lambda)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}\bar\lambda_\beta+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d(\lambda\gamma^d)_\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)}
\end{aligned}
\tag{B.36}
$$

Changing the order of the gamma-matrices as indicated above the curly brackets makes it for some of the terms possible to use Fierz identities of the form $(\gamma^a\lambda)_\alpha(\gamma_a\lambda)_\beta \propto (\lambda\gamma_a\lambda)\gamma^a_{\alpha\beta} = 0$ or $\zeta^a(\gamma_a\lambda)_\alpha = 0$ (2.17) and their complex conjugate counterparts. However, a few terms will arise where these identities will not be applicable. Nevertheless the change of order in the gamma-matrices is advantageous, as a multiplication by a λ from the right or a $\bar\lambda$ from the left will trigger these identities. So after replacing the products of γ-matrices by the expressions above the curly brackets, using the just mentioned identities wherever possible, and sorting the terms by collecting first those of the form $\lambda\otimes\lambda$, then

$\lambda \otimes \bar{\lambda}$ and so on, we obtain (before simplifying)

$$
\begin{aligned}
&\Pi^{\alpha}_{(f)\perp\beta}(\rho,\bar{\rho}) = f(\xi)\,\delta^{\alpha}_{\beta} + \\
&+\Bigg\{\tfrac{\left(1+\sqrt{1-\xi}\right)f'(\xi)\bar{f}(\xi)}{2f(\xi)} - \tfrac{\xi f'(\xi)\bar{f}(\xi)}{2f(\xi)} + \\
&-\tfrac{\bar{f}(\xi)}{2f(\xi)\left(1+\sqrt{1-\xi}\right)}\left(f(\xi)+\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+f'(\xi)\right)\xi\right) + \\
&+\tfrac{\xi\bar{f}(\xi)}{2\left(1+\sqrt{1-\xi}\right)^{2}f(\xi)}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}}+\xi f'(\xi)\right)\Bigg\}\frac{\lambda^{\alpha}\bar{\zeta}_{b}(\lambda\gamma^{b})_{\beta}}{(\lambda\bar{\lambda})} \\
&+\Bigg\{\xi f'(\xi) - \xi f'(\xi)\left(1+\sqrt{1-\xi}\right) - \tfrac{\xi}{\left(1+\sqrt{1-\xi}\right)^{2}}\left(f(\xi)+\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+f'(\xi)\right)\xi\right) + \\
&+\tfrac{\xi}{\left(1+\sqrt{1-\xi}\right)}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}}+\xi f'(\xi)\right)\Bigg\}\frac{\lambda^{\alpha}\bar{\lambda}_{\beta}}{(\lambda\bar{\lambda})} \\
&-\tfrac{f(\xi)}{8\left(1+\sqrt{1-\xi}\right)^{2}}\frac{\bar{\zeta}_{c}(\gamma^{c}\gamma_{b}\lambda)^{\alpha}\zeta^{d}(\bar{\lambda}\gamma^{b}\gamma_{d})_{\beta}}{(\lambda\bar{\lambda})} + \\
&+\Bigg\{\tfrac{1}{4}f'(\xi) - \tfrac{\xi f'(\xi)}{4\left(1+\sqrt{1-\xi}\right)} - \tfrac{1}{2}\left(\tfrac{f(\xi)}{4\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+\tfrac{1}{2}f'(\xi)\right) + \\
&+\tfrac{1}{4\left(1+\sqrt{1-\xi}\right)}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}}+\xi f'(\xi)\right)\Bigg\}\frac{\zeta^{a}\left(\bar{\lambda}\gamma_{a}\right)^{\alpha}\bar{\zeta}_{b}(\lambda\gamma^{b})_{\beta}}{(\lambda\bar{\lambda})} \\
&-\tfrac{1}{2}f(\xi)\frac{(\gamma_{a}\bar{\lambda})^{\alpha}(\lambda\gamma^{a})_{\beta}}{(\lambda\bar{\lambda})} + \\
&+\Bigg\{\tfrac{f'(\xi)f(\xi)\xi}{2\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)} - \tfrac{\xi f'(\xi)f(\xi)}{2f(\xi)} + \\
&-\tfrac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)\bar{f}(\xi)}\left(f(\xi)+\left(\tfrac{f(\xi)}{4\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+\tfrac{1}{2}f'(\xi)\right)2\xi\right) + \\
&+\tfrac{f(\xi)}{2\bar{f}(\xi)}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}}+\xi f'(\xi)\right)\Bigg\}\frac{\zeta^{a}\left(\gamma_{a}\bar{\lambda}\right)^{\alpha}\bar{\lambda}_{\beta}}{(\lambda\bar{\lambda})}
\end{aligned}
\tag{B.37}
$$

In a final effort we sort the terms in the curly brackets to terms that have an f' and those that have not and simplify the expressions. It turns out that many terms cancel, in particular the curly bracket before $\frac{\zeta^{a}\left(\gamma_{a}\bar{\lambda}\right)^{\alpha}\bar{\zeta}_{b}(\lambda\gamma^{b})_{\beta}}{(\lambda\bar{\lambda})}$ vanishes completely, as well as the one before $\frac{\zeta^{a}(\gamma_{a}\bar{\lambda})^{\alpha}\bar{\lambda}_{\beta}}{(\lambda\bar{\lambda})}$ and one ends up with the expression in (3.7) of the proposition.

Next we plug (2.18) and (2.19) into (3.6):

$$
\begin{aligned}
\pi^{\alpha\beta}_{(f)\perp}(\rho,\bar\rho) = & -\tfrac{1}{2}\tfrac{f(\xi)}{1+\sqrt{1-\xi}}\zeta^a\gamma_a^{\alpha\beta} + \\
& -2\xi f'(\xi)\,\frac{\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\alpha+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^a\left(\gamma_a\bar\lambda\right)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\beta+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^b\left(\bar\lambda\gamma_b\right)^\alpha\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)} + \\
& +f'(\xi)\,\frac{\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\alpha+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^a\left(\gamma_a\bar\lambda\right)^\alpha\right)\zeta^b\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}(\bar\lambda\gamma_b)^\beta+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d(\lambda\overbrace{\gamma^d\gamma_b}^{-\gamma_b\gamma^d+2\delta_b^d})^\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)} + \\
& +\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+\left(1-\sqrt{1-\xi}\right)f'(\xi)\right)\times \\
& \times\frac{\zeta^a\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}(\gamma_a\bar\lambda)^\alpha+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d(\overbrace{\gamma_a\gamma^d}^{-\gamma^d\gamma_a+2\delta_a^d}\lambda)^\alpha\right)\left(\frac{1+\sqrt{1-\xi}}{2f(\xi)\sqrt{1-\xi}}\lambda^\beta+\frac{1}{4\bar f(\xi)\sqrt{1-\xi}}\zeta^b(\bar\lambda\gamma_b)^\alpha\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)} + \\
& -\frac{1}{2\left(1+\sqrt{1-\xi}\right)}\left(\frac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}+f'(\xi)\right)\times \\
& \times\frac{\zeta^a\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}(\gamma_a\bar\lambda)^\alpha+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d(\overbrace{\gamma_a\gamma^d}^{-\gamma^d\gamma_a+2\delta_a^d}\lambda)^\alpha\right)\zeta^b\left(\frac{1+\sqrt{1-\xi}}{2\bar f(\xi)\sqrt{1-\xi}}(\bar\lambda\gamma_b)^\beta+\frac{1}{4f(\xi)\sqrt{1-\xi}}\bar\zeta_d(\lambda\overbrace{\gamma^d\gamma_b}^{-\gamma_b\gamma^d+2\delta_b^d})^\beta\right)}{\frac{\left(1+\sqrt{1-\xi}\right)}{2|f(\xi)|^2(1-\xi)}(\lambda\bar\lambda)}
\end{aligned}
\tag{B.38}
$$

Again we change the order of the gamma-matrices as indicated above the curly brackets for the same reason as before and use Fierz identities of the form $(\gamma^a\lambda)_\alpha(\gamma_a\lambda)_\beta \propto (\lambda\gamma_a\lambda)\gamma^a_{\alpha\beta} = 0$ (footnote 29) or $\zeta^a(\gamma_a\lambda)_\alpha = 0$ (2.17) and their complex conjugate counterparts. Then we sort the terms by collecting first those of the form $\lambda \otimes \lambda$, then $\lambda \otimes \bar{\lambda}$ and so on, and obtain (again before simplifying)

$$
\begin{aligned}
&\pi^{\alpha\beta}_{(f)\perp}(\rho,\bar{\rho}) = \\
&= \Bigg\{ -\tfrac{\xi f'(\xi)\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)}{f(\xi)} + \tfrac{f'(\xi)\bar{f}(\xi)\xi}{f(\xi)} + \tfrac{\xi \bar{f}(\xi)}{f(\xi)\left(1+\sqrt{1-\xi}\right)} \left(\xi f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}}\right) + \\
&\quad - \tfrac{\xi^2 \bar{f}(\xi)}{f(\xi)\left(1+\sqrt{1-\xi}\right)^2} \left(f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\right)\Bigg\} \frac{\lambda^\alpha \lambda^\beta}{(\lambda\bar{\lambda})} + \\
&\quad \Bigg\{ -\frac{1}{2}\xi f'(\xi) + \tfrac{f'(\xi)\left(1+\sqrt{1-\xi}\right)}{2} + \tfrac{\xi}{2\left(1+\sqrt{1-\xi}\right)^2} \left(\xi f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}}\right) + \\
&\quad - \tfrac{\xi}{2\left(1+\sqrt{1-\xi}\right)} \left(f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\right)\Bigg\} \frac{\lambda^\alpha \zeta^b \left(\bar{\lambda}\gamma_b\right)^\alpha}{(\lambda\bar{\lambda})} + \\
&\quad \Bigg\{ -\frac{1}{2}\xi f'(\xi) + \tfrac{\xi f'(\xi)}{2\left(1+\sqrt{1-\xi}\right)} + \tfrac{1}{2}\left(\xi f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}}\right) + \\
&\quad - \tfrac{\xi}{2\left(1+\sqrt{1-\xi}\right)} \left(f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\right)\Bigg\} \frac{\zeta^a \left(\gamma_a\bar{\lambda}\right)^\alpha \lambda^\beta}{(\lambda\bar{\lambda})} + \\
&\quad \Bigg\{ -\tfrac{\xi f'(\xi) f(\xi)}{4\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)} + \tfrac{f(\xi)f'(\xi)}{4\bar{f}(\xi)} + \tfrac{f(\xi)}{4\bar{f}(\xi)\left(1+\sqrt{1-\xi}\right)} \left(\xi f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}}\right) + \\
&\quad - \tfrac{f(\xi)}{4\bar{f}(\xi)} \left(f'(\xi) + \tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)}\right)\Bigg\} \frac{\zeta^a \left(\gamma_a\bar{\lambda}\right)^\alpha \zeta^b \left(\bar{\lambda}\gamma_b\right)^\alpha}{(\lambda\bar{\lambda})} + \\
&\quad - \tfrac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)} \zeta^a \gamma_a^{\alpha\beta}
\end{aligned}
\tag{B.39}
$$

Again, after collecting within the curly brackets the terms that contain f' and those that don't, and after simplifying, one arrives indeed at (3.8). $\square$

3. The fact that $\Pi_{(f)\perp}(\rho,\bar{\rho})$ and $\pi_{(f)\perp}(\rho,\bar{\rho})$ seen as linear endomorphisms map to spinors which are γ-orthogonal to $\lambda^\alpha \equiv P^\alpha_{(f)}(\rho,\bar{\rho})$ as claimed in (3.9) and (3.10) follows directly from the fact that $P^\alpha_{(f)}(\rho,\bar{\rho})$ is a pure spinor according to (2.5). I.e. the variation of (2.5) yields

$$
\begin{aligned}
0 &= \delta\left(P_{(f)}(\rho,\bar{\rho})\gamma^m P_{(f)}(\rho,\bar{\rho})\right) = && \text{(B.40)}\\
&= 2\delta\rho^\beta \partial_{\rho^\beta} P^\alpha_{(f)}(\rho,\bar{\rho}) \left(\gamma^m P_{(f)}(\rho,\bar{\rho})\right)_\alpha + \\
&\quad +2\delta\bar{\rho}_\beta \partial_{\bar{\rho}_\beta} P^\alpha_{(f)}(\rho,\bar{\rho}) \left(\gamma^m P_{(f)}(\rho,\bar{\rho})\right)_\alpha && \text{(B.41)}
\end{aligned}
$$

For this to hold for all variations $\delta\rho, \delta\bar{\rho}$, we necessarily need (3.9) and (3.10). One can also easily verify these equations directly by using the form (3.7) and (3.8) of the projector-matrices. $\square$

4. On the constraint surface of spinors $\rho^\alpha = \lambda^\alpha$ with $\lambda\gamma^m\lambda = 0$ we have $\zeta^a = 0$ and $\xi = 0$ (2.14). Plugging this into either (3.4), (3.6) or (3.7) and (3.8) and assuming that f and f' are non-singular at $\xi = 0$, one obtains immediately the claimed results (3.11) and (3.12). The remaining matrix on the constraint surface $\Pi_\perp \equiv \mathbb{1} - \frac{1}{2}\frac{(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^a)}{(\lambda\bar{\lambda})}$ finally is indeed idempotent:

$$
\begin{aligned}
\Pi_\perp^2 &= \mathbb{1} - \frac{(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^a)}{(\lambda\bar{\lambda})} + \tfrac{1}{4}\frac{(\lambda \overbrace{\gamma^a\gamma_b}^{-\gamma_b\gamma^a+2\delta^a_b} \bar{\lambda})(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^b)}{(\lambda\bar{\lambda})^2} = && \text{(B.42)}\\
&\overset{\text{Fierz}}{=} \mathbb{1} - \tfrac{1}{2}\frac{(\gamma_a\bar{\lambda})\otimes(\lambda\gamma^a)}{(\lambda\bar{\lambda})} = \Pi_\perp && \text{(B.43)}
\end{aligned}
$$

This last property is certainly already well-known, as $\Pi_\perp$ is the transposed of the projection matrix that maps the antighost $\omega_{z\alpha}$ to its gauge invariant part ([16, 18, 17]). $\Box$

5. The projection-properties (3.14) and (3.15) directly follow from derivatives of the projection property $P_{(f)} = P_{(f)} \circ P_{(f)}$ (2.6) :

$$\begin{aligned}
\partial_{\rho^\beta} P^\alpha_{(f)}(\rho,\bar\rho) &= \partial_{\rho^\beta} P^\alpha_{(f)}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right) = && \text{(B.44)}\\
&= \partial_{\rho^\beta} P^\gamma_{(f)}(\rho,\bar\rho)\ \partial_{\tilde\rho^\gamma} P^\alpha_{(f)}(\tilde\rho,\bar{\tilde\rho})\Big|_{\tilde\rho = P_{(f)}(\rho,\bar\rho)} + \\
&\quad + \partial_{\rho^\beta} \bar P_{(f)\perp\gamma}(\rho,\bar\rho)\ \partial_{\bar{\tilde\rho}_\gamma} P^\alpha_{(f)}(\tilde\rho,\bar{\tilde\rho})\Big|_{\tilde\rho = P_{(f)}(\rho,\bar\rho)} && \text{(B.45)}\\
&= \Pi^\alpha_{(f)\perp\gamma}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right) \Pi^\gamma_{(f)\perp\beta}(\rho,\bar\rho) + \\
&\quad + \underbrace{\pi^{\alpha\gamma}_{(f)\perp}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right)}_{=0 \quad (3.12)} \bar\pi_{(f)\perp\gamma\beta}(\rho,\bar\rho) && \text{(B.46)}
\end{aligned}$$

The last term vanishes (as indicated) according to (3.12). This then proves (3.14). In the same way we can write

$$\begin{aligned}
\partial_{\bar\rho_\beta} P^\alpha_{(f)}(\rho,\bar\rho) &= \partial_{\bar\rho_\beta} P^\alpha_{(f)}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right) = && \text{(B.47)}\\
&= \partial_{\bar\rho_\beta} P^\gamma_{(f)}(\rho,\bar\rho)\ \partial_{\tilde\rho^\gamma} P^\alpha_{(f)}(\tilde\rho,\bar{\tilde\rho})\Big|_{\tilde\rho = P_{(f)}(\rho,\bar\rho)} + \\
&\quad + \partial_{\bar\rho_\beta} \bar P_{(f)\perp\gamma}(\rho,\bar\rho)\ \partial_{\bar{\tilde\rho}_\gamma} P^\alpha_{(f)}(\tilde\rho,\bar{\tilde\rho})\Big|_{\tilde\rho = P_{(f)}(\rho,\bar\rho)} = && \text{(B.48)}\\
&= \Pi^\alpha_{(f)\perp\gamma}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right) \pi^{\gamma\beta}_{(f)\perp}(\rho,\bar\rho) + \\
&\quad + \underbrace{\pi^{\alpha\gamma}_{(f)\perp}\left(P_{(f)}(\rho,\bar\rho), \bar P_{(f)}(\rho,\bar\rho)\right)}_{=0 \quad (3.12)} \bar\Pi_{(f)\perp\gamma}{}^\beta(\rho,\bar\rho) && \text{(B.49)}
\end{aligned}$$

which proves (3.15). $\Box$

6. The trace of (3.4) is given by:

$$\begin{aligned}
&\Pi^\alpha_{(f)\perp\alpha}(\rho,\bar\rho) = \\
&= \quad 16 f(\xi) + f'(\xi)\frac{(\rho\gamma^b\rho)\bar\zeta_b}{\rho\bar\rho} - 2\xi f'(\xi) + \\
&\quad - \tfrac{1}{1+\sqrt{1-\xi}}\left(f(\xi)\,\delta^a_b + \tfrac{1}{2}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} + f'(\xi)\right)\zeta^a\bar\zeta_b\right)\frac{(\rho\,\overbrace{\gamma^b\gamma_a}^{-\gamma_a\gamma^b+2\delta^b_a}\,\bar\rho)}{\rho\bar\rho} + \\
&\quad + \tfrac{1}{1+\sqrt{1-\xi}}\left(\tfrac{f(\xi)}{2\sqrt{1-\xi}} + \xi f'(\xi)\right)\frac{\zeta^a(\bar\rho\gamma_a\bar\rho)}{\rho\bar\rho} = && \text{(B.50)}\\
&\overset{(2.13)}{=} \quad 16 f(\xi) + \\
&\quad - \tfrac{1}{1+\sqrt{1-\xi}}\left(f(\xi)\left(10 + \tfrac{\xi}{\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)} - \tfrac{\xi}{\sqrt{1-\xi}}\right) - 2\xi(\xi-1)f'(\xi)\right) = && \text{(B.51)}\\
&= \quad \left(16 - \tfrac{9+\sqrt{1-\xi}}{1+\sqrt{1-\xi}}\right) f(\xi) + \tfrac{2\xi(\xi-1)}{1+\sqrt{1-\xi}} f'(\xi) = && \text{(B.52)}\\
&= \quad \left(11 - \tfrac{4(1-\sqrt{1-\xi})}{1+\sqrt{1-\xi}}\right) f(\xi) - 2(1-\xi)\left(1-\sqrt{1-\xi}\right) f'(\xi) && \text{(B.53)}
\end{aligned}$$

It as a nice consistency check that the same result is obtained by taking the trace of (3.7). Note that this result is in general[34] not equal to 11 for $\xi \neq 0$. For $\xi = 0$ it clearly reduces to 11 for all f. This completes the proof of proposition 2. □

B.3 Proof of proposition 3

Proof of proposition 3. 1. From the explicit form of $\Pi_{(h)\perp}$ and $\pi_{(h)\perp}$ in (5.15) and (5.16), it is obvious that $\Pi_{(h)\perp}$ is Hermitian and $\pi_{(h)\perp}$ is symmetric. So what remains to show that h is the only function for which this is the case.

To do so, let us assume that f is any function defined and differentiable at least on $I = [0, b[$ (with $b \leq 1$) for which we have Hermiticity in this region (i.e. for all ρ^α for which $\xi \in I$). So starting from (3.7), we build the difference of

[34] For the toy model in the appendix, the choice $f = 1$ yields a proper projection matrix also off the constraint surface (see (A.58) on page 43)), whose trace is the same as on the surface. This is here not the case for $f = 1$. In order to find an f such that $\Pi^2_{(f)\perp} = \Pi_{(f)\perp}$ even off the surface, we would need at least that $\operatorname{tr}\Pi_{(f)\perp} = 11$ off the surface which gives a differential equation

$$\left(11 - \tfrac{4(1-\sqrt{1-\xi})}{1+\sqrt{1-\xi}}\right) f(\xi) - 2(1-\xi)\left(1 - \sqrt{1-\xi}\right) f'(\xi) \stackrel{!}{=} 11$$

This differential equation is slightly easier to solve in the parametrization $\tilde{\xi}$ of equation (2.23) where it turns into

$$(11 - 4\tilde{\xi})\tilde{f}(\tilde{\xi}) - (1-\tilde{\xi})\tilde{\xi}\tilde{f}'(\tilde{\xi}) \stackrel{!}{=} 11$$

The homogeneous equation $\frac{\tilde{f}'(\tilde{\xi})}{\tilde{f}(\tilde{\xi})} = \frac{(11-4\tilde{\xi})}{(1-\tilde{\xi})\tilde{\xi}}$ is solved by $\tilde{f}(\tilde{\xi}) = C \cdot \frac{\tilde{\xi}^{11}}{(1-\tilde{\xi})^7}$ with some constant C. In order to solve the inhomogeneous equation, one can promote C to a function $C(\tilde{\xi})$. Plugging the ansatz

$$\tilde{f}(\tilde{\xi}) = C(\tilde{\xi}) \cdot \tfrac{\tilde{\xi}^{11}}{(1-\tilde{\xi})^7}$$

back into the differential equation yields $C'(\tilde{\xi}) = -\frac{11(1-\tilde{\xi})^6}{\tilde{\xi}^{12}}$ which can be integrated to

$$C(\tilde{\xi}) = \frac{1}{210\tilde{\xi}^5} - \frac{1-\tilde{\xi}}{42\tilde{\xi}^6} + \frac{(1-\tilde{\xi})^2}{14\tilde{\xi}^7} - \frac{(1-\tilde{\xi})^3}{6\tilde{\xi}^8} + \frac{(1-\tilde{\xi})^4}{3\tilde{\xi}^9} - \frac{3\,(1-\tilde{\xi})^5}{5\tilde{\xi}^{10}} + \frac{(1-\tilde{\xi})^6}{\tilde{\xi}^{11}} + \text{const} \qquad \diamond$$

$\Pi_{(f)\perp}$ and its Hermitian conjugate and require it to vanish

$$
\begin{aligned}
0 &\stackrel{!}{=} \Pi_{(f)\perp}(\rho,\bar\rho) - \Pi^\dagger_{(f)\perp}(\rho,\bar\rho) = & \text{(B.54)}\\
&= \big(f(\xi)-\bar f(\xi)\big)\underbrace{\left(\mathbb{1}-\tfrac{1}{2}\frac{(\gamma^a\bar\lambda)\otimes(\lambda\gamma_a)}{(\lambda\bar\lambda)}\right)}_{\Pi_\perp^\dagger=\Pi_\perp}+\\
&\quad -\left(\tfrac{\bar f(\xi)}{2\left(1+\sqrt{1-\xi}\right)}-\tfrac{\bar f(\xi)f'(\xi)(1-\xi)}{f(\xi)}\right)\frac{\lambda\otimes\bar\zeta_c\,(\lambda\gamma^c)}{(\lambda\bar\lambda)}+\\
&\quad +\left(\tfrac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)}-\tfrac{f(\xi)\bar f'(\xi)(1-\xi)}{\bar f(\xi)}\right)\frac{\zeta^c\left(\gamma_c\bar\lambda\right)\otimes\bar\lambda}{(\lambda\bar\lambda)}+\\
&\quad -2(1-\xi)\left(1-\sqrt{1-\xi}\right)\left(f'(\xi)-\bar f'(\xi)\right)\underbrace{\frac{\lambda\otimes\bar\lambda}{(\lambda\bar\lambda)}}_{\Pi_\lambda^\dagger=\Pi_\lambda}+\\
&\quad -\tfrac{\left(f(\xi)-\bar f(\xi)\right)}{8\left(1+\sqrt{1-\xi}\right)^2}\underbrace{\frac{\bar\zeta_c(\gamma^c\gamma_b\lambda)\otimes\zeta^d(\bar\lambda\gamma^b\gamma_d)}{(\lambda\bar\lambda)}}_{\bar\zeta_c(\gamma^c\gamma_b\Pi_\lambda\gamma^b\gamma_d)\zeta^d} & \text{(B.55)}
\end{aligned}
$$

A priori this has to vanish for all ρ^α for which the projection matrices are defined (so for which $\xi \in I$). This is according to (2.18) and (2.17) equivalent to saying that it has to vanish for all pure spinors λ^α and for all ζ^a with $\zeta^a(\gamma_a\lambda)_\alpha = 0$ and with $\xi \in I$ and $f(\xi) \neq 0$ (the last one is in order for (2.18) to be well-defined). For simplicity let us first assume that

$$
f(\xi) \neq 0 \quad \forall \xi \in I \quad \text{(assumption)} \qquad \text{(B.56)}
$$

We will at the end of this proof relax this assumption. Note that the constraint $\zeta^a(\gamma_a\lambda)_\alpha = 0$ is no constraint on the modulus ξ, because to this constraint always exist solutions of the form $\zeta^a = (\lambda\gamma^a\chi)$ with some arbitrary spinor χ^α, and these solutions can be rescaled to obtain any ξ one wants. It is therefore necessary (not sufficient) that (B.55) has to hold for any pure spinor λ^α and for all $\xi \in I$.

Although it seems very plausible for a generic λ^α, it is not completely obvious if the 5 matrices $\mathbb{1}-\frac{(\gamma^a\bar\lambda)\otimes(\lambda\gamma_a)}{2(\lambda\bar\lambda)}$, $\frac{\lambda\otimes\bar\zeta_c(\lambda\gamma^c)}{(\lambda\bar\lambda)}$, $\frac{\zeta^c\left(\gamma_c\bar\lambda\right)\otimes\bar\lambda}{(\lambda\bar\lambda)}$, $\frac{\lambda\otimes\bar\lambda}{(\lambda\bar\lambda)}$ and $\frac{\bar\zeta_c(\gamma^c\gamma_b\lambda)\otimes\zeta^d(\bar\lambda\gamma^b\gamma_d)}{(\lambda\bar\lambda)}$ are all linearly independent, because there are many ways of rewriting them using Fierz identities. If they are independent, its coefficients have to vanish separately. From the first line of (B.55) this would already imply reality of the function f, i.e. $\bar f \stackrel{!}{=} f$, which in turn also would make vanish the last two lines of (B.55) and we would be left with a differential equation from the second (or equivalently from the third) line.

In order to be on the safe side, however, we could expand (B.55) in a basis from which we know about linear Independence, namely $\{\mathbb{1}, \gamma^{e_1e_2}, \gamma^{e_1e_2e_3e_4}\}$. An alternative way is to perform all kind of contractions of (B.55) until we have enough conditions on h to really make the full equation (B.55) hold. We will follow the latter path. Note that in the following we will use the identities $(\lambda\gamma^c\lambda) = (\bar\lambda\gamma^c\bar\lambda) = 0$ and $\zeta^c(\gamma_c\lambda) = \bar\zeta_c(\gamma^c\bar\lambda) = 0$ wherever possible without always mentioning this. Let us start with the most obvious contraction, namely taking the trace of (B.55):

$$
\begin{aligned}
0 \;&\stackrel{!}{=}\; 11\left(f(\xi)-\bar f(\xi)\right) - {\scriptstyle 2(1-\xi)\left(1-\sqrt{1-\xi}\right)}\left(f'(\xi)-\bar f'(\xi)\right) + \\
&\quad -\frac{\left(f(\xi)-\bar f(\xi)\right)}{8\left(1+\sqrt{1-\xi}\right)^2}\,\frac{\bar\zeta_c(\bar\lambda\gamma^b \overbrace{\gamma_d\gamma^c}^{\gamma_d{}^c+\delta^c_d} \gamma_b\lambda)\zeta^d}{(\lambda\bar\lambda)}
\end{aligned}
\tag{B.57}
$$

Using the identity

$$
\gamma^b\gamma^{e_1\dots e_l}\gamma_b = (-)^l(10-2l)\gamma^{e_1\dots e_l} \quad \forall l\in\{0,\dots,5\}
\tag{B.58}
$$

we can rewrite $\gamma^b(\gamma_d{}^c+\delta^c_d)\gamma_b = 6\gamma_d{}^c + 10\delta^c_d = -6\gamma^c\gamma_d + 16\delta^c_d$. Having in mind that $\zeta^c\bar\zeta_c = 2\xi$ and $\frac{\xi}{\left(1+\sqrt{1-\xi}\right)^2} = \frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}$, we arrive at

$$
\begin{aligned}
\underline{\mathrm{tr}\,(B.55):}\quad 0 \;&\stackrel{!}{=}\; \left(f(\xi)-\bar f(\xi)\right)\left(11-4\tfrac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}\right) + \\
&\quad -2\,(1-\xi)\left(1-\sqrt{1-\xi}\right)\left(f'(\xi)-\bar f'(\xi)\right)
\end{aligned}
\tag{B.59}
$$

This is the first condition on $f(\xi)$.

Next let us contract (B.55) $\bar\lambda$ from the left and λ from the right, which yields

$$
\underline{\left(\bar\lambda(B.55)\lambda\right):}\quad 0 \stackrel{!}{=} \left(f(\xi)-\bar f(\xi)\right)(\lambda\bar\lambda) - {\scriptstyle 2(1-\xi)\left(1-\sqrt{1-\xi}\right)}\left(f'(\xi)-\bar f'(\xi)\right)(\lambda\bar\lambda)
\tag{B.60}
$$

Using (B.59) one can eliminate f' and obtains

$$
0 \stackrel{!}{=} -\left(f(\xi)-\bar f(\xi)\right)\left(10-4\underbrace{\tfrac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}}_{\leq 1}\right)(\lambda\bar\lambda)
\tag{B.61}
$$

Because of the indicated inequality, the factor $10-4\frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}$ can never become zero. Further more the above condition has to hold for all pure spinors λ^α, in particular those with $(\lambda\bar\lambda)\neq 0$. This finally shows

$$
\bar f(\xi) \stackrel{!}{=} f(\xi) \quad \forall\xi\in I
\tag{B.62}
$$

This was what we suspected from the beginning (right after (B.55)), but now we are sure.

Finally let us plug the above reality result into (B.55) and contract the equation from the right with λ and with $\bar\zeta_d(\lambda\gamma^d)$ from the left. This yields

$$
\begin{aligned}
0 \;&\stackrel{!}{=}\; \left(\frac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)} - f'(\xi)\,(1-\xi)\right)\zeta^c\bar\zeta_d(\lambda\,\overbrace{\gamma^d\gamma_c}^{-\gamma_c\gamma^d+2\delta^d_c}\,\bar\lambda) = && \text{(B.63)}\\
&= \left(\frac{f(\xi)}{2\left(1+\sqrt{1-\xi}\right)} - f'(\xi)\,(1-\xi)\right)4\xi(\lambda\bar\lambda) && \text{(B.64)}
\end{aligned}
$$

Again this should hold for all pure spinors λ^α, so including those with $(\lambda\bar\lambda)\neq 0$. This is the case if and only if either $\xi=0$ or

$$
\frac{f'(\xi)}{f(\xi)} = \frac{1}{2(1-\xi)\left(1+\sqrt{1-\xi}\right)} \quad \forall\xi\in I/\{0\}
\tag{B.65}
$$

As we have assumed continuity of f' at $\xi = 0$ in the proposition, the equation will also hold for $\xi = 0$. We now have already extracted all information from (B.55), which is clear when plugging the above equation (together with $\bar{f} = f$) back into (B.55) and arriving at $0 \stackrel{!}{=} 0$. Using now that $f(0) = 1$, this can be finally uniquely integrated to $f(\xi) = h(\xi) = \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$, as it was defined in (5.1). Together with its derivative $h'(\xi) = \frac{1}{4\sqrt{1-\xi}^3}$ (5.9), this function indeed obeys the differential equation (B.65) and the boundary condition $h(0) = 1$ is met. So already Hermiticity of $\Pi_{(f)\perp}$fixes f uniquely to be h, so that no further conditions may come from symmetry of $\pi_{(f)\perp}$. And indeed, $\pi_{(h)\perp}$ with h of (5.1) is obviously symmetric in (5.16), as was noted already at the beginning of the proof.

In (B.56) we had made the assumption that $f(\xi) \neq 0$ for all $\xi \in I = [0, b[$. Now assume that there is a zero of f at $\xi_0 \in I/\{0\}$ (at $\xi = 0$ we necessarily have $f(0) = 1$). Then the above proof of $f = h$ holds only for the interval $[0, \xi_0[$, but because of continuity it has to hold also for ξ_0. And $h = \frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}$ does not have any zeros on $[0, 1[$ which disproves the assumption that f had a zero at $\xi_0 \in I$. $\square$

2. When $\Pi_{(h)\perp}$ and $\pi_{(h)\perp}$ are blocks of a Hermitian matrix, it means that $\partial_{\rho^\alpha} P^\beta_{(h)} - \partial_{\bar{\rho}_\beta} \bar{P}_{(h)\alpha} = 0$ as well as $\bar{\partial}^{[\alpha} P^{\beta]}_{(h)} = 0$ (with $\bar{\partial}^\alpha \equiv \frac{\partial}{\partial\bar{\rho}_\alpha}$). This means that $P_{(h)}$ regarded as a 1-form $\boldsymbol{P} \equiv P^\alpha_{(h)} \mathbf{d}\bar{\rho}_\alpha + \bar{P}_{(h)\alpha}\mathbf{d}\rho^\alpha$ is closed and thus locally exact, i.e. $\mathbf{d}\boldsymbol{P} = 0$. As this is a flat vector space, there are no global obstructions and one can find a potential for the projector. And indeed the potential $\Phi \equiv \frac{(\rho\bar{\rho})}{2}(1 + \sqrt{1-\xi})$ of equation (5.4) has the following derivatives

$$\partial_{\bar{\rho}_\alpha}\Phi(\rho,\bar{\rho}) = \frac{\rho^\alpha}{2}(1+\sqrt{1-\xi}) - \frac{(\rho\bar{\rho})}{2}\partial_{\bar{\rho}_\alpha}\xi \cdot \frac{1}{2\sqrt{1-\xi}} = \tag{B.66}$$

$$\stackrel{(B.29)}{=} \frac{\rho^\alpha}{2}\underbrace{\left(1+\sqrt{1-\xi}+\frac{\xi}{\sqrt{1-\xi}}\right)}_{\frac{1+\sqrt{1-\xi}}{\sqrt{1-\xi}}} - \frac{\zeta^a(\gamma_a\bar{\rho})^\alpha}{4\sqrt{1-\xi}} \tag{B.67}$$

This result indeed agrees with $P^\alpha_{(h)}(\rho,\bar{\rho})$ of equation (5.10). Via complex conjugation we finally obtain also $\bar{P}_{(h)\alpha} = \partial_{\rho^\alpha}\Phi$. $\square$

3. The absolute value squared of $P^\alpha_{(h)}(\rho,\bar{\rho})$ is given by

$$P^\alpha_{(h)}(\rho,\bar{\rho})\bar{P}_{(h)\alpha}(\rho,\bar{\rho}) =$$

$$\stackrel{(5.10)}{=} \left(\frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\rho^\alpha - \frac{\zeta^a(\bar{\rho}\gamma_a)^\alpha}{4\sqrt{1-\xi}}\right)\left(\frac{1+\sqrt{1-\xi}}{2\sqrt{1-\xi}}\bar{\rho}_\alpha - \frac{(\gamma^b\rho)_\alpha\bar{\zeta}_b}{4\sqrt{1-\xi}}\right) = \tag{B.68}$$

$$= \frac{\left(1+\sqrt{1-\xi}\right)^2}{4(1-\xi)}(\rho\bar{\rho}) - \frac{1+\sqrt{1-\xi}}{8(1-\xi)}\underbrace{(\rho\gamma^b\rho)}_{(\rho\bar{\rho})\zeta^b\ (5.11)}\bar{\zeta}_b +$$

$$-\frac{1+\sqrt{1-\xi}}{8(1-\xi)}\zeta^a\underbrace{(\bar{\rho}\gamma_a\bar{\rho})}_{(\rho\bar{\rho})\bar{\zeta}_a\ (5.11)} + \frac{1}{16(1-\xi)}\zeta^a(\bar{\rho}\underbrace{\gamma_a\gamma^b}_{-\gamma^b\gamma_a+2\delta^b_a}\rho)\bar{\zeta}_b = \tag{B.69}$$

$$\stackrel{(2.13)}{=} (\rho\bar{\rho})\left\{\frac{\left(1+\sqrt{1-\xi}\right)^2}{4(1-\xi)} - \frac{1+\sqrt{1-\xi}}{8(1-\xi)}\zeta^b\bar{\zeta}_b - \frac{1+\sqrt{1-\xi}}{8(1-\xi)}\zeta^a\bar{\zeta}_a + \frac{1}{8(1-\xi)}\zeta^a\bar{\zeta}_a\right\} \tag{B.70}$$

Using $\zeta^a\bar{\zeta}_a = 2\xi$ (5.11) and simplifying a bit, the expression in the curly bracket becomes $\frac{1}{2}(1 + \sqrt{1-\xi})$ and thus (via the definition (5.4) of Φ) indeed yields $P^\alpha_{(h)}(\rho,\bar{\rho})\bar{P}_{(h)\alpha}(\rho,\bar{\rho}) = \Phi(\rho,\bar{\rho})$. $\square$

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