MANY NON-EQUIVALENT REALIZATIONS OF THE ASSOCIAHEDRON

CESAR CEBALLOS, FRANCISCO SANTOS, AND GÜNTER M. ZIEGLER

Abstract. We show that three systematic construction methods for the n-dimensional associahedron,

- as the secondary polytope of a convex $(n+3)$-gon (by Gelfand–Kapranov–Zelevinsky),
- via cluster complexes of the root system A_n (by Chapoton–Fomin–Zelevinsky), and
- as Minkowski sums of simplices (by Postnikov)

produce substantially different realizations, independent of the choice of the parameters for the constructions.

The cluster complex and the Minkowski sum realizations were generalized by Hohlweg–Lange to produce exponentionally many distinct realizations, all of them with normal vectors in $\{0, \pm 1\}^n$. We present another, even larger, exponential family, generalizing the cluster complex construction — and verify that this family is again disjoint from the previous ones, with one single exception: The Chapoton–Fomin–Zelevinsky associahedron appears in both exponential families.

Contents

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1. Introduction

The n-dimensional associahedron is a simple polytope with C_{n+1} (the Catalan number) vertices, corresponding to the triangulations of a convex $(n+3)$-gon, and $n(n+3)/2$ facets, in bijection with the diagonals of the $(n+3)$-gon. It appears in Dov Tamari's unpublished 1951 thesis [32], and was described as a combinatorial object and realized as a cellular ball by Jim Stasheff in 1963 in his work on the associativity of H-spaces [30]. A realization as a polytope by John Milnor from the 1960s is lost; Huguet & Tamari claimed in 1978 that the associahedron can be realized as a convex polytope [18]. The first such construction, via an explicit inequality system, was provided in a manuscript by Mark Haiman from 1984 that remained unpublished, but is available as [15]. The first construction in print, which used stellar subdivisions in order to obtain the dual of the associahedron, is due to Carl Lee, from 1989 [20].

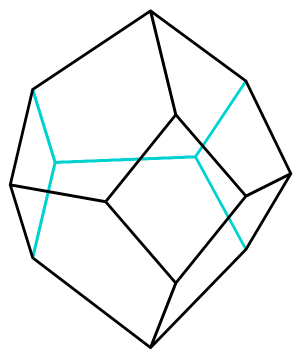

Figure 1. The 3-dimensional associahedron, realized as the secondary polytope of a regular hexagon.

Subsequently three systematic approaches were developed that produce realizations of the associahedra in more general frameworks and suggest generalizations:

- the associahedron as a secondary polytope due to Gelfand, Kapranov and Zelevinsky [13] [14] (see also [12, Chap. 7]),
- the associahedron associated to the cluster complex of type A_n, conjectured by Fomin and Zelevinsky [11] and constructed by Chapoton, Fomin and Zelevinsky [6], and
- the associahedron as a Minkowski sum of simplices introduced by Postnikov in [24]. Essentially the same associahedron, but described much differently, had been constructed independently by Shnider and Sternberg [28], (compare Stasheff and Shnider [31, Appendix B]), Loday [21], Rote, Santos and Streinu [26], and most recently Buchstaber [5]. Following [16] we reference it as the "Loday realization", as Loday obtained explicit vertex coordinates that were used subsequently.

The last two approaches were generalized by Hohlweg and Lange [16] and by Santos [27], who showed that they are particular cases of *exponentially many* constructions of the associahedron. The Hohlweg–Lange construction produces roughly 2^{n-3} distinct realizations, while the Santos construction produces about $\frac{1}{2(n+3)}C_{n+1} \approx 2^{2n+1}/\sqrt{\pi n^5}$ different ones; exact counts are in Sections 4 and 5. The construction by Santos appears in print for the first time in this paper, so we prove in detail that it actually works. For the others we rely on the original papers for most of the details.

The goal of this paper is to compare the constructions, showing that they produce essentially different realizations for the associahedron. Let us explain what we exactly mean by *different* (see more details in Section 2). Since the associahedron is simple, its realizations form an open subset in the space of $\frac{(n+3)n}{2}$-tuples of half-spaces in $\mathbb{R}^n$. Hence, classifying them by affine or projective equivalence does not seem the right thing to do. But most of the constructions of the associahedron (all the ones in this paper except for the secondary polytope construction) happen to have facet normals with very small integer coordinates. This suggests that one natural classification is by *linear isomorphism of their normal fans* or, as we call it, *normal isomorphism*.

The secondary polytope construction has a completely different flavor from the others. Coordinates for its vertices are computed from the actual coordinates of the $(n+3)$-gon used, which can be arbitrary, and a continuous deformation of the polygon produces a continuous deformation of the associahedron obtained. The rest of the constructions are more combinatorial in nature, with no need to give coordinates for the polygon. This is apparent comparing Figures 1 and 2. The first one shows the secondary polytope of a regular hexagon, and the second shows (affine images of) other constructions of the 3-associahedron.

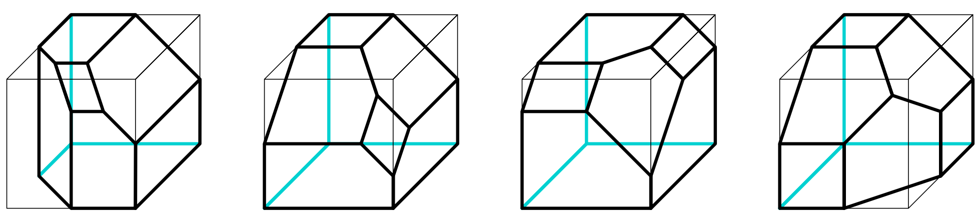

FIGURE 2. Four normally non-isomorphic realizations of the 3-dimensional associahedron. From left to right: The Postnikov associahedron (which is a special case of the Hohlweg–Lange associahedron), the Chapoton-Fomin-Zelevinsky associahedron (a special case of both Hohlweg–Lange and Santos) and the other two Santos associahedra. Since they all have three pairs of parallel facets, we draw them inscribed in a cube.

One way of pinning down this difference (and of testing, for example, whether two associahedra are normally isomorphic) is to look at which parallel facets arise, if any. We start doing this in Section 3, where we show that secondary polytope associahedra never have parallel facets (Theorem 3.5, but see Remark 3.6) while the Chapoton-Fomin-Zelevinsky and the Postnikov ones have n pairs of parallel facets each (Theorems 3.11 and 3.22).

In Sections 4 and 5 we present the families of realizations by Hohlweg–Lange and by Santos. The first one produces one n-associahedron for each sequence in $\{+,-\}^{n-1}$. The second one constructs one n-associahedron from each triangulation of the $(n+3)$-gon. We call them associahedra of *types* I and II.

Apart of reviewing the two constructions, we show they both provide exponentially-many normally non-isomorphic realizations of the n-dimensional associahedron with the following common features:

- ◦ They all have n pairs of parallel facets.
- ◦ In the basis given by the normals to those n pairs, all facet normals have coordinates in $\{0, \pm 1\}$.

For the Santos construction both properties follow from the definition, for Hohlweg–Lange we prove them in Sections 4.2 and 4.3. Put differently, all these constructions are (normally isomorphic to) polytopes obtained from the regular n-cube by cutting certain $\binom{n}{2}$ faces according to specified rules; for example, the last example of Figure 2 cannot be obtained by cutting faces lexicographically; the three faces, edges in this case, need to be cut at exactly the same depth.

In Section 6 we put together results from the previous two sections, and show that there is a single associahedron that can be obtained both with the Hohlweg–Lange and the Santos construction, namely the one by Chapoton-Fomin-Zelevinsky.

We also note that Hohlweg–Lange–Thomas [17] provided a generalization of the Hohlweg–Lange construction to general finite Coxeter groups; Bergeron–Hohlweg–Lange–Thomas [2] have provided a classification of the Hohlweg–Lange–Thomas c-generalized associahedra in Coxeter group theoretic language up to isometry, and also up to normal isomorphism [2, Cor. 2.6]. For type A, this specializes to a classification of the Hohlweg–Lange associahedra, which we obtain in Theorem 4.7 in a different, more combinatorial, setting. Besides the isometries of c-generalized associahedra presented in [2], normal isomorphisms of these polytopes are discussed earlier by Reading–Speyer [25] in the context of c-Cambrian fans. In particular, they obtained combinatorial isomorphisms of the normal fans, which are in general only piecewise-linear [25, Thm. 1.1 and Sec. 5].

One of the questions that remains is whether there is a common generalization of the Hohlweg–Lange and the Santos construction, which may perhaps produce even more examples of "combinatorial" associahedra. It has to be noted that the associahedron seems to be quite versatile as a polytope. For example, besides the four 3-associahedra of Figure 2 we have found another four 3-associahedra that arise by cutting three faces of a 3-cube (see Figure 3). Do these admit a natural combinatorial interpretation as well?

2. Some preliminaries

We start by recalling the definition of an n-dimensional associahedron in terms of polyhedral subdivisions of an $(n+3)$-gon.

Definition 2.1. Let P_{n+3} be a convex $(n+3)$-gon, whose vertices we label cyclically with the symbols 1 through $n+3$.

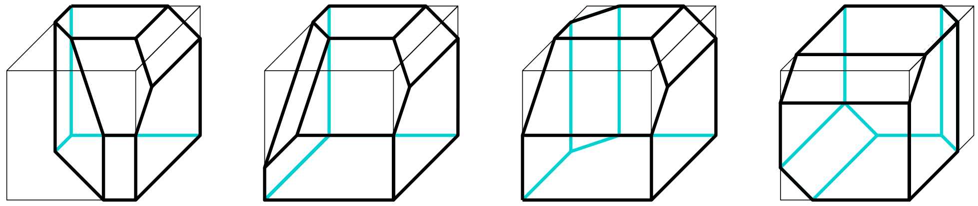

FIGURE 3. More 3-associahedra inscribed in a 3-cube. The 3-associahedron is the only simple 3-polytope with nine facets all of which are quadrilaterals or pentagons.

An *associahedron* Ass_n is an n-dimensional simple polytope whose poset of non-empty faces is isomorphic to the poset of non-crossing sets of diagonals of P_{n+3}, ordered by reverse inclusion.

Equivalently, the poset of non-empty faces of the associahedron is isomorphic to the set of polyhedral subdivisions of P_{n+3} (without new vertices), ordered by coarsening. The minimal elements (vertices of the associahedron) correspond to the *triangulations* of P_{n+3}.

For example, for the associahedron of dimension two we look at which diagonals of the pentagon cross each other. There are five diagonals, with five of the $\binom{5}{2}$ pairs of them crossing and the other five non-crossing. Thus, the poset of non-empty faces of the two-dimensional associahedron is isomorphic to the Hasse diagram of Figure 4, in which the five bottom elements correspond to the five triangulations of the pentagon and the top element corresponds to the "trivial" subdivision into a single cell, the pentagon itself.

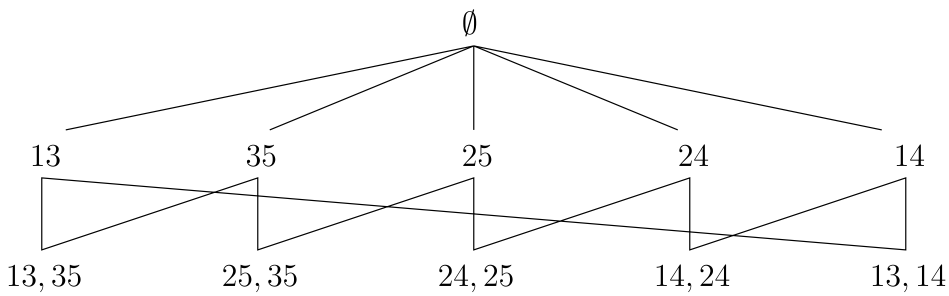

FIGURE 4. The Hasse diagram of the 2-dimensional associahedron.

This is also the Hasse diagram of the poset of non-empty faces of a pentagon, so the 2-dimensional associahedron is a pentagon. Figure 5 shows the associahedra of dimensions 0, 1, and 2.

The goal of this paper is to compare different types of constructions of the associahedron, saying which ones produce equivalent polytopes, in a suitable sense. The following notion reflects the fact that the main constructions that we are going to discuss produce associahedra whose normal vectors have small integer coordinates, usually 0 or ± 1. In these

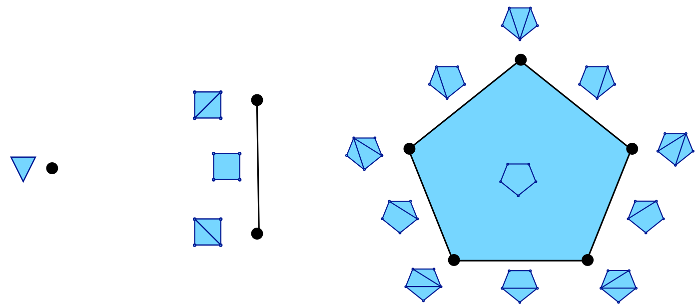

FIGURE 5. The associahedron Ass_n for $n = 0, 1$ and 2.

constructions the normal fan of the associahedron can be considered canonical, while there is still freedom in the right-hand sides of the inequalities. (See [33, Sec. 7.1] for a discussion of fans and of normal fans.) This leads us to use the following notion of equivalence.

Definition 2.2. Two complete fans in real vector spaces V and V' of the same dimension are *linearly isomorphic* if there is a linear isomorphism $V \to V'$ sending each cone of one to a cone of the other. Two polytopes P and P' are *normally isomorphic* if they have linearly isomorphic normal fans.

Normal isomorphism is weaker than the usual notion of *normal equivalence*, in which the two polytopes P and P' are assumed embedded *in the same space* and their normal fans are required to be exactly the same, not only linearly isomorphic.

The following lemma is very useful in order to prove (or disprove) that two associahedra are normally isomorphic. It implies that all linear (or combinatorial, for that matter) isomorphisms between associahedra come from isomorphisms between the $(n + 3)$-gons defining them.

Lemma 2.3. *The automorphism group of the face lattice of the associahedron* Ass_n *is the dihedral group of the* $(n + 3)$*-gon: All automorphisms are induced by symmetries of the* $(n + 3)$*-gon.*

Proof. Suppose φ is an automorphism of the face lattice of the associahedron Ass_n, and let D be the set of all diagonals of a convex $(n + 3)$-gon. φ induces a natural bijection

$$\widetilde{\varphi} : D \longrightarrow D$$

such that for any two diagonals $\delta, \delta' \in D$ we have:

$$\delta \text{ cross } \delta' \iff \widetilde{\varphi}(\delta) \text{ cross } \widetilde{\varphi}(\delta').$$

For a diagonal $\delta \in D$ denote by length(δ) the minimum between the lengths of the two paths that connect the two end points of δ on the boundary of the $(n + 3)$-gon. Then

$$\text{length}(\delta) = \text{length}(\widetilde{\varphi}(\delta)).$$

The reason is that the length of δ is determined by the number of diagonals that cross δ, and this property is invariant under the map $\widetilde{\varphi}$.

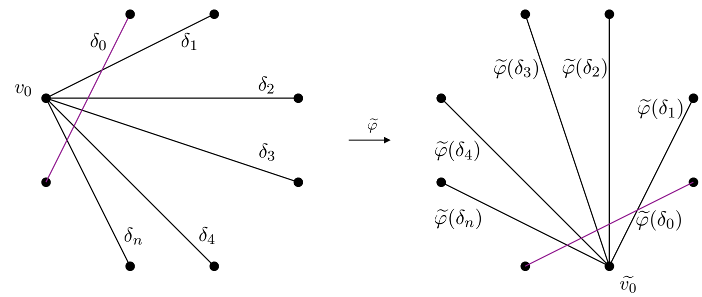

FIGURE 6. The situation in the proof of Lemma 2.3.

Let δ_0 be a diagonal of length 2, and $\widetilde{\varphi}(\delta_0)$ its image under $\widetilde{\varphi}$. The diagonals that cross δ_0 have a common intersection vertex v_0; from this vertex we label these diagonals in clockwise direction by $\delta_1, \dots, \delta_n$. Similarly, the diagonals that cross $\widetilde{\varphi}(\delta_0)$ have a common intersection vertex $\widetilde{v_0}$, and they are labeled by $\widetilde{\varphi}(\delta_1), \dots, \widetilde{\varphi}(\delta_n)$. For any non-empty interval $I \subset [n]$ there is an unique diagonal δ_I that intersects the diagonal δ_i if and only if $i \in I$. Applying the map $\widetilde{\varphi}$ we obtain diagonals $\widetilde{\varphi}(\delta_I)$ that intersect $\widetilde{\varphi}(\delta_i)$ if and only if $i \in I$. This task is possible only if the labelings $\widetilde{\varphi}(\delta_1), \dots, \widetilde{\varphi}(\delta_n)$ appear in either clockwise or counterclockwise direction. From this, we deduce that $\widetilde{\varphi}$ restricted to $\{\delta_1, \dots, \delta_n\}$ is equivalent to a reflection-rotation map. Moreover, this map coincides with $\widetilde{\varphi}$ for all other diagonals δ_I. □

3. THREE REALIZATIONS OF THE ASSOCIAHEDRON

3.1. **The Gelfand–Kapranov–Zelevinsky associahedron.** The secondary polytope is an ingenious construction motivated by the theory of hypergeometric functions as developed by I.M. Gelfand, M. Kapranov and A. Zelevinsky [12]. In this section we recall the basic definitions and main results related to this topic, which yield in particular that the secondary polytope of any convex $(n+3)$-gon is an n-dimensional associahedron. For more detailed presentations we refer to [7, Sec. 5] and [33, Lect. 9]. All the subdivisions and triangulations of polytopes that appear in the following are understood to be without new vertices.

The secondary polytope construction.

Definition 3.1 (GKZ vector/secondary polytope)**.** Let Q be a d-dimensional convex polytope with $n+d+1$ vertices. The *GKZ vector* $v(t) \in \mathbb{R}^{n+d+1}$ of a triangulation t of Q is

$$v(t) \;:=\; \sum_{i=1}^{n+d+1} \mathrm{vol}(\mathrm{star}_t(i))e_i \;=\; \sum_{i=1}^{n+d+1} \sum_{\sigma \in t \,:\, i \in \sigma} \mathrm{vol}(\sigma)e_i$$

The *secondary polytope* of Q is defined as

$$\Sigma(Q) \;:=\; \mathrm{conv}\{v(t) : t \text{ is a triangulation of } Q\}.$$

Theorem 3.2 (Gelfand–Kapranov–Zelevinsky [13]). *Let Q be a d-dimensional convex polytope with $m = n + d + 1$ vertices. Then the secondary polytope $\Sigma(Q)$ has the following properties:*

(i) *$\Sigma(Q)$ is an n-dimensional polytope.*
(ii) *The vertices of $\Sigma(Q)$ are in bijection with the regular triangulations of Q.*
(iii) *The faces of $\Sigma(Q)$ are in bijection with the regular subdivisions of Q.*
(iv) *The face lattice of $\Sigma(Q)$ is isomorphic to the lattice of regular subdivisions of Q, ordered by refinement.*

The associahedron as the secondary polytope of a convex $(n + 3)$-gon.

Definition 3.3. The *Gelfand–Kapranov–Zelevinsky associahedron* $\mathrm{GKZ}_n(Q) \subset \mathbb{R}^{n+3}$ is defined as the (n-dimensional) secondary polytope of a convex $(n + 3)$-gon $Q \subset \mathbb{R}^2$:

$$\mathrm{GKZ}_n(Q) := \Sigma(Q).$$

Corollary 3.4 ([13]). *$\mathrm{GKZ}_n(Q)$ is an n-dimensional associahedron.*

There is one feature that distinguishes the associahedron as a secondary polytope from all the other constructions that we mention in this paper: the absence of parallel facets. This property, in particular, will imply that the GKZ–associahedra are not normally isomorphic to the associahedra produced by the other constructions:

Theorem 3.5. *Let Q be a convex $(n + 3)$-gon. Then $\mathrm{GKZ}_n(Q)$ has no parallel facets for $n \geq 2$.*

Our proof is based on the understanding of the facet normals in secondary polytopes. Let Q be an arbitrary d-polytope with $n+d+1$ vertices $\{q_1, \ldots, q_{n+d+1}\}$, so that $\mathrm{GKZ}_n(Q)$ lives in $\mathbb{R}^{n+d+1}$, although it has dimension n. In the theory of secondary polytopes one thinks of each linear functional $\mathbb{R}^{n+d+1} \to \mathbb{R}$ as a function ω : vertices$(Q) \to \mathbb{R}$ assigning a value $\omega(q_i)$ to each vertex q_i. In turn, to each triangulation t of Q (with no additional vertices) and any such ω one associates the function $g_{\omega,t} : Q \to \mathbb{R}$ which takes the value $\omega(q_i)$ at each q_i and is affine linear on each simplex of t. That is, we use t to piecewise linearly interpolate a function whose values $(\omega(q_1), \ldots, \omega(q_n))$ we know on the vertices of Q. The main result we need is the following equality for every ω and every triangulation t (see, e.g., [7, Thm. 5.2.16]):

$$\langle \omega, v(t) \rangle = (d + 1) \int_Q g_{\omega,t}(x) dx.$$

In particular:

- ◦ If ω is affine-linear (that is, if the points $\{(q_1, \omega_1), \ldots, (q_{n+d+1}, \omega_{n+d+1})\} \subset \mathbb{R}^{n+d+1} \times \mathbb{R}$ lie in a hyperplane) then $\langle \omega, v(t) \rangle$ is the same for all t. Moreover, the converse is also true: The affine-linear ω's form the lineality space of the normal fan of $\mathrm{GKZ}_n(Q)$.
- ◦ An ω lies in the linear cone of the (inner) normal fan of $\mathrm{GKZ}_n(Q)$ corresponding to a certain triangulation t (that is, $\langle \omega, v(t) \rangle \leq \langle \omega, v(t') \rangle$ for every other triangulation t') if and only if the function $g_{\omega,t}$ is convex; that is to say, if its graph is a convex hypersurface.

Proof of Theorem 3.5. With the previous description in mind we can identify the facet normals of the secondary polytope of a polygon Q. For this we use the correspondence:

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{triangulations of } Q \\ \text{facets} & \longleftrightarrow & \text{diagonals of } Q \end{array}$$

For a given diagonal δ of Q, denote by F_δ the facet of $\mathrm{GKZ}_n(Q)$ corresponding to δ. The vector normal to F_δ is not unique, since adding to any vector normal to F_δ an affine-linear ω_0 we get another one. One natural choice is

$$\omega_\delta(q_i) := \mathrm{dist}(q_i, l_\delta),$$

where l_δ is the line containing δ and $\mathrm{dist}(\cdot,\cdot)$ is the Euclidean distance. Indeed, ω_δ lifts the vertices of Q on the same side of δ to lie in a half-plane in $\mathbb{R}^3$, with both half-planes having δ as their common intersection. That is, $g_{\omega_\delta,t}$ is convex for every t that uses δ. But another choice of normal vector is better for our purposes: choose one side of l_δ to be called positive and take

$$\omega_\delta^+(q_i) := \begin{cases} \mathrm{dist}(q_i, l_\delta) & \text{if } q_i \in l_\delta^+ \\ 0 & \text{if } q_i \in l_\delta^- \end{cases}.$$

For the end-points of δ, which lie in both l_δ^+ and l_δ^-, there is no ambiguity since both definitions give the value 0. Again, ω_δ^+ is a normal vector to F_δ since it lifts points on either side of l_δ to lie in a plane.

We are now ready to prove the theorem. If two diagonals δ and δ' of Q do not cross, then they can simultaneously be used in a triangulation. Hence, the corresponding facets F_δ and $F_{\delta'}$ meet, and they cannot be parallel. So, assume in what follows that δ and δ' are two crossing diagonals. Let $\delta = pr$ and $\delta' = qs$, with $pqrs$ being cyclically ordered along Q. Since $n \geq 2$ there is at least another vertex a in Q. Without loss of generality suppose a lies between s and p. Now, we call negative the side of l_δ and the side of $l_{\delta'}$ containing a, and consider the normal vectors ω_δ^+ and $\omega_{\delta'}^+$ as defined above. They take the following values on the five points of interest:

$$\begin{array}{lllll} \omega_\delta^+(a) = 0, & \omega_\delta^+(p) = 0, & \omega_\delta^+(q) > 0, & \omega_\delta^+(r) = 0, & \omega_\delta^+(s) = 0, \\ \omega_{\delta'}^+(a) = 0, & \omega_{\delta'}^+(p) = 0, & \omega_{\delta'}^+(q) = 0, & \omega_{\delta'}^+(r) > 0, & \omega_{\delta'}^+(s) = 0. \end{array}$$

Suppose that F_δ and $F_{\delta'}$ were parallel. This would imply that δ and δ' are linearly dependent or, more precisely, that there is a linear combination of them that gives an affine-linear ω (in the lineality space of the normal fan). But any (non-trivial) linear combination ω of ω_δ^+ and $\omega_{\delta'}^+$ necessarily takes the following values on our five points, which implies that ω is not affine-linear:

$$\omega(a) = 0, \quad \omega(p) = 0, \quad \omega(q) \neq 0, \quad \omega(r) \neq 0, \quad \omega(s) = 0.$$ □

Remark 3.6. The secondary polytope can be defined for any set of points $\{q_1, \dots, q_{n+3}\}$ in the plane, not necessarily the vertices of a convex polygon. In general this does not produce an associahedron, but there is a case in which it does: if the points are cyclically placed on the boundary of an m-gon with $m \leq n+3$ in such a way that no four of them lie on a boundary edge. By the arguments in the proof above, a necessary condition for the associahedron obtained to have parallel facets is that $m \leq 4$. For $m = 4$ we can obtain associahedra up to dimension 4 with exactly one pair of parallel facets (those corresponding to the main diagonals of the quadrilateral). For $m = 3$, we can obtain 2-dimensional associahedra with two pairs of parallel facets, and 3-dimensional associahedra with three pairs of parallel facets. The latter is obtained for six points $\{p, q, r, a, b, c\}$ with p, q and r being the vertices of a triangle and $a \in pq$, $b \in qr$ and $c \in ps$ intermediate points in the three sides. The associahedron obtained has the following three pairs of parallel facets:

$$F_{pq}||F_{ar}, \quad F_{qr}||F_{bs}, \quad F_{ps}||F_{cq}.$$

Remark 3.7. Rote, Santos and Streinu [26] introduce a *polytope of pseudo-triangulations* associated to each finite set A of m points (in general position) in the plane. This polytope lives in $\mathbb{R}^{2m}$ and has dimension $m+3+i$, where i is the number of points interior to $\operatorname{conv}(A)$. They show that for points in convex position their polytope is affinely isomorphic to the secondary polytope for the same point set. Their constructions uses rigidity theoretic ideas: the edge-direction joining two neighboring triangulations t and t' is the vector of velocities of the (unique, modulo translation and rotation) infinitesimal flex of the embedded graph of $t \cap t'$.

3.2. **The Postnikov associahedron.** We now review two further realizations of the associahedron: one by Postnikov [24] and one by Rote–Santos–Streinu [26] (different from the one in Remark 3.7). The main goal of this section is to prove that these two constructions produce affinely equivalent results. As special cases of these constructions one obtains, respectively, the realizations by Loday [21] and Buchstaber [5], which turn out to be affinely equivalent as well.

3.2.1. *The Postnikov associahedron.*

Definition 3.8. For any vector $\mathbf{a} = \{\mathrm{a}_{ij} > 0 : 1 \leq i \leq j \leq n+1\}$ of positive parameters we define the *Postnikov associahedron* as the polytope

$$\mathrm{Post}_n(\mathbf{a}) := \sum_{1 \leq i \leq j \leq n+1} \mathrm{a}_{ij} \Delta_{[i,\dots,j]},$$

where $\Delta_{[i,\dots,j]}$ denotes the simplex $\operatorname{conv}\{e_i, e_{i+1}, \dots, e_j\}$ in $\mathbb{R}^{n+1}$.

Proposition 3.9 (Postnikov [24, Sec. 8.2])**.** $\mathrm{Post}_n(\mathbf{a})$ *is an n-dimensional associahedron. In particular, for $\mathrm{a}_{ij} \equiv 1$ this yields the realization of Loday* [21].

In terms of inequalities the Postnikov associahedron is given as follows.

Lemma 3.10.

$$\mathrm{Post}_n(\mathbf{a}) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{p<i<q} x_i \geq f_{p,q} \quad \text{for } 0 \leq p < q \leq n+2, \\ x_1 + \cdots + x_{n+1} = f_{0,n+2}\},$$

where $f_{p,q} = \sum_{p<i\leq j<q} \mathrm{a}_{i,j}$.

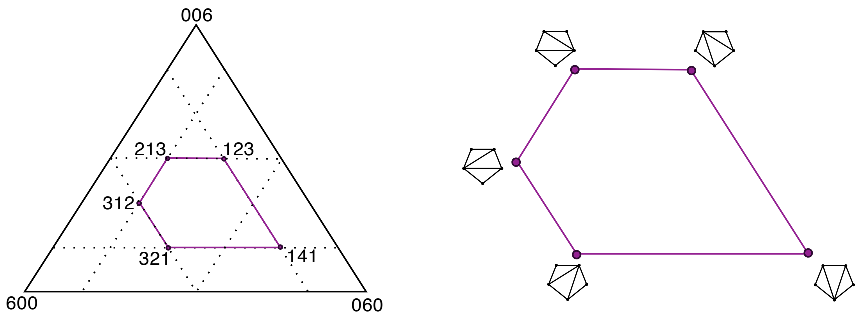

FIGURE 7. The Postnikov associahedron $\mathrm{Ass}_2^{\mathrm{II}}(\mathbf{1})$ with the coordinates of the vertices. This coincides with the realization of Loday.

The facet of $\mathrm{Post}_n(\mathbf{a})$ determined by the hyperplane with right hand side parameter $f_{p,q}$ corresponds to the diagonal pq of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. In particular:

Theorem 3.11. $\mathrm{Post}_n(\mathbf{a})$ *has exactly* n *pairs of parallel facets. These correspond to the pairs of diagonals* $(\{0, i+1\}, \{i, n+2\})$ *for* $1 \leq i \leq n$*, as illustrated in* Figure 8.

Proof. Two hyperplanes of the form $\sum_{i \in S_1} x_i \geq c_1$ and $\sum_{i \in S_2} x_i \geq c_2$ for $S_1, S_2 \subseteq [n+1]$, intersected with an affine hyperplane $x_1 + \cdots + x_{n+1} = c$ are parallel if and only if $S_1 \cup S_2 = [n+1]$ and $S_1 \cap S_2 = \emptyset$. Therefore two diagonals pq and rs correspond to parallel facets if and only if $pq = \{0, i+1\}$ and $qr = \{i, n+2\}$. □

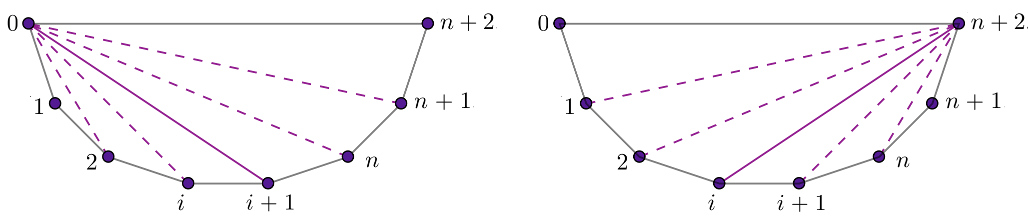

FIGURE 8. Diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of both $\mathrm{Post}_n(\mathbf{a})$ and $\mathrm{RSS}_n(\mathbf{g})$.

3.2.2. *The Rote–Santos–Streinu associahedron.* By "generalizing" the construction of Remark 3.7 to sets of points along a line, Rote, Santos and Streinu [26] obtain a second realization of the associahedron.

Definition 3.12. The *Rote–Santos–Streinu associahedron* is the polytope

$$\mathrm{RSS}_n(\mathbf{g}) = \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2} : y_j - y_i \geq g_{i,j} \text{ for } j > i,\ y_0 = 0,\ y_{n+1} = g_{0,n+1}\},$$

where $\mathbf{g} = (g_{i,j})_{0\le i<j\le n+1}$ is any vector with real coordinates satisfying

$$\begin{aligned} g_{i,l} + g_{j,k} > g_{i,k} + g_{j,l} & \quad \text{for all} \quad i < j \le k < l, \\ g_{i,l} > g_{i,k} + g_{k,l} & \quad \text{for all} \quad i < k < l. \end{aligned}$$

Proposition 3.13 (Rote–Santos–Streinu [26, Sec. 5.3])**.** *If the vector* $\mathbf{g}$ *satisfies the previous inequalities then* $\mathrm{RSS}_n(\mathbf{g})$ *is an* n*-dimensional associahedron.*

A particular example of valid parameters $\mathbf{g}$ is given by $\mathbf{g_0}$: $g_{i,j} = i(i-j)$. In this case we get the realization of the associahedron introduced by Buchstaber in [5, Lect. II Sec. 5].

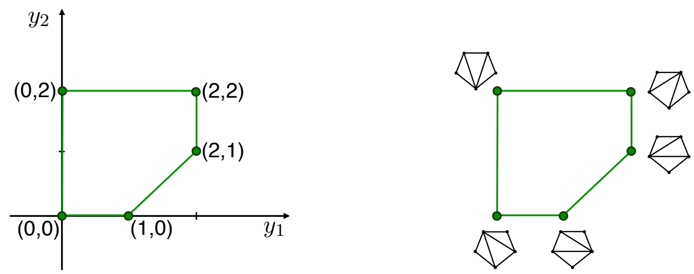

Figure 9. The Rote–Santos–Streinu associahedron $\mathrm{RSS}_2(\mathbf{g_0})$ with the coordinates of the vertices. This coincides with the realization of Buchstaber.

The facet of $\mathrm{RSS}_n(\mathbf{g})$ related to $y_j - y_i \ge g_{i,j}$ corresponds to the diagonal $\{i, j+1\}$ of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. One can also see that with this specified combinatorics of the facets, the conditions on the vector $\mathbf{g}$ are also necessary for the proposition to hold.

Theorem 3.14. $\mathrm{RSS}_n(\mathbf{g})$ *has exactly* n *pairs of parallel facets. They correspond to the pairs of diagonals* $(\{0, i+1\}, \{i, n+2\})$ *for* $1 \le i \le n$*, as illustrated in* Figure 8.

Rote, Santos and Streinu stated in [26, Sec. 5.3] that $\mathrm{RSS}_n(\mathbf{g})$ is not affinely equivalent to neither the associahedron as a secondary polytope nor the associahedron from the cluster complex of type A. Next we prove that $\mathrm{RSS}_n(\mathbf{g})$ is affinely isomorphic to $\mathrm{Post}_n(\mathbf{a})$. Furthermore, we prove, in Corollary 4.8 and Theorem 6.1, that these two polytopes are not normally isomorphic to the associahedron as a secondary polytope or the associahedron from the cluster complex of type A.

3.2.3. *Affine equivalence.*

Theorem 3.15. *Let* φ *be the affine transformation*

$$\begin{array}{cccc} \varphi: & \mathbb{R}^{n+1} & \to & \mathbb{R}^n \\ & (x_1, \dots, x_{n+1}) & \to & (y_1, \dots, y_n) \end{array}$$

defined by $y_k = \sum_{i=1}^{k}(x_i - i)$*. Then* φ *maps* $\mathrm{Post}_n(\mathbf{a})$ *bijectively to* $\mathrm{RSS}_n(\mathbf{g})$*, for* $\mathbf{g}$ *given by* $g_{i,j} - \frac{(i+j+1)(j-i)}{2} = f_{i,j+1}(\mathbf{a})$*. In particular,* φ *maps the Loday associahedron* $\mathrm{Post}_n(\mathbf{1})$ *to the Buchstaber associahedron* $\mathrm{RSS}_n(\mathbf{g_0})$*.*

Proof.

$$\begin{aligned} y_j - y_i &\geq g_{i,j} \\ (x_{i+1} + \cdots + x_j) + ((i+1) + \cdots + j) &\geq g_{i,j} \\ x_{i+1} + \cdots + x_j &\geq g_{i,j} - \tfrac{(i+j+1)(j-i)}{2}. \end{aligned}$$ □

Corollary 3.16 (Minkowski sum decomposition of $\mathrm{RSS}_n(\mathbf{g})$)**.** *Every Rote–Santos–Streinu associahedron can be written as*

$$\mathrm{RSS}_n(\mathbf{g}) = \sum_{1 \leq i \leq j \leq n} b_{i,j} \widetilde{\Delta}_{i,j},$$

for certain $(b_{i,j})$ *with* $b_{i,j} > 0$ *whenever* $i < j$*, and* $b_{i,i}$ *possibly negative. Here* $\widetilde{\Delta}_{i,j} = \mathrm{conv}\{u_i, u_{i+1}, \ldots, u_j\}$ *and* $u_i = (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{R}^n$ *is a* $0/1$*-vector with* i *zeros.*

3.3. **The Chapoton–Fomin–Zelevinsky associahedron.**

3.3.1. *The associahedron associated to a cluster complex.* Cluster complexes are combinatorial objects that arose in the theory of cluster algebras [9] [10] initiated by Fomin and Zelevinsky. They correspond to the normal fans of polytopes known as generalized associahedra because the particular case of type A_n yields to the classical associahedron. This polytope was constructed by Chapoton, Fomin and Zelevinsky in [6]. We refer to [11], [8] and [6] for more detailed presentations.

3.3.2. *The cluster complex of type* A_n. The *root system of type* A_n is the set $\Phi := \Phi(A_n) = \{e_i - e_j,\ 1 \leq i \neq j \leq n+1\} \subset \mathbb{R}^{n+1}$. The *simple roots* of type A_n are the elements of the set $\Pi = \{\alpha_i = e_i - e_{i+1}, i \in [n]\}$, the set of *positive roots* is $\Phi_{>0} = \{e_i - e_j : i < j\}$, and the set of *almost positive roots* is $\Phi_{\geq -1} := \Phi_{>0} \cup -\Pi$.

There is a natural correspondence between the set $\Phi_{\geq -1}$ and the diagonals of the $(n+3)$-gon P_{n+3}: We identify the negative simple roots $-\alpha_i$ with the diagonals on the *snake* of P_{n+3} illustrated in Figure 10.

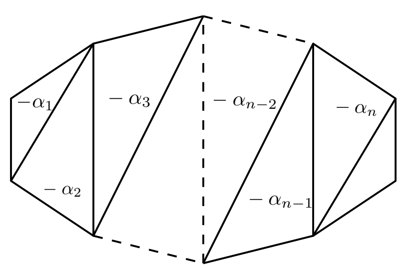

FIGURE 10. Snake and negative roots of type A_n.

Each positive root is a consecutive sum

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \qquad 1 \leq i \leq j \leq n,$$

and thus is identified with the unique diagonal of P_{n+3} crossing the (consecutive) diagonals that correspond to $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$.

Definition 3.17 (Cluster complex of type A_n)**.** Two roots α and β in $\Phi_{\geq -1}$ are *compatible* if their corresponding diagonals do not cross. The *cluster complex* $\Delta(\Phi)$ of type A_n is the clique complex of the compatibility relation on $\Phi_{\geq -1}$, i.e., the complex whose simplices correspond to the sets of almost positive roots that are pairwise compatible. Maximal simplices of $\Delta(\Phi)$ are called *clusters*.

In this case, the cluster complex satisfies the following correspondence, which is dual to the complex of the associahedron:

vertices	$\longleftrightarrow$	diagonals of a convex $(n+3)$-gon
simplices	$\longleftrightarrow$	polyhedral subdivisions of the $(n+3)$-gon (viewed as collections of non-crossing diagonals)
maximal simplices	$\longleftrightarrow$	triangulations of the $(n+3)$-gon (viewed as collections of n non-crossing diagonals)

Theorem 3.18 ([11, Thms. 1.8, 1.10])**.** *The simplicial cones $\mathbb{R}_{\geq 0}C$ generated by all clusters C of type A_n form a complete simplicial fan in the ambient space*

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

Theorem 3.19 ([6, Thm. 1.4])**.** *The simplicial fan in* Theorem 3.18 *is the normal fan of a simple n-dimensional polytope P.*

Theorem 3.18 is the case of type A_n of [11, Thm. 1.10]. It allows us to think of the cluster complex as the complex of a complete simplicial fan. Theorem 3.19 was conjectured by Fomin and Zelevinsky [11, Conj. 1.12] and subsequently proved by Chapoton, Fomin, and Zelevinsky [6]. For an explicit description by inequalities see [6, Cor. 1.9]. These two theorems are special cases of Theorems 5.1 and 5.2, proved in Section 5.

3.3.3. *The Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}_n(A_n)$*.*

Definition 3.20. The *Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}_n(A_n)$ is any polytope whose normal fan is the fan with maximal cones $\mathbb{R}_{\geq 0}C$ generated by all clusters C of type A_n.

Proposition 3.21 ([11, 6])**.** $\mathrm{CFZ}_n(A_n)$ *is an n-dimensional associahedron.*

A polytopal realization of the associahedron $\mathrm{CFZ}_2(A_2)$ is illustrated in Figure 11; note how the facet normals correspond to the almost positive roots of A_2.

Theorem 3.22. $\mathrm{CFZ}_n(A_n)$ *has exactly n pairs of parallel facets. These correspond to the pairs of roots $\{\alpha_i, -\alpha_i\}$, for $i = 1, \ldots, n$, or, equivalently, to the pairs of diagonals $\{\alpha_i, -\alpha_i\}$ as indicated in* Figure 12. □

4. Exponentially many realizations, by Hohlweg–Lange

4.1. **The Hohlweg–Lange construction.** In this section we give a short description of the first, “type I”, exponential family of realizations of the associahedron, as obtained by Hohlweg and Lange in [16]. We prove that the number of normally non-isomorphic

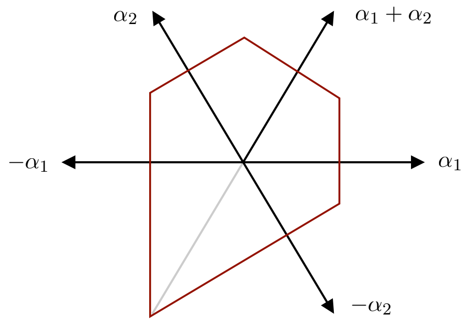

FIGURE 11. The complete simplicial fan of the cluster complex of type A_2 and an associahedron $\mathrm{CFZ}_2(A_2)$.

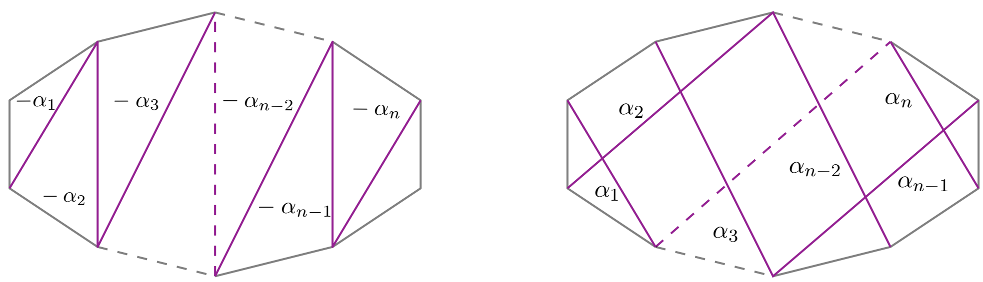

FIGURE 12. The diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of $\mathrm{CFZ}_n(A_n)$.

realizations obtained this way is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal. This number is equal to $2^{n-3}+2^{\lfloor\frac{n-3}{2}\rfloor}$ for $n\geq 3$ (see [29, Sequence A005418]).

Let $\sigma\in\{+,-\}^{n-1}$ be a sequence of signs on the edges of an horizontal path on n nodes. We identify $n+3$ vertices $\{0,1,\dots,n+1,n+2\}$ with the signs of the sequence $\widetilde{\sigma}=\{+,-,\sigma,-,+\}$, and place them in convex position from left to right so that all positive vertices are above the horizontal path, and all negative vertices are below it. These vertices form a convex $(n+3)$-gon that we call $P_{n+3}(\sigma)$. Figure 4.1 illustrates the example $P_7(\{+,-,+\})$, where $n=4$.

Definition 4.1. For a diagonal ij $(i<j)$ of $P_{n+3}(\sigma)$, we denote by $R_{ij}(\sigma)$ the set of vertices strictly below it. We define the set $S_{ij}(\sigma)$ as the result of replacing 0 by i in $R_{ij}(\sigma)$ if $0\in R_{ij}(\sigma)$, and replacing $n+2$ by j if $n+2\in R_{ij}(\sigma)$.

The *Hohlweg–Lange associahedron* $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is the polytope

$$\mathrm{Ass}_n^{\mathrm{I}}(\sigma)=\Big\{(x_1,\dots x_{n+1})\in\mathbb{R}^{n+1}:\ \sum_{i\in S_\delta(\sigma)}x_i\geq\tfrac{1}{2}|S_\delta(\sigma)|(|S_\delta(\sigma)|+1)\text{ for all diagonals }\delta,$$
$$x_1+\cdots+x_{n+1}=\tfrac{(n+1)(n+2)}{2}\Big\}.$$

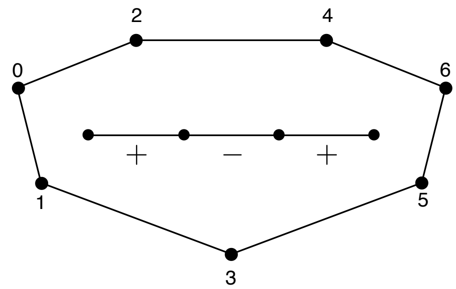

FIGURE 13. $P_7(\{+,-,+\})$.

Remark 4.2. If in $\widetilde{\sigma} = \{+,-,\sigma,-,+\}$ we interchange the first two signs and/or the last two signs, the sets $S_\delta(\sigma)$ do not change and the construction will produce the same associahedron $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$.

Proposition 4.3 ([16, Thm. 1.1])**.** $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *is an n-dimensional associahedron.*

Proposition 4.4 ([16, Remarks 1.2 and 4.3])**.** $\mathrm{Ass}_n^{\mathrm{I}}(\{-,-,\dots,-\})$ *produces the Postnikov (Loday) associahedron* $\mathrm{Post}_n(\mathbf{1})$*, and* $\mathrm{Ass}_n^{\mathrm{I}}(\{+,-,+,-,\dots\})$ *is normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron* $\mathrm{CFZ}_n(A_n)$*.*

Proof. For the first part we note that for $\sigma = \{-,-,\cdots-\}$, the set $S_{p,q}(\sigma)$ of a diagonal pq is given by $S_{p,q} = \{i : p < i < q\}$, and that the description of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ coincides with that of $\mathrm{Post}_n(\mathbf{a})$ in Lemma 3.10 for $\mathbf{a} = 1$. For the second part let $\sigma = \{+,-,+,-,\dots\}$. We write S_δ instead of $S_\delta(\sigma)$ for simplicity, and denote by $I_S \in \mathbb{R}^{n+1}$ the 0/1 vector with ones in the positions of a set $S \subseteq [n+1]$. The snake triangulation is given by the set of diagonals of the form $i, i+1$, for $1 \le i \le n$ (in the case where $n, n+1$ is not a diagonal we interchange vertices $n+1$ and $n+2$; this doesn't change the associahedron we get, see Remark 4.2). We denote by $-\alpha_i = I_{S_{i,i+1}}$ the normal vector associated to the diagonal $i, i+1$, and by $n_{i,j} = I_{S_{i-1,j+2}}$ $(i \le j)$ the normal vector associated to the diagonal crossing $\{-\alpha_i, -\alpha_2, \dots, -\alpha_j\}$. We need to prove that

$$n_{i,j} \equiv \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mod (1,\dots,1).$$

The reason is that our polytope lies in an affine hyperplane orthogonal to the vector $(1,\dots,1)$, and so we must consider the normal vectors modulo $(1,\dots,1)$. To this end, note that

$$n_{i,i} = \alpha_i + (1,\dots,1)$$

and

$$n_{i,j+1} = \begin{cases} n_{i,j} + (1,\dots,1) + \alpha_{j+1} & \text{if } j \text{ is odd,} \\ n_{i,j} + \alpha_{j+1} & \text{if } j \text{ is even.} \end{cases}$$ □

Remark 4.5. The Postnikov associahedron was defined as a Minkowski sum of certain faces Δ_S of the standard simplex $\Delta_{[n+1]}$. The question arises whether such Minkowski sum descriptions exist for $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ in general. A partial answer is as follows. Postnikov [24] introduced *generalized permutahedra* as the polytopes with facet normals contained in those of the standard permutahedron such that the collection of right hand side parameters of the defining inequalities belongs to the deformation cone of the standard permutahedron. This includes all the Minkowski sums $\sum_{S\subseteq[n+1]} a_S\Delta_S$ for which the coefficients a_S are non-negative. Ardila et al. [1] have shown that by dropping the deformation cone condition every polytope of the resulting family admits a (unique) expression as a Minkowski *sum and difference* of faces of the standard simplex. These decompositions, for the case of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$, are studied by Lange in [19]. A different decomposition arises from the work of Pilaud and Santos [23], who show that the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are the "brick polytopes" of certain sorting networks. As such, they admit a decomposition as the Minkowski sum of the $\binom{n}{2}$ polytopes associated to the individual "bricks". However, these summands need not be simplices.

4.2. **Parallel facets.**

Theorem 4.6. $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ *has exactly n pairs of parallel facets. They correspond to the diagonals of the quadrilaterals with vertices $\{i, j, j+1, k\}$ for $j = 1, \dots, n$, where*

$$i = \max\{0 \le r < j : \mathrm{sign(r)} \cdot \mathrm{sign(j)} = -\}$$
$$k = \min\{j+1 < r \le n+2 : \mathrm{sign(r)} \cdot \mathrm{sign(j+1)} = -\}$$

Proof. Two diagonals δ and δ' correspond to two parallel facets of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ if and only if the sets S_δ and $S_{\delta'}$ satisfy $S_\delta \cup S_{\delta'} = [n+1]$ and $S_\delta \cap S_{\delta'} = \emptyset$. These two properties hold if and only if δ and δ' are the diagonals of the quadrilateral $\{i, j, j+1, k\}$ for $j = 1, \dots, n$, and i and k satisfying the conditions of the theorem. □

Associated to a sequence σ we define two operations, reflection and reversal. The reflection of σ is the sequence $-\sigma$, and the reversal σ^t is the result of reversing the order of the signs in σ.

Theorem 4.7. *Let $\sigma_1, \sigma_2 \in \{+,-\}^{n-1}$. Then the two realizations $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_1)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_2)$ are normally isomorphic if and only if σ_2 can be obtained from σ_1 by reflections and reversals.*

Proof. Suppose there is a linear isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_1)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma_2)$. It induces an automorphism of the face lattice of the associahedron that, by Lemma 2.3, corresponds to a certain reflection-rotation of the polygon. We denote this reflection-rotation by $\varphi : P_{n+3}(\sigma_1) \to P_{n+3}(\sigma_2)$. Any linear isomorphism of the normal fans preserves the property of a pair of facets being parallel, so φ maps the "parallel" pairs of diagonals of $P_{n+3}(\sigma_1)$, to the "parallel" pairs of diagonals of $P_{n+3}(\sigma_2)$. Furthermore, for both realizations there are exactly four diagonals that cross at least one diagonal of every parallel pair; they are $\{0, n+1\}, \{0, n+2\}, \{1, n+1\}$ and $\{1, n+2\}$. The set of these four diagonals is also preserved under φ. This is possible only if φ is a reflection-rotation that corresponds to a composition of reflections and reversals of the sequence $\widetilde{\sigma_1} = \{+, -, \sigma_1, -, +\}$.

It remains to prove that $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is normally-isomorphic to both $\mathrm{Ass}_n^{\mathrm{I}}(-\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma^t)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(-\sigma)$ is given by multiplication by -1, since $S_\delta(-\sigma) = [n] - S_\delta(\sigma)$. The isomorphism between the normal fans of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{I}}(\sigma^t)$ is given by the permutation of coordinates $\tau(i) = n + 1 - i$, as $S_\delta(\sigma^t) = \tau(S_\delta(\sigma))$. □

Corollary 4.8. *The Postnikov associahedron is not normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron.*

Proof. The Postnikov associahedron is produced by the sequence $\sigma_1 = \{-, -, \ldots, -\}$, and the Chapoton–Fomin–Zelevinsky associahedron is normally isomorphic to the one produced by the sequence $\sigma_2 = \{+, -, +, -, \ldots\}$. The two sequences are not equivalent under reflections and reversals. □

4.3. **Facet vectors.** We now show that $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ can (modulo normal isomorphism) be embedded in $\mathbb{R}^n$ so that its facet normals are a subset of $\{0, -1, +1\}^n$ and contain the n standard basis vectors and their negatives among them. That is, it can be obtained from a cube by cutting certain faces, as in Figures 2 and 3.

Obviously, the basis vectors and their negatives will correspond to the n pairs of parallel facets that we identified in Theorem 4.6. Each such pair consists of a diagonal with positive slope and one with negative slope. We choose as "positive basis vector" the one with positive slope, which can be characterized as follows:

Lemma 4.9. *Let* $\{i, j, j+1, k\}$ *for* $j = 1, \ldots, n$ *be as in* Theorem 4.6. *Let*

$$\begin{aligned} a &:= \max\{0 \le r \le j : \mathrm{sign(r)} = -\}, \\ b &:= \min\{j + 1 \le r \le n + 2 : \mathrm{sign(r)} = +\}. \end{aligned}$$

Then ab *is one of the diagonals of the quadrilateral with vertices* $\{i, j, j+1, k\}$ *and it has positive slope.*

Proof. By construction, $\{i, j, j+1, k\}$ has two positive points and two negative points (i and j have opposite sign, as have $j+1$ and k). Our definition of a and b is equivalent to: a is the negative point in $\{i, j\}$ and b is the positive point in $\{j + 1, k\}$. □

As customary, we call *characteristic vector* of a set $S \subset [n+1]$ the vector in $\{0, 1\}^{n+1}$ with 1's in the coordinates of the elements of S. We denote it e_S. In particular, the i-th standard basis vector is $e_i = e_{\{i\}}$.

For each $j = 1, \ldots, n$, let $X_j = e_{S_{a,b}(\sigma)}$, where a and b are as in Lemma 4.9 and $S_{a,b}(\sigma)$ is from Definition 4.1. Then X_j is normal to the facet of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ corresponding to the diagonal ab, one of the facets in the j-th parallel pair. By convention, let $X_{n+1} = e_\emptyset = (0, \ldots, 0)$ and $X_0 = e_{[n+1]} = (1, \ldots, 1)$.

Theorem 4.10. *For every* $S \subset [n+1]$, *the characteristic vector of* S *is a linear combination of* $\{X_0, \ldots, X_{n+1}\}$ *with coefficients in* $\{0, +1, -1\}$.

Proof. Since

$$e_S = \sum_{j \in S} e_j,$$

the statement follows from the formula

$$e_j = X_{j-1} - X_j, \qquad \forall j \in [n],$$

which we prove distinguishing the case of j being positive or negative (the cases $j = 1$ and $j = n + 1$ need separate treatment, but the formula holds for them too). Let a and b be as in Lemma 4.9 and let a' and b' be the same, but computed for $j - 1$ instead of j. That is, let X_{j-1} be the characteristic vector of $S_{a',b'}$. If j is positive, then $a = a'$, $b' = j$ and b is the next positive point after j. If j is negative, then $b = b'$, $a = j$ and a' is the previous negative point before j. □

Definition 4.1 says that the characteristic vector of $S_\delta(\sigma)$ is a normal vector to the facet of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ corresponding to a certain diagonal δ. Since $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ is not full-dimensional, the normal to each facet is not unique. Others are obtained adding multiples of $e_{[n+1]} = (1, \dots, 1)$ to it. Put differently, the normal fan of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ lives naturally in $(\mathbb{R}^{n+1})^*/\langle X_0 \rangle$. For the basis $\{X_1, \dots, X_n\}$ in this space, Theorem 4.10 yields the following.

Corollary 4.11. *With respect to the basis $\{X_1, \dots, X_n\}$, the normal vectors of $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ are all in $\{0, +1, -1\}^n$ and include the $2n$ vectors $\{\pm X_1, \dots, \pm X_n\}$.*

5. Catalan many realizations, by Santos

In this section we describe a generalization of the Chapoton–Fomin–Zelevinsky construction of the associahedron (Section 3.3), originally presented at a conference in 2004 [27]. We prove that the number of normally non-isomorphic realizations obtained this way, our "type II exponential family", is equal to the number of triangulations of an $(n + 3)$-gon modulo reflections and rotations. Interest in this number goes back to Motzkin (1948) [22]. An explicit formula for it is

$$\tfrac{1}{2(n+3)} C_{n+1} + \tfrac{1}{4} C_{(n+1)/2} + \tfrac{1}{2} C_{\lfloor (n+1)/2 \rfloor} + \tfrac{1}{3} C_{n/3},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \in \mathbb{Z}$ and $C_n = 0$ otherwise [29, Sequence A000207].

Let $\alpha_1, \dots, \alpha_n$ denote a linear basis of an n-dimensional real vector space $V \cong \mathbb{R}^n$, and let T_0 be a certain triangulation of the $(n + 3)$-gon, fixed once and for all throughout the construction. We call T_0 the *seed triangulation.* The CFZ associahedron will arise as the special case where $V = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i = 0\}$, $\alpha_i = e_i - e_{i+1}$, and T_0 is the *snake triangulation* of Figure 10.

Let $\{\delta_1, \dots, \delta_n\}$ denote the n diagonals present in the seed triangulation T_0. To each diagonal pq out of the $\frac{n(n+3)}{2}$ possible diagonals of the n-gon we associate a vector v_{pq} as follows:

- ◦ If $pq = \delta_i$ for some i (that is, if pq is used in T_0) then let $v_{pq} = -\alpha_i$.

∘ If $pq \notin T_0$ then let
$$v_{pq} := \sum_{pq \text{ crosses } \delta_i} \alpha_i.$$

As a running example, consider the triangulation $\{123, 345, 156, 135\}$ of a hexagon with its vertices labelled cyclically. Let $\delta_1 = 13$, $\delta_2 = 35$ and $\delta_3 = 15$. Written with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ the nine vectors v_{pq} that we get are as follows (see Figure 14):

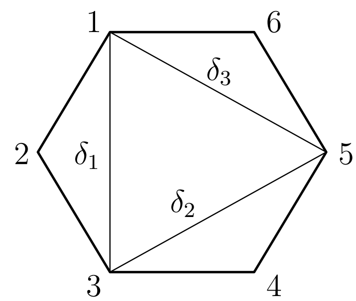

FIGURE 14. A seed triangulation for Santos' construction.

$$\begin{array}{lll} v_{13} = -\alpha_1 = (-1,0,0), & v_{35} = -\alpha_2 = (0,-1,0), & v_{15} = -\alpha_3 = (0,0,-1), \\ v_{25} = \alpha_1 = (1,0,0), & v_{14} = \alpha_2 = (0,1,0), & v_{36} = \alpha_3 = (0,0,1), \\ v_{46} = \alpha_2 + \alpha_3 = (0,1,1), & v_{26} = \alpha_1 + \alpha_3 = (1,0,1), & v_{24} = \alpha_1 + \alpha_2 = (1,1,0). \end{array}$$

With a slight abuse of notation, for each subset of diagonals of the polygon we denote with the same symbol the set of diagonals and the set of vectors associated with them. For example, $\mathbb{R}_{\geq 0} T_0 = \mathbb{R}_{\geq 0}\{-\alpha_1, \dots, -\alpha_n\}$ is the negative orthant in V (with respect to the basis $[\alpha_i]_i$). More generally, for each triangulation T of the $(n+3)$-gon consider the cone $\mathbb{R}_{\geq 0} T$. We claim the following generalizations of Theorems 3.18 and 3.19, and Proposition 3.21:

Theorem 5.1. *The simplicial cones $\mathbb{R}_{\geq 0} T$ generated by all triangulations T of the $(n+3)$-gon form a complete simplicial fan $\mathcal{F}_{T_0}$ in the ambient space V.*

Theorem 5.2. *This fan $\mathcal{F}_{T_0}$ is the normal fan of an n-dimensional associahedron.*

5.1. **Proof of Theorem 5.1.** The statement follows from the following two claims:

(1) $\mathbb{R}_{\geq 0} T_0$ is a simplicial cone and is the only cone in $\mathcal{F}_{T_0}$ that intersects (the interior of) the negative orthant.
(2) If T_1 and T_2 are two triangulations that differ by a flip, let $v_1 \in T_1$ and $v_2 \in T_2$ be the diagonals removed and inserted by the flip. That is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$. Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign (and different from zero) in the elements v_1 and v_2.

The first assertion is obvious, and the second one is Lemma 5.3 below. Before proving it let us argue why these two assertions imply Theorem 5.1. Suppose that we have two triangulations T_1 and T_2 related by a flip as in the second assertion, and suppose that we already know that one of them, say T_1, spans a full-dimensional cone (that is, we know that T_1 considered as a set of vectors is independent). Then assertion (2) implies that T_2 spans a full-dimensonal cone as well and that $\mathbb{R}_{\geq 0}T_1$ and $\mathbb{R}_{\geq 0}T_2$ lie in opposite sides of their common facet $\mathbb{R}_{\geq 0}(T_1 \cap T_2)$. This, together with the fact that there is some part of V covered by exactly one cone (which is why we need assertion (1)) implies that we have a complete fan. (See, for example, [7, Cor. 4.5.20], where assertion (2) is a special case of "property (ICoP)" and assertion (1) a special case of "property (IPP)".)

Lemma 5.3. *Let T_1 and T_2 be two triangulations that differ by a flip, and let v_1 and v_2 be the diagonals removed and inserted by the flip from T_1 to T_2, respectively (that is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$). Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign in the elements v_1 and v_2.*

Proof. Let p, q, r and s be the four points involved by the two diagonals v_1 and v_2, in cyclic order. That is, the diagonals removed and inserted are pr and qs. We claim that one (and exactly one) of the following things occurs (see Figure 15):

(a) There is a diagonal in the seed triangulation T_0 that crosses two opposite edges of the quadrilateral $pqrs$.
(b) One of pr and qs is used in the seed triangulation T_0.
(c) There is a triangle abc in T_0 with a vertex in $pqrs$ and the opposite edge crossing two sides of $pqrs$ (that is, without loss of generality $p = a$ and bc crosses both qr and rs).
(d) There is a triangle abc in T_0 with an edge in common with $pqrs$ and with the other two edges of the triangle crossing the opposite edge of the quadrilateral (that is, without loss of generality, $p = a$, $q = b$ and rs crosses both ac and bc).

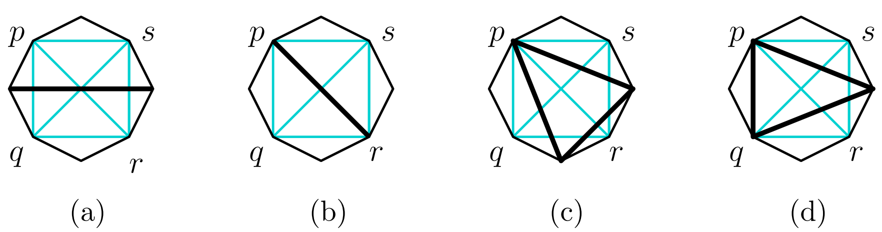

FIGURE 15. The four cases in the proof of Lemma 5.3.

To prove that one of the four things occurs we argue as follows. It is well-known that in any triangulation of a k-gon one can "contract a boundary edge" to get a triangulation of a $(k-1)$-gon. Doing that in all the boundary edges of the seed triangulation T_0 except those incident to either p, q, r or s we get a triangulation $\widetilde{T_0}$ of a polygon $\widetilde{P}$ with at most eight vertices: the four vertices p, q, r and s and at most one extra vertex between each two of them. We embed $\widetilde{P}$ having as vertex a subset of the vertices of a regular octagon, with $pqrs$ forming a square. We now look at the position of the center of the octagon $\widetilde{P}$ with respect to the triangulation $\widetilde{T_0}$: If it lies in the interior of an edge, then this edge is a diameter of the octagon and we are in cases (a) or (b). If it lies in the interior of a triangle of $\widetilde{T_0}$, then we are in cases (c) or (d). See Figure 15 again.

Now we show explicitly the linear dependences involved in $T_1 \cup T_2$ in each case.

(a) Suppose T_0 has a diagonal crossing pq and rs. Then

$$v_{pr} + v_{qs} = v_{pq} + v_{rs}, \tag{1}$$

because every diagonal of T_0 intersecting the two (respectively, one; respectively none) of pr and qs intersects also the two (respectively, one; respectively none) of pq and rs.

(b) If T_0 contains the diagonal pr, let a and b be vertices joined to pr in T_0, with a on the side of q and b on the side of s. We define the following vectors w_a and w_b:
- ◦ w_a equals 0, v_{pq} or v_{qr} depending on whether a equals q, lies between p and q, or lies between q and r.
- ◦ w_b equals 0, v_{ps} or v_{rs} depending on whether a equals s, lies between p and s, or lies between s and r.

We claim that in the nine cases we have the equality

$$v_{pr} + v_{qs} = w_a + w_b. \tag{2}$$

This is so because $v_{pr} + v_{qs}$ now equals the sum of the α_i's corresponding to the diagonals of $T_0 \setminus \{pr\}$ crossing qs, and we have that:
- ◦ The diagonals of T_0 crossing qs in the q-side of pr are none, the same as those crossing pq, or the same as those crossing qr in the three cases of the definition of w_a, and
- ◦ The diagonals of T_0 crossing qs in the s-side of pr are none, the same as those crossing ps, or the same as those crossing rs in the three cases of the definition of w_b

(c) If T_0 contains a triangle pbc with bc crossing both qr and rs then we have the equality

$$2v_{pr} + v_{qs} = v_{qr} + v_{rs}, \tag{3}$$

because in this case the diagonals of T_0 crossing pr are the same as those crossing both qr and rs, while the ones crossing qs are those crossing one, but not both, of qr and rs.

(d) If T_0 contains a triangle pqc with rs crossing both pc and qc then we have the equality

$$v_{pr} + v_{qs} = v_{rs} \tag{4}$$

because the diagonals of T_0 crossing rs are the same as those crossing pr and the same as those crossing qs. □

Observe that when T_0 is a snake triangulation (the CFZ case) or, more generally, when the dual tree of T_0 is a path, cases (c) and (d) do not occur.

5.2. **Proof of Theorem 5.2.** To prove that $\mathcal{F}_{T_0}$ is the normal fan of a polytope we use the following characterization.

Lemma 5.4. *Let $\mathcal{F}$ be a complete simplicial fan in a real vector space V and let A be the set of generators of $\mathcal{F}$ (more precisely, A has one generator of each ray of $\mathcal{F}$). Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is the normal fan of a polytope.*

(2) *There is a map* $\omega : A \to \mathbb{R}_{>0}$ *such that for every pair* (C_1, C_2) *of maximal adjacent cones of* $\mathcal{F}$ *the following happens: Let* $\lambda : A \to \mathbb{R}$ *be the (unique, up to a scalar multiple) linear dependence with support in* $C_1 \cup C_2$*, with its sign chosen so that* λ *is positive in the generators of* $C_1 \setminus C_2$ *and* $C_2 \setminus C_1$*. Then the scalar product* $\lambda \cdot \omega = \sum_v \lambda(v)\omega(v)$ *is strictly positive.*

Proof. One short proof of the lemma is that both conditions are equivalent to "$\mathcal{F}$ is a regular triangulation of the vector configuration A" [7]. But let us show a more explicit proof of the implication from (2) to (1), which is the one we need. What we are going to show is that if such an ω exists and if we consider the set of points

$$\widetilde{A} := \left\{ \tfrac{v}{\omega(v)} : v \in A \right\},$$

then the convex hull of $\widetilde{A}$ is a simplicial polytope with the same face lattice as the complete fan $\mathcal{F}$. (We think of $\widetilde{A}$ as points in an affine space, rather than as vectors in a vector space.) Hence $\mathcal{F}$ is the *central fan* of $\operatorname{conv}(\widetilde{A})$, which coincides with the normal fan of the polytope polar to $\operatorname{conv}(\widetilde{A})$.

To show the claim on $\operatorname{conv}(\widetilde{A})$ we argue as follows. Consider the simplicial complex Δ with vertex set $\widetilde{A}$ obtained by embedding the face lattice of $\mathcal{F}$ in it. That is, for each cone C of $\mathcal{F}$ we consider the simplex with vertex set in $\widetilde{A}$ corresponding to the generators of C. Since $\mathcal{F}$ is a complete fan and since the elements of $\widetilde{A}$ are generators for its rays (they are positive scalings of the elements of A), Δ is the boundary of a star-shaped polyhedron with the origin in its kernel. The only thing left to be shown is that this polyhedron is strictly convex, that is, that for any two adjacent maximal simplices σ_1 and σ_2 the origin lies in the same side of σ_1 as $\sigma_2 \setminus \sigma_1$ (or, equivalently, in the same side of σ_2 as $\sigma_1 \setminus \sigma_2$). Equivalently, if we understand σ_1 and σ_2 as subsets of $\widetilde{A}$, we have to show that the unique affine dependence between the points $\{O\} \cup \sigma_1 \cup \sigma_2$ has opposite sign in O than in σ_1 and σ_2.

Now the proof is easy. The coefficients in the *linear* dependence among the *vectors* in $\sigma_1 \cup \sigma_2$ are the vector

$$(\lambda(v)\omega(v))_{v \in A}.$$

To turn this into an *affine* dependence of points involving the origin we simply need to give the origin the coefficient $-\sum_v \lambda(v)\omega(v)$ which is, by hypothesis, negative. □

So, in the light of Lemma 5.4, to prove Theorem 5.2 we simply need to choose weights ω_{ij} for the diagonals of the polygon with the property that, for each of the linear dependences exhibited in equations (1), (2), (3), and (4), the equation $\sum_{ij} \omega_{ij}\lambda_{ij} > 0$ holds.

As a first approximation, let $\omega_{ij} = 2$ if ij is in T_0 and $\omega_{ij} = 1$ otherwise. This is good enough for equations (3) and (4) in which all the ω's in the dependence are 1 and the sum of the coefficients in the left-hand side is greater than in the right-hand side. It also works for equations (2), in which we have

$$\omega_{pr} = 2, \qquad \omega_{qs} = 1, \qquad \lambda_{pr} = 1, \qquad \lambda_{qs} = 1,$$

so that the sum $\sum_{ij} \omega_{ij}\lambda_{ij}$ for the left-hand side is three, while that of the right-hand side can be 0, -1 or -2 depending on the cases for the points a and b.

The only (weak) failure is that in equation (1) we have

$$\lambda_{pr} = 1, \qquad \lambda_{qs} = 1, \qquad \lambda_{pq} = -1, \qquad \lambda_{rs} = -1$$

and all the ω's are 1, so we get $\sum_{ij} \omega_{ij}\lambda_{ij} = 0$. We solve this by slightly perturbing the ω's. A slight perturbation will not change the correct signs we got for equations (2), (3), and (4). For example, for each ij not in T_0 change ω_{ij} to

$$\omega_{ij} = 1 + \varepsilon g_{ij}$$

for a sufficiently small $\varepsilon > 0$ and for a vector $(g_{ij})_{ij}$ satisfying

$$g_{ik} + g_{jl} > \max\{g_{ij} + g_{kl}, g_{il} + g_{jk}\} \quad \text{for all } i, j, k, l,\ 1 \le i < j < k < l \le n + 3.$$

This holds (for example) for $g_{ij} := (j - i)(n + 3 + i - j)$.

5.3. **Distinct seed triangulations produce distinct realizations.** Let $\mathrm{Ass}_n^{\mathrm{II}}(T)$ denote the n-dimensional associahedron obtained with the construction of the previous section starting with a certain triangulation T. (This is a slight abuse of notation, since the associahedron depends also in the weight vector ω that gives the right-hand sides for an inequality definition of our associahedron. Put differently, by $\mathrm{Ass}_n^{\mathrm{II}}(T)$ we denote the normal fan rather than the associahedron itself.) We want to classify the associahedra $\mathrm{Ass}_n^{\mathrm{II}}(T)$ by normal isomorphism.

In principle, it looks like we have as many associahedra as there are triangulations (that is, Catalan-many) but that is not the case because, clearly, changing T by a rotation or a reflection does not change the associahedron obtained. The question is whether this is the only operation that preserves $\mathrm{Ass}_n^{\mathrm{II}}(T)$, modulo normal isomorphism. The answer is yes, as we show below.

Lemma 5.5. $\mathrm{Ass}_n^{\mathrm{II}}(T_0)$ *has exactly n pairs of parallel facets, each pair consisting of (the facet of) one diagonal in T_0 and the diagonal obtained from it by a flip in T_0.*

Proof. As always, a necessary condition for the facets corresponding to two diagonals to be parallel is that the diagonals cross; if the diagonals do not cross, they are present in some common triangulation which implies the corresponding facets intersect.

So, let pr and qs be two crossing diagonals. Since $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is full-dimensional, their facets are parallel only if v_{pr} and v_{qs} are linearly dependent. By definition of the vectors v_{ij} this only happens when $\{v_{pr}, v_{qs}\} = \{\pm\alpha_i\}$ for some i, which is the case of the statement. □

Lemma 5.6. *Let Q be an $(n+3)$-gon, with $n \ge 2$. For each triangulation T of Q let B_T denote the set consisting of the n diagonals in T plus the n diagonals that can be introduced by a single flip from T. Then for every $T_1 \ne T_2$ we have $B_{T_1} \ne B_{T_2}$.*

Proof. Suppose that T_1 and T_2 had $B_{T_1} = B_{T_2}$. We claim that T_2 is obtained from T_1 by a set of "parallel flips". That is, by choosing a certain subset of diagonals of T_1 such that no two of them are incident to the same triangle and flipping them simultaneously. This is so because every diagonal pr in T_2 but not in T_1 intersects a single diagonal qs of T_1. If

pqr and prs were not triangles in T_2, then let a be a vertex joined to pr in T_2, different from q or s. One of pa and ra intersects the diagonal qs of T_1 and one of the edges pq, qr, rs and pr of T_1.

Once we have proved this for T_2, the statement is obvious. For every T_2 different from T_1 but with all its diagonals in B_{T_1} there is a diagonal that we can flip to get one that is not in B_{T_1} (same argument, let pr be a diagonal in T_2 but not in T_1; let pq, qr, rs and pr be the other sides of the two triangles of T_2 containing pq. Flipping any of them, say pq, gives a diagonal that crosses pq and qs, which are both in T_1). □

Corollary 5.7. *Let T_1 and T_2 be two triangulations of a convex $(n+3)$-gon. Then $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic if and only if T_1 and T_2 are equivalent under rotation-reflection.*

Proof. If T_1 and T_2 are equivalent under rotation-reflection then the resulting associahedra are clearly the same. Now suppose that $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$ are normally isomorphic. By Lemma 2.3 the automorphism of the associahedron face lattice induced by the isomorphism corresponds to a rotation-reflection of the polygon. Now, normal isomorphism preserves the property of a pair of facets being parallel, so using the previous lemma we get that this rotation-reflection sends T_1 to T_2. □

However, the same is not true if we only look at the set of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T)$:

Proposition 5.8. *Let T_1 and T_2 be two triangulations of the $(n+3)$-gon. Let $A(T_1)$ and $A(T_2)$ be the sets of normal vectors of $\mathrm{Ass}_n^{\mathrm{II}}(T_1)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T_2)$. Then $A(T_1)$ and $A(T_2)$ are linearly equivalent if, and only if, T_1 and T_2 have isomorphic dual trees.*

Proof. Let $\mathcal{T}$ be the dual tree of a triangulation T. Observe that the edges of $\mathcal{T}$ correspond bijectively to the inner diagonals in T. Moreover, the diagonals of the polygon not used in T correspond bijectively to the possible paths in $\mathcal{T}$. More precisely: for every pair of nodes of $\mathcal{T}$, the two corresponding triangles of T have the property that one edge of each triangle "see each other". Let p and q be the vertices of the two triangles opposite (equivalently, not incident) to those two edges. Then the diagonals of T crossed by pq correspond to the path in $\mathcal{T}$ joining the two nodes.

This means that, if we label the edges of $\mathcal{T}$ with the numbers 1 through n in the same manner as we labelled the diagonals of T we have that

$$A(T) = \{-\alpha_i : i \in [n]\} \cup \{\sum_{i\in p} \alpha_i : p \text{ is a path in } \mathcal{T}\}.$$

In particular, $A(T)$ can be recovered knowing only $\mathcal{T}$ as an abstract graph. For the converse, observe that if two trees are not isomorphic then there is no bijection between their edges that sends paths to paths. For example, knowing only the sets of edges that form paths we can identify the (stars of) vertices of the tree as the sets of edges such that every two of them form a path. □

In particular, this gives us exponentially many ways of embedding the associahedron of dimension n with facet normals in the root system of A_n:

Corollary 5.9. *Let T_0 be a triangulation whose dual tree is a path. Let its diagonals be numbered from 1 to n in the order they appear in the path. Then, taking $\alpha_i = e_{i+1} - e_i$, we have that $A(T_0)$ is the set of almost positive roots in the root system A_n.*

The number of normally non-isomorphic classes of associahedra, for which the dual tree of the seed triangulation T_0 is a path, is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal.

It is surprising that the number of realizations that we get in this way is exactly the same as we got in the previous section. Nevertheless, we prove in Theorem 6.2 that the two sets of realizations are (almost) disjoint.

6. How many associahedra?

We have presented several constructions of the associahedron. We call associahedra of types I and II the associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$ studied in the previous two sections. Associahedra of type I include the Postnikov (or Rote–Santos–Streinu, or Loday, or Buchstaber) associahedron, and both types I and II include the Chapoton–Fomin–Zelevinsky associahedron. They all have pairs of parallel facets while the secondary polytope on an n-gon (according to Section 3.1) does not. This implies that:

Theorem 6.1. *The associahedron as a secondary polytope is never normally isomorphic to any associahedron of type I or type II. In particular, it is not normally isomorphic to the Postnikov associahedron or the Chapoton–Fomin–Zelevinsky associahedron.*

Both types I and II produce exponentially many normally non-isomorphic realizations. The number of normally non-equivalent associahedra of type I is asymptotically 2^{n-3}, while for type II is asymptotically $2^{2n+1}/\sqrt{\pi n^5}$. Explicit computations up to dimension 15 are given in Table 1.

$n=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathrm{Ass}_n^{\mathrm{I}}$	1	1	1	2	3	6	10	20	36	72	136	272	528	1056	2080	4160
$\mathrm{Ass}_n^{\mathrm{II}}$	1	1	1	3	4	12	27	82	228	733	2282	7528	24834	83898	285357	983244

Table 1. The number of normally non-isomorphic realizations of the associahedron of types I and II up to dimension 15.

Surprisingly, the realizations of types I and II are (almost) disjoint:

Theorem 6.2. *The only associahedron that is normally isomorphic to both one of type I and one of type II is the Chapoton–Fomin–Zelevinsky associahedron.*

Proof. Suppose that a sequence $\sigma \in \{+,-\}^{n-1}$ and a triangulation T produce normally isomorphic associahedra $\mathrm{Ass}_n^{\mathrm{I}}(\sigma)$ and $\mathrm{Ass}_n^{\mathrm{II}}(T)$. The induced automorphism between the face lattice of these two associahedra comes from a reflection-rotation map on the $(n+3)$-gon, by Lemma 2.3, so there is no loss of generality in assuming that this reflection-rotation is the identity.

Denote by B_σ and B_T the $2n$ diagonals corresponding to the n pairs of parallel facets in both constructions respectively. The diagonals of B_T consist of the diagonals of T together with its flips. Since normal isomorphisms preserve pairs of parallel facets, $B_T = B_\sigma$.

We consider the $(n+3)$-gon drawn in the Hohlweg–Lange fashion (with vertices placed along two x-monotone chains, the positive and the negative one, placed in the x-order indicated by σ). The crucial property we use is that B_σ contains only diagonals between vertices of opposite signs. Knowing this we conclude:

- *Every triangle in T contains a boundary edge in one of the chains. (That is, the dual tree of T is a path).* Suppose, in the contrary, that T has a triangle pqr with no boundary edge. Then the three diagonals pq, pr and qr lie in $B_T = B_\sigma$. This is impossible since at least two of p, q and r must have the same sign.
- *The third vertex of each triangle is in the opposite chain. (That is, the dual path of T separates the two chains).* Otherwise the three vertices of a certain triangle lie in the same chain. This is impossible, because (at least) one of the three edges of each triangle is a diagonal, hence it is in B_σ.
- *No two consecutive boundary edges in one chain are joined to the same vertex in the opposite chain. (That is, the dual tree of T alternates left and right turns).* Otherwise, let abp and bcp be two triangles in T with ab and bc consecutive boundary edges in one of the chains. Then the flip in bp inserts the edge ac, so that $ac \in B_\sigma$. This is impossible, since a and c are in the same chain.

These three properties imply that T is the snake triangulation, so $\mathrm{Ass}_n^{\mathrm{II}}(T)$ is the Chopoton–Fomin–Zelevinsky associahedron. $\square$

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