Generalizing Ramanujan's J Functions

Jerome Malenfant
American Physical Society
Ridge, NY
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We generalize Ramanujan's expansions of the fractional-power Euler functions $(q^{1/5})_\infty = [\ J_1 - q^{1/5} + q^{2/5} J_2\ ](q^5)_\infty$ and $(q^{1/7})_\infty = [\ J_1 + q^{1/7} J_2 - q^{2/7} + q^{5/7} J_3\ ](q^7)_\infty$ to $(q^{1/N})_\infty$, where N is a prime number greater than 3. We show that there are exactly $(N+1)/2$ non-zero J functions in the expansion of $(q^{1/N})_\infty$, that one of these functions has the form $\pm q^{X_0}$, that all others have the form $\pm q^{X_k} \times$ the ratio of two Ramanujan theta functions, and that the product of all the non-zero J's is $\pm q^Z$, where Z and the X's denote non-negative integers.

I. INTRODUCTION

In his study of the congruence-5 properties of the partition function $p(n)$, Ramanujan [1] made the replacement $q \to q^{1/5}$ in its generating-function equation,

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty}, \tag{1}$$

where

$$(q)_\infty \equiv \prod_{k=1}^{\infty} (1 - q^k) \tag{2}$$

is the Euler function. Then, using Euler's pentagonal number theorem,

$$(q)_\infty = \sum_{m=-\infty}^{\infty} (-1) q^{m(3m-1)/2}, \tag{3}$$

he made the expansion

$$\frac{(q^{1/5})_\infty}{(q^5)_\infty} = J_1 - q^{1/5} + q^{2/5} J_2. \tag{4}$$

In this equation, the J functions denote power series expansions in q with integer exponents and coefficients. These functions can be expressed as the ratios [2]

$$J_1(q) = \frac{f(-q^2, -q^3)}{f(-q, -q^4)}, \quad J_2(q) = -\ \frac{f(-q, -q^4)}{f(-q^2, -q^3)}, \tag{5}$$

(sequences A003823 and A007325 in OEIS [3], respectively), where $f(a,b)$ is the Ramanujan theta function:

$$f(a,b) = f(b,a) \equiv \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{6}$$

Ramanujan then showed that

$$\frac{1}{J_1 - q^{1/5} + q^{2/5} J_2} \ = \ \frac{J_1^4 + 3qJ_2 + q^{1/5}(J_1^3 + 2qJ_2^2) + q^{2/5}(2J_1^2 + qJ_2^3) + q^{3/5}(3J_1 + qJ_2^4) + 5q^{4/5}}{J_1^5 - 11q + q^2 J_2^5} \tag{7}$$

by rationalizing the denominator on the left and using the identity

$$J_1 J_2 = -1. \tag{8}$$

From eqs. (1), (7), and another identity,

$$J_1^5 - 11q + q^2 J_2^5 = \frac{(q)_\infty^6}{(q^5)_\infty^6}, \tag{9}$$

it follows that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\ \frac{(q^5)_\infty^5}{(q)_\infty^6} \tag{10}$$

and therefore that $p(5n+4) \equiv 0 \bmod 5$.

In like manner, in studying the congruence-7 properties of $p(n)$, Ramanujan wrote down the expansion

$$\frac{(q^{1/7})_\infty}{(q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3 \tag{11}$$

and showed that these $N = 7$ J functions satisfy (among others; see Section IV) the identity

$$J_1 J_2 J_3 = -1. \tag{12}$$

In this article we generalize these expansions to $(q^{1/N})_\infty$, where N will denote a prime number greater than 3, and we will derive explicit formulas for the J_p functions. In this, we will be using a slightly different, more convenient notation than that used by Ramanujan, in that the subscript for J_p will correspond to its associated fractional exponent. I.e., our expansion will read

$$\frac{(q^{1/N})_\infty}{(q^N)_\infty} = J_0 + q^{1/N}J_1 + q^{2/N}J_2 + \cdots + q^{(N-1)/N}J_{N-1}. \tag{13}$$

This equation is equivalent to "multisecting" the power series for $(q)_\infty$; see the article by Somos [2].

II. EXPANSION OF $(q^N)_\infty/(q^{1/N})_\infty$

We make the replacement $q \to q^{1/N}$ in the Euler function and write the identity

$$\frac{1}{(q^{1/N})_\infty} = \frac{1}{(q^{1/N})_\infty} \times \frac{\prod_{p=1}^{N-1}(\omega^p q^{1/N})_\infty}{\prod_{p=1}^{N-1}(\omega^p q^{1/N})_\infty} = \frac{\prod_{p=1}^{N-1}(\omega^p q^{1/N})_\infty}{\prod_{p=0}^{N-1}(\omega^p q^{1/N})_\infty}, \tag{14}$$

where $\omega \equiv e^{2\pi i/N}$ is an N-th root of unity. We consider the product in the denominator in this expression, with the replacement $x = q^{1/N}$:

$$\prod_{p=0}^{N-1}(\omega^p x)_\infty = \prod_{p=0}^{N-1}\prod_{k=1}^{\infty}\left(1-(\omega^p x)^k\right). \tag{15}$$

Now make the change of index $k = nN + a$, $1 \le a \le N$, to get

$$\begin{aligned}
\prod_{p=0}^{N-1}(\omega^p x)_\infty &= \prod_{p=0}^{N-1}\prod_{n=0}^{\infty}\prod_{a=1}^{N}\left(1-\omega^{pa}x^{nN+a}\right) \\
&= \prod_{n=0}^{\infty}\prod_{p=0}^{N-1}\left(1-x^{nN+N}\right)\prod_{a=1}^{N-1}\left(1-\omega^{pa}x^{nN+a}\right) \\
&= \prod_{n=0}^{\infty}\left(1-x^{nN+N}\right)^N \prod_{p=0}^{N-1}\prod_{a=1}^{N-1}\left(1-\omega^{pa}x^{nN+a}\right) \\
&= \left(x^N\right)_\infty^N \prod_{n=0}^{\infty}\prod_{a=1}^{N-1}\prod_{p=0}^{N-1}\left(1-\omega^{pa}x^{nN+a}\right).
\end{aligned} \tag{16}$$

Since N is prime, $1, \omega^a, \omega^{2a}, \ldots, \omega^{(N-1)a}$ are, for fixed a, all distinct, and so $\{1, \omega^a, \omega^{2a}, \ldots, \omega^{(N-1)a}\} = \{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$. We can therefore make the replacement $\omega^{pa} \to \omega^p$ in the product over p, since this amounts to simply a re-ordering of the factors:

$$\begin{aligned}
\prod_{p=0}^{N-1} (\omega^p x)_\infty &= \left(x^N\right)_\infty^N \prod_{n=0}^{\infty} \prod_{a=1}^{N-1} \prod_{p=0}^{N-1} \left(1 - \omega^p x^{nN+a}\right) \\
&= \left(x^N\right)_\infty^N \prod_{n=0}^{\infty} \prod_{a=1}^{N-1} \left(1 - x^{N(nN+a)}\right) \\
&= \left(x^N\right)_\infty^N \prod_{n=0}^{\infty} \frac{\prod_{a=1}^{N} \left(1 - x^{N(nN+a)}\right)}{1 - (x^{N^2})^{n+1}} = \frac{\left(x^N\right)_\infty^{N+1}}{\left(x^{N^2}\right)_\infty},
\end{aligned} \tag{17}$$

where we've used

$$\prod_{p=0}^{N-1} (1 - \omega^p X) = 1 - X^N. \tag{18}$$

And so we have

$$\prod_{p=0}^{N-1} (\omega^p q^{1/N})_\infty = \frac{(q)_\infty^{N+1}}{(q^N)_\infty} \tag{19}$$

as the denominator on the right side of eq. (14).

We next consider the numerator in this equation. We make the replacement $q^{1/N} \to \omega^p q^{1/N}$ in (13) and take the product of $(\omega^p q^{1/N})_\infty$ over $p = 1, \ldots, N-1$:

$$\prod_{p=1}^{N-1} (\omega^p q^{1/N})_\infty = (q^N)_\infty^{N-1} \prod_{p=1}^{N-1} \left(J_0 + \omega^p q^{1/N} J_1 + \omega^{2p} q^{2/N} J_2 + \cdots + \omega^{(N-1)p} q^{(N-1)/N} J_{N-1}\right). \tag{20}$$

We can use the fact that the product $\prod_{p=0}^{N-1} \left(x_0 + \omega^p x_1 + \cdots + \omega^{(N-1)p} x_{N-1}\right)$ is equal to the determinant of a circulant matrix:

$$\prod_{p=0}^{N-1} \left(x_0 + \omega^p x_1 + \cdots + \omega^{(N-1)p} x_{N-1}\right) = \begin{vmatrix} x_0 & x_{N-1} & \cdots & x_2 & x_1 \\ x_1 & x_0 & \cdots & x_3 & x_2 \\ x_2 & x_1 & \cdots & x_4 & x_3 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{N-2} & x_{N-3} & \cdots & x_0 & x_{N-1} \\ x_{N-1} & x_{N-2} & \cdots & x_1 & x_0 \end{vmatrix}. \tag{21}$$

We add columns 1 through $(N-1)$ to the N-th columnn and write the determinant as

$$\begin{aligned}
\begin{vmatrix} x_0 & x_{N-1} & \cdots & x_2 & x_1 \\ x_1 & x_0 & \cdots & x_3 & x_2 \\ x_2 & x_1 & \cdots & x_4 & x_3 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{N-2} & x_{N-3} & \cdots & x_0 & x_{N-1} \\ x_{N-1} & x_{N-2} & \cdots & x_1 & x_0 \end{vmatrix} &= \begin{vmatrix} x_0 & x_{N-1} & \cdots & x_2 & x_0 + \cdots + x_{N-1} \\ x_1 & x_0 & \cdots & x_3 & x_0 + \cdots + x_{N-1} \\ x_2 & x_1 & \cdots & x_4 & x_0 + \cdots + x_{N-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{N-2} & x_{N-3} & \cdots & x_0 & x_0 + \cdots + x_{N-1} \\ x_{N-1} & x_{N-2} & \cdots & x_1 & x_0 + \cdots + x_{N-1} \end{vmatrix} \\
&= (x_0 + \cdots + x_{N-1}) \begin{vmatrix} x_0 & x_{N-1} & \cdots & x_2 & 1 \\ x_1 & x_0 & \cdots & x_3 & 1 \\ x_2 & x_1 & \cdots & x_4 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{N-2} & x_{N-3} & \cdots & x_0 & 1 \\ x_{N-1} & x_{N-2} & \cdots & x_1 & 1 \end{vmatrix}.
\end{aligned} \tag{22}$$

And so the product from $p = 1$ to $N - 1$ is

$$\prod_{p=1}^{N-1} \left(x_0 + \omega^p x_1 + \cdots + \omega^{(N-1)p} x_{N-1}\right) = \begin{vmatrix} x_0 & x_{N-1} & \cdots & x_2 & 1 \\ x_1 & x_0 & \cdots & x_3 & 1 \\ x_2 & x_1 & \cdots & x_4 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{N-2} & x_{N-3} & \cdots & x_0 & 1 \\ x_{N-1} & x_{N-2} & \cdots & x_1 & 1 \end{vmatrix}. \tag{23}$$

Upon the replacements $x_k \to q^{k/N} J_k$ we have

$$\prod_{p=1}^{N-1} (\omega^p q^{1/N})_\infty = \left(q^N\right)_\infty^{N-1} \begin{vmatrix} J_0 & q^{(N-1)/N} J_{N-1} & \cdots & q^{2/N} J_2 & 1 \\ q^{1/N} J_1 & J_0 & \cdots & q^{3/N} J_3 & 1 \\ q^{2/N} J_2 & q^{1/N} J_1 & \cdots & q^{4/N} J_4 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ q^{(N-2)/N} J_{N-2} & q^{(N-3)/N} J_{N-3} & \cdots & J_0 & 1 \\ q^{(N-1)/N} J_{N-1} & q^{(N-2)/N} J_{N-2} & \cdots & q^{1/N} J_1 & 1 \end{vmatrix} \tag{24}$$

and, together with (14) and (19),

$$\begin{aligned} \frac{(q^N)_\infty}{(q^{1/N})_\infty} &= \frac{1}{J_0 + q^{1/N} J_1 + q^{2/N} J_2 + \cdots + q^{(N-1)/N} J_{N-1}} \\ &= \frac{\left(q^N\right)_\infty^{N+1}}{(q)_\infty^{N+1}} \begin{vmatrix} J_0 & q^{(N-1)/N} J_{N-1} & \cdots & q^{2/N} J_2 & 1 \\ q^{1/N} J_1 & J_0 & \cdots & q^{3/N} J_3 & 1 \\ q^{2/N} J_2 & q^{1/N} J_1 & \cdots & q^{4/N} J_4 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ q^{(N-2)/N} J_{N-2} & q^{(N-3)/N} J_{N-3} & \cdots & J_0 & 1 \\ q^{(N-1)/N} J_{N-1} & q^{(N-2)/N} J_{N-2} & \cdots & q^{1/N} J_1 & 1 \end{vmatrix}. \end{aligned} \tag{25}$$

III. MAIN RESULTS

Theorem 1 *Let N be a prime number greater than 3, let A be an integer $\in [0, (N-1)/2]$, and let*

$p \equiv \dfrac{(N-6A)^2 - 1}{24} \mod N.$

Then

(I) *the expansion*

$\dfrac{(q^{1/N})_\infty}{(q^N)_\infty} = J_0(q) + q^{1/N} J_1(q) + q^{2/N} J_2(q) + \cdots + q^{(N-1)/N} J_{N-1}(q)$

has exactly $(N+1)/2$ non-zero terms;

(II) *for A=0,*

$J_p(q) \;=\; (-1)^{\lfloor (N+1)/6 \rfloor} q^{\lfloor (N^2-1)/24N \rfloor};$

(III) *for A>0,*

$J_p(q) \;=\; (-1)^{A + \lfloor (N+1)/6 \rfloor} q^{\lfloor [(N-6A)^2-1]/24N \rfloor} \;\dfrac{f(-q^{2A}, -q^{N-2A})}{f(-q^A, -q^{N-A})}.$

Proof:

Prime numbers greater than 3 can be expressed as $N = |6m-1|$ where m is a positive or a negative integer with absolute value $|m| = \lfloor (N+1)/6 \rfloor$.

Proof of (I): We expand $(q^{1/N})_\infty$ as

$$(q^{1/N})_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2N}. \tag{26}$$

Set $n = kN + a$, with $-\infty < k < \infty$ and $a = 0, \ldots, N-1$. Then

$$(q^{1/N})_\infty \;=\; \sum_{a=0}^{N-1} (-1)^a q^{a(3a-1)/2N} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3kN-6a+1)/2} \tag{27}$$

We now define an equivalence relation on the integers $a \in [0, N-1]$ such that $a_1 \sim a_2$ iff

$$\frac{a_1(3a_1-1)}{2} \bmod N \equiv \frac{a_2(3a_2-1)}{2} \bmod N. \tag{28}$$

We will denote a particular equivalence class either by listing its elements or as $\{p\}$, where p is defined by

$$p \equiv \frac{a(3a-1)}{2} \bmod N \tag{29}$$

for any $a \in \{p\}$. From eqs. (13) and (27), each equivalence class corresponds to a non-zero term, with subscript p, in the expansion of $(q^{1/N})_\infty$. I.e.,

$$(q^N)_\infty J_p(q) = \left[(-1)^a \; q^{\lfloor a(3a-1)/2N \rfloor} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3kN-6a+1)/2} \right]_{a\in\{p\}} \tag{30}$$

where the expression inside the brackets is to be evaluated over the element(s) of the equivalence class $\{p\}$.

Let $a_1, a_2 \in [0, N-1]$. From eq. (28), $a_2 \sim a_1$ iff

$$\frac{(a_2-a_1)[\;3(a_2+a_1)-1\;]}{2} \equiv 0 \bmod N. \tag{31}$$

This requires, since N is prime and $|a_2 - a_1|$ is less than N, that N divides either $[3(a_2+a_1)-1]$ or $[3(a_2+a_1)-1]/2$, depending on whether $a_2 + a_1$ is even or odd.

Case 1: $a_2 + a_1$ is even. Then

$$a_2 + a_1 = \pm 2Km + \frac{1 \mp K}{3} \quad \text{for } N = \pm(6m-1) \tag{32a}$$

for some positive integer K. The only solutions for this equation for a_1, a_2 in this interval are:

$a_2 = 2m - a_1$ if $m > 0$ and $a_1 \leq 2m$;
$a_2 = 2N - |2m| - a_1$ for $m < 0$ and $a_1 \geq N - |2m| + 1$.

Case 2: $a_2 + a_1$ is odd. Then N divides $(3(a_2 + a_1) - 1)/2$, and

$$a_2 + a_1 = \pm 4Km + \frac{1 \mp 2K}{3}. \tag{32b}$$

The only allowed solution is $a_2 = N + 2m - a_1$ for $2m + 1 \leq a_1 \leq N + 2m$.

Summarizing, for $m > 0$,

$$a \sim \begin{cases} 2m - a & \text{for } \; a \in [0, 2m], \\ N + 2m - a & \text{for } \; a \in [2m+1, N-1]. \end{cases} \tag{33a}$$

while for $m < 0$,

$$a \sim \begin{cases} N + 2m - a & \text{for } \; a \in [0, N - |2m|], \\ 2N + 2m - a & \text{for } \; a \in [N - |2m| + 1, N-1]. \end{cases} \tag{33b}$$

For $m > 0$, the first equation is trivial when $a = m$. Therefore, the equivalence class that contains m has only one distinct element. Similarly, for $m < 0$, the class containing $N + m$ has just one element. All other equivalence classes contain exactly 2 elements. If M is the number of equivalence classes, the N values of a are thus grouped into one 1-element class and $(M-1)$ 2-element classes: $N = 1+2(M-1)$. Therefore, $M = (N+1)/2$, which proves (I).

Proof of (II): The index p for the 1-element equivalence class, either $\{a = m\}$ for $N = 6m - 1$ or $\{a = N + m\}$ for $N = -6m + 1$, is

$$p \equiv \frac{m(3am-1)}{2} \bmod N = \frac{N^2-1}{24} \bmod N. \tag{34}$$

For $N = 6m - 1, a = m$, we have from eq. (30) that

$$(q^N)_\infty J_p(q) = (-1)^m q^{\lfloor m(3m-1)/2N \rfloor} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3kN-6m+1)/2}$$
$$= (-1)^{\lfloor (N+1)/6 \rfloor} q^{\lfloor (N^2-1)/24N \rfloor} \sum_{k=-\infty}^{\infty} (-1)^k q^{kN(3k-1)/2} \qquad (35)$$

The result then follows, since the sum over k is $(q^N)_\infty$ by the pentagonal number theorem.
The proof for $N = -6m+1, a = N+m$ follows a similar calculation, with the substitution $k = \rightarrow k-1$ in the sum.

Proof of (III): The 2-element equivalence classes are:

$$m > 0: \begin{cases} \{0, 2m\},\ \{1, 2m-1\},\ \cdots, \{m-1, m+1\}, & \text{(I)} \\ \{2m+1, N-1\},\ \{2m+2, N-2\},\ \cdots, \{\tfrac{1}{2}(N+2m-1), \tfrac{1}{2}(N+2m+1)\}; & \text{(II)} \end{cases}$$

$$m < 0: \begin{cases} \{0, N+2m\},\ \{1, N+2m-1\},\ \cdots, \{\tfrac{1}{2}(N+2m-1), \tfrac{1}{2}(N+2m+1)\}, & \text{(II)} \\ \{N+2m+1, N-1\},\ \{N+2m+2, N-2\},\ \cdots,\ \{N+m-1, N+m+1\}. & \text{(I)} \end{cases}$$

For a given m, they thus break into two groups, which are characterized by the evenness (group I) or the oddness (group II) of $a_2 - a_1$. To each $\{a_1, a_2\}$, with $a_1 < a_2$, we assign an integer A, defined as

$$A = \begin{cases} (a_2 - a_1)/2 & \text{if } a_2 - a_1 \text{ is even,} \\ (N - a_2 + a_1)/2 & \text{if } a_2 - a_1 \text{ is odd.} \end{cases} \qquad (36)$$

For the 1-element class we set $A = 0$. It is easy to see from the list of classes above that each equivalence class corresponds to a different value of A between 0 and $(N-1)/2$.
The 2-element classes thus give 4 cases to consider, which we can characterize as:

Case 1: $m > 0$, group I: $a_1 = m - A$, $a_2 = m + A$;
Case 2: $m > 0$, group II: $a_1 = m + A$, $a_2 = N + m - A$;
Case 3: $m < 0$, group I: $a_1 = N + m - A$, $a_2 = N + m + A$;
Case 4: $m < 0$, group II: $a_1 = m + A$, $a_2 = N + m - A$.

Expressed in the variables N and A however, eq. (29) for p and eq. (30) for $J_p(q)$ takes on the same form in all 4 cases:

$$p \equiv \frac{(N-6A)^2 - 1}{24} \bmod N; \qquad (37)$$

$$J_p(q) = (-1)^{A + \lfloor (N+1)/6 \rfloor} \frac{q^{\lfloor [(N-6A)^2-1]/24N \rfloor}}{(q^N)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{Nk(3k-1)/2} \left[q^{3kA} + q^{(1-3k)A} \right], \quad (A > 0). \qquad (38)$$

Now consider the identity [5],

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty; \quad (a; q)_\infty \equiv \prod_{n=0}^{\infty} (1 - aq^n). \qquad (39)$$

With this identity, the ratio of the theta functions in part (III) of the theorem is

$$\frac{f(-q^{2A}, -q^{N-2A})}{f(-q^A, -q^{N-A})} = \prod_{n=0}^{\infty} \frac{(1 - q^{nN+2A})(1 - q^{nN+N-2A})}{(1 - q^{nN+A})(1 - q^{nN+N-A})}. \qquad (40)$$

We separate the factors in the numerator into those with even n and with odd n:

$$\begin{aligned}\prod_{n=0}^{\infty}\frac{(1-q^{nN+2A})(1-q^{nN+N-2A})}{(1-q^{nN+A})(1-q^{nN+N-A})} &= \prod_{n=0}^{\infty}\frac{(1-q^{2nN+2A})(1-q^{(2n+1)N+2A})(1-q^{2nN+N-2A})(1-q^{(2n+1)N+N-2A})}{(1-q^{nN+A})(1-q^{nN+N-A})}\\ &= \prod_{n=0}^{\infty}(1+q^{nN+A})(1-q^{2nN+N+2A})(1-q^{2nN+N-2A})(1+q^{nN+N-A})\\ &= \prod_{n=1}^{\infty}(1+q^{(n-1)N+A})(1+q^{nN-A})(1-q^{(2n-1)N-2A})(1-q^{(2n-1)N+2A}). \qquad (41)\end{aligned}$$

The product on the right, aside from a factor of $(q^N)_\infty$, is in the form of the product in the quintuple product identity [4], which we write in the form,

$$\prod_{n=1}^{\infty}(1-q^n)(1-aq^{n-1})(1-a^{-1}q^n)(1-a^2q^{2n-1})(1-a^{-2}q^{2n-1}) = \sum_{k=-\infty}^{\infty} q^{k(3k-1)/2}\left[a^{3k}-a^{1-3k}\right], \qquad (42)$$

under the substitutions $q \to q^N$ and $a \to -q^A$. So we have

$$\frac{f(-q^{2A},-q^{N-2A})}{f(-q^A,-q^{N-A})} = \frac{1}{(q^N)_\infty}\sum_{k=-\infty}^{\infty}(-1)^k q^{Nk(3k-1)/2}\left[q^{3kA}+q^{(1-3k)A}\right]. \qquad (43)$$

Part (III) of the theorem then follows by comparing the above equation to eq. (38).
QED.

For $N = 7$, A takes on the values $0, 1, 2, 3$, corresponding to the functions

$$\begin{aligned}J_2 &= -1,\\ J_0 &= \frac{f(-q^2,-q^5)}{f(-q,-q^6)},\\ J_1 &= -\frac{f(-q^4,-q^3)}{f(-q^2,-q^5)},\\ J_5 &= \frac{f(-q^6,-q,)}{f(-q^3,-q^4)}.\end{aligned}$$

The identity (12) then follows trivially. This identity and that of (8) can be written (in our notation) as

$$J_0J_1J_2 = 1, \quad (N=5); \quad J_0J_1J_2J_5 = 1, \quad (N=7). \qquad (44)$$

The generalization of these relations is given by the theorem below:

Theorem 2 *Let $S = \{p_1, \ldots, p_{(N+1)/2}\}$ be the set of indices corresponding to non-zero J functions in the expansion of $(q^{1/N})_\infty$. Then*

$$\prod_{p\in S} J_p(q) = (-1)^{|m|(|m|-1)/2} q^Z,$$

where Z is the non-negative integer

$$Z = \frac{(N-1)(N+1)^2}{48N} - \sum_{p\in S}\frac{p}{N}.$$

Proof: From parts (II) and (III) of Theorem 1 we have

$$\prod_{p\in S} J_p(q) = (-1)^{\sum_A (m+A)} q^{\sum_A \lfloor[(N-6A)^2-1]/24N\rfloor} \prod_{\{a_1,a_2\}} \frac{f(-q^{2A},-q^{N-2A})}{f(-q^A,-q^{N-A})}. \qquad (45)$$

The sums over A go from 0 to $(N-1)/2$, while the product on the right is over all 2-element equivalence classes, since the 1-element class contributes only a factor of 1. Consider the numerator in this product:

$$\prod_{\{a_1,a_2\}} f(-q^{2A},-q^{N-2A}). \qquad (46)$$

There are $(N-1)/2$ factors in this product and each factor contains two distinct positive integers less than N; i.e., the exponents $2A$ and $N-2A$, with A ranging from 1 to $(N-1)/2$. The set of exponents in this product is therefore the set of positive integers less than N, and the factors in the product can be reordered as

$$\prod_{\{a_1,a_2\}} f(-q^{2A},-q^{N-2A}) = f(-q,-q^{N-1})f(-q^2,-q^{N-2})\cdots f(-q^{(N-1)/2},-q^{(N+1)/2}). \tag{47}$$

By a similar argument, the $(N-1)$ exponents in the product in the denominator also equal the set $\{1,2,\ldots,N-1\}$. The denominator can thus also be reordered as above and cancels with the numerator.

The sum over $\lfloor[(N-6A)^2-1]/24N\rfloor$ is found by writing

$$\lfloor[(N-6A)^2-1]/24N\rfloor \;=\; \frac{(N-6A)^2-1}{24N} - \frac{1}{N}\,\frac{(N-6A)^2-1}{24} \bmod N = \frac{(N-6A)^2-1}{24N} - \frac{p}{N}. \tag{48}$$

We have then

$$Z \;=\; \sum_{A=0}^{(N-1)/2} \frac{(N-6A)^2-1}{24N} - \sum_{p\in S}\frac{p}{N} = \frac{(N+1)^2(N-1)}{48N} - \sum_{p\in S}\frac{p}{N}. \tag{49}$$

To find the exponent of (-1), we consider the cases $m>0$ and $m<0$ separately:

$m>0$: A goes from 0 to $(N-1)/2 = 3m-1$;

$$\sum_{A=0}^{3m-1}(m+A) = m(3m) + \frac{(3m-1)(3m)}{2} = 3\;\frac{5m^2-m}{2}. \tag{50a}$$

But $(-1)^{3(5m^2-m)/2} = (-1)^{m(m-1)/2}$.

$m<0$: A goes from 0 to $|3m|$;

$$\sum_{A=0}^{|3m|}(m+A) = m(|3m|+1) + \frac{|3m|(|3m|+1)}{2} = \frac{3m^2-m}{2}. \tag{50b}$$

In this case, $(-1)^{(3m^2-m)/2} = (-1)^{m(m+1)/2}$. Therefore, both cases are covered by the factor $(-1)^{|m|(|m|-1)/2}$.
QED.

IV. SOME ADDITIONAL REMARKS

To derive the identities in eqs. (8) and (9), Ramanujan cubed both sides of eq. (4), used Jacobi's identity,

$$(q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{51}$$

to expand the left-hand side in fractional powers of q, and then equated terms. Another way of arriving at eq. (9) is to use eq. (19) and express the product on the left side as the determinant of a circulant matrix as in eq. (21). Setting $N=5$ in eq. (19) and dividing by $(q^5)_\infty^5$, we have

$$\begin{aligned}\frac{(q)_\infty^6}{(q^5)_\infty^6} = \frac{1}{(q^5)_\infty^5}\prod_{p=0}^{4}(\omega^p q^{1/5})_\infty \;&=\; \begin{vmatrix} J_0 & 0 & 0 & q^{2/5}J_2 & -q^{1/5} \\ -q^{1/5} & J_0 & 0 & 0 & q^{2/5}J_2 \\ q^{2/5}J_2 & -q^{1/5} & J_0 & 0 & 0 \\ 0 & q^{2/5}J_2 & -q^{1/5} & J_0 & 0 \\ 0 & 0 & q^{2/5}J_2 & -q^{1/5} & J_0 \end{vmatrix} \\ &=\; J_0^5 + q(5J_0J_2 - 1 - 5J_0^2J_2^2) + q^2J_2^5 \end{aligned} \tag{52}$$

Now substituting $J_0J_2 = -1$ into this equation gives the identity in (9).

Clearly, we can continue in this fashion. E.g., for $N = 7$, this becomes

$$\frac{(q)_\infty^8}{(q^7)_\infty^8} = \frac{1}{(q^7)_\infty^7}\prod_{p=0}^{6}(\omega^p q^{1/7})_\infty = \begin{vmatrix} J_0 & 0 & q^{5/7}J_5 & 0 & 0 & -q^{2/7} & q^{1/7}J_1 \\ q^{1/7}J_1 & J_0 & 0 & q^{5/7}J_5 & 0 & 0 & -q^{2/7} \\ -q^{2/7} & q^{1/7}J_1 & J_0 & 0 & q^{5/7}J_5 & 0 & 0 \\ 0 & -q^{2/7} & q^{1/7}J_1 & J_0 & 0 & q^{5/7}J_5 & 0 \\ 0 & 0 & -q^{2/7} & q^{1/7}J_1 & J_0 & 0 & q^{5/7}J_5 \\ q^{5/7}J_5 & 0 & 0 & -q^{2/7} & q^{1/7}J_1 & J_0 & 0 \\ 0 & q^{5/7}J_5 & 0 & 0 & -q^{2/7} & q^{1/7}J_1 & J_0 \end{vmatrix}$$

$$= J_0^7 + q(\ J_1^7 + 7J_0J_1^5 + 14J_0^2J_1^3 + 7J_0^4J_1^2J_5 + 7J_0^3J_1 + 7J_0^5J_5\ )$$

$$+q^2(\ 7J_0J_1^4J_5^2 + 7J_1^3J_5 + 7J_0^2J_1^2J_5^2 + 14J_0J_1J_5 + 14J_0^3J_5^2 - 1\ )$$

$$+q^3(\ 14J_1^2J_5^3 + 7J_0^2J_1J_5^4 + 7J_0J_5^3\ ) + 7q^4J_1J_5^5 + q^5J_5^7. \tag{53}$$

Using $J_0J_1J_5 = -1$, this simplifies to

$$\frac{(q)_\infty^8}{(q^7)_\infty^8} = J_0^7 + qJ_1^7 + q^5J_5^7 + 7q(\ J_0J_1^5 + J_5J_0^5 + q^3J_1J_5^5\ ) + 14q(\ J_0^2J_1^3 + qJ_5^2J_0^3 + q^2J_1^2J_5^3\ ) - 8q^2. \tag{54}$$

We can further simplify this expression by using some of the other Ramanujan $N = 7$ identities [1]:

$$\frac{J_0^2}{J_5} + \frac{J_1}{J_5^2} = q; \tag{55a}$$

$$J_0^7 + qJ_1^7 + q^5J_5^7 = \frac{(q)_\infty^8}{(q^7)_\infty^8} + 14q\frac{(q)_\infty^4}{(q^7)_\infty^4} + 57q^2; \tag{55b}$$

$$J_0^3J_1 + qJ_1^3J_5 + q^2J_5^3J_0 = -\frac{(q)_\infty^4}{(q^7)_\infty^4} - 8q; \tag{55c}$$

$$J_0^2J_1^3 + qJ_5^2J_0^3 + q^2J_1^2J_5^3 = -\frac{(q)_\infty^4}{(q^7)_\infty^4} - 5q. \tag{55d}$$

[Note that we have corrected a misprint in [1] in the last term on the right in eq. (55b).] Substituting the left-hand sides of eqs. (55b) and (55d) into eq. (54), we get the additional identity,

$$J_0J_1^5 + J_5J_0^5 + q^3J_1J_5^5 = 3q. \tag{56}$$

[1] Berndt and Ono,"Ramanujan's Unpublished Manuscript on the Partition and Tau Functions with Proofs and Commentary". http://www.math.wisc.edu/∼ono/reprints/044.pdf.

[2] Somos, M. "A Multisection of q-Series" 2010. http://cis.csuohio.edu/∼somos/multiq.html.

[3] The On-Line Encyclopedia of Integer Sequences. http://oeis.org.

[4] Carlitz and Subbarao, "A Simple Proof of the Quintuple Product Identity", Proceedings of the American Mathematical Society vol. 32, Number 1, March 1972.

[5] Weisstein, Eric, W. "Ramanujan Theta Functions." http://mathworld.wolfram.com/RamanujanThetaFunctions.html.