TRIANGULATIONS AND SEVERI VARIETIES

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Abstract. We consider the problem of constructing triangulations of projective planes over Hurwitz algebras with minimal numbers of vertices. We observe that the numbers of faces of each dimension must be equal to the dimensions of certain representations of the automorphism groups of the corresponding Severi varieties. We construct a complex involving these representations, which should be considered as a geometric version of the (putative) triangulations.

1. Introduction

Compare the following two statements, one from complex projective geometry, the other one from combinatorial topology.

Theorem 1 (Zak, 1982). *Let $X^d \subset \mathbb{P}^{N-1}$ be a smooth irreducible complex projective variety of dimension d.*

(1) *If $N < 3\frac{d}{2} + 3$, then the secant variety of X fills out the ambient space, $Sec(X) = \mathbb{P}^{N-1}$.*
(2) *If $N = 3\frac{d}{2} + 3$, then either $Sec(X) = \mathbb{P}^{N-1}$, or $d = 2, 4, 8, 16$.*

Recall that the secant variety $Sec(X)$ is obtained by taking the union of the lines joining any two points of X, and passing to the Zariski closure.

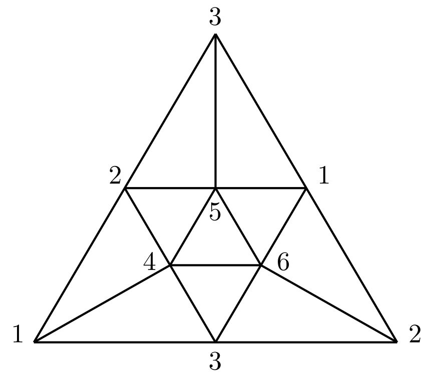
The only exceptions to the second statement are the *Severi varieties*, the complexifications $\mathbb{AP}^2_{\mathbb{C}}$ of the projective planes over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, the four normed algebras (see e.g. [Ba02]).

Theorem 2 (Brehm-Kühnel, 1987). *Let X^d be a combinatorial manifold of dimension d, having N vertices.*

(1) *If $N < 3\frac{d}{2} + 3$, then X is topologically a sphere.*
(2) *If $N = 3\frac{d}{2} + 3$, then either X is a sphere, or $d = 2, 4, 8$, or 16.*

Possible exceptions to the second statement are the *real Severi varieties*, the projective planes $\mathbb{AP}^2$ over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

More precisely, there is a classical triangulation of the real projective plane $\mathbb{RP}^2$ with 6-vertices, described in the picture below where opposite sides of the big triangle must be identified.



There is a unique triangulation of the complex projective plane $\mathbb{CP}^2$ with only 9-vertices [BaK83]. Over the quaternions the situation is not completely clear: it was shown in [BrK92] that there exists three different combinatorial triangulations with 15 vertices of an eight-dimensional manifold which is "like the quaternionic projective plane", but the authors could not decide whether this topological manifold was indeed $\mathbb{HP}^2$ or a fake quaternionic plane. Finally, there is no candidate for a combinatorial triangulation of $\mathbb{OP}^2$ with 27 vertices. We will see that the number of maximal faces of such a triangulation should be 100386!

Recall that the homology of $\mathbb{AP}^2$ is

$$H_i(\mathbb{AP}^2, k) = \begin{cases} k & \text{if } i = 0, a, 2a, \\ 0 & \text{otherwise,} \end{cases}$$

if k stands for $\mathbb{Z}$ when $a \geq 2$, and for $\mathbb{Z}_2$ when $a = 1$. A more uniform statement is that each $\mathbb{AP}^2$ has a Morse function with only three critical points. The precise statement of the second assertion of the theorem of Brehm and Kühnel is that X, if not a sphere, must admit such a Morse function. Manifolds with this property, which are *like projective planes*, were studied systematically in [EK62]. Among other topological restrictions, the fact that the dimension of such a manifold must be $2, 4, 8$ or 16 is established there.

The goal of our paper is to explore the relationships between the two statements above. Our main observation will be that the numbers of faces of each dimension in a (putative) triangulation of a projective plane $\mathbb{AP}^2$, must be equal to the dimensions of certain linear representations of the automorphism groups of the corresponding Severi varieties $\mathbb{AP}^2_{\mathbb{C}}$. Moreover, we will construct complexes involving these representations, which we conjecture to be closely related with the face complexes of the triangulations. Over the complex and quaternionic numbers we will reconsider the results of [BrK87] and [BrK92] and check that they correctly fit with our perspective. We then present an observation concerning the links of vertices in the triangulations. In a final section we elaborate on possible extensions to projective spaces of higher dimensions.

2. The Severi varieties

We briefly recall the main geometric properties of the Severi varieties $\mathbb{AP}^2_{\mathbb{C}}$ (see e.g. [Ba02] and references therein). In all the sequel we denote by $a = 1, 2, 4, 8$ the dimension of the Hurwitz algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ as a real vector space. Recall that each $\mathbb{A}$ has a natural involution generalizing the usual complex conjugation; it can be defined as the orthogonal symmetry with respect to the unity. Then consider the space $J_3(\mathbb{A})$ of Hermitian 3×3 matrices with coefficients in $\mathbb{A}$. This is a real vector space of dimension $3a+3$, endowed with a structure of Jordan algebra defined by symmetrization of the ordinary matrix multiplication.

The automorphism group of the Jordan algebra $J_3(\mathbb{A})$ will be denoted $SO(3,\mathbb{A})$. It preserves the cubic form defined by the determinant, which exists even over the octonions. The group of invertible linear transformations of the vector space $J_3(\mathbb{A})$ preserving this determinant will be denoted $SL(3,\mathbb{A})$.

Let $J_3(\mathbb{A}_\mathbb{C})$ denote the complexification of $J_3(\mathbb{A})$. The Severi variety $\mathbb{A}\mathbb{P}^2_\mathbb{C}\subset \mathbb{P}J_3(\mathbb{A}_\mathbb{C})$ can be defined as the cone over the set of rank one matrices, where having rank one is defined by the property that all the derivatives of the determinant vanish (these derivatives are the analogues of 2×2 minors). Geometrically, this means that the Severi variety is the singular locus of the determinantal hypersurface. We will only mention a few of its many remarkable properties:

(1) $\mathbb{A}\mathbb{P}^2_\mathbb{C}$ is smooth of dimension $2a$.
(2) $\mathbb{A}\mathbb{P}^2_\mathbb{C}$ is homogeneous under the action of the complexified group $SL(3,\mathbb{A}_\mathbb{C})$.
(3) The secant variety of $\mathbb{A}\mathbb{P}^2_\mathbb{C}$ is the determinantal hypersurface. (The sum of two rank one matrices has rank at most two!) In particular $\mathbb{A}\mathbb{P}^2_\mathbb{C}$ is *secant defective*, in the sense that the secant variety is smaller that expected.

Polarizing the determinant, one obtains a quadratic map $c: J_3(\mathbb{A}_\mathbb{C})\to J_3(\mathbb{A}_\mathbb{C})^\vee$ which we call the *comatrix map*. Let

$$W(\mathbb{A})=\mathbb{C}\oplus J_3(\mathbb{A}_\mathbb{C})\oplus J_3(\mathbb{A}_\mathbb{C})^\vee\oplus\mathbb{C}.$$

One can define the projective variety $LG(3,\mathbb{A}_\mathbb{C})$ as the image of the rational map from $J_3(\mathbb{A}_\mathbb{C})$ to $\mathbb{P}W(\mathbb{A})$ mapping $x\in J_3(\mathbb{A}_\mathbb{C})$ to the line generated by $1+x+c(x)+\det(x)$. Denote by p the point of $LG(3,\mathbb{A}_\mathbb{C})$ defined by $x=0$. The following properties do hold:

(1) $LG(3,\mathbb{A}_\mathbb{C})$ is smooth of dimension $3a+3$.
(2) $LG(3,\mathbb{A}_\mathbb{C})$ is homogeneous under the action of a simple Lie group $Sp(6,\mathbb{A}_\mathbb{C})$.
(3) The lines through p contained in $LG(3,\mathbb{A}_\mathbb{C})$ generate a cone over $\mathbb{A}\mathbb{P}^2_\mathbb{C}$.

The groups we met have the following types:

$\mathbb{A}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$SO(3,\mathbb{A}_\mathbb{C})$	A_1	A_2	C_3	F_4
$SL(3,\mathbb{A}_\mathbb{C})$	A_2	$A_2\times A_2$	A_5	E_6
$Sp(6,\mathbb{A}_\mathbb{C})$	C_3	A_5	D_6	E_7

This table is a chunk of the famous Tits-Freudenthal magic square. (For more on this see the survey paper [LM04] and references therein.)

3. Faces and representations

3.1. **Special properties of minimal triangulations.** The triangulations of $\mathbb{R}\mathbb{P}^2$, $\mathbb{C}\mathbb{P}^2$ and (supposedly) $\mathbb{H}\mathbb{P}^2$ with minimal numbers of vertices have very peculiar properties [BrK87, BrK92], among which:

(1) (Tightness) Each face of dimension a or less is part of the triangulation.
(2) (Duality) A face of dimension $a+j$ is part of the triangulation if and only if the complementary face of dimension $2a-j+1$ is not.

(3) (Secant defectivity) Any two maximal simplices intersect along a simplex of dimension at least $a-1$.

Note that tightness is a consequence of duality, since there is no face of dimension $2a+1$. Moreover, it was noticed by Marin that the duality property is imposed by the algebra structure of the $\mathbb{Z}_2$-valued cohomology (see [AM91], and [BrK92, Proposition 2]).

Since the number of vertices is $3a+3$, the dimension of the intersection of two simplices of dimension $2a$ is at least $a-2$, and should in general be equal to $a-2$. In our triangulations any two maximal simplices meet in dimension $a-1$, and this means that their linear span has dimension $3a+1$ rather than the expected $3a+2$. This is why this property should be understood as the combinatorial version of the secant defectivity property of the Severi varieties.

3.2. **Numbers of faces.** The numbers of faces f_k of each dimension k in a triangulation of a smooth manifold are not independent. For example a codimension one face has to belong to exactly two codimension zero faces, hence the relation

$$(d+1)f_d = 2f_{d-1}$$

if d denotes the dimension. More generally, the Dehn-Sommerville equations for simplicial complexes, or rather their extension by Klee (see [NS09, Theorem 5.1]) to a context including combinatorial manifolds, imply that the numbers of faces of dimension smaller than half of d determine the remaining numbers of faces. The precise statement is the following. Let $f_{-1}=1$, and consider the h-vector, which is the sequence $h_0,\dots,h_{d+1}$ defined by the identity

$$\sum_{i=0}^{d+1} h_i x^{d+1-i} = \sum_{j=0}^{d+1} f_{j-1}(x-1)^{d+1-j}.$$

For a simplicial complex, the classical Dehn-Sommerville equations assert that the h-vector is symmetric, that is $h_i = h_{d+1-i}$. More generally, for a triangulation of a smooth manifold X, the h-vector is such that

$$h_{d+1-i} - h_i = (-1)^i \binom{d+1}{i} \Big(\chi_{top}(X) - \chi_{top}(S^d)\Big),$$

where $\chi_{top}(X)$ denotes the topological Euler characteristic.

Let us denote by f_k^a the number of faces of dimension k in a tight triangulation of $\mathbb{AP}^2$ with $3a+3$ vertices. The tightness property means that

$$f_k^a = \binom{3a+3}{k+1} \qquad \text{for} \quad 0 \le k \le a.$$

The generalized Dehn-Sommerville equations show that all the numbers f_k^a are then completely determined. These numbers are given in the following table:

	$\mathbb{RP}^2$	$\mathbb{CP}^2$	$\mathbb{HP}^2$	$\mathbb{OP}^2$
#vertices	6	9	15	27
	15	36	105	351
$\#2-\dim$	10	84	455	2925
		90	1365	17550
$\#4-\dim$		36	3003	80730
			4515	296010
			4230	888030
			2205	2220075
$\#8-\dim$			490	4686825
				8335899
				12184614
				14074164
				12301200
				7757100
				3309696
				853281
$\#16-\dim$				100386

Main observation. *For each k, the number f_k^a of k-dimensional faces of a minimal triangulation of $\mathbb{AP}^2$ is the dimension of a representation of $SL(3, \mathbb{A}_{\mathbb{C}})$.*

3.3. **A connection with Severi varieties.** One can be much more precise. We will describe a recipe which allows to understand a priori which representation of $SL(3, \mathbb{A}_{\mathbb{C}})$ has dimension f_k^a. Note that it will not be an irreducible representation in general, but it will have very few irreducible components, and never more than three.

Consider the variety $LG(3, \mathbb{A}_{\mathbb{C}}) = G/P$, where $G = Sp(6, \mathbb{A}_{\mathbb{C}})$ and P is the stabilizer of the base point p. As any rational homogeneous variety does, $LG(3, \mathbb{A}_{\mathbb{C}})$ has a cellular decomposition defined by the Schubert cells. If $B \subset P \subset G$ is a Borel subgroup, recall that the Schubert cells can be defined as the B-orbits inside $LG(3, \mathbb{A}_{\mathbb{C}})$. Their closures are called Schubert varieties and usually denoted X_u, where u belongs to some index set W_P defined in terms of the combinatorics of the root system of G. It is clear from the definition that the boundary of any Schubert variety is a finite union of smaller Schubert varieties. This allows to define an oriented graph, which we call the *Hasse diagram.* The vertices of this graph are in bijection with the Schubert varieties, that is with W_P. Moreover, there is an arrow $u \to v$ if X_u is an irreducible component of the boundary of X_v (or equivalently, X_u is a codimension one subvariety of X_v). The Hasse diagram is obviously ranked by the dimension of the Schubert varieties. Moreover, Poincaré duality implies that it is symmetric with respect to the middle dimension. We denote the operation of Poincaré duality on the Hasse diagram by π.

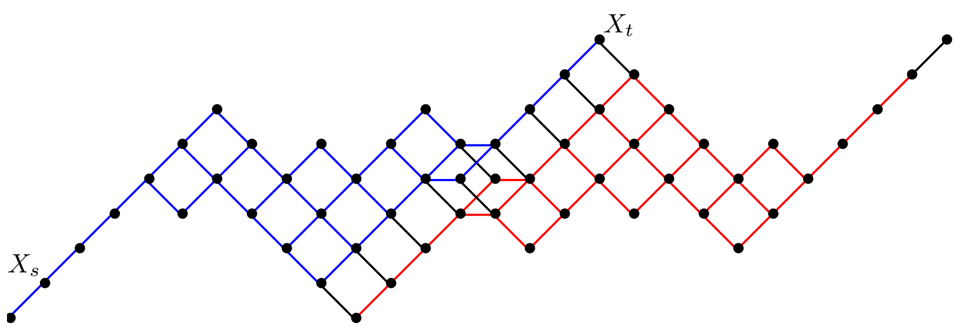
Proposition 1. *One has the following properties:*

1. *The cone over $\mathbb{AP}^2_{\mathbb{C}}$ is a Schubert variety of $LG(3, \mathbb{A}_{\mathbb{C}})$.*
2. *The Hasse diagram of $\mathbb{AP}^2_{\mathbb{C}}$ embeds in the Hasse diagram of $LG(3, \mathbb{A}_{\mathbb{C}})$, as an interval I.*

(3) *The Hasse diagram of* $LG(3,\mathbb{A}_{\mathbb{C}})$ *is the disjoint union of the interval* I*, the Poincaré dual* $\pi(I)$*, and the two extremities given by the fundamental class and the punctual class.*

Proof. The first claim is clear since the cone over $\mathbb{AP}^2_{\mathbb{C}}$ is the union of the lines in $LG(3,\mathbb{A}_{\mathbb{C}})$ passing through p. In particular it is stabilized by P, hence by B, which implies that it is a Schubert variety X_t since there are only finitely many B-orbits. The second claim follows immediately: there is only one one-dimensional Schubert variety X_s (a line), and since the Schubert subvarieties of $LG(3,\mathbb{A}_{\mathbb{C}})$ contained in $\mathbb{AP}^2_{\mathbb{C}}$ are exactly the cones over the Schubert subvarieties of the rational homogeneous variety $\mathbb{AP}^2_{\mathbb{C}}$, the interval $I=[s,t]$ is isomorphic with the Hasse diagram of $\mathbb{AP}^2_{\mathbb{C}}$. Finally, the third claim was first observed in [CMP07] in connection with certain unexpected symmetry properties of quantum cohomology. □

The picture below shows the Hasse diagram of $LG(3,\mathbb{O})$, the *Freudenthal variety*, which is a homogeneous space of exceptional type, with automorphism group $Sp(6,\mathbb{O}_{\mathbb{C}})$ of type E_7. The interval I is in blue while $\pi(I)$ is in red.



Remark. There is another connection between these Hasse diagrams. By Birkhoff's theorem, the Hasse diagram of $LG(3,\mathbb{A}_{\mathbb{C}})$, being a distributive lattice, is the lattice of upper ideals of a poset P. This poset is precisely the poset encoded by the Hasse diagram of $\mathbb{AP}^2_{\mathbb{C}}$. In particular the vertices of the latter can be associated with the join-irreducibles of the former Hasse diagram. That the Hasse diagram of $LG(3,\mathbb{A}_{\mathbb{C}})$ is a distributive lattice is a consequence of the fact that this is a minuscule homogeneous space [Hil82].

3.4. **Wedge powers of the Jordan algebra.** On the other hand, consider the following problem: decompose the wedge powers of $J_3(\mathbb{A}_{\mathbb{C}})$ into irreducible components, with respect to the action of $SL(3,\mathbb{A}_{\mathbb{C}})$. We shall see shortly that this decomposition is multiplicity free. This allows to define an oriented graph $G(J_3(\mathbb{A}_{\mathbb{C}}))$ as follows. The vertices are in bijection with the components of the wedge powers of $J_3(\mathbb{A}_{\mathbb{C}})$. There is an edge between a component U of $\wedge^k J_3(\mathbb{A}_{\mathbb{C}})$ and a component V of $\wedge^{k+1} J_3(\mathbb{A}_{\mathbb{C}})$ if the composite map

$$V\otimes J_3(\mathbb{A}_{\mathbb{C}})^\vee \hookrightarrow \wedge^{k+1} J_3(\mathbb{A}_{\mathbb{C}})\otimes J_3(\mathbb{A}_{\mathbb{C}})^\vee \to \wedge^k J_3(\mathbb{A}_{\mathbb{C}}) \to U$$

is non-zero. Here the morphism $\wedge^{k+1}J_3(\mathbb{A}_\mathbb{C})\otimes J_3(\mathbb{A}_\mathbb{C})^\vee \to \wedge^k J_3(\mathbb{A}_\mathbb{C})$ is the natural contraction map, and the map $\wedge^k J_3(\mathbb{A}_\mathbb{C}) \to U$ is the projection with respect to the other irreducible components.

Proposition 2. *The graph $G(J_3(\mathbb{A}_\mathbb{C}))$ coincides with the Hasse diagram of $LG(3,\mathbb{A}_\mathbb{C})$.*

Before giving the proof, we need to recall certain properties of the relationship between $\mathbb{AP}^2_\mathbb{C}$ and $LG(3,\mathbb{A}_\mathbb{C})$. First, the latter being minuscule, the Lie algebra $\mathfrak{g}=\mathfrak{sp}(6,\mathbb{A})$ of its automorphism group has an associated three-step grading

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1,$$

where $\mathfrak{g}_0$ is a reductive Lie algebra with one dimensional center and with semi-simple part $\mathfrak{sl}(3,\mathbb{A}_\mathbb{C})$, while $\mathfrak{g}_1$ is isomorphic with $J_3(\mathbb{A}_\mathbb{C})$ as a $\mathfrak{sl}(3,\mathbb{A}_\mathbb{C})$-module. The positive part $\mathfrak{g}_0\oplus\mathfrak{g}_1$ of the grading is the Lie algebra of the parabolic subgroup P of $Sp(6,\mathbb{A}_\mathbb{C})$ such that $G/P = LG(3,\mathbb{A}_\mathbb{C})$. This parabolic is always maximal, hence associated to a simple root α_0. Its Weyl group W^P can then be defined as the stabilizer, inside the Weyl group W of $Sp(6,\mathbb{A})$, of the associated fundamental coweight $\omega_0^\vee$. The orthogonal hyperplane to $\omega_0^\vee$ cuts the root system Φ of $Sp(6,\mathbb{A})$ along the root system Φ_0 of $SL(3,\mathbb{A}_\mathbb{C})$, which is the root subsystem generated by the simple roots except α_0. The positive roots which do not belong to Φ_0 are those that appear in $\mathfrak{g}_1$, that is, they are exactly the weights of $J_3(\mathbb{A}_\mathbb{C})$. This module is again minuscule, which means that W^P acts transitively on the roots having positive evaluation on $\omega_0^\vee$.

Proof of the proposition. We have mentioned the fact that the vertices of the Hasse diagram are indexed by a set W_P, a subset of the Weyl group of $Sp(6,\mathbb{A})$. This is the set of minimal length representatives of W/W^P, and it can be characterized as follows:

$$W_P=\{w\in W,\quad w(\alpha)\in\Phi^+\ \forall\alpha\in\Phi_0^+\}.$$

Now, the cotangent space to $LG(3,\mathbb{A}_\mathbb{C})$ at the point p is nothing else than $J_3(\mathbb{A}_\mathbb{C})$, not only as a vector space but as a P-module, hence also as an $SL(3,\mathbb{A}_\mathbb{C})$-module since $SL(3,\mathbb{A}_\mathbb{C})$ is the semi-simple part of P. (In fact the action of the unipotent radical of P is trivial, because $LG(3,\mathbb{A}_\mathbb{C})$ is *minuscule*). Under such favourable circumstances, the decomposition of the bundle of k-forms has been obtained by B. Kostant [Ko61]:

$$\wedge^k J_3(\mathbb{A}_\mathbb{C})=\bigoplus_{v\in W_P,\ell(v)=k} V_{\rho-v^{-1}(\rho)},$$

where ρ denotes the half-sum of the positive roots in the root system of $Sp(6,\mathbb{A}_\mathbb{C})$. In particular, the vertices of $G(J_3(\mathbb{A}_\mathbb{C}))$ are in bijection with W_P, hence with the vertices of the Hasse diagram of $LG(3,\mathbb{A}_\mathbb{C})$

There remains to check that the edges are the same. In the Hasse diagram, consider u of length k and v of length $k+1$. There is an edge $u\to v$ if and only if $v=su$ for some reflection s in the Weyl group W. We claim that this is equivalent to the condition that $u^{-1}(\rho)-v^{-1}(\rho)$ is a weight of $J_3(\mathbb{A}_\mathbb{C})$. Admitting this, we conclude the proof as follows. If $u^{-1}(\rho)-v^{-1}(\rho)$ is not a weight of $J_3(\mathbb{A}_\mathbb{C})$, then $V_{\rho-v^{-1}(\rho)}$ cannot be a component of $V_{\rho-u^{-1}(\rho)}\otimes J_3(\mathbb{A}_\mathbb{C})$ by [Zh73, section 131]. If $u^{-1}(\rho)-v^{-1}(\rho)$ is a weight of $J_3(\mathbb{A}_\mathbb{C})$, then the fact that $V_{\rho-v^{-1}(\rho)}$ is a component of $V_{\rho-u^{-1}(\rho)}\otimes J_3(\mathbb{A}_\mathbb{C})$ is a special case of the PRV conjecture, proved in [Ku88].

There remains to prove our claim. First recall that in the minuscule setting the strong Bruhat order coincides with the weak Bruhat order [LW90, Lemma 1.14].

This means that the reflexion s must be the simple reflection s_i associated to a simple root α_i. Since $\ell(v) = \ell(s_i u) = \ell(u) + 1$ we must have $u(\alpha_i) > 0$. We also need that v belongs to W_P, which means that any positive root in the subsystem Φ_0 must be sent to a positive root. Since this is already the case for u, and since the only positive root that s_i sends to a negative root is α_i, this is equivalent to the condition that the positive root $\beta = u^{-1}(\alpha_i)$ does not belong to Φ_0^+. Since Φ_0 is the set of roots in Φ orthogonal to $\omega_0^\vee$, our condition can be restated as $\omega_0^\vee(\beta) > 0$. But this means that the root β appears in $\mathfrak{g}_1$, hence that it is a weight of $J_3(\mathbb{A}_\mathbb{C})$. Since $u^{-1}(\rho) - v^{-1}(\rho) = \beta$, our claim follows. □

A nice consequence is that the interval $I = [s, t]$ defines a submodule of the exterior algebra of $J_3(\mathbb{A}_\mathbb{C})$, namely

$$L^k = \bigoplus_{\substack{v \in W_P, \ell(v)=k, \\ s \leq v \leq t}} V_{\rho - v^{-1}(\rho)}.$$

Our refined version of the main observation is the following:

Proposition 3.

$$f_k^a = \dim L^{k+1}.$$

This is straightforward to check case by case. A conceptual proof would probably require an interpretation of the generalized Dehn-Sommerville equations in representation theoretic terms. We have no idea of what could be such an interpretation.

As we already mentioned, although not always irreducible, L^k has only a very small number of irreducible components. More precisely, it contains at most three components, as is apparent on the Hasse diagrams of $LG(3, \mathbb{A}_\mathbb{C})$. For k small enough $\wedge^k J_3(\mathbb{A}_\mathbb{C})$ is irreducible, hence by symmetry L^k is also irreducible when $2a + 1 - k$ is small. A natural question to ask is, when the representation is not irreducible, whether there is any natural way to split the faces into subsets of the corresponding dimensions.

Maximal faces. In particular, the number of faces of maximal dimension is the dimension of the irreducible module L^{2a+1}. This module can be interpreted as follows. Recall that each point of $\mathbb{AP}^2_\mathbb{C}$ defines an $\mathbb{A}$-line on the dual plane in $\mathbb{P}J_3(\mathbb{A}_\mathbb{C})^\vee$. This $\mathbb{A}$-line is a quadric of dimension a, whose linear span is projective space of dimension $a + 1$. Hence an equivariant map

$$\pi : \mathbb{AP}^2_\mathbb{C} \to G(a + 2, J_3(\mathbb{A}_\mathbb{C})^\vee)$$

If this map is of degree d, in the sense that $\pi^*\mathscr{O}(1)$ is equal to the d-th power of the hyperplane line bundle on $\mathbb{AP}^2_\mathbb{C}$, we get a non-zero equivariant map

$$H^0(G(a+2, J_3(\mathbb{A}_\mathbb{C})^\vee), \mathscr{O}(1)) = \wedge^{a+2} J_3(\mathbb{A}_\mathbb{C}) \xrightarrow{\pi^*} H^0(\mathbb{AP}^2_\mathbb{C}, \mathscr{O}(d)) = (J_3(\mathbb{A}_\mathbb{C})^\vee)^{(d)},$$

the d-th Cartan power of $J_3(\mathbb{A}_\mathbb{C})^\vee$. Dualizing, we get an inclusion of $J_3(\mathbb{A}_\mathbb{C})^{(d)}$ inside $\wedge^{a+2} J_3(\mathbb{A}_\mathbb{C})^\vee = \wedge^{2a+1} J_3(\mathbb{A}_\mathbb{C})$. In fact $d = a/2 + 1$ and

$$L^{2a+1} = J_3(\mathbb{A}_\mathbb{C})^{(a/2+1)}.$$

(This still makes sense for $a = 1$ because the hyperplane class of $\mathbb{RP}^2_\mathbb{C}$, the Veronese surface, is divisible by two.)

In general the number of irreducible components of L^k behaves as follows:

$$\#\text{irred } L^{k+1} = \begin{cases} 1 & \text{if } 0 \leq k \leq \frac{a}{2} - 1 \text{ or } \frac{3a}{2} + 1 \leq k \leq 2a, \\ 2 & \text{if } \frac{a}{2} \leq k \leq a - 1 \text{ or } a + 1 \leq k \leq \frac{a}{2}, \\ 3 & \text{if } k = a. \end{cases}$$

(There is a strange similarity with the homology of $\mathbb{AP}^2$.)

Tightness. Note that $\pi(I) = [\pi(s), \pi(t)]$ where the dimension of the Schubert variety $X_{\pi(t)}$ is $a+1$, being complementary to the dimension of X_t, which is $2a+1$ since it is a cone over $\mathbb{AP}^2_{\mathbb{C}}$. Since the whole Hasse diagram is the disjoint union of $I, \pi(I)$ and the two extremities, this implies that

$$L^k = \wedge^k J_3(\mathbb{A}_{\mathbb{C}}) \qquad \text{for} \quad 1 \leq k \leq a+1.$$

This is the algebraic version of tightness.

Duality. Duality can also be interpreted in the representation theoretic setting. Indeed, there exists an exterior automorphism of $SL(3, \mathbb{A})$ exchanging the representations $J_3(\mathbb{A}_{\mathbb{C}})$ and its dual $J_3(\mathbb{A}_{\mathbb{C}})^\vee$. Since

$$\wedge^k J_3(\mathbb{A}_{\mathbb{C}}) \simeq \wedge^{3a+3-k} J_3(\mathbb{A}_{\mathbb{C}})^\vee,$$

the graph $G(J_3(\mathbb{A}_{\mathbb{C}}))$ has an induced symmetry which can be seen to coincide with Poincaré duality. Since, once again, the disjoint union of I and $\pi(I)$ is the whole Hasse diagram minus its two extremities, we must have the identity

$$\wedge^k J_3(\mathbb{A}_{\mathbb{C}}) = L^k \oplus (L^{3a+3-k})^\vee.$$

This is the algebraic version of duality. Indeed, taking dimensions, we conclude that the number of $(k-1)$-dimensional faces is equal to the number of $(3a+2-k)$-dimensional "non-faces".

Secant defectivity. In [EPW00] the minimal triangulation Δ of $\mathbb{RP}^2$ is considered. The Stanley-Reisner ideal I_Δ defines an arrangement of 10 hyperplanes in $\mathbb{P}^5$. Over a field k of characteristic two, the corresponding scheme is Gorenstein and its canonical bundle is 2-torsion. Moreover, this scheme can be flatly deformed into a family of special smooth Enriques surfaces in $\mathbb{P}^5$. This family is defined in terms of Lagrangian subspaces in $\wedge^3 k^6$, endowed with the quadratic form (characteristic two !) induced by the wedge product.

In terms of representations (and over $\mathbb{C}$), the relevant property is that

$$\wedge^3(Sym^2\mathbb{C}^3) = Sym^3\mathbb{C}^3 \oplus (Sym^2\mathbb{C}^3)^\vee$$

where both components are Lagrangian (with respect to the skew-symmetric form induced by the wedge product). In the other cases we have the following substitute:

Proposition 4. *Consider $(L^{2a+1})^\vee$ as an irreducible component of $\wedge^{a+2} J_3(\mathbb{A}_{\mathbb{C}}) \simeq \wedge^{2a+1} J_3(\mathbb{A}_{\mathbb{C}})^\vee$. Then the natural map*

$$(L^{2a+1})^\vee \otimes (L^{2a+1})^\vee \to \wedge^{2a+4} J_3(\mathbb{A}_{\mathbb{C}}) = \wedge^{a-1} J_3(\mathbb{A}_{\mathbb{C}})^\vee$$

is zero.

Otherwise stated, $(L^{2a+1})^\vee$ is isotropic with respect to a whole system of bilinear forms parametrized by $\wedge^{a-1} J_3(\mathbb{A}_{\mathbb{C}})$, which is a very strong property. Moreover these forms are symmetric for $a \geq 2$, exactly as for $a = 1$ in characteristic two (but skew-symmetric for $a = 1$ in characteristic zero...).

Proof. Since $(L^{2a+1})^\vee$ is irreducible, it is enough to prove that $\omega \wedge \omega' = 0$ when ω, ω' are two highest weight vectors. Since L^{2a+1} is a Cartan power of $J_3(\mathbb{A}_\mathbb{C})$, these highest weight vectors correspond to two points p, p' of the dual $\mathbb{AP}^2_\mathbb{C}$. Moreover, we have seen that the associated points in $\wedge^{a+2} J_3(\mathbb{A}_\mathbb{C})$ correspond to the linear spans of the $\mathbb{A}$-lines on $\mathbb{AP}^2_\mathbb{C}$ defined by p and p'. But two such $\mathbb{A}$-lines always meet non-trivially (we are dealing with a plane projective geometry!), and this implies that $\omega \wedge \omega' = 0$. □

4. Complexes

4.1. **A subcomplex of the Koszul complex.** Recall that the wedge powers of $J_3(\mathbb{A}_\mathbb{C})$ can be put together into a Koszul complex: for any non-zero linear form $\phi \in J_3(\mathbb{A}_\mathbb{C})^\vee$, the contraction by ϕ,

$$\cdots \to \wedge^{k+1} J_3(\mathbb{A}_\mathbb{C}) \xrightarrow{\phi} \wedge^k J_3(\mathbb{A}_\mathbb{C}) \to \cdots$$

defines an exact complex $K^\bullet(\phi)$. By their very definition, the contraction map by any linear form ϕ maps L^{k+1} to L^k and we get a subcomplex $L^\bullet(\phi)$

$$0 \to L^{2a+1} \to \cdots \to L^{k+1} \xrightarrow{\phi} L^k \to \cdots \to L^1 \to 0.$$

This complex is not exact. Indeed, suppose that $\phi \in J_3(\mathbb{A}_\mathbb{C})^\vee$ is general, in the sense that it does not belong to the determinantal hypersurface. The stabilizer $SO(\phi)$ of ϕ in $SL(3, \mathbb{A})$ is then a conjugate of $SO(3, \mathbb{A}) = Aut(J_3(\mathbb{A}_\mathbb{C}))$ (such that ϕ becomes the identity of the twisted Jordan structure). The complex $L^\bullet(\phi)$ is $SO(\phi)$-equivariant. In particular we consider its Euler characteristic as an element of the representation ring of $SO(\phi)$. A direct check with LiE [LiE] yields:

Proposition 5. *The Euler characteristic of the complex $L^\bullet(\phi)$ is*

$$\chi(L^\bullet(\phi)) = \chi_{top}(\mathbb{AP}^2)\,[\mathbb{C}],$$

where $[\mathbb{C}]$ denotes the class of the trivial representation of $SO(\phi)$. In particular the Euler characteristic is $SO(\phi)$-invariant.

One can also check that the $SO(\phi)$-invariants of the complex are

$$(L^{k+1}(\phi))^{SO(3,\mathbb{A})} = \begin{cases} \mathbb{C} & \text{if } k = 0, a, 2a, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of these invariants can be seen as follows. Inside $L^1 = J_3(\mathbb{A}_\mathbb{C})$ there is the invariant hyperplane $J_3(\mathbb{A}_\mathbb{C})_\phi = \phi^\perp$. This is an irreducible $SO(\phi)$-module, and therefore it admits a unique invariant supplementary line $\ell_\phi \subset J_3(\mathbb{A}_\mathbb{C})$. Since $J_3(\mathbb{A}_\mathbb{C}) = J_3(\mathbb{A}_\mathbb{C})_\phi \oplus \ell_\phi$ as $SO(\phi)$-modules, we have for any integer k

$$S^k J_3(\mathbb{A}_\mathbb{C}) = \oplus_{\ell=0}^k S^\ell J_3(\mathbb{A}_\mathbb{C})_\phi.$$

It turns out that a similar statement holds for Cartan powers:

$$J_3(\mathbb{A}_\mathbb{C})^{(k)} = \oplus_{\ell=0}^k J_3(\mathbb{A}_\mathbb{C})_\phi^{(\ell)}.$$

In particular there is always a unique line of $SO(\phi)$-invariants inside $J_3(\mathbb{A}_\mathbb{C})^{(k)}$, hence inside $L^{2a+1} = J_3(\mathbb{A}_\mathbb{C})^{(a/2+1)}$. Moreover this line is contained in $\wedge^{2a+1} J_3(\mathbb{A}_\mathbb{C})_\phi$, and since $J_3(\mathbb{A}_\mathbb{C})_\phi$ is self-dual of dimension $3a+2$, there is an induced line of $SO(\phi)$-invariants inside $\wedge^{a+1} J_3(\mathbb{A}_\mathbb{C})_\phi \subset \wedge^{a+1} J_3(\mathbb{A}_\mathbb{C}) = L^{a+1}$.

Proposition 5 suggests the following conjecture:

Conjecture. *Let $\phi \in J_3(\mathbb{A}_\mathbb{C})^\vee$ be general. Then the inclusion of $L^\bullet(\phi)^{SO(\phi)}$ inside $L^\bullet(\phi)$ is a quasi-isomorphism.*

Proposition 5 also shows that $L^\bullet(\phi)$ has one of the main properties of the face complex of a triangulation of $\mathbb{AP}^2$.

4.2. **The main conjecture.** Let Δ be a simplicial complex. Associate to each vertex v of Δ a variable x_v. Let $I_\Delta \subset k[x_v, v \in \Delta_0]$ denote the ideal generated by all the square-free monomials $x_{v_1} \cdots x_{v_r}$ such that $(v_1, \ldots, v_r)$ is not a face of Δ. Then $R = k[x_v, v \in \Delta_0]/I_\Delta$ is the *Stanley-Reisner* ring of Δ [BH93]. When Δ is a spherical complex, R is a Cohen-Macaulay ring. If Δ is a triangulation of a topological manifold (not necessarily a sphere), then R is only Buchsbaum [NS09].

The *face complex* $C^\bullet_\Delta$ is defined by

$$C^k_\Delta = \bigoplus_{v_1,\ldots,v_k} R_{x_{v_1}\cdots x_{v_k}},$$

where the sum is over all $(k-1)$-dimensional faces. (In order to define the morphisms one has to chose an ordering of Δ_0.) This complex computes the local cohomology of R at the maximal ideal. For Buchsbaum modules the local cohomology is closely connected with the socle (see [NS09], in particular Corollary 3.5).

Remark. Note that a consequence of Δ not being Cohen-Macaulay is that the h-vector is not symmetric. As explained in [NS09], the symmetry can be recovered by changing the h-vector into a h''-vector, the modification taking into account the Betti numbers of the manifold triangulated by Δ. For $\mathbb{AP}^2$ we would get

$$h''_k = h''_{2a-k} = \binom{a+k+1}{k} \qquad \text{for} \quad 0 \le k \le a.$$

These numbers are the dimensions of the graded part of a Gorenstein Artinian ring ([NS09], Conjecture 7.3), and by Macaulay's theorem one can associate a polynomial F_a to this ring. What is the significance of F_a?

We will now define a variant of the face complex. Considering a space V endowed with a basis e_v indexed by vertices of Δ. We can then define

$$L^k_\Delta = \bigoplus_{v_1,\ldots,v_k} \mathbb{C} e_{v_1} \wedge \cdots \wedge e_{v_k} \subset \wedge^k V,$$

the sum being again over all $(k-1)$-dimensional faces. Since every subset of a face is a face, each contraction map by a linear form $\phi \in V^*$ sends L^k_Δ to L^{k-1}_Δ.

Conjecture. *There exists a degeneration of $L^\bullet$ to $L^\bullet_\Delta$, for some triangulation Δ of $\mathbb{AP}^2_\mathbb{C}$ with $3a+3$ vertices.*

More precisely, such a degeneration should exist inside the Koszul complex of $J_3(\mathbb{A}_\mathbb{C})$, which means that we do not need to care about the morphisms, but only to prove the existence of a degeneration L^k_t of each L^k to L^k_Δ inside the Grassmannian parametrizing subspaces of $\wedge^k J_3(\mathbb{A}_\mathbb{C})$ of the same dimension. Of course we require that for any $\phi \in V^\vee = J_3(\mathbb{A}_\mathbb{C})^\vee$, the contraction map by ϕ sends L^k_t to L^{k-1}_t. It would even be natural to require that for all k,

$$L^k_t = Im(L^{2a+1}_t \otimes \wedge^{2a+1-k} J_3(\mathbb{A}_\mathbb{C})^\vee \to \wedge^k J_3(\mathbb{A}_\mathbb{C})).$$

Then we would only have to degenerate L^{2a+1}, subject to the condition that these contractions maps have constant rank.

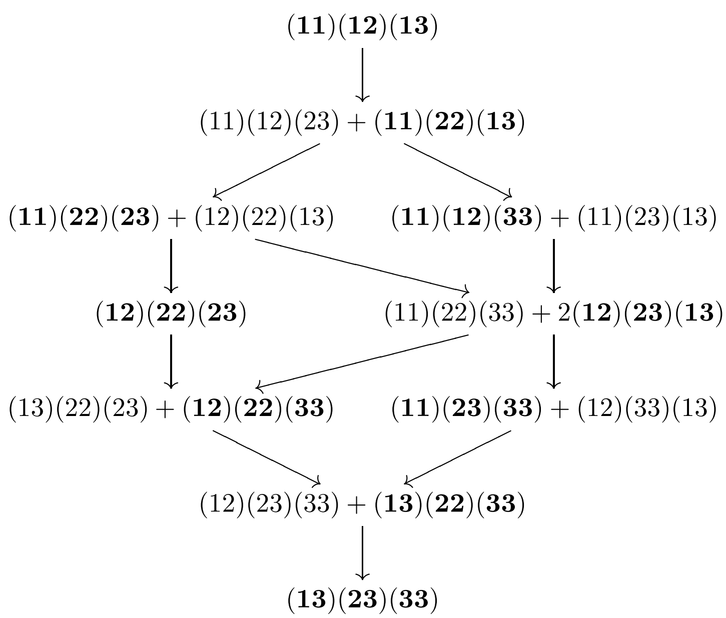
Proposition 6. *The conjecture is true for $a = 1$.*

Proof. In this case we only have a three term complex to deal with:

$$L^3 \to L^2 = \wedge^2 V \to L^1 = V.$$

Here $V = S^2U$ for U of dimension three. In particular there is only L^3 to degenerate in the Grassmannian of ten-dimensional subspaces of $\wedge^3 V$, subject to the condition that the contraction map to $\wedge^2 V$ is surjective. Since this is an open condition, we can certainly degenerate it to the space L^3_Δ defined by the classical triangulation Δ of $\mathbb{RP}^2$. □

It turns out that something rather special happens. Let u_1, u_2, u_3 be a basis of U, and consider the Borel subgroup of $GL(U)$ defined by this basis. Then $L^3 = S_{411}U$ is the submodule of $\wedge^3(S^2U)$ with highest weight vector $u_1^2 \wedge u_1u_2 \wedge u_1u_3$ with respect to our Borel subgroup. We denote this vector by (11)(12)(13). A basis of L^3, consisting in eigenvectors of the maximal torus defined by the basis, can then be obtained by applying successively the root vectors associated to the opposite of the two simple roots of sl_3. We get the following diagram.



Now we may consider each basis vector $u_iu_j = (ij)$ of V as a vertex of a triangulation. Then a decomposable tensor $(ij)(kl)(mn)$ in $\wedge^3 V$ encodes a two-dimensional face. Not all the vectors in our basis are decomposable, but those that are not are the sum of only two decomposable vectors, and there is a unique way to choose one among these two, for each of the seven non decomposable vectors, in such a way that the ten faces that we obtain define a triangulation Δ of $\mathbb{RP}^2$ (the minimal triangulation). The terms corresponding to these ten faces are indicated in bold on our diagram. In particular we can get a degeneration of L^3 to L^3_Δ just by rescaling the terms that are not in bold.

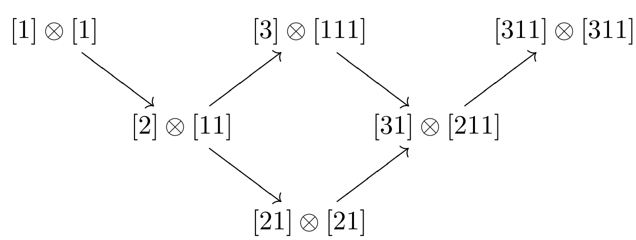
5. The minimal triangulation of $\mathbb{CP}^2$, revisited

In [BrK87] the authors exhibited a triangulation of $\mathbb{CP}^2$ with nine vertices. The list of its 36 maximal faces was obtained with the help of a computer:

12456	45789	12378
23456	56789	12389
13456	46789	12379
12459	34578	12678
23567	15689	23489
13468	24679	13579
23469	35679	13689
13457	14678	12479
12568	24589	23578
13569	34689	23679
12467	14579	13478
23458	25678	12589

The symmetry group G of this triangulation has order 54 and acts transitively on the vertices. More specifically, the permutations (147)(258)(369) and (123)(456)(789) generate a subgroup H of the symmetry group isomorphic with $\mathbb{Z}_3 \times \mathbb{Z}_3$, and this subgroup acts simply transitively on the vertices. Note also that H has index two in its normalizer $N_G(H)$, which is generated by H and the involution $\tau = (12)(46)(89)$. The involutions in G are all conjugate.

Let us review how this can be connected to our approach. For $\mathbb{A} = \mathbb{C}$, the Jordan algebra $J_3(\mathbb{A}_\mathbb{C})$ can be identified with the tensor product $A \otimes B$ of two vector spaces of dimension three. The terms of the Koszul complex are given by the Cauchy formula, and the subcomplex $L^\bullet$ is encoded in the following graph:



Our notation here is the following: by $[\mu] \otimes [\nu]$ we mean the tensor product of Schur powers $S_\mu A \otimes S_\nu B$, plus the symmetric term $S_\nu A \otimes S_\mu B$ if $\mu \neq \nu$. We have $L^k = \wedge^k(A \otimes B)$ for $1 \leq k \leq 3$, corresponding to the first three columns of the complex. On the extreme right, $L^5 = S_{311}A \otimes S_{311}B = S^2A \otimes \det A \otimes S^2 B \otimes \det B$ has dimension 36.

If we choose a basis a_1, a_2, a_3 of A and a basis b_1, b_2, b_3 of B, we get a basis $a_i \otimes b_j = (ij)$ of $A \otimes B$ and an induced basis of its wedge powers. Note that $L^5 = [311] \otimes [311]$ is a multiplicity free module. As a submodule of $\wedge^5(A \otimes B)$, it is generated by the highest weight vector (11)(12)(13)(21)(31). Taking into account the action of the Weyl group $W = S_3 \times S_3$, we get nine decomposable vectors. Our principle is that each weight vector (ij) should be identified with a vertex of the triangulation, and each decomposable vector to a face of this triangulation. Starting from the configuration of the nine faces that have to be associated to the nine decomposable vectors, we are led to the following identification between our weight vectors and the vertices of the Brehm-Kühnel triangulation:

1	2	3	4	5	6	7	8	9
(23)	(32)	(11)	(21)	(33)	(12)	(22)	(31)	(13)

We can then make several observations:

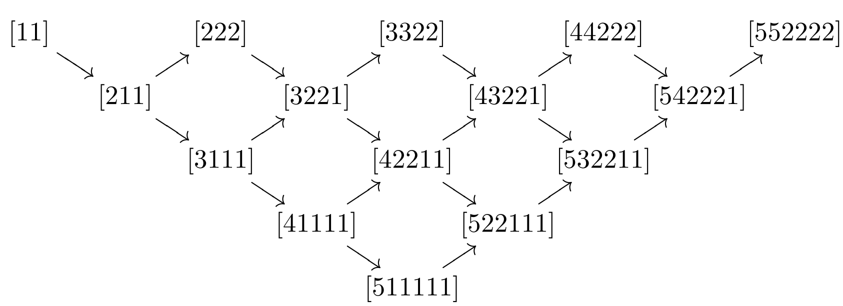
(1) The transitive action of H on the vertices coincides with the natural action of the subgroup $A_3 \times A_3$ of the Weyl group $S_3 \times S_3$.
(2) The involution τ coincides with the external symmetry $(ij) \mapsto (ji)$.
(3) The maximal faces split into four H-orbits of nine elements, corresponding to the four W-orbits among the weights of L^5.
(4) Each weight space is generated by a vector which is the sum of one, two or four decomposable vectors, and exactly one of these decomposable vectors correspond to a maximal face of the triangulation.

The minimal triangulation of $\mathbb{CP}^2$ and its symmetries thus become much more transparent when interpreted in our representation theoretic setting.

6. The quaternionic case, revisited

In [BrK92], three combinatorial triangulations were constructed of a manifold "like a quaternionic projective plane". One of these triangulations is more symmetric than the two others: its automorphism group, the icosahedral group A_5, acts transitively on the 15 vertices. It can be characterized as the unique tight triangulation of a manifold with this symmetry property. The authors conjectured that the underlying manifold is really the quaternionic projective plane $\mathbb{HP}^2$, but up to our knowledge this conjecture remains open.

For $\mathbb{A} = \mathbb{H}$, the Jordan algebra $J_3(\mathbb{H}_{\mathbb{C}})$ can be identified with the second wedge power $\wedge^2 A$ of a vector space a of dimension six. The subcomplex $L^\bullet$ of the Koszul complex is encoded in the following graph:



Here again we denote by $[\lambda]$ the Schur power $S_\lambda A$. In particular [11] denotes the minuscule representation $\wedge^2 A$, whose fifteen weights (with respect to some fixed maximal torus) should represent the fifteen vertices of the Brehm-Kühnel triangulation. This suggests in particular that the action of the icosahedral group A_5 on these vertices, which is produced in [BrK92] by exhibiting an explicit embedding in S_{15}, is in fact induced by a much more simple embedding in S_6.

This is indeed the case, and we consider this fact as a strong hint that our insights should be correct. Consider the permutations defined in cyclic notation by

$$\begin{aligned} p &= (1)(23456), \\ r_1 &= (1)(2)(36)(45), \\ s &= (156)(243), \\ r_2 &= (4)(5)(36)(12). \end{aligned}$$

There is an induced action on the set of pairs of distinct integers, that we put in correspondence with integers between 1 and 15 by identifying the following tables:

45	36	12		1	6	11
56	24	13		2	7	12
26	35	14	vs	3	8	13
23	46	15		4	9	14
34	25	16		5	10	15

It is then straightforward to check that the resulting permutations of S_{15} coincide with the permutations denoted P, R_1, S, R_2 in [BrK92], pp. 170-171.

The maximal simplices defining the Brehm-Kühnel triangulation (see [BrK92], Table 2 p. 174) can then be identified with sets of nine pairs of integers. In particular, the simplex denoted M_1 corresponds to (12)(13)(14)(15)(16)(23)(24)(25)(26). This expression can be seen as defining a highest weight vector of the representation [552222] inside $\wedge^9[11]$, in complete agreement with our expectations.

7. Spherical links

Assuming that the vertex-transitive action by a symmetry group, which exists for the known triangulations of $\mathbb{RP}^2$ and $\mathbb{CP}^2$ and for the conjectural triangulation of $\mathbb{HP}^2$, also exists for the hypothetical case of $\mathbb{OP}^2$, one can consider the link of an arbitrary vertex in one of these triangulations. This link does not depend on the chosen vertex, up to isomorphism, and defines a triangulation of a sphere of dimension $2a-1$.

Knowing the number of simplices in the triangulation of $\mathbb{AP}^2$, one can compute the number of simplices in this triangulated sphere by a double counting argument. First note that in order to count simplices of the link, one can study the star instead of the link. Consider now the set of pairs (v, f) where v is a vertex in the triangulation of $\mathbb{AP}^2$ and f is a simplex in the star of v. Every k-dimensional simplex of the triangulation of $\mathbb{AP}^2$ belongs exactly to the links of its $k+1$ elements, hence will appear $k+1$ times in the set of pairs (v, f). One can then count pairs (v, f) such that f is k-dimensional in two different ways.

The results for the spherical triangulations are listed below by increasing dimensions. The spherical link in $\mathbb{RP}^2$ is a pentagonal circle. According to [BrK92, §5], the spherical link in $\mathbb{CP}^2$ is a non-polytopal 3-sphere, called the Brückner-Grünbaum sphere, and the spherical link in $\mathbb{HP}^2$ is a non-polytopal 7-sphere.

	$\mathbb{RP}^2$	$\mathbb{CP}^2$	$\mathbb{HP}^2$	$\mathbb{OP}^2$
#vertices	5	8	14	26
#1 − dim	**5**	28	91	325
		40	364	2600
#3 − dim		**20**	1001	14950
			1806	65780
			1974	230230
			1176	657800
#7 − dim			**294**	1562275
				3087370
				4964102
				6255184
				5922800
				4022200
				1838720
				505648
#15 − dim				**63206**

Observation. *The number of maximal faces in the spherical triangulation is*

$$\frac{3a+2}{a+2}\binom{2a+1}{a+1}.$$

This is the dimension of the irreducible representation of the Lie algebra $\mathfrak{so}(a+4)$, whose highest weight is $a\omega$, where ω is the fundamental weight defining the vector representation of dimension $a+4$.

The meaning of this observation remains unclear to us. Moreover we could not find similar interpretations for the other numbers of faces.

As a curiosity, one can note that the numbers of maximal faces also appear in the sequence A129869 of the On-Line Encyclopedia of Integer Sequences (oeis.org), which counts tilting modules for quivers of type D.

8. Higher ranks

Much of what we have explained in the previous section remains true for higher rank, that is, for the projective spaces $\mathbb{AP}^n = \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$ with $n \geq 3$. Their complex versions $\mathbb{AP}^n_{\mathbb{C}}$ are homogeneous under the action of a group $SL(n+1,\mathbb{A})$. Moreover there exists a bigger homogeneous variety $LG(n+1,\mathbb{A})$, with automorphism group $Sp(2n+2,\mathbb{A})$, such that $\mathbb{AP}^n_{\mathbb{C}}$ can be identified with the space of lines in $LG(n+1,\mathbb{A})$ passing through a prescribed point p.

$\mathbb{A}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{AP}^n_{\mathbb{C}}$	$v_2(\mathbb{P}^n)$	$\mathbb{P}^n \times \mathbb{P}^n$	$G(2,2n+2)$
$SL(n+1,\mathbb{A})$	$SL(n+1)$	$SL(n+1)\times SL(n+1)$	$SL(2n+2)$
$LG(n+1,\mathbb{A})$	$LG(n+1,2n+2)$	$G(n+1,2n+2)$	$\mathbb{S}_{2n+2}$
$Sp(2n+2,\mathbb{A})$	$Sp(2n+2)$	$SL(2n+2)$	$SO(4n+4)$

Exactly as for $n=2$, this allows to embed the Hasse diagram of $\mathbb{AP}^n_{\mathbb{C}}$ into that of $LG(n+1,\mathbb{A})$, as an interval $I=[s,t]$. Moreover the Hasse diagram of $LG(n+1,\mathbb{A})$ is canonically identified to the graph constructed from the wedge powers of $J_{n+1}(\mathbb{A}_{\mathbb{C}})$, the complexification of the space of Hermitian matrices of size $n+1$ with coefficients in $\mathbb{A}$. These wedge powers are given by the following classical formulas.

$\mathbb{A}=\mathbb{R}$. Then $J_{n+1}(\mathbb{R}_{\mathbb{C}}) = Sym^2 U$ where U has dimension $n+1$. We have

$$\wedge^k J_{n+1}(\mathbb{R}_{\mathbb{C}}) = \bigoplus_{\substack{|\lambda|=k,\\ h(\lambda)\leq n}} S_{d_+(\lambda)}U,$$

where the sum is over *strict partitions* $\lambda = (\lambda_1 > \cdots > \lambda_\ell > 0)$ of size k and of height $h(\lambda)=\lambda_1$ at most n. Moreover $d_+(\lambda)$ is the partition of size $2k$ obtained by putting together λ and its conjugate:

$$d_+(\lambda) = (\lambda_1, \lambda_2+1, \ldots, \lambda_\ell+\ell-1, \ell^{\lambda_\ell}, (\ell-1)^{\lambda_{\ell-1}-\lambda_\ell-1}, \ldots, 1^{\lambda_1-\lambda_2-1}),$$

where powers mean repetitions.

$\mathbb{A}=\mathbb{C}$. Then $J_{n+1}(\mathbb{C}_{\mathbb{C}}) = U\otimes V$ where U and V have dimension $n+1$. We have

$$\wedge^k J_{n+1}(\mathbb{C}_{\mathbb{C}}) = \bigoplus_{\substack{|\lambda|=k,\\ \ell(\lambda),h(\lambda)\leq n+1}} S_\lambda U\otimes S_{\lambda^\vee}V,$$

where the sum is over partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$ of size k and of height $h(\lambda)=\lambda_1$ and length $\ell(\lambda)=\ell$ at most $n+1$.

$\mathbb{A}=\mathbb{H}$. Then $J_{n+1}(\mathbb{H}_{\mathbb{C}}) = \wedge^2 U$ where U has dimension $2n$. We have

$$\wedge^k J_{n+1}(\mathbb{H}_{\mathbb{C}}) = \bigoplus_{\substack{|\lambda|=k,\\ h(\lambda)\leq 2n}} S_{d_-(\lambda)}U,$$

where the sum is over *strict partitions* $\lambda = (\lambda_1 > \cdots > \lambda_\ell > 0)$ of size k and of height $h(\lambda)=\lambda_1$ at most $2n$. Moreover $d_-(\lambda)$ is the conjugate partition to $d_+(\lambda)$.

In each case the Hasse diagram of $LG(n+1,\mathbb{A})$ coincides with the graph of partitions with the partial order defined by the inclusion relation. The minimal element s of I corresponds to the partition of size one, while t corresponds to the partition $(n), (n+1, 1^n), (2n+1, 2n)$ respectively. This yields

$$L^{an+1} = J_{n+1}(\mathbb{A}_{\mathbb{C}})^{(d)} \qquad \text{where} \quad d = a\frac{n-1}{2}+1.$$

The other terms are then easy to write down explicitly:

$\mathbb{A}=\mathbb{R}$. Then $L^k = S_{k+1,1^{k-1}}U$ for $1\leq k\leq n+1$. This implies that

$$f_k^{1,n} = \frac{1}{2}\binom{n+k+2}{k+1}\binom{n+1}{k+1}.$$

$\mathbb{A}=\mathbb{C}$. Here $L^k = \oplus_{i+j=k-1} S_{i+1,1^j}U\otimes S_{j+1,1^i}U$ for $1\leq k\leq 2n+1$. Therefore

$$f_k^{2,n} = \Big(\frac{n+1}{k+1}\Big)^2 \sum_{i+j=k} \binom{n+i+1}{i}\binom{n}{i}\binom{n+j+1}{j}\binom{n}{j}.$$

$\mathbb{A}=\mathbb{H}$. Here $L^k = \oplus_{i+j=k, i>j} S_{d_-(i,j)}U$.

If we fix a generic element $\phi\in J_{n+1}(\mathbb{A}_{\mathbb{C}})^\vee$, its stabilizer is a conjugate of $SO(n+1,\mathbb{A}) = Aut(J_{n+1}(\mathbb{A}_{\mathbb{C}}))$. We expect that the complex $L^\bullet(\phi)$ should be quasi-isomorphic with the complex of $SO(n+1,\mathbb{A})$-invariants, with trivial arrows. Moreover, we have:

Proposition 7.

$$(L^{k+1})^{SO(n+1,\mathbb{A})} = \begin{cases} \mathbb{C} & \text{if } k = 0, a, \ldots, na, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider for example the case where $a = 4$. It follows from the branching rules from SL to Sp [Li40] that a Schur module $S_\mu U$ has a $Sp(2n)$-invariant if and only if the conjugate partition $\mu^\vee$ has only even parts, in which case this invariant is unique up to scalars. For $\mu = d_-(i,j)$, hence $\mu^\vee = d_+(i,j)$, this means that $i = j+1$ and j is even. Hence $i+j-1 = 2j$ must be divisible by four, and the claim follows. □

There is therefore an intriguing relation between these modules and the cohomology of $\mathbb{AP}^n$, confirmed by the following statement:

Proposition 8. *For any $a = 1, 2, 4$ and any $n \geq 2$, one has*

$$\sum_{k=0}^{na} (-1)^k f_k^{a,n} = \begin{cases} \frac{1+(-1)^n}{2} & \text{if } a = 1, \\ n+1 & \text{if } a = 2 \text{ or } a = 4. \end{cases}$$

An optimistic guess would be that some degeneration of the complex $L^\bullet(\phi)$ should be the Stanley-Reisner complex of some triangulation of $\mathbb{AP}^n$. This triangulation would have $f_k^{a,n}$ faces of dimension k, in particular it would have exactly $a\binom{n+1}{2} + n + 1$ vertices. Unfortunately, this is definitely over-optimistic: it was proved in [AM91] that $\mathbb{RP}^3$ does not admit any triangulation with only 10 vertices!

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