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Strings from Feynman Graph counting : without large N

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ABSTRACT

A well-known connection between n strings winding around a circle and permutations of n objects plays a fundamental role in the string theory of large N two dimensional Yang Mills theory and elsewhere in topological and physical string theories. Basic questions in the enumeration of Feynman graphs can be expressed elegantly in terms of permutation groups. We show that these permutation techniques for Feynman graph enumeration, along with the Burnside counting lemma, lead to equalities between counting problems of Feynman graphs in scalar field theories and Quantum Electrodynamics with the counting of amplitudes in a string theory with torus or cylinder target space. This string theory arises in the large N expansion of two dimensional Yang Mills and is closely related to lattice gauge theory with S_n gauge group. We collect and extend results on generating functions for Feynman graph counting, which connect directly with the string picture. We propose that the connection between string combinatorics and permutations has implications for QFT-string dualities, beyond the framework of large N gauge theory.

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1 Introduction

There is a basic connection between winding states of strings and partitions of numbers. If we have a string wound around a circle, it can have a winding number which is a positive integer. For multiple strings, the winding numbers can be added to define a total winding number. In the sector of the multi-string Hilbert space with total winding number n, the partitions of n, denoted $p(n)$, correspond to the different types of string states. The number $p(n)$ also plays an important role in connection with the symmetric group S_n, of order $n!$, consisting of the re-arrangements of the integers $\{1, 2 \cdots, n\}$. It is the number of conjugacy classes in S_n.

In generic string theories, a string state is characterized by a winding number along with additional vibrational and momentum quantum numbers. In simple cases, the winding numbers completely characterize the string state. A particularly striking example of this simplification occurs in the string dual of two dimensional Yang Mills theory (2dYM) on a Riemann surface Σ_G with genus G. This theory, for $U(N)$ gauge group, can be solved exactly [1]. The coefficients in the large N expansion of amplitudes in 2dYM defined on Riemann surfaces can be expressed in terms of sums over symmetric group elements. This combinatoric data has an interpretation in terms of branched covers of Σ_G by a string worldsheets Σ_h, Riemann surfaces of genus h related to the order in the $1/N$ expansion [2, 3, 4]. This provides a beautiful realization of the ideas of [5]. These spaces of covers are also called Hurwitz spaces and they are spaces of holomorphic maps from Σ_h to Σ_G. These spaces are reviewed from the point of view of 2dYM in [6].

Of particular interest are manifolds $\Sigma_{G,B}$, with genus G and B boundaries. The observables of 2dYM are defined as functions of the boundary holonomies taking values in the gauge group $U(N)$. The gauge-invariant functions of these group elements are multi-traces, which are also classified for fixed degree n, by partitions. This leads to an interpretation of the boundary partition functions in terms of covering spaces of the manifold with boundary. Generically, the partition function sums over branched covers. The derivative of the map at some points on the worldsheet can vanish, and the images of such points are called *branch points*. A particularly simple situation occurs when the Yang Mills theory is defined on a cylinder, and the area is taken to zero. Then the partition function sums only over *unbranched covers*.

The large N expansion of 2dYM can also be expressed in terms of lattice gauge theory with S_n gauge group which can be formulated following Wilson and letting edge holonomies take values in S_n, with an appropriate plaquette action. This theory is topological and will be called S_n TFT (topological field theory). So along with the original $U(N)$ description, there is the picture of S_n TFT, as well as the Hurwitz space description. The Hurwitz space connects most directly to the equations formulated on the string worldsheet, since the holomorphy condition is the localization locus for the string path integral. Explicit worldshseet actions have been proposed [6, 7, 8, 9, 10]. The 2d S_n TFT can be viewed as the string field theory of this Hurwitz space string theory. In this paper we will find the $2d$ S_n TFT particularly useful.

Other examples where the string-permutation connection is important is Matrix string theory[17]. An orbifold superconformal field theory with target space $S^N(\mathbb{R}^8)$ arises as an IR limit of $N = 8$ SYM in two dimensions. Conjugacy classes in S_N correspond to string winding numbers which are dual to momenta of gravitons in the Matrix Theory interpretation. Yet another example is the connection between Belyi maps (a special class of branched covers with sphere target space) and correlators of the Gaussian Hermitian matrix model [18, 19].

From the above we take the lesson that the connection between permutations and strings is rather generic. In many cases, it is a tool to implement the ideas of strings emerging from the large N expansion in quantum field theories. Now a large number of counting problems can be formulated in terms of permutations acting on various types of sets. Some of these counting problems, such as the ones relevant to 2d Yang Mills and to Belyi maps, involve permutations acting on each other, for example by group multiplication. It will not surprise anybody that the question of counting Feynman graphs in quantum field theories, *without large* N, can be formulated in terms of symmetric groups acting on some sets having to do with vertices and edges.

Investigating the counting of Feynman graphs more carefully reveals that, in fact, these counting problems can be formulated in terms of rather intrinsic properties of the symmetric groups themselves. The nature of the specific interactions determine the form of the vertices, and in turn certain *subgroups* of symmetric groups, which can be viewed as the symmetry of all the vertices. Then we draw in an idea from the study of Belyi maps and their associated ribbon graphs, which is called *cleaning*. This is the construction of associating to a given graph, a new graph obtained by introducing a new type of vertex in the middle of the existing edges, dividing them into half-edges. This allows the description of Wick contractions in terms of permutations which are pairings. This was exploited recently in a physics setting in [19].

This line of thinking culminates in an elegant way to compute numbers of Feynman graphs of real scalar fields, which coincides with the results in a classic in the mathematical literature on graph counting [20]. Our derivation does not rely on the notion of graph superposition used in [20] and appears more direct, at least when approaching these graphs from the perspective of Wick contractions in perturbative quantum field theory. We also show that the formulae counting Feynman graphs, as well as those describing symmetry factors of individual graphs, can be interpreted as observables in the 2d S_n TFT. The number n will depend on the number of vertices and edges in the Feynman graph calculation. We extend our considerations to QED, deriving new counting results on Feynman graphs. Again we obtain an expression of Feynman graph counting problems in terms of observables in the 2d S_n TFT. An interesting aside is a surprising connection between the counting of QED Feynman graphs and the counting of ribbon graphs.

While the string-permutation connections underlie the emergence of strings from large N 2d YM, our results suggest that these same connections could also lead to the emergence of strings from quantum field theories such as that of a scalar field or QED.

We now give an outline of the paper. In section 2, we review the connection between strings and permutations, which plays a prominent role in the string theory of large N 2d Yang Mills theory with $U(N)$ gauge group. We describe the perspective of lattice gauge theory with S_n gauge group and explain its topological nature by drawing on existing 2dYM literature. We also review the Hermitian matrix model and its connection to Belyi maps, as another realization of the string-permutations connection. Finally, we mention the Burnside Lemma from combinatorics, which is used very generally for counting orbits of group actions. In section 3, we review Feynman rules for ϕ^4 theory. We recall how the symmetry factors get multiplied with additional group theory factors associated with global or gauge symmetry groups, and space-time integrals. Our main focus will be the combinatorics of the graphs and their symmetry factors.

In section 4, we describe a method of enumerating Feynman graphs and calculating their symmetry factors, which relies on a pair of combinatoric objects. The first of the pair, which we call Σ_0, captures the form of the vertices. The second piece of data, which we call Σ_1, is associated with the Wick contractions. Given a graph, this data can be constructed by putting a new vertex in the middle of each edge. We can imagine these vertices to be coloured differently from the ones already present in the graph. This splits each existing edge into a pair of half-edges. We associate labels $\{1, 2, \cdots\}$ with each of these half-edges. This is explained in the context of real scalar field theory, in the first instance, with vacuum graphs in ϕ^4 theory. This formulation leads to simple one-liners in GAP (mathematical software of choice for group theory) [22] for calculations of symmetry factors and enumeration. This is explained with examples in Appendix C

In section 5, we use the Burnside Lemma to count Feynman graphs, and obtain the first hints of stringy geometry of maps to a torus. The picture of maps to a torus has some intricacies, and it turns out that a deeper understanding of the combinatorics leads to a simpler picture in terms of strings covering a cylinder. This is is achieved by first recalling from the maths literature that graph counting formulae are expressed in terms of a certain double coset [20]. We find that the formulation of graph counting in terms of the pair (Σ_0, Σ_1) finds a natural meaning in terms of the double coset. This allows us to exhibit the counting of Feynman graphs in scalar field theory as an observable in 2d S_n TFT on a cylinder. Following standard constructions in covering space theory, the S_n data is used to construct unbranched covers of the cylinder. Some of the counting formulae that follow from the double coset picture can be understood using the idea of Fourier transformation on symmetric groups. Calculations explaining this are in Appendix B. Interestingly very similar calculations are relevant to recent results in correlators of BPS operators in $N = 4$ super-Yang-Mills [21].

We pursue, following [20], the application of the double coset picture to derive generating functions for graph counting in section 6. We also explain here how the formulae change when we generalize our considerations from ϕ^4 to ϕ^3 and other interactions. Generalizations away from vacuum Feynman graphs to the case with external edges is also explained. The double coset formulae lead most directly to the set of all Feynman graphs, included disconnected ones. Counting formulae for the connected ones are obtained here. Appendix D contains explicit lists for Feynman graph counting. The first few terms agree with existing physics literature. A few of the series we consider are listed in the Online Encyclopaedia of Integer sequences [23], but the majority are not listed there.

In section 7, we show that these approaches work in a simple way for the ribbon graphs which arise in doing the $1/N$ expansion. While the ribbon graphs for large N are traditionally drawn using double lines, they can also be defined in a way closer to ordinary graphs, with single lines but with the difference that the vertices have a cylic order (see e.g. [24] [25]). This allows cyclic symmetry of the half-edges at the vertices in testing equivalence of differently labelled ribbon graphs, but not arbitrary permutations. Taking this into account, we express the question of how many Feynman diagrams correspond to the same ribbon graph (embedded graph) in group theoretic terms, associated with the symmetry breaking from permutation group to cyclic group.

In section 8, we apply these ideas to QED or Yukawa theory, giving the connection to 2d S_n TFT and deriving generating functions. In section 9, we adapt the counting to QED, with the vanishings due to Furry's theorem taken into account. Again we obtain the 2d S_n TFT connection and the generating functions. Manipulations of the double coset description for QED leads to a somewhat surprising connection between QED graph counting and ribbon graph counting. We explain this by describing a bijection between these graphs. The arrows on electron loops provide the cyclic symmetry which is key to ribbon graphs. A consequence is that the number of QED/Yukawa vacuum graphs with $2v$ vertices is equal to the number of ribbon graphs with v edges.

In section 10, we discuss our results with a view to extracting some lessons for gauge string duality. Ribbon graphs give a vivid physical picture of how strings arise from quantum field theory, but an equally compelling and arguably simpler physical picture is that of strings winding on a circle being described by permutations. This latter picture has been exploited here to give new connections between QFT Feynman diagrams and string counting. The natural question is whether this connection can be extended to extract from Feynman graphs of QFT, without large N, something more than stringy combinatorics to include space-time dependences of correlators and S-matrices.

In section 11, we summarize the paper and discuss some concrete avenues for future work. Appendix A describes a semi-direct product structure of the Automorphism groups, whose orders give symmetry factors of Feynman graphs, and points out a difference between the notion of Automorphism most commonly used in the mathematics literature on graphs and the one relevant to symnmetry factors of Feynman graphs.

2 Review of strings and permutations

Consider a string wrapping a circle. The winding number is a topological characteristic of the map. In string theory, the string is part of the worldsheet, the circle is viewed as part of target space. Let the target circle be parametrized by X with $X \sim X + 2\pi$. Let the string be parametrized by σ with $\sigma \sim \sigma + 2\pi$. A string with winding number k is described by

$$X = k\sigma \tag{2.1}$$

For multiple strings, we may have distinct winding numbers. Adding the winding numbers gives the degree of the map from the strings.

Given such a configuration of winding strings, we may label the inverse images of a fixed point on the circle from 1 to n. Following the inverse images as we move round the target circle yields a permutation. The two possible winding configurations at $n = 2$ are shown in Figure 1.

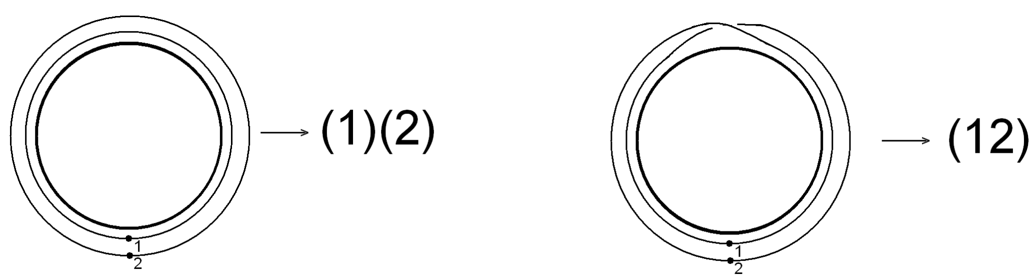

Figure 1: Winding strings and permutations

In ordinary string theory, the winding number is one of the quantum numbers specifying a string state. In some topological settings, the string winding number is the only relevant quantum number.

2.1 Review of results from 2d Yang Mills

Recall that 2d Yang Mills on a target space with a boundary has the partition function

$$Z = \sum_R (\mathrm{Dim} R)^{2-2G-B} e^{-AC_2(R)} \prod_i \chi_R(U_i) \tag{2.2}$$

G is the genus of the surface, B is the number of boundaries, U_i are the holonomies at the boundaries and A is the area of the target and $C_2(R)$ is the quadratic Casimir [1]. The choice of fixed boundary holonomies is illustrated in Figure 2.

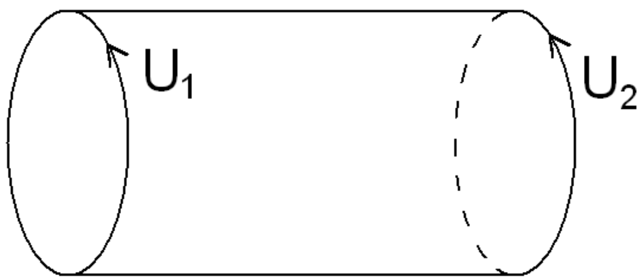

Figure 2: The 2dYM partition function with boundary holonomies is known exactly

For a target cylinder, we have

$$Z(U_1, U_2) = \sum_R \chi_R(U_1)\chi_R(U_2)e^{-AC_2(R)} \tag{2.3}$$

in terms of characters of $U(N)$ elements. The power of the dimension is zero because $G = 0, B = 2$. The characters have an expansion in multi-traces. These traces can be constructed from permutations as follows

$$tr(\sigma U) \equiv U^{i_1}_{i_{\sigma(1)}} U^{i_1}_{i_{\sigma(2)}} \cdots U^{i_1}_{i_{\sigma(n)}} \tag{2.4}$$

This is a very useful formula when developing a string interpretation for Wilson loops in 2dYM [26].

Using these observables for the cylinder, we define

$$Z(\sigma_1, \sigma_2) = \int dU_1 dU_2 tr(\sigma_1 U_1^\dagger) tr(\sigma_2 U_2^\dagger) Z(U_1, U_2) \tag{2.5}$$

This integral can be done, and in the zero area limit $A = 0$ we find

$$Z(\sigma_1, \sigma_2) = \sum_{R \vdash n} \chi_R(\sigma_1)\chi_R(\sigma_2) \tag{2.6}$$

The notation $R \vdash n$ means that R is running over partitions of n (row lengths of Young diagrams) which parametrize irreducible representations of S_n, and the $\chi_R(\sigma)$ are characters of S_n elements. Using the invariance of the character under conjugation, we can replace

$$\chi_R(\sigma_2) = \frac{1}{n!} \sum_{\gamma \in S_n} \chi_R(\gamma \sigma_2 \gamma^{-1}) \tag{2.7}$$

This gives

$$Z(\sigma_1,\sigma_2) = \frac{1}{n!}\sum_{\gamma\in S_n}\sum_{R\vdash n}\chi_R(\sigma_1)\chi_R(\gamma\sigma_2\gamma^{-1}) \tag{2.8}$$

Since $\sum_\gamma \gamma\sigma_2\gamma^{-1}$ is a central element in the group algebra of S_n, we can use Schur's Lemma to combine the product of characters into a single character

$$Z(\sigma_1,\sigma_2) = \frac{1}{n!}\sum_{\gamma\in S_n}\sum_{R\vdash n} d_R\chi_R(\sigma_1\gamma\sigma_2\gamma^{-1}) \tag{2.9}$$

Now use the Fourier expansion on the symmetric group, which allows the delta function to be written in terms of characters, to obtain

$$Z(\sigma_1,\sigma_2) = \sum_{\gamma}\delta(\sigma_1\gamma\sigma_2\gamma^{-1}) \tag{2.10}$$

When the gauge group is a product $U(N)\times U(N)\times U(N)$, we have to specify a holonomy for each gauge group in the product. Denote them by U, V, W. Recalling that the character of a direct product is the product of the characters we have the zero area partition function for a cylinder

$$Z(U_1,V_1,W_1;U_2,V_2,W_2) = \sum_{R,S,T}\chi_R(U_1)\chi_S(V_1)\chi_T(W_1)\chi_R(U_2)\chi_S(V_2)\chi_T(W_2)) \tag{2.11}$$

The boundary observables in this case, in a basis appropriate for a string interpretation, are $tr(\sigma U)tr(\rho V)tr(\tau W)$. In terms of permutations

$$Z(\sigma_1,\rho_1,\tau_1;\sigma_2,\rho_2,\tau_2) = \sum_{\gamma\in S_n\times S_n\times S_n}\delta(\sigma_1\circ\rho_1\circ\tau_1\ \gamma\ \sigma_2\circ\rho_2\circ\tau_2\ \gamma^{-1}) \tag{2.12}$$

We have discussed above the classification of string maps to a circle target space, in motivating the choice of boundary observables in large N 2dYM. The full interpretation of this result requires an extension of our considerations to a Riemann surface target. In this case, the strings-permutations connection has deeper implications captured in the *Riemann existence theorem*. Consider the equivalence classes of holomorphic maps from the worldsheet (Σ_h) to the target (Σ_G) with two maps f_1 and f_2 defined to be equivalent if these exists a holomorphic invertible map $\phi : \Sigma_h \to \Sigma_G$ such that $f_1 = f_2\circ\phi$. Given a holomorphic map (branched cover) with L branch points and of degree n, we can obtain a combinatoric description by picking a generic base point and labeling the inverse images as $\{1, 2, \cdots, n\}$. By following the inverse images of a closed path starting at the base point and encircling each branch point, we can get a sequence $\sigma_1, \sigma_2, ..., \sigma_L$ of permutations. For $G > 0$, there are also permutations for the a, b cycles of Σ_G, denoted

s_i, t_i for $i = 1 \cdots G$. Two equivalent holomorphic maps are described by permutations $\sigma_1, \sigma_2, ..., \sigma_L, s_1, t_1, \cdots, s_G, t_G$ and $\sigma_1', \sigma_2', ..., \sigma_L', s_1', t_1', \cdots, s_G', t_G'$ which are related by conjugation $\sigma_i = \alpha \sigma_i' \alpha^{-1}, s_i = \alpha s_i' \alpha^{-1}, t_i = \alpha t_i' \alpha^{-1}$. This correspondence between sequences of permutations and holomorphic maps is captured in the Riemann existence theorem (see for example [25]). Relations in the fundamental group of the Riemann surface punctured at the branch points are reflected in the sequence of permutations.

The delta function in (2.10) enforces a relation in the fundamental group of the cylinder. When this partition function is generalized beyond zero area, the sum (2.10) is modified to include additional permutations which can be interpreted as a counting of branched covers. Even at zero area, but for other Riemann surfaces, there are additional permutations associated with branch points. We will come back to this point in the discussion of (2.16).

2.2 Lattice topological field theory for S_n

In lattice gauge theory[27] for two dimensions, we discretize (e.g triangulate or give a more general cell decomposition for) the Riemann surface. To each edge we associate a group element. To each plaquette, we associate a weight which depends on the product of group elements along the boundary of the plaquette. Let us choose the plaquette weight to be

$$Z_P(\sigma) = \delta(\sigma) = \sum_{R \vdash n} \frac{d_R}{n!} \chi_R(\sigma) \tag{2.13}$$

where σ is the product of group elements around the boundary of the plaquette. The partition function for the discretized manifold is the product of weights for all the plaquettes. The partition function can be shown to be invariant under refinement of the lattice so that the result depends only on the topology of the Riemann surface. This is proved in [28]. The language used there is that of continuous groups, but by replacing integrals with sums, it applies equally well to finite groups. In [28] surfaces with a finite area A are considered. For the topological aspects of the theory that interest us, it is enough to consider $A = 0$. For manifolds with boundary one fixes the group elements at the boundaries. As a simple example, consider a lattice theory defined on a disc. The target might be discretized with either one plaquette or two plaquettes as shown in Figure 3. Starting from the discretization given in (a) we find

$$\begin{aligned} Z(\sigma_1, \sigma_2) &= \sum_\gamma Z_P(\sigma_1 \gamma^{-1}) Z_P(\sigma \gamma) \\ &= \sum_\gamma \delta(\sigma_1 \gamma^{-1}) \delta(\sigma_2 \gamma) \\ &= \delta(\sigma_1 \sigma_2) \end{aligned} \tag{2.14}$$

which is precisely the partition function obtained using the discretization shown in (b). The generalization to finer discretization works in exactly the same way.

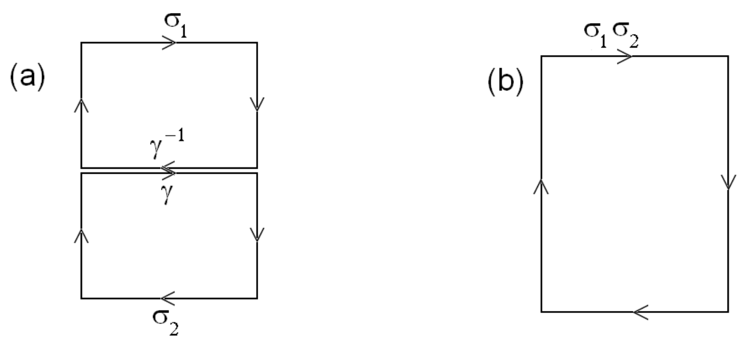

Figure 3: Two discretizations of the disc with the same boundary condition.

For the cylinder

$$Z(\sigma_1, \sigma_2) = \sum_{\gamma} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1}) \tag{2.15}$$

which is of course (2.10). For a 3-holed sphere,

$$\begin{aligned} Z(\sigma_1, \sigma_2, \sigma_3) \qquad &= \frac{1}{(n!)^2} \sum_{\gamma_1, \gamma_2} \delta(\sigma_1 \gamma_2 \sigma_2 \gamma_1^{-1} \gamma_2 \sigma_3 \gamma_2^{-1}) \\ &= \frac{1}{n!} \sum_{R} \frac{\chi_R(\sigma_1) \chi_R(\sigma_2) \chi_R(\sigma_3)}{d_R} \end{aligned} \tag{2.16}$$

This is also derived in [29] from a more axiomatic approach.

These formulae from lattice gauge theory of S_n arise from the leading the large N limit of the $U(N)$ gauge theory at zero area. The cylinder case is special in that this is the full answer to all orders in the $1/N$ expansion. The answer in (2.16) is obtained after using the leading large approximation of Ω, which is a sum over elements in S_n weighted by powers of N. See for example 6.1.2 of [6] for a discussion of Ω. At subleading orders, the Ω factor contains a sum over permutations, which can be interpreted in terms of sources of curvature in the S_n bundle. From the Hurwitz space string interpretation of 2dYM, there are branch points in the interior.

The S_n lattice gauge theory perspective for $U(N)$ gauge theory at large N is emphasized in [30]. Computations of 2dYM partition functions for general Riemann surfaces with boundaries expressed in the symmetric group basis, and the connection with Hurwitz space counting is given in [4, 6, 26]. The connection to topological field theory with S_n gauge group was also observed in [31].

To summarize, the large N expansion of 2dYM with $U(N)$ gauge group on a Riemann surface $\Sigma_{G,B}$ (genus G, with B boundaries) can be expressed in terms of symmetric groups. This combinatoric data arising has an interpretation as 2d gauge theory with S_n gauge group. This connects directly with the string theory interpretation in terms of maps from a string worldsheet Σ_h (genus h related to the order in the $1/N$ expansion) using classic results relating the space of branched covers (Hurwitz space) to symmetric groups.

We will find in this paper that counting problems of Feynman graphs can be expressed in terms of certain generalizations of the 2dYM results, which can be expressed elegantly in terms of the 2d S_n gauge theory perspective, and also admit a connection to branched covers.

2.3 Permutations and Strings beyond 2dYM

The connection between strings and permutations also has interesting implications for the correlators of the Gaussian hermitian one-Matrix model [19]. The computation of correlators can be mapped to the counting of certain triples of permutations which multiply to the identity. These count holomorphic maps from world-sheet to sphere target with three branch points on the target. Holomorphic maps with three branch points are related, by Belyis theorem, to curves and maps defined over algebraic numbers. Thus, the string theory dual of the one-matrix model at generic couplings has worldsheets defined over the algebraic numbers and a sphere target. For related ideas see [32, 33].

Finally, yet another connection between strings and permutations that is relevant to our present discussion is provided by Matrix String Theory, defined by the IR limit of two-dimensional $N = 8$ SYM. This limit is strongly coupled and a nontrivial conformal field theory describes the IR fixed point. The conformal field theory is the $N = 8$ supersymmetric sigma model on the orbifold target space $(\mathbb{R}^8)^N/S_N$, formed from the eigenvalues of the Higgs fields X_I of the theory [17]. If we go around the space-like S^1 of the world-sheet, the eigenvalues can be interchanged. Concretely

$$X_I(\sigma + 2\pi) = X_{g(I)}(\sigma)$$

where g takes value in the Weyl group of $U(N)$ which is the symmetric group S_N. These twisted sectors correspond to configurations of strings with various lengths. Consequently, twisted sectors with given winding number are labeled by the conjugacy classes of the orbifold group S_N.

2.4 A useful theorem in combinatorics

Let G be a finite group that acts on a set X. For each g in G let X^g denote the set of elements in X that are fixed by g. Burnside's lemma asserts the following formula for the number of orbits

$$\text{Number of orbits of the } G\text{-action on } X = \frac{1}{|G|}\sum_{g\in G}|X^g|. \tag{2.17}$$

Thus the number of orbits is equal to the average number of points fixed by an element of G. This is called the Burnside Lemma or the Burnside counting theorem. Useful references for the Burnside Lemma are [41] [42], the former also provides other useful combinatoric background.

3 Review of Feynman graph combinatorics

There are many excellent articles and textbooks that deal with the Feynman graph expansion of perturbative quantum field theory. Here we will simply review those aspects most relevant for our goals. For a relevant reference that has more details see [43].

Perturbative quantum field theory expresses quantities of interest (for example, an amplitude A) as a power series expansion in the small coupling g

$$A(g) = \sum_{k=0}^{\infty} A_v g^v \tag{3.1}$$

The coefficients A_v are obtained by summing Feynman graphs with v vertices D_v

$$A_v = \sum_{D_v} C_{D_v} N_{D_v} F_{D_v} \tag{3.2}$$

C_{D_v} is the symmetry factor of graph D_v, N_{D_v} is a group theory factor coming from the color combinatorics of global or gauge symmetry groups and F_{D_v} is the result of integrating over the loop momenta in D_v. For concreteness assume that g is the strength of a ϕ^4 interaction. In this case the factor N_{D_v} is 1. If we canonically quantize the theory, we expand about the free theory using Wick's Theorem. The Feynman graphs are used to compute the sum over all possible Wick contractions. Different ways of doing the Wick contractions can lead to the same Feynman graph. The symmetry factors C_{D_v} keep track of this. Perturbation theory expands the exponential of the $g\phi^4$ interaction. Consequently, g^v come with a $1/v!$ from the Taylor series of the exponential. Accounting for this factor, the number of Wick contractions leading to Feynman graph D_v is $v!C_{D_v}$. This observation can be used to generate an interesting sum rule for the symmetry factors. The sum of graphs with v vertices and $E = 2n$ external lines reproduces the complete set of Wick contractions for $4v + 2n$ fields. The total number of Wick contractions is equal to the number of pairs that can be formed from the $4v+2n$ fields and is also equal to the sum over D_v of $v! C_{D_v}$. Consequently

$$v! \sum_{D_v} C_{D_v} = (4v + 2n - 1)!! \tag{3.3}$$

Note that for this sum rule to hold we must not subtract vacuum graphs. This sum rule provides a good test to check the symmetry factors. One way to count how many Wick contractions will produce the same graph, is to count the number of ways of interchanging components which don't modify the graph. These are the automorphisms of the graph. Indeed, the standard recipe [44] for computing the C_{D_k} starts by giving a set of rules to determine these automorphisms. For $g\phi^4$ theory the rules are

- for a closed propagator (one with both ends attached to the same vertex) there is a swap which exchanges the endpoints.
- for p propagators which each start at the same vertex and end at the same vertex (the starting and ending points might not be distinct) there are $p!$ transformations that permute the propagators.
- for n vertices that have the same structure there are $n!$ transformations that permute the vertices.
- for d identical graphs there are $d!$ transformations that permute the graphs.

Denote the automorphism group of a Feynman graph D by $\mathrm{Aut}(D)$. The order $|\mathrm{Aut}(D)|$ is the product of the numbers obtained applying each rule above. A Feynman graph built using v vertices comes with a coefficient

$$C_{D_v} = \frac{(4!)^v}{|\mathrm{Aut}(D)|} \tag{3.4}$$

Applying the rules to compute symmetry factors can be tricky. The excellent textbook[44] suggests that "When in doubt, you can always determine the symmetry factor by counting equivalent contractions". As v grows, this quickly becomes hopeless. One of the results of this paper is give conceptually simple permutation group algorithms for symmetry factors (Appendix C).

One basic quantity of interest is the number of graphs

$$N_v = \sum_{D_v} 1 \tag{3.5}$$

contributing, since it is an important effect in determining the behavior of field theory at large orders in perturbation theory [45]. For questions of this type the value of F_{D_v} is unimportant (we are implicitly assuming they are all roughly the same size) so that we may, for simplicity, work in zero dimensions which amounts to setting $F_{D_v} = 1$.

Instead of focusing the discussion on any particular amplitude, it is useful and standard, to study the generating functional of correlation functions

$$Z[J] = \int [D\phi]\, e^{iS+i\int d^4xJ\phi} \tag{3.6}$$

where

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^4 \right) \tag{3.7}$$

The value of Z at $J = 0$ computes the sum of all vacuum graphs. Derivatives of Z with respect to J compute the sum of all Feynman graphs with E external legs

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_E)|0\rangle = \frac{1}{i^E}\left.\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_E)}\right|_{J=0} \tag{3.8}$$

Another quantity of interest is the logarithm of Z, $W = \ln Z$. The value of W at $J = 0$ computes the sum of all connected vacuum graphs. Derivatives of W with respect to J, at $J = 0$, compute the sum of all connected Feynman graphs with E external legs. This suggests another interesting question: how many connected Feynman graphs N_v^{conn} are there? The connection between Z and W holds for graphs with the symmetry factor C_{D_v} included. The relation between N_v^{conn} and N_v (which does not include the symmetry factors) is different.

4 Feynman graphs in terms of the pair (Σ_0, Σ_1)

Consider a vacuum Feynman graph in ϕ^4 theory. Let v be the number of vertices. At $v = 1$ we have a single graph, given in figure 4. At $v = 2$ we have three graphs given

Figure 4: One vertex vacuum graph in ϕ^4 theory

in figure 5. To get a systematic counting, we describe these graphs in terms of some numbers. One way to do this is to label the vertices $1, 2, \cdots, v$, and list the edges as $[ij]$.

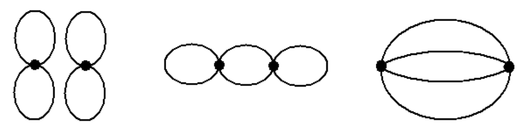

Figure 5: Two vertex vacuum graphs in ϕ^4 theory

This method is well-known in graph theory and quickly leads to elegant results for graphs where there are no edges connecting a vertex to itself, and no multiple edges between a given pair of vertices. In the case at hand, we do have loops and we do have multiple edges between the same vertices. For this reason, the graphs at hand are sometimes called multi-graphs in the mathematics literature.

For the graphs we consider, it is more convenient to give a combinatoric description, by putting a new vertex in the middle of each edge. Each edge is thus divided into two half-edges. Each half-edge is labeled by a number from $1, 2, \cdots 4v$. Using this labeling, the graphs shown in figure 4 and figure 5 have been labeled in figure 6 and figure 7 respectively. We have drawn the original vertices as black dots and the new vertices as white dots.

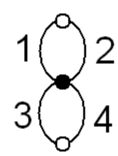

Figure 6: Numbering the half-edges

Now the graphs can be described by specifying the list of 4-tuples of numbers at the black vertices and the pairs of numbers at the white vertices. We thus introduce two quantities Σ_0, Σ_1. For the $v = 1$ graph, with the labeling chosen in figure 6

$$\begin{aligned} \Sigma_0 &= \ < 1, 2, 3, 4 > \\ \Sigma_1 &= \ (12)(34) \end{aligned} \tag{4.1}$$

For the three graphs at $v = 2$, inspection of figure 7 shows that we have fixed

$$(a) \qquad \begin{aligned} \Sigma_0 &= \ < 1, 2, 3, 4 >< 5, 6, 7, 8 > \\ \Sigma_1 &= \ (12)(34)(56)(78) \end{aligned}$$

$$(b) \qquad \Sigma_0 = \ < 1, 2, 3, 4 >< 5, 6, 7, 8 >$$

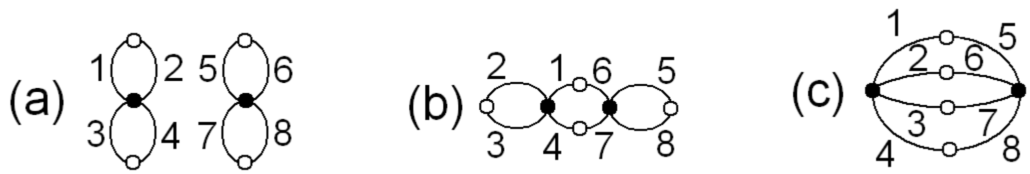

Figure 7: Numbering the half-edges

$$\Sigma_1 = (23)(16)(47)(58)$$

$$(c) \qquad \begin{aligned} \Sigma_0 &= <1,2,3,4><5,6,7,8> \\ \Sigma_1 &= (15)(26)(37)(48) \end{aligned} \tag{4.2}$$

Clearly, we could have labeled things differently producing different pairs Σ_0, Σ_1. These relabelings are elements of S_4 in the case $v = 1$ and elements of S_8 in the case $v = 2$. More generally, we have S_{4v}. It is useful to consider breaking the half-edges, leaving us with $4v$ half-edges dangling from v 4-valent vertices. And $4v$ half-edges from $2v$ 2-valent white vertices. We imagine we did not remove the labels as we broke the half-edges, so we know how to put the graph back together: by connecting half-edges with identical numbers. After breaking the half edges in figures 6 and 7 we obtain figures 8 and 9.

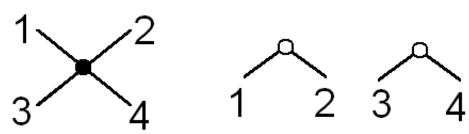

Figure 8: Splitting the half-edges

The symmetry group of the 4-valent vertices includes the 4! permutations of each vertex, along with the $v!$ permutations of the vertices themselves. These permutations form a subgroup of S_{4v} called the wreath product $S_v[S_4]$. It is the semi-direct product of $(S_4)^v \rtimes S_v$. The physics reader is not required to have any prior knowledge of these groups. All the relevant facts we will need from the math literature will be quoted as we need them.

The symmetry group of the Feynman graph, whose order appears in the denominator as the symmetry factor, is obtained from the action of the symmetric group S_{4v} on the pairs (Σ_0, Σ_1). A permutation in S_{4v} is a map from integers $\{1, 2, \cdots 4v\}$ to these integers $\{1, 2, \cdots, 4v\}$, the map being one-one and onto. It can be written as an ordered list of

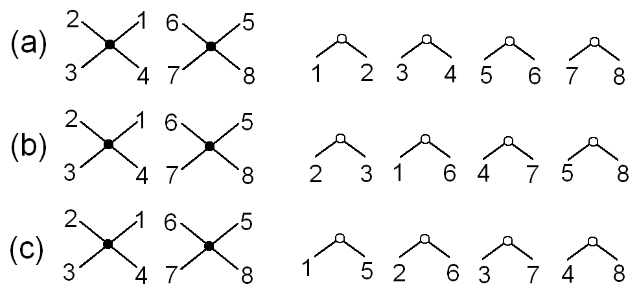

Figure 9: Splitting the half-edges

the images of $\{1, 2, \cdots, 4v\}$, e.g a cyclic permutation of $\{1, 2, 3, 4\}$ can be written as

$$2341 \tag{4.3}$$

Alternatively we use cycle notation, (1234) for the example at hand. The image of every integer in the bracket is the one to the right, the image of the last is the first.

A permutation σ acts on Σ_0, Σ_1 by taking the integers i in these tuples to $\sigma(i)$. We denote this action by

$$\sigma : (\Sigma_0, \Sigma_1) \to (\sigma(\Sigma_0), \sigma(\Sigma_1)) \tag{4.4}$$

If $(\Sigma_0', \Sigma_1') = (\sigma(\Sigma_0), \sigma(\Sigma_1))$ for some σ, then both pairs (Σ_0', Σ_1') and (Σ_0, Σ_1) represent the same Feynman graph. We may say that **Feynman graphs correspond to orbits of the S_{4v} action on the pairs** (Σ_0, Σ_1).

As we saw in the examples above we can fix the form of Σ_0. The subgroup of S_{4v} which preserves this is the $S_v[S_4]$. Then different Feynman graphs correspond to orbits of $S_v[S_4]$ acting on the pairings Σ_1. These are nothing but elements of the conjugacy class $[2^{2v}]$ in S_{4v}. This leads directly to the computation of numbers of Feynman graphs and symmetry factors using GAP software, which is the mathematical package for group theory. See Appendix C.

5 ϕ^4 theory and string theory

In this section we develop a number of key ideas. As we saw in the last section, to generate all possible Feynman graphs, we can fix Σ_0 and allow Σ_1 to vary. Using the action

$$\sigma : (\Sigma_0, \Sigma_1) \to (\sigma(\Sigma_0), \sigma(\Sigma_1)) \tag{5.1}$$

of $\sigma \in S_{4v}$, the condition $\sigma(\Sigma_0) = \Sigma_0$ fixes σ to be in the subgroup $S_v[S_4]$ of S_{4v}. Permutations Σ_1 that can not be related by $\sigma \in S_v[S_4]$ are distinct Feynman graphs. This leads to the realization of Feynman graphs as orbits of the vertex symmetry group acting on Wick contractions. A simple application of the Burnside Lemma in section 5.1 then gives an explicit formula (5.4) for the number of Feynman graphs. The formula has an immediate interpretation as a sum over covers of a torus. This suggests an interpretation as string worldsheets mapping to a torus. However, the sum over maps to torus have some constraints which seem non-trivial to implement geometrically. This motivates a deeper look at the combinatorics, which reveals a somewhat simpler connection to strings with a cylinder target space. In section 5.2 we explain that Feynman graphs are elements of a double coset. This description is already present in [20], where it is derived using an operation called graph superposition. Our derivation does not use this operation and is obtained more directly using the half-edges introduced in Section 4. The pair (Σ_0, Σ_1) encountered there finds a natural interpretation in terms of the double coset. This is a key result with a number of implications. It allows us to express the counting of Feynman graphs as well as their symmetry factors in terms of data in S_n TFT. In turn this S_n TFT data is used to construct covers of the cylinder. The figure 10 summarizes the key message of this section. This will set the stage for Section 6 where generating functions and counting sequences will be given.

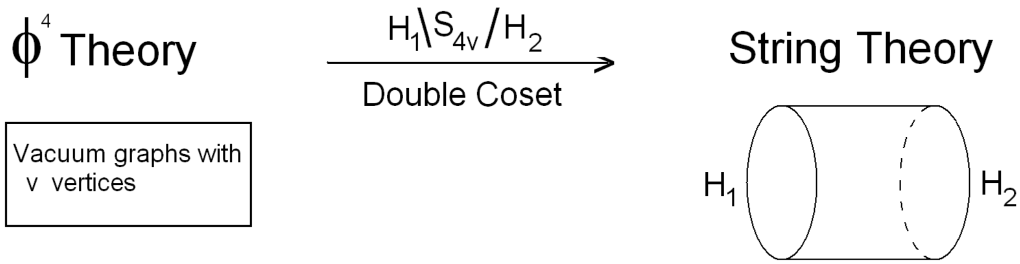

Figure 10: Double coset connection

5.1 Commuting permutations

A vacuum Feynman graph in ϕ^4 theory is constructed by starting with v vertices, each having 4 edges attached, and connecting up the edges, as shown in Figure 11. Without any loss of generality label the edges of the first vertex $(1, 2, 3, 4)$, the edges of the second vertex as $(5, 6, 7, 8)$, etc. We could have chosen different numbers (from $(1..4v)$ to go with each vertex), but we have made a choice. This is a choice of

$$\Sigma_0 = \langle 1, 2, 3, 4\rangle\langle 5, 6, 7, 8\rangle \cdots \tag{5.2}$$

where the angled brackets are completely symmetric and the different brackets can be freely interchanged. The symmetry of Σ_0 is $S_v[S_4]$.[1]

To complete the diagram we need to give the Wick contractions. The Wick contractions are specified by choosing an element Σ_1 in $[2^{2v}]$. For example

$$(12)(34)..(4v-1,4v) \tag{5.3}$$

corresponds to the disconnected graph with a bunch of eights.

To construct all possible Feynman graphs we need to allow Σ_1 to run over all possible Wick contractions holding Σ_0 fixed to the choice we made above. This is easily done by acting on Σ_1 with $S_v[S_4]$. It follows that the distinct Feynman graphs correspond to orbits of the vertex symmetry group $S_v[S_4]$ acting in the set of pairings $[2^{2v}]$, which are elements of a conjugacy class in S_{4v}. If we apply the Burnside Lemma, we see that the number of Feynman graphs is equal to

$$\text{Number of Feynman Graphs } = \frac{1}{4^v v!}\sum_{\gamma\in H_1}\sum_{\sigma\in[2^{2v}]}\delta(\gamma\sigma\gamma^{-1}\sigma^{-1}) \tag{5.4}$$

This is a sum over certain covers of the torus of degree $4v$. The worldsheet is also a torus. One of the monodromies is constrained to lie in the subgroup $H_1 = S_v[S_4]$, the other lies in the conjugacy class $C = [2^{2v}]$. Constrained sums of this sort where the permutations are constrained to lie in a given conjugacy class are naturally motivated from holomorphic maps. Here we are constraining one permutation to lie in a subgroup and one to lie in a conjugacy class. Such a constraint is entirely possible in S_n TFT, but the the geometrical interpretation in terms of Hurwitz space is, at this point, less clear. Subsequent formulae we derive will show the emergence of a cylinder rather than a torus. The formulae then become more symmetric with respect to exchange of H_1 and H_2.

For purposes of counting applications, the equation (5.4) is easy to implement in GAP, although related techniques involving cycle indices introduced in Section 6 will be more efficient.

Apart from the number of Feynman graphs, it is also interesting to compute the automorphism group of a given graph. This appears, for example, in the denominator for the formula (3.4) for the symmetry factor. With our choice of a fixed Σ_0, a graph is uniquely specified by Σ_1. For this reason we will denote the automorphism group of a given graph by $\text{Aut}([\Sigma_1]_{H_1})$. The $[\Sigma_1]_{H_1}$ denotes an equivalence class under conjugation by H_1. The automorphism group is given by those elements of H_1 which leave Σ_1 invariant. The order of the automorphism group is

$$|\text{Aut}([\Sigma_1]_{H_1})| = \sum_{\gamma\in H_1}\delta(\gamma\Sigma_1\gamma^{-1}\Sigma_1^{-1}) \tag{5.5}$$

[1] Some authors prefer the notation $S_4 \wr S_v$. We follow [20] for the reason he gives : that the $S_v[S_4]$ notation connects nicely with the substitution formula for cycle indices

A symmetry γ and a Wick contraction Σ_1 specify an unbranched cover of the torus, of degree $4v$ with γ constrained to lie in H_1 and Σ_1 constrained to be made of 2-cycles.

Recall from section 3 that the number of Wick contractions leading to a particular Feynman graph is

$$v!C_{D_v} = v!\frac{|(S_4)^v|}{|\mathrm{Aut}([\Sigma_1]_{H_1})|} \tag{5.6}$$

Noting that $\mathrm{Aut}(\Sigma_0) = (S_4)^v \rtimes S_v$ we find

$$v!\frac{|(S_4)^v|}{|\mathrm{Aut}([\Sigma_1]_{H_1})|} = \frac{|\mathrm{Aut}(\Sigma_0)|}{|\mathrm{Aut}(D)|} \tag{5.7}$$

This formula has a very natural interpretation: start by making a choice of Σ_0. To determine a Wick contraction, we need to specify the cycle Σ_1. Any other Wick contraction contributing to the same graph can be obtained by swapping vertices or swapping the half edges at a given vertex. The full set of these transformations is performed by $\mathrm{Aut}(\Sigma_0)$. If we have an element of $\mathrm{Aut}(\Sigma_0)$ that leaves Σ_1 invariant (this by definition belongs to $\mathrm{Aut}(D)$) we do not get a new Wick contraction. Thus, the formula (5.6) is an application of the Orbit-stablizer theorem [34].

5.2 Double Cosets and the meaning of the pair (Σ_0, Σ_1).

Read [20] derives explicit counting formulae for graphs by developing a formulation in terms of double cosets. He arrives at the double cosets by a procedure of graph superposition. We will arrive at the same double coset descriptions using the procedure of separating the edges into half-edges, and keeping track of the permutations which contain the information of how the half-edges are glued to make up the graph (See Figure 9).

There are two related double coset descriptions relevant to ϕ^4 graphs. The first one is a double coset of $S_n \times S_n$. The second is a double coset of S_n and can be obtained by a gauge fixing of the $S_n \times S_n$ picture. Let us start with this description. The elements of the double coset are pairs

$$(\sigma_1, \sigma_2) \in (S_n \times S_n) \tag{5.8}$$

with the equivalence

$$(\sigma_1, \sigma_2) \sim (\alpha\sigma_1\beta_1, \alpha\sigma_2\beta_2) \tag{5.9}$$

where $n = 4v, \alpha \in S_{4v}, \beta_1 \in H_1 = S_v[S_4]$ and $\beta_2 \in H_2 = S_{2v}[S_2]$.

To understand why this double coset counts Feynman graphs of ϕ^4 theory consider graphs with v 4-valent vertices which will have a Σ_0, Σ_1 description

$$(\Sigma_0, \Sigma_1) \in (< 4^v >, [2^{2v}]) \tag{5.10}$$

As explained in section 4 the graph splits up into $2v$ bi-valent white vertices and v 4-valent black vertices. Label the edges connected to the white vertices $1 \cdots 4v$, so we have edges $\{W_1, W_2 \cdots W_{4v}\}$. Label the edges connected to the black vertices $\{1, 2, \cdots, 4v\}$, so we have $\{B_1, B_2, \cdots B_{4v}\}$. All possible relabellings of the edges of the white vertices are paramatrized by $\sigma_1 \in S_{4v}$. All possible relabellings of the edges of the 4-valent black vertices are parametrized by $\sigma_2 \in S_{4v}$. Given any (σ_1, σ_2) we can construct a graph by gluing

$$W_{\sigma_1(i)} \leftrightarrow B_{\sigma_2(i)} \tag{5.11}$$

for $i = 1 \cdots 4v$.

Clearly by considering all possible (σ_1, σ_2) we can get all possible graphs.

If we replace the i by $\alpha(i)$, we get the same graph, since the labelings do not matter. So we learn that $(\alpha\sigma_1, \alpha\sigma_2)$ and $(\sigma\sigma_1, \sigma\sigma_2)$ produce the same graph. Hence we are interested in equivalence classes

$$(\sigma_1, \sigma_2) \sim (\alpha\sigma_1, \alpha\sigma_2) \tag{5.12}$$

We also know that the disconnected graph of bi-valent vertices has a symmetry of $H_2 = S_{2v}[S_2]$, so that σ_2 and $\sigma_2\beta_2$ with $\beta_2 \in H_2$ is the same graph. In the same way, $\sigma_1 \sim \sigma_1\beta_1$ with $\beta_1 \in H_1$. This leads us to the conclusion that Feynman graphs are equivalence classes of the equivalence relation

$$(\sigma_1, \sigma_2) \sim (\alpha\sigma_1\beta_1, \alpha\sigma_2\beta_2) \tag{5.13}$$

completing the demonstration that Feynman graphs are in one-to-one correspondence with elements of a double coset.

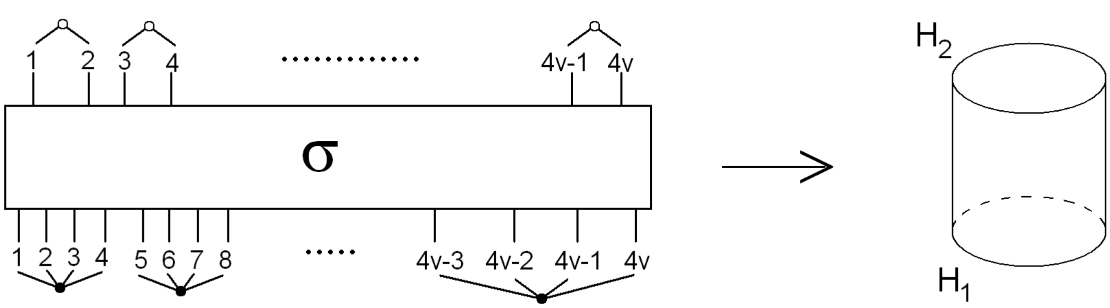

Figure 11: Double coset connection

If we choose $\alpha = \beta_1^{-1}\sigma_1^{-1}$, then any pair is mapped to

$$(1, \beta_1^{-1}\sigma_1^{-1}\sigma_2\beta_2) \tag{5.14}$$

So now only the combination $\sigma_1^{-1}\sigma_2$ appears which is in S_n, and we have equivalences by right multiplication with H_2 and left multiplication with H_1. We learn that

$$S_n \setminus (S_n \times S_n)/(H_1 \times H_2) = H_1 \setminus S_n/H_2 \tag{5.15}$$

From (5.13) we can consider first doing the coset by H_1 and H_2, to we have an action of S_n on $S_n/H_1 \times S_n/H_2$. This is the description we are using when we impose S_n relabeling equivalences on (Σ_0, Σ_1) which are nothing but a parametrization of $S_n/H_1 \times S_n/H_2$. So we have now learnt the group theoretic meaning of the Σ_0, Σ_1, which we wrote earlier as a combinatoric description of the graph that followed (at the beginning of section 4) immediately from introducing the white vertices to separate the existing vertices and keep track of the Wick contractions.

This line of argument makes the generalization to ribbon graphs clear. Not any permutation of the legs connecting to a black vertex is allowed: to preserve the genus of the ribbon the cyclic order of the labeling must be respected. Consequently, $H_1 = S_v[S_4]$ must be replaced by $H_1 = S_v[\mathbb{Z}_4]$. We pursue this direction in section 7.

5.3 Cycle Index formulae related to double cosets

In this section we would like to make a connection with the classic results of Read [20] on the counting of locally restricted graphs. The starting point of [20] it to count equivalence classes of pairs (a_1, a_2) in $(S_n \times S_n)$, where the pair (a_1, a_2) is equivalent to (b_1, b_2) if

$$(b_1, b_2) = (xa_1g_1, xa_2g_2) \tag{5.16}$$

where $x \in S_n$ and $g_1 \in H_1, g_2 \in H_2$. This is equivalent to counting elements of the double coset

$$S_n \setminus (S_n \times S_n)/(H_1 \times H_2) = H_1 \setminus S_n/H_2 \tag{5.17}$$

The number of equivalence classes is counted by $N(Z(H_1) * Z(H_2))$, where the Z's are the cycle indices, and N is a star-product for multivariable polynomials defined as follows: Consider two polynomials P and Q in variables $f_1, f_2..f_n$

$$\begin{aligned} P &= \sum_{j \vdash n} P_{j_1, j_2.., j_n} f_1^{j_1} f_2^{j_2} \cdots f_n^{j_n} \\ Q &= \sum_{j \vdash n} Q_{j_1, j_2.., j_n} f_1^{j_1} f_2^{j_2} \cdots f_n^{j_n} \end{aligned} \tag{5.18}$$

where j is a partition of n : $j_1 + 2j_2 + \cdots + nj_n = n$. We abbreviate the coefficients as P_j, Q_j. The function N of a star product is now defined by

$$N(P * Q) = \sum_{j \vdash n} P_j Q_j \mathrm{Sym}(j) \tag{5.19}$$

where $\mathrm{Sym}(j)$ is the symmetry of the conjugacy class corresponding to partition j

$$\mathrm{Sym}(j) = \prod_{i=1}^{n} (j_i)^i i!$$

For our applications we will sometimes want to change the numerator group, in which case the symmetry of the conjugacy class will change. This generalization of Read's formula is discussed in section 8.1.

Read [20] also proves

$$N(Z(H_1) * Z(H_2)) = \sum_{p \vdash 4v} Z_p^{S_v[S_4]} Z_p^{S_v[S_2]} Sym(p) = \frac{1}{|H_1||H_2|n!} \sum_{a_1, a_2 \in S_{4v}} \nu(a_1, a_2) \quad (5.20)$$

where $\nu(a_1, a_2)$ is the order of the intersection $a_1 H_1 a_1^{-1} \cup a_2 H_2 a_2^{-1}$. We can rewrite the above as

$$\begin{aligned} N(Z(H_1) * Z(H_2)) &= \frac{1}{|H_1||H_2|n!} \sum_{a_1, a_2 \in S_{4v}} \sum_{u_1 \in H_1} \sum_{u_2 \in H_2} \delta(a_1 u_1 a_1^{-1} a_2 u_2^{-1} a_2^{-1}) \\ &= \frac{1}{|H_1||H_2|} \sum_{b \in S_{4v}} \sum_{u_1 \in H_1} \sum_{u_2 \in H_2} \delta(u_1 b u_2^{-1} b^{-1}) \end{aligned} \quad (5.21)$$

which is reminiscent of a *cylinder partition function* for gauge theory with S_{4v} gauge symmetry, and with holonomies at the boundaries restricted being summed in H_1 and H_2 respectively.

Since these formulae will come up, in slight variations, repeatedly, we make some comments on notation. We will denote the number of points in $H_1 \setminus G / H_2$ as $\mathcal{N}(H_1, H_2)$ when G is just the symmetric group S_n, or $\mathcal{N}(H_1, H_2; G)$ more generally. The function of N of star products will also sometimes written to make the numerator group explicit. So the previous formulae can be expressed as

$$\mathcal{N}(H_1, H_2) = N(Z(H_1) * Z(H_2)) \quad (5.22)$$

or

$$\mathcal{N}(H_1, H_2; G = S_n) = N(Z(H_1) * Z(H_2); G = S_n) \quad (5.23)$$

Note that we can also write (5.21) as

$$\frac{1}{|H_1||H_2|} \sum_{R \vdash S_{4v}} \sum_{u_1 \in H_1} \sum_{u_2 \in H_2} \chi_R(u_1) \chi_R(u_2) \quad (5.24)$$

The sums over u_1 and u_2 produce projection operators which project onto the trivial representation so that the only representations R which contribute are the representations of S_{4v} which contain the trivial of $S_v[S_4]$ and the trivial of $S_{2v}[S_2]$. Each such R contributes the product of the multiplicities with which the trivial of H_1 and H_2 appear.

$$\text{Number of Feynman Graphs } = \sum_{R \vdash 4v} \mathcal{M}^R_{\mathbf{1}_{H_1}} \mathcal{M}^R_{\mathbf{1}_{H_2}} \tag{5.25}$$

We have used the notation $\mathcal{M}^R_{\mathbf{1}_{H_1}}$ for the multiplicity of the one-dimensional representation of H_1 when the irreducible representation R of S_{4v} is decomposed into representations of the subgroup H_1.

5.4 Action of Vertex symmetry group on Wick contractions

We now have two different ways to compute the number of Feynman graphs: as the number of orbits of the vertex symmetry group acting on Wick contractions, which leads to

$$\frac{1}{(4!)^v v!} \sum_{u_1 \in S_v[S_4]} \sum_{\sigma \in [2^{2v}]} \delta(u_1 \sigma u_1^{-1} \sigma^{-1}) \tag{5.26}$$

or in terms of the star product of the cycle indices $N(Z(H_1)*Z(H_2))$. We will demonstrate, using some general group theory, the equivalence of (5.26)and (5.21).

Noting that $H_1 = S_v[S_4]$, (5.26) and (5.21) are equivalent if

$$\sum_{\sigma \in [2^{2v}]} \delta(u_1 \sigma u_1^{-1} \sigma^{-1}) = \frac{1}{|H_2|} \sum_{b \in S_{4v}} \sum_{u_2 \in H_2} \delta(u_1 b u_2^{-1} b^{-1}) \tag{5.27}$$

To see that this relation holds, suppose there is a solution to the first delta function. σ is conjugate by some b to $\sigma_0 \equiv (12)(34)..(4v-1, 4v)$

$$b^{-1} \sigma_0 b = \sigma \tag{5.28}$$

We know that all elements commuting with σ_0 are in $H_2 \equiv S_{2v}[S_2]$. From (5.27) we have

$$u_1 \sigma u_1^{-1} = \sigma \tag{5.29}$$

So

$$u_1 b^{-1} \sigma_0 b u_1^{-1} = b^{-1} \sigma_0 b \tag{5.30}$$

It follows that

$$b u_1 b^{-1} = u_2 \tag{5.31}$$

for some u_2 in H_2. The b that takes σ_0 to σ can be multiplied by an arbitrary element of H_2. So if we replace the sum over $\sigma \in [2^{2v}]$ by a sum over b, u_2, we will be over counting by $|H_2|$. So we conclude that the equality (5.27) is correct.

It is instructive to note that the set of elements $[2^{2v}]$ can be thought as the set of cosets S_{4v}/H_2, where $H_2 = S_{2v}[S_2]$. To see this, start by noting that S_{4v} acts on σ_0 by conjugation to generate all the elements in $[2^{2v}]$. For any $\sigma \in [2^{2v}]$, we have

$$\sigma = \alpha\sigma_0\alpha^{-1} \tag{5.32}$$

for some $\alpha \in S_{4v}$. If we multiply α on the right by $\beta \in S_{2v}[S_2]$, we get the same σ. We can write

$$[2^{2v}] = S_{4v}/H_2 \tag{5.33}$$

Thus, we can rewrite (5.26) as

$$\text{Number of Feynman Graphs } = \frac{1}{(4!)^v v!}\sum_{u_1\in H_1}\sum_{\sigma\in S_{4v}/H_2}\delta(u_1\sigma u_1^{-1}\sigma^{-1}) \tag{5.34}$$

A similar argument implies that

$$< 4^v >= S_{4v}/H_1 \tag{5.35}$$

Here $< 4^v >$ stands for the space of Σ_0's.

Using that fact that the expression (5.21) is symmetric under the exchange of H_1 and H_2, we can also write

$$\text{Number of Feynman Graphs } = \frac{1}{2^v(2v)!}\sum_{u_2\in H_2}\sum_{\Sigma_0\in S_{4v}/H_1}\delta(u_2(\Sigma_0),\Sigma_0) \tag{5.36}$$

Thus, by Burnside's Lemma the number of Feynman graphs is also equal to the number of orbits of $S_{2v}[S_2]$ acting on the set of vertex labels. Note that (5.36) looks slightly different from (5.34) because Σ_1 is a permutation, but Σ_0 is not. The action of substituting $i \to u(i)$ for some permutation $u \in S_{4v}$ can be achieved by conjugation for Σ_1 but not for Σ_0.

5.5 Number of Feynman graphs from strings on a cylinder

In this subsection we show that the formula (5.24) for counting Feynman graphs in ϕ^4 theory is computing an observable in S_n TFT. Recall from section 2.1 that the expectation value of the observables $tr(\sigma_1 U_1^\dagger), tr(\sigma_2 U_2^\dagger)$ on the cylinder with boundary holonomies U_1 and U_2 are

$$Z(\sigma_1,\sigma_2) = \int dU_1 dU_2 tr(\sigma_1 U_1^\dagger)tr(\sigma_2 U_2^\dagger)Z(U_1,U_2) = \sum_\gamma \delta(\sigma_1\gamma\sigma_2\gamma^{-1}) \tag{5.37}$$

Summing this expectation value over permutations in the subgroups H_1, H_2

$$\begin{aligned} Z(H_1, H_2) \quad &\equiv \frac{1}{|H_1||H_2|} \sum_{\mu_1 \in H_1} \sum_{\mu_2 \in H_2} Z(\mu_1, \mu_2) \\ &= \frac{1}{|H_1||H_2|} \sum_{\mu_1 \in H_1} \sum_{\mu_2 \in H_2} \sum_{\sigma \in S_n} \delta(\mu_1 \sigma \mu_2 \sigma^{-1}) \end{aligned} \tag{5.38}$$

we recover the formula (5.21) for counting Feynman graphs in ϕ^4 theory.

There is an interesting subtlety we should comment on. In the large N expansion of $U(N)$ 2dYM, we encounter observables which can be parametrized using permutations. For observables constructed from gauge-invariant polynomials of degree n in the holonomy U, the partition functions are functions only of the conjugacy class of σ in S_n. In the above expression (5.38), the sums over μ_i do not run over entire S_n conjugacy classes, but are restricted to H_i.

As explained in section 2.2 the observables in the $\Omega \to 1$ approximation of large N 2dYM, for Riemann surfaces with or without boundary, can be expressed in terms of physical observables in S_n TFT. The quantity in (5.38) is a generalization of the observables one gets from the large N expansions of $U(N)$ 2dYM, but it is still an observable in the lattice S_n TFT. The boundary observables are not invariant under conjugation by S_n elements at the boundary. The S_n conjugation symmetry is broken to H_1 and H_2 respectively. There is a general principle of Schur-Weyl duality which relates the symmetric group constructions to unitary groups (see [6, 35, 36, 37] for applications in gauge-string duality). The wreath products of symmetric groups have also appeared in connection with symmetrised traces in the context of constructing eighth-BPS operators [38]. We therefore expect that these more general S_n TFT observables can also be expressed in terms of some construction with gauge theory involving unitary groups. We will leave this clarification for the future.

As mentioned in Section 2, the S_n TFT is closely related to covering space theory. In the next section we will encounter expressions of the form (5.38) but without the sum over σ. We will show how cutting and gluing constructions of covering spaces use the data μ_1, μ_2, σ.

5.6 The symmetry factor of a Feynman graph from strings on a cylinder

We have already argued that the Feynman graphs of ϕ^4 theory correspond to elements of the double coset,

$$S_{4v} \setminus (S_{4v} \times S_{4v}) / (\ S_v[S_4] \times S_{2v}[S_2]\ ) \tag{5.39}$$

The S_{4v} on the left is the diagonal subgroup of the product group, i.e pairs of the form (σ, σ). The symmetry factor of a Feynman graph corresponding to an orbit with representative (σ_1, σ_2) is the size of the stabilizer group. This can be computed by calculating

$$\sum_{\gamma \in S_{4v}} \sum_{\mu_1 \in S_v[S_4]} \sum_{\mu_2 \in S_{2v}[S_2]} \delta(\gamma \sigma_1 \mu_1 \sigma_1^{-1}) \delta(\gamma \sigma_2 \mu_2 \sigma_2^{-1}) \tag{5.40}$$

Using one of the delta functions to perform the sum over γ we obtain

$$\sum_{\mu_1 \in S_v[S_4]} \sum_{\mu_2 \in S_{2v}[S_2]} \delta(\sigma_1 \mu_1^{-1} \sigma_1^{-1} \sigma_2 \mu_2 \sigma_2^{-1}) \tag{5.41}$$

By defining $\sigma = \sigma_1^{-1} \sigma_2$, we can write the formula for the symmetry factor as

$$\mathrm{Sym}(\sigma) = \sum_{\mu_1 \in S_v[S_4]} \sum_{\mu_2 \in S_{2v}[S_2]} \delta(\mu_1^{-1} \sigma \mu_2 \sigma^{-1}) \tag{5.42}$$

This expression relates more directly to the equivalent description of the double coset (5.39) as

$$S_v[S_4] \setminus S_{4v} / \times S_{2v}[S_2] \tag{5.43}$$

Comparing to (2.10) we see that this is an observable in S_n TFT, with H_1, H_2 observables on the boundaries. This is illustrated in Fig. 12.

Consider a cover of the cylinder. Choose a point on one boundary circle and label the inverse images of that point $\{1, 2, \cdots, n\}$. Following the inverse images of these points along the boundary circle leads to a permutation μ_1 which is constrained to be in H_1. Likewise there is a permutation μ_2 of $\{1', 2', \cdots, n'\}$ at the other boundary. Following a path on the cylinder which joins the two points will produce a permutation σ. We are fixing σ to lie in a fixed element of $H_1 \setminus S_n / H_2$.

From the point of view of S_n lattice TFT, the line joining the two chosen points on the boundary circle, associated with a fixed permutation σ is a Wilson line. Given the close relation, between S_n lattice TFT and Hurwitz spaces, we can map the Wilson line observable to a construction in covering spaces of the cylinder.

First notice that the cylinder can be obtained by gluing two ends of a square $[AA'B'B]$ in Figure 13. This is a topological quotient which identifies the edge AB with $A'B'$. To construct a covering space associated with the data μ_1, σ, μ_2 subject to $\mu_1^{-1} \sigma \mu_2 \sigma^{-1} = 1$, we consider cutting the square further into the rectangles $[AA'C'C]$ and $[DD'B'B]$ (Figure 14). The cylinder is recovered by the gluings (Fig 14)

$$\begin{aligned} AC &= A'C' \\ DB &= D'B' \end{aligned}$$

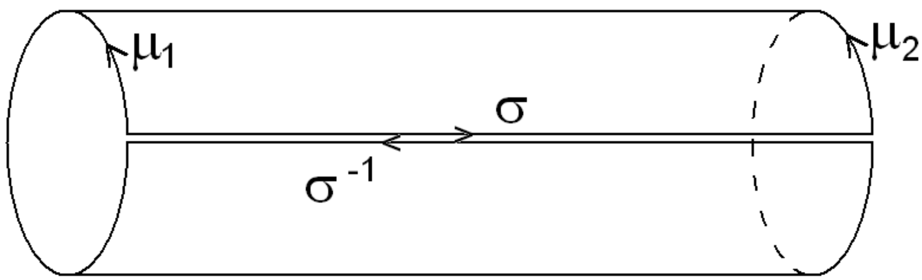

Figure 12: The symmetry factor of a graph is an open Wilson line S_n lattice TFT observable

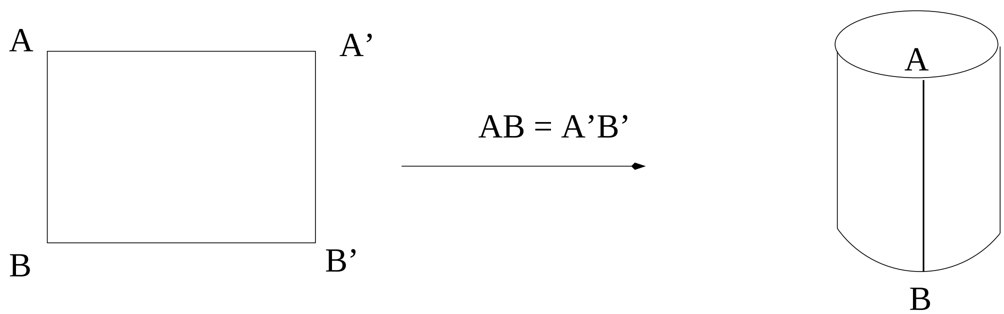

Figure 13: Gluing a square to get cylinder

$$CC' = DD' \tag{5.44}$$

Now take n copies of these pairs of rectangles, with labels $[A_i A'_i C'_i C_i]$ and $[D_i D'_i B'_i B_i]$, with i ranging from 1 to n, as in Figure 15. For the application to (5.38) we have $n = 4v$.

The union of rectangles $\{[A_1 A'_1 C'_1 C_1], \cdots, [A_n A'_n C'_n C_n]\}$ is quotiented by identifying edges $A_i C_i$ to the edges $A'_i C'_i$ by the permutation μ_1.

$$A'_i C'_i = A_{\mu_1(i)} C_{\mu_1(i)} \tag{5.45}$$

The rectangles $[D_i D'_i B'_i B_i]$ are quotiented by identifications

$$D'_i B'_i = D_{\mu_2(i)} B_{\mu_2(i)} \tag{5.46}$$

Finally we identify

$$D_i D'_i = C_{\sigma(i)} C'_{\sigma(i)} \tag{5.47}$$

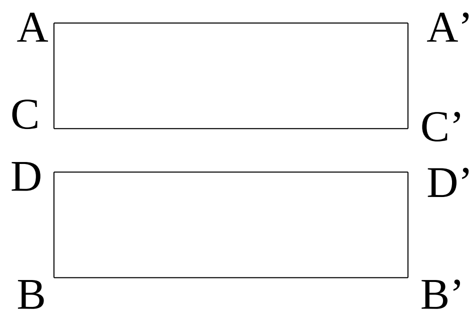

Figure 14: Gluing a pair of rectangles to get cylinder

The covering map is specified by mapping each of the n rectangles, to the original rectangles without labels in the obvious way (see Figure 15)

The condition $\mu_1^{-1}\sigma\mu_2\sigma^{-1} = 1$ ensures that if we consider the inverse image of the closed contractible path shown in blue in Figure 16, the inverse image is also contractible.

This type of construction is a standard part of covering space theory in connection with the Riemann existence theorem (see for example [39]).

The generalizations of (2.8) visible in (5.38) and (5.42) involve restricting the sums over μ_1, μ_2 to specific subgroups H_1, H_2 and leaving the σ unsummed. From the point of view of S_n TFT these constraints are easily implemented, since the degrees of freedom on each edge form the whole group, where elements and subgroups can be chosen. However the boundary observables are not invariant under conjugation by S_n and have no dual in the large N 2dYM that we yet know how to construct. Given the close connection between S_n elements and monodromies of covers, it is not surprising that a definite covering space construction for specific μ_1, μ_2, σ exists. We have provided such an explicit cutting-and-gluing construction here. There are clear analogies between the construction given and the implementation of twisted boundary conditions in D-branes on orbifolds [40]. The presence of a Wilson line defect located at a line joining the 2 boundaries of the cylinder in the S_n TFT picture suggests that there should be a D-brane interpretation. We leave this as a future direction of research. Developing the connection of the lattice S_n TFT to the axiomatic approach to TFT and branes in [46] maybe a useful approach.

In the standard discussions of the Riemann existence theorem for (branched) covers, one has a two-way relation. On the geometrical side, there are maps f with the equivalence $f = f \circ \phi$ for holomorphic automorphisms ϕ (see more details in [6, 25]). On the combinatoric side there are permutations with conjugation equivalence. From permutations we can construct covers, and vice versa. The equivalence classes map to each other. Here we have permutations, but with some restrictions having to do with choices H_1, H_2. We can still construct some covers with this data. We have not fully articulated the correct equivalences on the geometrical side for a 1-1 correspondence. The construction we

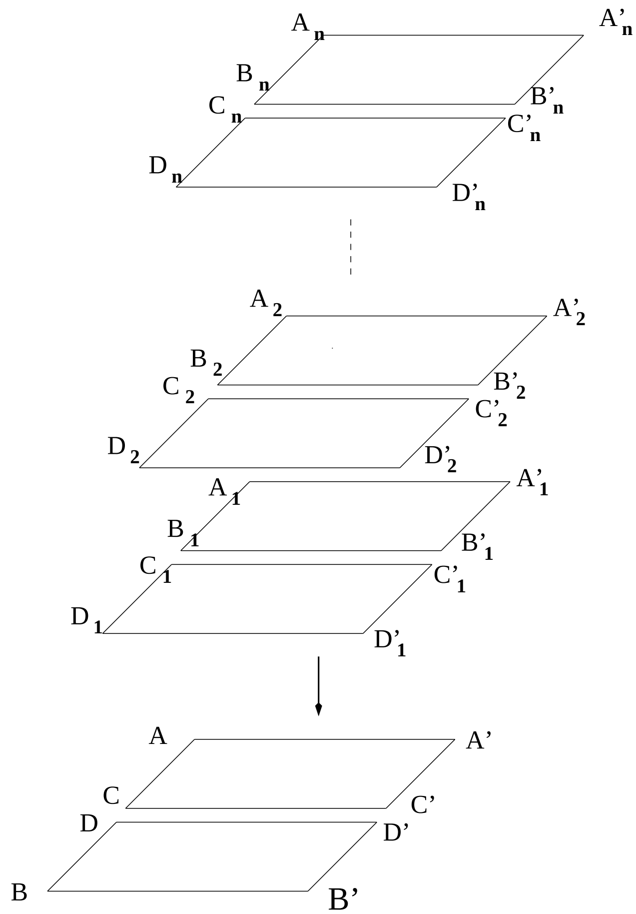

Figure 15: Gluing copies of rectangle pairs to get covering of cylinder - identifications determined by μ_1, μ_2, σ

gave should provide useful hints for the precise definitions on the geometrical side, which will provide the equivalence. We leave this as a problem for the future.

6 The number of Feynman graphs

In section 5.3 a formula expressing the number of Feynman graphs in terms of the star products of two cycle indices was given. This result is useful because formulas for the cycle index of the wreath product of two symmetric groups are known. Using these results we will write down rather explicit formulas for the number of vacuum graphs in ϕ^4 theory in section 6.1.

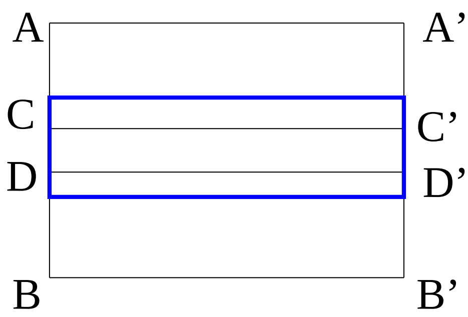

Figure 16: Closed contractible path

6.1 Some analytic expressions exploiting generating functions

Let d_i denote the total number of vacuum Feynman graphs in ϕ^4 theory with i vertices. To obtain an analytic formula for d_i we will need the cycle indices of two wreath products, $S_n[S_4]$ and $S_n[S_2]$. The known generating functions for these wreath products are

$$\mathcal{Z}^{S_\infty[S_4]}[t,\vec{x}] \equiv \sum_n t^n Z^{S_n[S_4]}(\vec{x}) = e^{\sum_{i=1}^{\infty} \frac{t^i}{24i}(x_i^4 + 8x_{3i}x_i + 6x_{2i}x_i^2 + 3x_{2i}^2 + 6x_{4i})} \tag{6.1}$$

$$\mathcal{Z}^{S_\infty[S_2]}[t,\vec{x}] \equiv \sum_n t^n Z^{S_n[S_2]}(\vec{x}) = e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}(x_i^2 + x_{2i})} \tag{6.2}$$

To compute $N(Z(H_1) * Z(H_2))$ take the product of the coefficients of $\prod_i x_i^{p_i}$ in the two cycle indices, and then further take the product with $\prod_i i^{p_i} p_i!$. This can be accomplished if we make the substitution $x_i \to \sqrt{i} y_i$, to obtain

$$\begin{aligned} \tilde{Z}^{S_\infty[S_2]}[t, y_i] &= \mathcal{Z}^{S_\infty[S_2]}[t, x_i = \sqrt{i} y_i] \\ \tilde{Z}^{S_\infty[S_4]}[t, y_i] &= \mathcal{Z}^{S_\infty[S_4]}[t, x_i = \sqrt{i} y_i] \end{aligned} \tag{6.3}$$

and then replace the $(y_i \bar{y}_i)^{p_i}$ with $p_i!$. This leads to the following formula for the number of ϕ^4 vacuum graphs with v vertices

$$d_v = \oint \frac{t_1^{-2v} dt_1}{t_1} \oint \frac{t_2^{-v} dt_2}{t_2} \prod_{i=1}^{\infty} \int \frac{dy_i d\bar{y}_i}{2\pi} e^{-\sum_k y_k \bar{y}_k} \tilde{Z}^{S_\infty[S_2]}[t_1, y_i] \tilde{Z}^{S_\infty[S_4]}[t_2, \bar{y}_i] \tag{6.4}$$

The integrals over y_i and $\bar{y}_i$ ensure that only terms with equal powers of y_i and $\bar{y}_i$ contribute, and they implement the substitution $(y_i \bar{y}_i)^{p_i} \to p_i!$. The contours of integration over t_1 and t_2 are both counter clockwise and they both encircle the origin.

We could contemplate many ways of refining this result. For example, can we determine how many of these vacuum graphs are connected? Before turning to this question, it is instructive to ask how we can identify if a graph is connected or disconnected from its description in terms of the pair Σ_0, Σ_1. A Feynman graph (Σ_0, Σ_1) with v vertices is disconnected if for some subgroup $G = S_{4v-p} \times S_p$ with $p > 0$ we have $\sigma(\Sigma_1) \in G$ for all $\sigma \in \mathrm{Aut}(\Sigma_0)$. Let c_i denote the number of connected Feynman graphs with i vertices. The total number of vacuum graphs with i vertices is the coefficient of g^i in the partition function

$$Z \equiv 1 + \sum_{i=1}^{\infty} d_i g^i = \prod_{i=1}^{\infty} \left[\frac{1}{1-g^i} \right]^{c_i} \tag{6.5}$$

This formula, given for example in [20], can be used to determine the c_i once the d_i have been computed. It is now straight forward to determine the number of vacuum diagrams. See Appendix D for numerical results.

6.2 Scalar fields with external legs

The next natural generalization is to consider Feynman graphs with E external legs. Summing Feynman graphs with E external legs produces an E-point correlation function. For the generic case when all E-points corresponds to different spacetime events, graphs obtained by permuting labels of the external points give distinct contributions to the correlation function. For this reason the automorphism group of the graph does not include any elements that permute the labels of external legs.

For a graph with v vertices and E external legs, we obtain a total of $4v + E$ half edges. Σ_1 is now an element of the conjugacy class $[2^{2v+\frac{1}{2}E}]$ of S_{4v+E}, while Σ_0 contains v 4-tuples of numbers specifying how the half edges connecting to the vertices are labeled, as well as E numbers specifying how the half edges connecting to external points are labeled. Since we do not have automorphisms that permute the half edges connecting to external points, $H_1 = \mathrm{Aut}(\Sigma_0) = S_v[S_4] \times (S_1)^E$. We can also write the subgroup as $S_v[S_4]$, with the understanding that it is acts by keeping fixed the last E integers from the set $\{1, 2, \cdots, 4v + E\}$ that S_{4v+E} acts on. Consequently, Feynman graphs in ϕ^4 theory with v vertices and E external points are in one-to-one correspondence with elements of the double coset

$$(S_v[S_4] \times S_1^E) \setminus S_{4v+E} / S_{\frac{(4v+E)}{2}}[S_2] \tag{6.6}$$

To compute the number of graphs the only new cycle index we need is

$$Z^{S_v[S_4] \times S_1^E}(x_1, \cdots, x_n) = x_1^E Z^{S_v[S_4]}(x_1, \cdots, x_n) \tag{6.7}$$

It is now straight forward to obtain the number of Feynman graphs in ϕ^4 theory with E external lines

$$\mathcal{N}(H_1, H_2; G) = N(Z(H_1) * Z(H_2); S_{4v+E}) = \sum_{p \vdash (4v+E)} Z_p^{S_n[S_4] \times S_1^E} Z_p^{S_{2v+\frac{E}{2}}[S_2]} \mathrm{Sym}(p) \tag{6.8}$$

The notation $p \vdash (4v+E)$ indicates that p runs over all partitions of $4v+E$.

We can again refine this counting by asking how many of these graphs are connected. For concreteness, focus on the case $E=2$. Let $d_{2,i}$ denote the number of Feynman graphs with two external lines and i vertices and let $c_{2,i}$ denote the number of these graphs that are connected. We can again write down a partition function which gives the total number of graphs with two external lines and i vertices as the coefficient of g^i. In this formula the number of vacuum graphs c_i and d_i defined in the last subsection participate. We find

$$\prod_{i=1}^{\infty}\left[\frac{1}{1-g^i}\right]^{c_i}\left(\sum_{k=0}^{\infty} c_{2,k} g^k\right) = \left(\sum_{k=1}^{\infty} d_{2,i} g^i\right) \tag{6.9}$$

This formula can be used to determine the $c_{2,i}$ once the $d_{2,i}$ have been computed using (6.8). See Appendix D for numerical results.

6.3 Generalization to ϕ^3 theory and other interactions

Although our discussion has focused on ϕ^4 theory it should be clear that our methods are general. To illustrate this we will now consider a ϕ^3 interaction. A graph with v vertices has $3v$ half edges. Σ_1 is now an element of the conjugacy class $[2^{\frac{3v}{2}}]$ of S_{3v}, while Σ_0 contains v 3-tuples of numbers specifying how the half edges connecting to the vertices are labeled. We have $H_1 = S_v[S_3]$ and $H_2 = S_{\frac{3v}{2}}[S_2]$. Vacuum Feynman graphs in ϕ^3 theory with v vertices are in one-to-one correspondence with elements of the double coset

$$S_v[S_3] \setminus S_{3v}/S_{\frac{3v}{2}}[S_2] \tag{6.10}$$

To compute the number of graphs the only new cycle index we need can be read from the generating function

$$\begin{aligned} Z^{S_\infty[S_3]}[t; x_1, x_2, \cdots] \quad &= \sum_{n=0}^{\infty} t^n Z^{S_n[S_3]}(x_1, x_2, \cdots, x_n) \\ &= e^{\sum_{i=1}^{\infty} \frac{t^i}{6i}(x_i^3 + 3x_{2i}x_i + 2x_{3i})} \end{aligned} \tag{6.11}$$

It is now straight forward to obtain the number of vacuum Feynman graphs in ϕ^3

$$N(Z(H_1) * Z(H_2)) = \sum_{p \vdash 3v} Z_p^{S_v[S_3]} Z_p^{S_{\frac{3v}{2}}[S_2]} \mathrm{Sym}(p) \tag{6.12}$$

Feynman graphs with E external legs are in one-to-one correspondence with elements of the double coset

$$(S_v[S_3] \times S_1^E) \setminus S_{3v+E}/S_{\frac{(3v+E)}{2}}[S_2] \tag{6.13}$$

Using the generating function

$$Z^{S_n[S_3]\times S_1^E}(x_1,\cdots,x_n) = x_1^E Z^{S_n[S_3]}(x_1,\cdots,x_n) \tag{6.14}$$

it is straight forward to compute the number of Feynman graphs with v vertices and E edges

$$\mathcal{N}(H_1,H_2;S_{3v+E})) = N(Z(H_1)*Z(H_2);S_{3v+E}) = \sum_{p\vdash(3v+E)} Z_p^{S_v[S_3]\times S_1^E} Z_p^{S_{\frac{3v+E}{2}}[S_2]}\mathrm{Sym}(p) \tag{6.15}$$

The notation $p\vdash(3v+E)$ indicates that p runs over partitions of $(3v+E)$. See Appendix D for numerical results. Cubic graphs have recently played a role in studies of $N=8$ SUGRA [49] as well as in the classification of $N=2$ 4-dimensional gauge theories [48]. The ϕ^3 theory in 6 dimensions is known to be asymptotically free [50]. Hence this case is of special interest.

Finally, consider a theory with both cubic and quartic vertices. To count the vacuum graphs having v_3 cubic and v_4 quartic vertices, we would repeat the above discussion with $S_{v_3}[S_3]\times S_{v_4}[S_4]$ replacing the groups $S_{v_3}[S_3]$ (for ϕ^3) or $S_{v_4}[S_4]$ (for ϕ^4).

6.4 Interpretation in terms of 2dYM string

We have already argued that the number of vacuum graphs in ϕ^4 theory is computing an observable in the Lattice TFT with S_n gauge group. The counting of Feynman graphs in this section also has a formulation in Lattice TFT: One simply uses (5.38) with the appropriate H_1, H_2 described above.

The formula (5.42) for the symmetry factor as an amplitude is also directly applicable here.

7 Feynman graphs as orbits: a view of ribbon graphs

The identification of Feynman graphs as elements of the double coset $S_n\backslash(S_n\times S_n)/(H_1\times H_2)$ is a nice unifying picture. The groups H_1 and H_2 are the symmetries of the interaction (black) and bivalent (white) vertices respectively. This allows a simple generalization from ordinary graphs to ribbon graphs: symmetries must by definition preserve the genus of the ribbon graph. The cyclic order of the labels at a black vertex must be respected if the genus is to be preserved. Thus, for example, the replacement $H_1 = S_v[S_4]\to H_1 = S_v[\mathbb{Z}_4]$ takes us from counting Feynman graphs in ϕ^4 theory to counting ribbon graphs in $\mathrm{Tr}\phi^4$ theory.

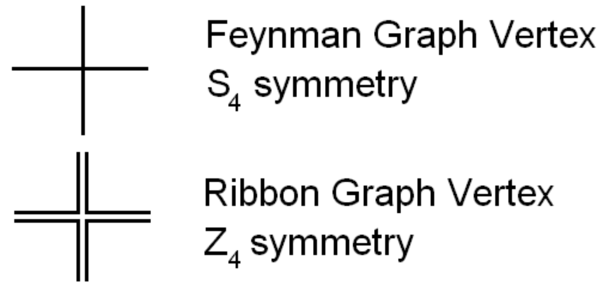

Figure 17: Cyclic versus symmetric for Feynman vertex versus ribbon vertex

It is now clear that ribbon graphs can also identified with elements of a double coset. For example, the vacuum graphs with v vertices of a matrix model with $\mathrm{Tr}\phi^4$ interaction are elements of the coset

$$S_{4v} \setminus (S_{4v} \times S_{4v})/(S_v[\mathbb{Z}_4] \times S_{2v}[S_2]) \tag{7.1}$$

where we have identified $H_1 = S_v[\mathbb{Z}_4]$ and $H_2 = S_{2v}[S_2]$.

7.1 Number of ribbon graphs using commuting pairs

Using the insights of the previous subsection and repeating the argument of section 5, we can generate all possible ribbon graphs by allowing $S_v[\mathbb{Z}_4]$ to act on the set of all possible Wick contractions, that is, the conjugacy class $[2^{2v}]$ of S_{4v}. Each orbit of $S_v[\mathbb{Z}_4]$ is a distinct ribbon graph. Thus, the number of ribbon graphs can be obtained, using Burnside's Lemma, as

$$\begin{aligned} \text{Number of ribbon graphs} &= \frac{1}{4^v v!} \sum_{\gamma \in S_v[\mathbb{Z}_4]} \sum_{\sigma \in [2^{2v}]} \delta(\sigma\gamma\sigma^{-1}\gamma^{-1}) \\ &= \frac{1}{|H_1|} \sum_{\gamma \in H_1} \sum_{\sigma \in S_{4v}/H_2} \delta(\sigma\gamma\sigma^{-1}\gamma^{-1}) \end{aligned} \tag{7.2}$$

Using the results of section 5.4, we can also write

$$\begin{aligned} \text{Number of ribbon graphs} &= \frac{1}{2^{2v}(2v)!} \sum_{\gamma \in S_{2v}[S_2]} \sum_{\sigma \in [4^v]} \delta(\sigma\gamma\sigma^{-1}\gamma^{-1}) \\ &= \frac{1}{|H_2|} \sum_{\gamma \in H_2} \sum_{\sigma \in S_{4v}/H_1} \delta(\sigma\gamma\sigma^{-1}\gamma^{-1}) \end{aligned} \tag{7.3}$$

There is also a way of writing this that is manifestly symmetric between exchange of H_1, H_2

$$\begin{aligned}\text{Number of ribbon graphs} &= \frac{1}{|H_1||H_2|n!}\sum_{R\vdash n}\sum_{\gamma_1\in H_1}\sum_{\gamma_2\in H_2}\chi_R(\gamma_1)\chi_R(\gamma_2) \\ &= \sum_{R\vdash n}\mathcal{M}^R_{\mathbf{1}_{H_1}}\mathcal{M}^R_{\mathbf{1}_{H_2}} \end{aligned} \tag{7.4}$$

Recall the notation $\mathcal{M}^R_{\mathbf{1}_{H_1}}$ for the multiplicity of the one-dimensional representation of H_1 when the irrep R of S_{4v} is decomposed into representations of the subgroup H_1.

7.2 How many ribbon graphs for the same Feyman graph ?

Ribbon graphs come equipped with a natural notion of a genus. For this reason not all of the contractions leading to a given Feynman graph will produce the same ribbon graph. As an example, consider vacuum graphs in ϕ^4 theory with $v = 1$ vertex. All three contractions give the same Feynman graph, but two of them give a genus zero ribbon graph and one gives a genus one ribbon graph : see Figure 18.

The line of thinking developed above allows an elegant answer to the question of how many ribbon graphs are there for a given Feynman graph. In this case Σ_0 can be chosen as a permutation made of 4-cycles

$$\Sigma_0 = (1234)(5678)...(4v-3, 4v-2, 4v-1, 4v) \tag{7.5}$$

The symmetry group of this permutation $S_v[\mathbb{Z}_4]$. This is the set of permutations in S_{4v} which commutes with Σ_0. The Wick contractions are described by Σ_1 which are pairings, i.e permutations in the conjugacy class $[2^{2v}]$ of S_{4v}. Ribbon graphs are orbits of $S_v[\mathbb{Z}_4]$ action (by conjugation) on the permutations in $[2^{2v}]$. We can first decompose the set of all possible the pairings in $[2^{2v}]$ into orbits of the group $S_v[S_4]$. Each such orbit is a Feynman graph. Then decompose each orbit into orbits of the subgroup $S_v[\mathbb{Z}_4]$ of $S_v[S_4]$. The elements within one orbit of $S_v[S_4]$ will fall generically into multiple orbits of $S_v[\mathbb{Z}_4]$, corresponding to multiple ribbon graphs. We can view the *genus* of the worldsheet described by the ribbon graph as an invariant of the orbits of $S_v[\mathbb{Z}_4]$ acting on 2-cycles. If two 2-cycles are in the same orbit, they must be associated with the same genus. Consider the permutation obtained by multiplying Σ_0 with any Wick contraction which belongs to a fixed orbit of $S_v[\mathbb{Z}_4]$. Denote this permutation by σ_3 and the number of cycles in this permutation by c. The genus g of each $S_v[\mathbb{Z}_4]$ orbit is related to the number of cycles in the permutation c and number of vertices v by

$$2 - 2g = c - v \tag{7.6}$$

This is obtained by applying the Riemann Hurwitz formula to maps the case of maps with three branch points determined by $\Sigma_0, \Sigma_1, \sigma_3$ [19].

It is straightforward to perform, in software such as GAP, the decomposition into orbits of $S_v[S_4]$, and then refine the decomposition according to $S_v[\mathbb{Z}_4]$.

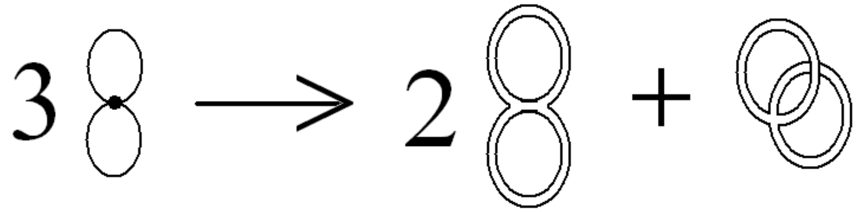

Figure 18: At $v = 1$ the three possible Wick contractions give the same Feynman graph. Two of them give a genus zero ribbon graph and one gives a genus one ribbon graph.

A number of interesting directions can be contemplated. Different ribbon graphs with the same underlying Feynman graph will have the same space-time integrals but possibly different genus. Our methods provide information about how the space-time dependence allows a certain range of genera. When we are dealing with vacuum graphs, each Feynman graph contributes a number. However, our techniques can be applied to graphs that have external legs in which case we have explicit space-time dependences. Presumably a Feynman graph with a certain "complexity" - reflected in its space-time dependence, will allow a certain range of genera. The more complex it becomes, the more genera it will allow. This could be studied quantitatively with the current set-up.

7.3 Using cycle indices for ribbon graphs

In this section we would like to count the number of ribbon graphs using cycle indices. For 4-valent ribbon graphs, the total number of graphs for v vertices, is obtained by using the formula

$$N(\ Z(S_v[\mathbb{Z}_4]) * Z(S_{2v}[S_2])\ ) \tag{7.7}$$

Thus, we need the cycle index of $S_v[\mathbb{Z}_4]$. The cycle index of any cyclic group is

$$Z^{\mathbb{Z}_n}(\vec{x}) = \frac{1}{n} \sum_{d \vdash n} \varphi(d) x_d^{n/d} \tag{7.8}$$

where $\varphi(d)$ is the Euler totient function. For $\mathbb{Z}_4$ we find

$$Z^{\mathbb{Z}_4}(\vec{x}) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4) \tag{7.9}$$

Using known results for the cycle index of wreath products we now find

$$\mathcal{Z}^{S_\infty[\mathbb{Z}_4]}[t,\vec{x}] \equiv \sum_n t^n Z^{S_n[\mathbb{Z}_4]}(\vec{x}) = e^{\sum_{i=1}^{\infty} \frac{t^i}{4i}(x_i^4 + x_{2i}^2 + 2x_{4i})} \tag{7.10}$$

It is now straight forward to obtain explicit answers for the number of ribbon graphs.

The methods at hand also apply to ribbon graphs with arbitrary numbers and types of traces. For a ribbon graph with v_1 single traces, v_2 double traces, v_3 triple traces etc. we would use the group $S_{v_1} \times S_{v_2}[Z_2] \times S_{v_3}[Z_3] \cdots$ to count graphs.

8 QED or Yukawa theory

We have focused on real fields corresponding to particles which are neutral. In this section we explain how our methods can be extended to the case of complex fields corresponding to charged particles. We will refer to the charged particle as an electron with a view to applications to QED. Since are just counting Feynman graphs, we do not track minus signs coming from the anti-commuting nature of fermion fields. As far as the number of diagrams or their symmetry factors goes, there is no difference between QED or Yukawa theory. Of course, in QED certain diagrams vanish automatically due to Furry's Theorem. In the next section we will consider the problem of counting the QED diagrams that remain after Furry's Theorem is applied.

The fact that we are dealing with charged particles implies that each edge will now have a preferred direction, determined by the flow of charge. The model we have in mind has a cubic vertex in which the charged particle interacts with a neutral particle. This structure of the vertex matches both the Yukawa interaction and the coupling of charged fermions to a gauge field. We will refer to the neutral particle as the photon. Thus, each vertex has 3 edges: an incoming electron, an outgoing electron and a photon. Consider vacuum Feynman graphs made of $2v$ vertices. We again clean the graph, by introducing new vertices in the middle of each edge. Label the vertices so that $1, ..., 2v$ are attached to the incoming electrons, $2v+1, \cdots, 4v$ go with the outgoing electrons and $4v+1 \cdots 6v$ go with the photons.

So we can describe the vertices by

$$\Sigma_0 = \prod_{i=1}^{2v} < i, 2v+i, 4v+i > \tag{8.1}$$

The Wick contractions are pairings of the form

$$\Sigma_1 = \prod_{i=1}^{2v} (i, 2v + \tau_1(i)) \cdot \sigma_1 \tag{8.2}$$

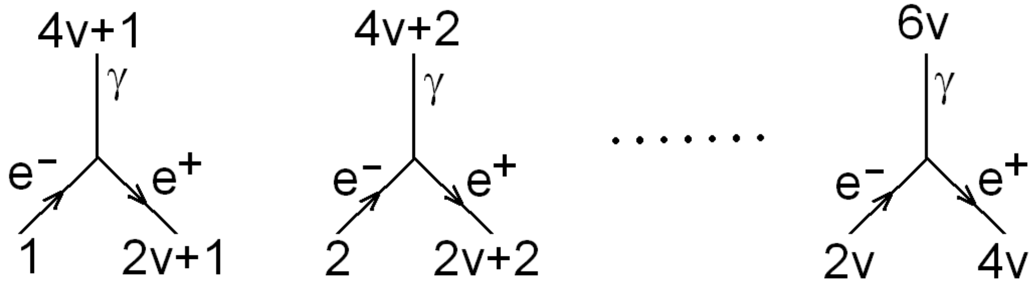

Figure 19: QED vertices

The permutation σ_1 is in $[2^v]$ inside the S_{2v} which acts on the last $2v$ among the $\{1, \cdots, 6v\}$. The permutation τ_1 is a general permutation of $2v$ objects. It tells us which of the first $2v$ (incoming electrons) go with which of the $2v + 1 \cdots 4v$ (outgoing electrons). The symmetry γ of Σ_0 is just S_{2v} of exchanging the $2v$ brackets. Because all legs are distinct there is no symmetry exchanging the legs at a vertex. Given the form of Σ_0 we see that this is the diagonal S_{2v} of the $S_{2v} \times S_{2v} \times S_{2v}$ acting on the electron, anti-electron and photon labels. We will call this $\mathrm{Diag}_3(S_{2v})$.

The automorphisms of the Feynman graph are those elements $\gamma \in \mathrm{Aut}(\Sigma_0) = \mathrm{Diag}_3(S_{2v})$ which also leave fixed the Σ_1 pairing determined by (σ_1, τ_1). The order of this group again determines the symmetry factor of the Feynman graph.

Distinct Feynman graphs are orbits of the permutations

$$\mathrm{Diag}_3(S_{2v}) \to S_{2v} \times S_{2v} \times S_{2v} \tag{8.3}$$

acting on $\Sigma_1(\sigma_1, \tau_1)$, where the σ_1, τ_1 are described above. We know that the stabilizer of $\sigma_1 \in [2^v]$ in S_{2v} is conjugate to $S_v[S_2]$. This leads to

$$S_{2v}/S_v[S_2] = [2^v] \tag{8.4}$$

The permutation τ_1 mixes the first $2v$ incoming electrons with the next $2v$ outgoing electrons. The stabilizer of such a permutation is the diagonal S_{2v} which simultaneously moves the first $2v$ with the next $2v$, so that τ_1 is running over the coset

$$(S_{2v} \times S_{2v})/\mathrm{Diag}_2(S_{2v}) \tag{8.5}$$

We use the subscript 2 because it is the diagonal of the two S_{2v}. This is enough to conclude that the Feynman graphs are in one-to-one correspondence with the points of the double coset

$$\mathrm{Diag}_3(S_{2v}) \setminus (S_{2v} \times S_{2v} \times S_{2v})/(\mathrm{Diag}_2(S_{2v}) \times S_v[S_2]) \tag{8.6}$$

The Diag_2 is the diagonal of the first two S_{2v} in $S_{2v} \times S_{2v} \times S_{2v}$.

8.1 Double cosets of product symmetric groups

To count the number of Feynman graphs, we need to count the number of points in the double coset

$$H_1 \setminus (S_{2v} \times S_{2v} \times S_{2v})/H_2 \tag{8.7}$$

where $H_1 = \mathrm{Diag}_3(S_{2v})$ and $H_2 = \mathrm{Diag}_2(S_{2v}) \times S_v[S_2]$. It is instructive to approach this problem by first considering cosets

$$H_1 \setminus G/H_2 \tag{8.8}$$

where G is a general product of symmetric groups $S_{n_1} \times S_{n_2} \cdots \times S_{n_k}$, and H_1, H_2 are general subgroups. Let us denote by

$$\mathcal{N}(H_1, H_2; G) \tag{8.9}$$

the number of points in this double coset. The case $\mathcal{N}(H_1, H_2; G = S_n) = N(H_1 * H_2)$ is the one we already discussed in sections 5.2 and 5.3. In the case at hand, we need

$$\mathcal{N}(H_1, H_2; S_{2v} \times S_{2v} \times S_{2v}) \tag{8.10}$$

As a consequence of the fact that the numerator group has changed, when we compute the star product, it will turn out that we should multiply the coefficients of $Z(H_1), Z(H_2)$, and now weight them by the symmetry factors appropriate for $S_{n_1} \times S_{n_2} \times \cdots S_{n_k}$. This is a generalization of Read's original formula [20] so it is worth discussing in some detail.

When the numerator group G is a product of symmetric groups, a subgroup such as H_1 will have some number of elements in each conjugacy class of G, which is specified by three partitions.

$$\{p^{(1)}, p^{(2)} \cdots p^{(k)}\} \tag{8.11}$$

For the case at hand $k = 3$. The symmetry of such a conjugacy class is

$$\mathrm{Sym}(\{p^{(i)}\}) = \prod_i \mathrm{Sym}(p^{(i)}) = \prod_{i=1}^{3} \prod_{j=1}^{2v} j^{p_j^{(i)}} p_j^{(i)}! \tag{8.12}$$

We will restrict the discussion to $k = 3$, but the generalization to any k is immediate. For a subgroup H of G it is natural to define

$$Z^{H \to G} = \sum_{p^{(1)}, p^{(2)}, p^{(3)}} Z^{H \to G}_{p^{(1)}, p^{(2)}, p^{(3)}} \prod_{i,j,k} x_i^{p_i^{(1)}} y_j^{p_j^{(2)}} z_k^{p_k^{(3)}} \tag{8.13}$$

where the coefficients $Z^{H\to G}_{p^{(1)},p^{(2)},p^{(3)}}$ keep track of the number of permutations in the subgroup with the cycle structure specified by the 3 partitions. For two subgroups H_1, H_2, we can use the two cycle indices (with respect to G) to define a generalization of the star product in (5.19) as follows

$$\mathcal{N}(H_1, H_2; G) = \sum_p Z^{H_1\to G}_{p^{(1)},p^{(2)},p^{(3)}} Z^{H_2\to G}_{p^{(1)},p^{(2)},p^{(3)}} \mathrm{Sym}(\{p^{(i)}\}) \tag{8.14}$$

We can understand this formula by adapting the reasoning in [20]. For any $\rho \in G$, the product $h_1\rho\, h_2$ gives $|H_1| \times |H_2|$ elements, that all belong to the same equivalence class in the coset (8.9), as h_1 runs over H_1 and h_2 runs over H_2. Some elements will be repeated. When this happens

$$h_1\rho h_2 = \tilde{h}_1\rho\tilde{h}_2 \tag{8.15}$$

which implies that

$$\tilde{h}_1^{-1}h_1 = \rho\tilde{h}_2 h_2^{-1}\rho^{-1} \tag{8.16}$$

Clearly $\tilde{h}_1^{-1}h_1 \in H_1 \cap \rho H_2\rho^{-1}$. Given $\tilde{h}_1$, $\tilde{h}_2$ and an element of $H_1 \cap \rho H_2 \rho^{-1}$, h_1 and h_2 are uniquely determined. Consequently the number of times that ρ is repeated, $\nu(\rho)$, is $|H_1 \cap \rho H_2\rho^{-1}|$. Every element in G which appears in the equivalence class of ρ comes with this same multiplicity in $H_1\rho H_2$. Hence, the number of distinct elements in the equivalence class of ρ

$$n(\rho) = \frac{|H_1||H_2|}{\nu(\rho)} \tag{8.17}$$

Consider

$$\sum_{\rho\in G} \nu(\rho) \tag{8.18}$$

For an equivalence class containing ρ, we get a contribution $n(\rho)\nu(\rho) = |H_1||H_2|$. This is independent of the equivalence class. If there are N_d equivalence classes, we get

$$\sum_{\rho\in G} \nu(\rho) = N_d|H_1||H_2| \tag{8.19}$$

Rearranging we find

$$N_d = \frac{1}{|H_1||H_2|}\sum_{\rho\in G} \nu(\rho) \tag{8.20}$$

To compute the sum over ρ of $\nu(\rho)$ we can choose $u_1 \in H_1$ and $u_2 \in H_2$ and count how many ρ's obey

$$u_1 = \rho u_2 \rho^{-1} \tag{8.21}$$

Since u_1 and u_2 are conjugate, they have the same cycle structure. Thus, to compute the sum over ρ of $\nu(\rho)$ we need to fix a cycle structure, multiply the number of elements in H_1 with this cycle structure by the number of elements in H_2 with this cycle structure, then multiply by the number of elements of G that leave this cycle structure invariant, and then finally sum over cycle structure. This is precisely what (8.14) is computing.

8.2 Analytic expressions for number of Feynman graphs in QED

In this section we will apply the formulas obtained in the previous section to obtain explicit results for the number of Feynman graphs. We need two cycle indices. For $H_1 = \mathrm{Diag}_3(S_{2v})$ we have

$$Z^{H_1 \to G} = \sum_{p \vdash 2v} \frac{1}{\mathrm{Sym}(p)} \prod_{j=1}^{2v} (x_j y_j z_j)^{p_j} \tag{8.22}$$

For $H_2 = \mathrm{Diag}_2(S_{2v}) \times S_v[S_2]$, we have

$$Z^{H_2 \to G} = \sum_{q \vdash 2v} \sum_{r \vdash 2v} \frac{1}{\mathrm{Sym}(q)} Z_r^{S_v[S_2]} \prod_{j=1}^{2v} (x_j y_j)^{q_j} z_j^{r_j} \tag{8.23}$$

To calculate the $\mathcal{N}(H_1, H_2; G)$, we need to multiply like terms in the (8.22) and (8.23), weighted by an appropriate symmetry factor. Picking up like terms forces $q = p = r$ so that

$$\begin{aligned} \mathcal{N}(H_1, H_2; G) &= \sum_{p \vdash 2v} \frac{1}{\mathrm{Sym}(p)} \frac{Z_p^{S_v[S_2]}}{\mathrm{Sym}(p)} \cdot (\mathrm{Sym}(p))^3 \\ &= \sum_{p \vdash 2v} Z_p^{S_v[S_2]} \mathrm{Sym}(p) \end{aligned} \tag{8.24}$$

For the wreath product

$$\begin{aligned} Z^{S_\infty[S_2]}[t; x_1, x_2, \cdots] &\equiv \sum_{n=0}^{\infty} Z^{S_n[S_2]}(x_1, \cdots, x_{2n}) t^n \\ &= e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}(x_i^2 + x_{2i})} \end{aligned} \tag{8.25}$$

To get the desired counting from this, we need to replace $\prod_i x_i^{p_i}$ wth $i^{p_i} p_i!$. Equivalently, do a replacement $x_i \rightarrow iy_i$ and expand in y_i replacing $y_i^{p_i}$ with $p_i!$. In terms of

$$\begin{aligned} \tilde{Z}[t; y_i] &= Z^{S_\infty}[t; x_i = iy_i] \\ &= e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}(iy_i^2 + 2y_{2i})} \end{aligned} \tag{8.26}$$

the counting of vacuum graphs with $2v$ vertices can be written as

$$\mathcal{N}(H_1(v), H_2(v); G(v)) = \oint \frac{t^{-2v} dt}{t} (\prod_i \int_0^\infty dy_i e^{-y_i}) \tilde{Z}[t; y_1, y_2 \cdots] \tag{8.27}$$

See Appendix D for explicit numerical results obtained using these formulas.

8.3 QED and TFT for product symmetric groups

The number of Feynman graphs is counted by the formula

$$\frac{1}{|H_1||H_2||G|} \sum_{u_1 \in H_1} \sum_{u_2 \in H_2} \sum_{\gamma \in G} \delta(u_1 \gamma u_2 \gamma^{-1}) \tag{8.28}$$

We have already established that

$$\begin{aligned} G &= S_{2v} \times S_{2v} \times S_{2v} \\ H_1 &= \text{Diag}_3(S_{2v}) \\ H_2 &= \text{Diag}_2(S_{2v}) \times S_v[S_2] \end{aligned} \tag{8.29}$$

This is a cylinder partition function in $S_\infty \times S_\infty \times S_\infty$ gauge theory which is Schur-Weyl dual to the large N limit of $U(N) \times U(N) \times U(N)$ gauge theory. As before the correspondence holds in the zero area limit.

9 QED with the Furry's theorem constraint

Above we have counted the total number of Feynman diagrams in QED or Yukawa theory. In the case of QED, Furry's Theorem proves a fermion loop punctuated with an odd number of photons vanishes [44]. In this the number of QED Feynman graphs, not vanishing by the Furry constraint, are counted. Enumeration of some Feynman graphs with their symmetry factors for this case is given in [43].

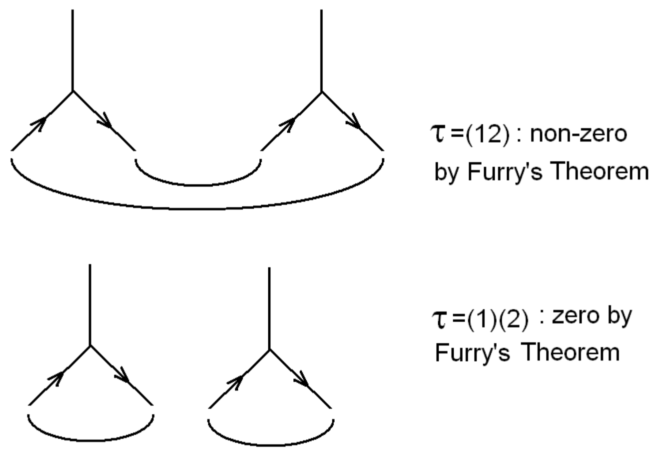

Figure 20: Even cycle lengths and Furry's theorem

9.1 Furry's theorem and a constraint to even cycles

Following the discussion of the last section, each QED Feynman graph is specified by

$$\begin{aligned} \Sigma_0 &= \prod_{i=1}^{2v} < i; 2v+i; 4v+i > \\ \Sigma_1(\sigma_1, \tau_1) &= \prod_{i=1}^{2v} (i, 2v+\tau_1(i)) \ \cdot \ \sigma_1 \end{aligned} \tag{9.1}$$

Recall that τ_1 specifies how incoming and outgoing electrons are connected. Consequently, the number of cycles appearing in τ_1 is equal to the number of fermion loops in the graph and the length of each cycles is equal to the number of photons decorating the loop. It is clear that the constraint that each fermion loop has only an even number of photons is easily imposed in the (Σ_0, Σ_1) description by requiring that τ_1 consists of permutations which only have cycles of even length, as shown in Fig 20.

The orbits of the group $G = S_{2v} \times S_{2v} \times S_{2v}$ acting on the pairs (Σ_0, Σ_1) correspond to the inequivalent Feynman graphs. The set of $\gamma \in G$ which preserve the pair (Σ_0, Σ_1) generates the automorphism group of the graph and the order of this automorphism group is the symmetry factor. Preserving Σ_0 forces γ to be in the diagonal S_{2v} of G :

$$\begin{aligned} \text{For } 1 \ \leq \ i \ &\leq \ 2v \\ i \ &\rightarrow \ \gamma(i) \\ 2v+i \ &\rightarrow \ 2v+\gamma(i) \\ 4v+i \ &\rightarrow \ 4v+\gamma(i) \end{aligned} \tag{9.2}$$

We can write this as $\gamma \circ \gamma \circ \gamma$, where the first γ acts on the first $2v$ according to the second line of (9.2), while leaving the subset $\{2v+1, \cdots, 6v\}$ fixed ; the second γ moves only the subset $\{2v+1, \cdots, 4v\}$ according to the 3rd line; and the third γ moves only the subset $\{4v+1, \cdots, 6v\}$ according to the 4th line.

The symmetry factor of a Feynman graph specified by the standard Σ_0 and a general Σ_1 is

$$|\mathrm{Aut}(\left[\Sigma_1\right]_{H_1})| = \sum_{\gamma \in S_{2v}} \delta(\gamma \circ \gamma \circ \gamma \ \Sigma_1 \ \gamma^{-1} \circ \gamma^{-1} \circ \gamma^{-1} \Sigma_1^{-1}) \tag{9.3}$$

Σ_1 is not a permutation in G, since the τ_1 mixes the first $2v$ with the second $2v$. The multiplication can be viewed in S_{6v} or $S_{4v} \times S_{2v}$

A simple application of the Burnside Lemma implies that the number of Feynman graphs remaining after Furry's Theorem is applied is

$$\frac{1}{(2v)!} \sum_{\gamma \in S_{2v}} \sum_{\sigma_1 \in [2^v] \in S_{2v}} \quad \sum_{\tau_1 \in S_{2v} : [\tau_1] \text{ even}} \delta \left(\ \gamma \circ \gamma \circ \gamma \ \ \Sigma_1(\sigma_1, \tau_1) \gamma^{-1} \circ \gamma^{-1} \circ \gamma^{-1} \ \ \Sigma_1^{-1}(\sigma_1, \tau_1) \ \right) \tag{9.4}$$

Another approach to the same counting problem would be to implement the Furry constraint in the double coset language. Recall the double coset relevant for QED is

$$\mathrm{Diag}_3(S_{2v}) \setminus (S_{2v} \times S_{2v} \times S_{2v}) / \mathrm{Diag}_2(S_{2v}) \times S_v[S_2] \tag{9.5}$$

Using (5.21) with the G, H_1, H_2 identified according to (9.5) the size F of this double coset is counted by

$$\begin{aligned} F = & \frac{1}{(2v)!(2v)!2^v v!} \sum_{\sigma \in S_{2v}} \sum_{\tau \in S_{2v}} \sum_{\rho \in S_v[S_2]} \sum_{b_1, b_2, b_3 \in S_{2v}} \\ & \delta_{S_{2v} \times S_{2v} \times S_{2v}} (\sigma \circ \sigma \circ \sigma \ \ b_1 \circ b_2 \circ b_3 \ \ \tau \circ \tau \circ \rho \ \ b_1^{-1} \circ b_2^{-1} \circ b_3^{-1}) \end{aligned} \tag{9.6}$$

b_1 acts on the incoming electrons, b_2 on the outgoing electrons and b_3 on the photons. σ permutes vertices in the graph, τ permutes electron lines and ρ permutes photon lines. This can be simplified to

$$\begin{aligned} F = & \frac{1}{(2v)!(2v)!2^v v!} \sum_{\sigma \in S_{2v}} \sum_{\tau \in S_{2v}} \sum_{\rho \in S_v[S_2]} \sum_{b_1, b_2, b_3 \in S_{2v}} \\ & \delta_{S_{2v} \times S_{2v}} (\sigma \circ \sigma \ \ b_1 \circ b_2 \ \ \tau \circ \tau \ \ b_1^{-1} \circ b_2^{-1}) \\ & \delta_{S_{2v}} (\sigma b_3 \rho b_3^{-1}) \end{aligned} \tag{9.7}$$

Solving the second delta function for σ, we can plug back into the first delta function and do a redefinition of the sums $\sum_{b_1, b_2}$ to absorb the b_3, so as to get

$$F \ = \ \frac{1}{(2v)!2^v v!} \sum_{\tau \in S_{2v}} \sum_{\rho \in S_v[S_2]} \sum_{b_1, b_2 \in S_{2v}}$$

$$
\begin{aligned}
&\delta_{S_{2v}\times S_{2v}}(\rho\circ\rho \quad b_1\circ b_2 \quad \tau\circ\tau \quad b_1^{-1}\circ b_2^{-1}) \\
= \quad & \frac{1}{(2v)!2^v v!}\sum_{\rho\in S_{2v}}\sum_{\tau_1\in S_v[S_2]}\sum_{b_1,b_2\in S_{2v}}\delta_{S_{2v}}(\rho b_1\tau b_1^{-1})\delta_{S_{2v}}(\rho b_2\tau b_2^{-1}) \\
= \quad & \frac{1}{(2v)!2^v v!}\sum_{\rho\in S_v[S_2]}\sum_{b_1,b_2\in S_{2v}}\delta_{S_{2v}}(b_2^{-1}\rho^{-1}b_2 b_1^{-1}\rho b_1) \\
= \quad & \frac{1}{2^v v!}\sum_{\rho\in S_v[S_2]}\sum_{\tau_1\in S_{2v}}\delta_{S_{2v}}(\tau_1\rho\tau_1^{-1}\rho^{-1})
\end{aligned} \tag{9.8}
$$

In the last line we have recognized that, given the interpretation of b_1, b_2 it is clear that $\tau_1 = b_1^{-1}b_2$ is the permutation τ_1 in (9.1). The Furry constraint is now easily implemented. Thus, the number of Feynman graphs remaining after the Furry constraint is implemented is given by

$$
\begin{aligned}
&\text{Number of Feynman graphs for QED with Furry constraint} \\
&= \frac{1}{2^v v!}\sum_{\tau_1\in S_{2v}:even}\sum_{\rho\in S_v[S_2]}\delta_{S_{2v}}(\tau_1\rho\tau_1^{-1}\rho^{-1})
\end{aligned} \tag{9.9}
$$

The permutation τ_1 is constrained to have even cycles only and we are summing over elements of $S_v[S_2]$. For each element of $S_v[S_2]$, the weight is the number of permutations with even cycles only, which commute with the given permutation in $S_v[S_2]$.

From the first line of (9.8) we see that an equivalent double coset description of the counting is

$$
S_{2v}\setminus(S_{2v}\times S_{2v})/\mathrm{Diag}_2(S_v[S_2]) \tag{9.10}
$$

This gives a slightly simpler connection to observables for 2d Yang-Mills with cylinder target space. Namely we have a connection to $U(\infty)\times U(\infty)$ rather than $U(\infty)\times U(\infty)\times U(\infty)$ as described earlier.

9.2 From QED to ribbon graphs

Comparing the expressions (9.8) and (9.9) with (7.4) for ribbon graph counting, it becomes clear that the counting of QED Feynman graphs can be matched with counting ribbon graphs. The restriction to vertices of valency 4 in (7.4) is being relaxed to allow arbitrary valencies in (9.8) and arbitrary even valencies in (9.9). We will now explain the bijection between ribbon graphs and QED/Yukawa graphs which explains the equality of counting formulae.

The general QED/Yukawa vacuum graph has loops with vertices where the photon joins the loop. These loops determine $2v$ cycles which form a permutation in S_{2v}. To build a bijection to ribbon graphs think of the photon labels in Figure 19 as attached to the vertices. The photon contractions determine an element of $[2^{2v}]$. At the centre of each electron loop draw a point and radiate edges (spokes) to intersect the edges of the loop, one spoke between each pair of electron-photon vertices. Use the arrow on the electron propagator to move each end of the photon propagator along the electron loop towards the intersection of the spoke. Erase the electron loop, leaving the spokes and the photon propagators connecting them. This gives a graph with vertices of arbitrary valency, but with the vertices being equipped with a cyclic order, which is nothing but a ribbon graph, completing the construction of the bijection. This process is illustrated in Figure 21. The bijection demonstrates that the number of QED vacuum graphs with $2v$ vertices is the same as the number of ribbon graphs with v edges (the photon propagators become edges).

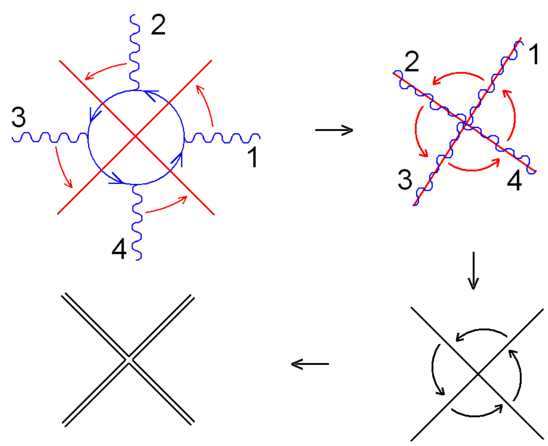

Figure 21: QED graphs to ribbon graphs

This means that the total valency of the vertices in the ribbon graph is $2v$. Each vertex can have any integer valency compatible with this constraint. Thus, following the discussion (7.3) and (7.7), the number of QED/Yukawa vacuum graphs with $2v$ vertices is

$$\text{Number of Feynman graphs for QED } = \sum_{p \vdash 2v} N(Z(H_1) * Z(H_p)) \tag{9.11}$$

where p is a partition of $2v$, the group H_1 is $S_v[S_2]$ and the group H_p is

$$S_{p_1}[\mathbb{Z}_1] \times S_{p_2}[\mathbb{Z}_2] \times \cdots S_{p_{2v}}[\mathbb{Z}_{2v}] \tag{9.12}$$

The function N of two cycle index polynomials is as defined in (5.19). For QED with the Furry constraint implemented, the valencies are constrained to be even so that

Number of Feynman graphs for QED with Furry constraint

$$= \sum_{\substack{p \vdash 2v \\ p \ \ even}} N(Z(H_1) * Z(H_p)) \tag{9.13}$$

Given the connection between ribbon graphs and Belyi pairs [19] we see that QED counts the number of clean Belyi pairs (bi-partite embedded graphs with bivalent white vertices) with degree v and black vertices of any valency. QED with the Furry constraint counts the number of such Belyi pairs with black vertices of even order only.

10 Discussion

10.1 Permutations and Strings : a new perspective on gauge-string dualities.

We have seen that the counting of Feynman diagrams can be interpreted in terms of string amplitudes with a cylinder target space. The nature of the interaction vertices determines a permutation symmetry group H_1 and the Wick contractions determine a permutation symmetry group H_2. The counting could also be written in terms of commuting pairs of permutations, where one permutation lives in one of these subgroups, and another one lives in a coset involving the other group. This latter realization can be interpreted in terms of covering maps of a torus target space, such as those that arise in the string theory of 2d Yang Mills for torus target space [3, 4], except that there is a constraint on the two permutations associated with the cover, by following the inverse image of paths along the a and b cycle.

In 2d Yang Mills, one encounters a sum over commuting permutations s_1, s_2 which is equivalent to counting the set of all covers of the torus by a torus. This sum is invariant under the $SL(2, Z)$ of the space-time torus, which acts on these permutations by

$$\begin{aligned} S : \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad &\rightarrow \begin{pmatrix} s_2 \\ s_1^{-1} \end{pmatrix} \\ T : \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad &\rightarrow \begin{pmatrix} s_1 s_2 \\ s_2 \end{pmatrix} \end{aligned} \tag{10.1}$$

and obeys the $SL(2, Z)$ relations :

$$\begin{aligned} S^2 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad &\rightarrow \begin{pmatrix} s_1^{-1} \\ s_2^{-1} \end{pmatrix} \\ (ST)^3 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad &\rightarrow \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \end{aligned} \tag{10.2}$$

In the case at hand, the $SL(2, Z)$ of space-time is broken in that s_1 is summed over H_1 while s_2 is summed over $S_{4v}/H_2 = [2^{2v}]$.

While making two choices of observable at each end of a cylinder seems like a natural construction for string theory with a cylinder, the constraints on the covering maps of the torus look rather intricate, and are not of any sort that has been discussed in the literature on topological strings. Nevertheless, there might well be a string construction which sums over maps with constraints of this sort.

In any case, the Feynman diagram combinatorics suggests two emergent dimensions of cylinder or torus, with one formulation possibly more appealing than the other. In the construction of correlators of general trace operators in the Hermitian Matrix model, there is another 2-dimensional target space emerging, which is the sphere with three punctures [19].

In the application of Feynman graphs to QFT, the Feynman graph counting is only part of the answer. When there are external edges, each edge is associated with a space-time position (or momentum), each of which takes values in $\mathbb{R}^4$. The computation of Greens functions $\langle \phi(x_1)\phi(x_2)\cdots\phi(x_E)\rangle$ is a sum over Feynman diagrams, which live in

$$DC(v,E) = (S_v[S_4] \times S_1^E) \setminus S_{4v+E}/S_{2v+E/2}[S_2] \tag{10.3}$$

This has an action of S_E. Projecting down the S_E orbits gives another double coset. There is a fibration

$$\begin{array}{l} (S_v[S_4] \times S_1^E) \setminus S_{4v+E}/S_{2v+E/2}[S_2] \leftarrow S_E \\ \qquad\qquad \downarrow \\ (S_v[S_4] \times S_E) \setminus S_{4v+E}/S_{2v+E/2}[S_2] \end{array} \tag{10.4}$$

We may write

$$G(x_1, x_2, \cdots, x_E) = \sum_{\Sigma} G(x_1, x_2, \cdots x_E; \Sigma) \tag{10.5}$$

where Σ lives in the double coset (10.3). The Bose symmetry of $G(x_1, x_2, \cdots, x_E)$ arises as an S_E invariance of the summands

$$G(x_{\sigma(1)}, x_{\sigma(2)}, \cdots x_{\sigma(E)}; \sigma(\Sigma)) \tag{10.6}$$

This shows that in some sense the space-time coordinates $(\mathbb{R}^4)^E$ and the double coset $DC(v,E)$ associated with covers of a cylinder may usefully be viewed as being on an equal footing when we work out the physics. It is tempting to infer that this is indicative of an underlying $\mathbb{R}^4 \times$ Cylinder or $\mathbb{R}^4 \times$ Torus in four dimensional QFTs. We have shown that the extra two dimensions emerge from the Feynman graph combinatorics, but it remains to be seen if there is a dynamical six dimensional picture along these lines, and how it might relate to other appearances of six dimensions in 4D QFTs such as [51].

A very interesting problem is to interpret the full correlators or S-matrices, in terms of the cylinder (or torus) emerging from combinatorics, with the $\mathbb{R}^4$ which is manifestly there to begin with. This would be an example of a holographic dual of a quantum field theory, analogous to AdS/CFT, where from the point of view of the dual gauge theory, an emergent S^1 along with an emergent radial direction, resulting in $M_5 \times S^1$, with the original $\mathbb{R}^4$ of the gauge theory realized as the boundary of the M_5.

If this is indeed possible with ϕ^4 theory, it would be an example of gauge-string duality with an interesting difference from all the examples known so far. It would not rely on the large N expansion, and ribbon graphs thickening into string worldsheets. Rather it would be a realization through the fundamental link between permutations and strings, of the physical expectation of holography [52, 53]. This makes it very important to know whether this works or not. For some general recent discussions of the scope of holography see [54]. Another direction for relating Feynman graphs to string amplitudes, by directly identifying the integrand on the moduli space of complex structures of string worlsheets, has been advocated in [55]. The ideas emerging from the permutations-strings connection may well have interesting overlaps with this programme, which we will leave to future investigation.

If there are indeed new full-fledged gauge-string dualities for theories such as ϕ^4 in four dimensions, or ϕ^3 in six dimensions or indeed QED or QED-like theories, there are some obvious questions to be addressed. What is the map between physical parameters? In the ϕ^4 theory, Feynman graphs with v vertices are weighted with g^v. We have associated them with cylinder worldsheets covering a target space cylinder, with maps of degree $4v$. If the cylinder has area A, the Nambu-Goto action will weight such strings with e^{-nA}. This suggests that the QFT coupling constant g should be identified with e^{-A}.

Another obvious question : What is the string coupling of this dual string theory? Since all the details of the field theory interaction translate into the group H_1 at a boundary of the cylinder amplitude, interaction in the field theory do not translate directly into interactions of the string. A related question is whether there is some modification of the Feynman graph counting problem on the field theory side which leads to higher genus string worldsheets covering the cylinder. This requires further exploration of the sort of string amplitudes that come from various QFT Feynman diagrams. The added motivation for these investigations is that the stringy picture leads directly to powerful counting results, such as those in Appendix D.

10.2 Topological Strings and worldsheet methods

We have argued that there is a link between Feynman graph counting and string amplitudes by recognizing the combinatorics of worldsheet maps to a cylinder in the Feynman graph combinatorics. The string theory in question involves the one that appears in the large N expansion of 2dYM with $U(N)$ gauge group. This string theory also has a spacetime description in terms of topological lattice theory with S_n (for all n, hence probably better described as S_∞) as gauge group. The observables needed to describe Feynman graph combinatorics are slightly more general than the ones that appear in $U(N)$ 2dYM. They involve boundary observables which are not invariant under the entire S_n in the lattice TFT description, but only certain specified subgroups H_1, H_2.

Several worldsheet approaches to the string theory of 2dYM have been proposed [6, 7, 8, 9, 10]. It will be very interesting to develop, in these approaches, the boundary observables which count the Feynman graphs and their symmetries. The geometrical description in terms of gluing world-sheets in section 5.6 should be useful. An interesting goal would be to find new worldsheet methods for calculating Feynman graph combinatorics, which may be efficient for obtaining asymptotic results.

10.3 Collect results on orbits

We have developed a method to think about the combinatorics of Feynman graphs, involving pairs (Σ_0, Σ_1) of combinatoric data. This was discussed from the point of view of double cosets, which provide a link to generating functions. From the point of view of constructing the Feynman graphs, perhaps the most immediate lesson extracted from the study of the pairs (Σ_0, Σ_1) is that

A Feynman graph is an orbit of the vertex symmetry group acting on the Wick contractions.

It is a simple exercise to obtain both the vertex symmetry group and the set of Wick contractions given a quantum field theory. We have listed this data in the table below for the theories that we have considered here. The data is relevant to vacuum graphs - there are simple generalizations including external edges.

Group	acts on	Feynman Graph problem
$S_v[S_4]$	permutations in $[2^{2v}]$ of S_{2v}	ϕ^4 theory
$S_v[S_3]$	permutations in $[2^{\frac{3v}{2}}]$ of $S_{3v/2}$	ϕ^3 theory
$S_{v_4}[S_4] \times S_{v_3}[S_3]$	permutations in $[2^{\frac{(3v_3+4v_4)}{2}}]$ of $S_{3v_3+4v_4}$	$\phi^3 + \phi^4$ theory
$S_v[S_2]$	All permutations in S_{2v}	Yukawa/QED
$S_v[S_2]$	All even-cycle permutations in S_{2v}	Furry QED
$S_v[\mathbb{Z}_4]$	permutations in $[2^{2v}]$ of S_{2v}	Large N expansion of Matrix ϕ^4

The double coset picture allows us to recognize that one can also consider the symmetry of the Wick contractions (e.g $S_{2v}[S_2]$ in case of ϕ^4) acting on the cosets associated with the vertices ($S_{4v}/S_v[S_4]$ in this case).

10.4 Galois Group and Feynman Graphs

We showed that the counting of vacuum graphs in QED/Yukawa and QED with Furry constraint implemented can be mapped to counting of ribbon graphs. In the case of QED with Furry constraint, there is an even-only restriction on the vertices. We know, from the theory of Dessins d'Enfants [12], that there is an action of the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on ribbon graphs. (For recent applications of Dessins d'Enfants to Hermitian Matrix Model correlators, supersymmetric gauge theories and extensive references to the Mathematical literature see [13, 19, 14, 15].) This means that there is an action of the Galois group on the QED vacuum diagrams. The list of orders of the vertices is a Galois invariant [16], so the Furry theorem restriction continues to allow the Galois action to close.

A related fact is that sums over permutations restricted by conjugacy classes are known to be sums over Galois group orbits.

$$\sum_{\sigma_1\in T_1}\sum_{\sigma_2\in T_2}\sum_{\sigma_3\in T_3}\delta(\sigma_1\sigma_2\sigma_3) \tag{10.7}$$

Given the link we have observed between counting ribbon graphs and counting QED graphs, we have Galois actions on the QED/Yukawa and Furry QED vacuum graphs. We do not know if such is the case for the vacuum graphs of scalar field theories. The answer depends on whether sums of the following form

$$\sum_{\sigma_1\in H_1}\sum_{\sigma_2\in H_2}\sum_{\sigma_3\in S_n}\delta(\sigma_1\sigma_3^{-1}\sigma_2\sigma_3) \tag{10.8}$$

for $H_1 = S_v[S_4], H_2 = S_v[S_2]$ are related to Galois theory. We know if $H_1 = S_v[\mathbb{Z}_4]$, because the connection between QED and ribbon graphs we described, these sums count Dessins. For $H_1 = S_v[S_4]$ we do not know an argument to relate to Galois actions.

11 Summary and Outlook

We have shown that the counting of Feynman graphs and their symmetry factors in scalar field theory, e.g ϕ^4 or ϕ^3 theories, as well as QED, can be mapped to string amplitudes. The string amplitudes are of the type that appear in the string dual of large N Yang Mills theory, for which there are several proposed worldsheet constructions. The large N 2dYM observables are known to be related to S_n TFT. The counting problems related to Feynman graphs have been expressed as observables in this S_n TFT. We used this S_n TFT data to construct covers of the cylinder. The covers are inrepreted as string world-sheets, which are also of cylinder topology. The form of the interactions in the QFT determines the observables at the two ends of the spacetime cylinder, which constrain the windings of the string worldsheets to belong to certain subgroups of the permutation group.

The formulation in terms of string amplitudes is directly related to some classic formulae on counting of graphs in papers by Read[20]. We have found it useful to think about these formulae in terms of an operation of introducing an additional vertex in the middle of each edge of the Feynman graph. This separates each edge into a pair of half-edges. We label the half-edges with numbers $1, 2, \cdots, n$, and associate a quantity Σ_0 describing the vertices of the Feynman graph. The newly-added vertices are described by a quantity Σ_1. This operation is called "cleaning" in the context of ribbon graphs and related Belyi maps. In that case two permutations σ_0, σ_1 play the role of Σ_0, Σ_1. In the case at hand, Σ_0 is generically not a permutation, but can be viewed as labeling a coset of permutation groups.

There are groups H_1, H_2 which are symmetry groups of Σ_0 and Σ_1 respectively. The counting of Feynman graphs is equivalent to the counting of elements in the double coset of the form

$$H_1 \setminus \ (\text{ Permutation group })/H_2 \tag{11.1}$$

For the case of scalar field theories the permutation group is something of the form S_n. In QED, it is a product group.

The double coset connects directly to a string amplitude with cylinder target space, where H_1, H_2 are associated with the boundaries. The similarities in the current approach between Feynman graphs and ribbon graphs, allows us to formulate in group theoretic terms, and derive nice formulae for questions such as the total number of types of ribbon graphs (summed over genus). Further it allows us to express as a group theory problem the counting of the number of ribbon graphs which correspond to the same Feynman graph. We may interpret this by saying that large N ribbon graphs (say of matrix ϕ^4 theory) arise from ordinary Feynman graphs (of ϕ^4 theory) by a symmetry breaking of $S_v[S_4]$ to $S_v[\mathbb{Z}_4]$.

The formulae we obtain for QED Feynman graph counting turn out, upon simplification, to be related to the counting of ribbon graphs. We give an explanation of this relation by mapping QED graphs to ribbon graphs. The key point is the cyclic nature of ribbon graph vertices which map to orientations of fermion loops, in the correspondence we describe between ribbon graphs and QED graphs.

We outline some avenues for extensions of this work. It will be desirable to understand in terms of double cosets and string amplitudes and to derive generating functions for the refinements of graph counting that occur in QFT. We have made some steps in the direction of connected graphs, and a more systematic understanding for the case of multiple external legs will be useful. In QFT, it is a familiar fact that the generating functional of one-particle irreducible graphs is related to the generateing functional of connected graphs by a Legendre transform. Clarifying the implications for double cosets and strings will be desirable. Extending to other quantum field theories is an obvious direction. This would give new counting results for the relevant Feynman graphs, and could also help address some conceptual issues on the meaning of the string amplitude interpretation, with a view to exploring the possibility that the stringy combinatorics we have uncovered is merely the tip of an iceberg, the bulk of which is a full-fledged QFT-string duality, which does not involve large N.

With regard to precise information on the asymptotic growth of amplitudes in perturbation theory, the counting sequences of Feynman graphs and their asymptotics are of interest. For the case of vacuum graphs in ϕ^4 theory, there is an asymptotic result [57]

$$e^{15/4}\frac{(4v)!}{(4!)^v(2v)!2^{2v}v!} \tag{11.2}$$

For general ϕ^r, with v vertices, we need rv to be even for non-vanishing vacuuum diagrams, i.e $rv = 2m$ for some m, and the result [57] is

$$e^{-(r^2-1)/4}\frac{(2m)!}{2^m m! r!^v v!} \tag{11.3}$$

We have given a string interpretation of these sequences. Strings at large quantum numbers often become classical. Can the above asymptotic result, e.g in ϕ^4, be explained by an appropriate semi-classical string ? For QED/Yukawa Feynman graphs or for QED, with the Furry constraint, we hope that the counting sequences and analytic formulae we have described, along with techniques such as those of [57], will allow the determination of the asymptotics.

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A Semi-direct product structure of Feynman graph symmetries.

In this section we will explain that the group of automorphisms of a Feynman graph can be realized as the semi-direct product of two subgroups defined shortly. Towards this end, it is useful to recall the definition of the semi-direct product. Given two groups G_1 and G_2, and a group homomorphism $\psi : G_2 \to \mathrm{Aut}(G_1)$, the semi-direct product of G_1 and G_2 with respect to ψ is denoted $G_1 \rtimes_\psi G_2$. As a set $G_1 \rtimes_\psi G_2$ is the Cartesian product $G_1 \times G_2$. Multiplication is defined using ψ as

$$(g_1, h_1) * (g_2, h_2) = (g_1 \psi_{h_1}(g_2), h_1 h_2) \tag{A.1}$$

for all $g_1, g_2 \in G_1$ and $h_1, h_2 \in G_2$. The identity element e is (e_{G_1}, e_{G_2}) and

$$(g, h)^{-1} = (\psi_{h^{-1}}(g^{-1}), h^{-1}) \tag{A.2}$$

The set of group elements (g_1, e_{G_2}) for a normal subgroup of $G_1 \rtimes_\phi G_2$ isomorphic to G_2, while the set of elements (e_{G_1}, h) form a subgroup isomorphic to G_1.

To make our discussion concrete, again consider $g\phi^4$ theory. Recall from Section 4 that any graph can be specified, after introducing a new type of vertex (say white when the original vertices are colored black) between existing edges and labelling the resulting half-edges, by data Σ_0 associated with the vertices and Σ_1 with the edges. For a graph with v vertices, Σ_0 is a collection of v 4-tuples of numbers

$$\Sigma_0 = \prod_{r=1}^{v} \Sigma_0^{(r)} \tag{A.3}$$

Σ_1 is a product of two cycles. The symmetric group S_{4v} acts by permuting the half edges.

We can define the Automorphism group of the graph using the data Σ_0 and Σ_1. It is the group of permutations $\gamma \in S_{4v}$ which have the property

$$\begin{aligned} \gamma(\Sigma_0) &= \Sigma_0 \\ \gamma(\Sigma_1)\gamma^{-1} &= \Sigma_1 \end{aligned} \tag{A.4}$$

By $\gamma(\Sigma_0)$ we mean the operation which acts on, on each factor of Σ_0, as follows

$$< i\ j\ k\ l > \to < \gamma(i)\ \gamma(j)\ \gamma(k)\ \gamma(l) > \tag{A.5}$$

The action on Σ_1 can also be written as above, or equivalently in terms of conjugation in S_{4v}. In testing the first equality, we treat each angled-bracket as completely symmetric.

Two types of actions can be automorphisms : vertices (i.e. black dots) can be swapped, and propagators (i.e. white dots) can be swapped. Construct the group G_E which acts only on the propagators and the group G_V that acts on the vertices. The elements of G_V act as a non-trivial permutation on the r index running over the vertices. The elements of G_E act trivially on the r index.

For any given Feynman graph we can argue that $\mathrm{Aut}(D) = G_E \rtimes_\psi G_V$. Towards this end we will prove that G_V acts as an automorphism of G_E. For each element $\nu \in G_V$ there is a permutation $\gamma \in S_v$ which permutes the vertices

$$\nu(\Sigma_0^{(r)}) = \Sigma_0^{(\gamma(r))}$$

The elements of G_E leave the vertices fixed

$$\epsilon(\Sigma_0^{(r)}) = \Sigma_0^{(r)} \qquad \epsilon \in G_E$$

We have

$$\nu^{-1}\epsilon\nu(\Sigma_0^{(r)}) = \nu^{-1}\epsilon((\Sigma_0^{(\gamma(r))}) = \nu^{-1}((\Sigma_0^{(\gamma(r))}) = \Sigma_0^{(r)}$$

Thus, $\nu^{-1}\epsilon\nu \in G_E$. Defining $\psi_\nu(\epsilon) = \nu^{-1}\epsilon\nu$, it follows that $\psi_{\nu_1}\psi_{\nu_2}(\epsilon) = \psi_{\nu_1\nu_2}(\epsilon)$ which shows that we indeed have a homomorphism $\psi : G_V \to \mathrm{Aut}(G_E)$.

To understand why $\mathrm{Aut}(D) = G_E \rtimes_\psi G_V$, consider the product law for the semi-direct product

$$(\epsilon_1, \nu_1) * (\epsilon_2, \nu_2) = (\epsilon_1\psi_{\nu_1}(\epsilon_2), \nu_1\nu_2) = (\epsilon_1\nu_1\epsilon_2\nu_1^{-1}, \nu_1\nu_2)$$

Acting on the graph D means acting on Σ_0 and Σ_1 as defined above. The left hand side applies $\epsilon_1\nu_1\epsilon_2\nu_2$ to D. The right hand side applies $\epsilon_1\psi_{\nu_1}(\epsilon_2)\nu_1\nu_2 = \epsilon_1\nu_1\epsilon_2\nu_1^{-1}\nu_1\nu_2 = \epsilon_1\nu_1\epsilon_2\nu_2$ completing the demonstration.

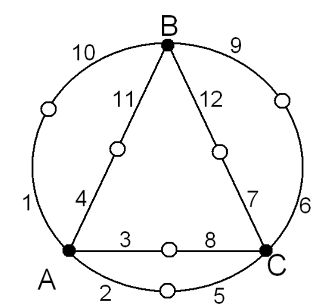

Figure 22: For this Feynman graph $v = 3$.

An example is in order. For the graph in Figure 22, we have

$$\begin{aligned} \Sigma_0 &= <1\ 2\ 3\ 4><5\ 6\ 7\ 8><9\ 10\ 11\ 12> \\ \Sigma_1 &= (1\ 10)(4\ 11)(3\ 8)(2\ 5)(6\ 9)(7\ 12) \end{aligned} \tag{A.6}$$

From the figure we read off the generators (1 4)(10 11), (2 3)(5 8), (9 12)(6 7) for G_E. Since these generators commute, G_E is a group of order 8. The group G_V will permute the vertices A, B and C of the Feynman graph. There are 6 possible permutations of the vertices. To obtain the generators of G_V consider (for example) the permutation which swaps B and C, shown in figure 23. The relevant permutation is σ_{BC} =(1 2)(3 4)(5 10)(6 9)(7 12)(8 11). This is indeed an endomorphism of G_E since

$$\sigma_{BC}(1\ 4)(10\ 11)\sigma_{BC}^{-1} = (2\ 3)(5\ 8) \qquad \sigma_{BC}(2\ 3)(5\ 8)\sigma_{BC}^{-1} = (1\ 4)(10\ 11)$$

$$\sigma_{BC}(9\ 12)(6\ 7)\sigma_{BC}^{-1} = (6\ 7)(9\ 12)$$

The same is true of the other elements of G_V. G_V has order 6 and G_E has order 8 so that $\mathrm{Aut}(D)$ is order 48. This Feynman graph thus comes with a coefficient $(4!)^3/48 = 288$.

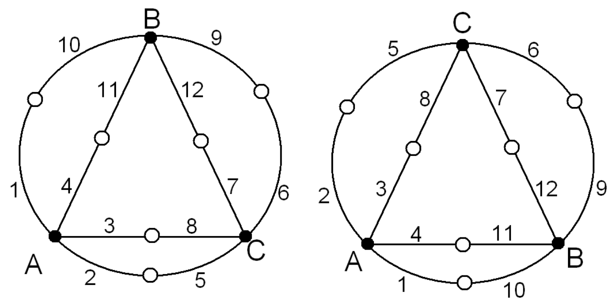

Figure 23: The two Feynman graphs shown are related by swapping vertices B and C.

Note that there is a notion of Graph automorphism used in graph theory [58]. This only includes G_V, since the standard labelling of graphs is to label the vertices and list the edges as pairs of numbers associated with the vertices they are incident on. The distinction is discussed in [59].

B Functions on the double-coset

In this Appendix we will explain how to build a complete set of functions on $S_n \setminus (S_n \times S_n)/(H_1 \times H_2)$. This Appendix makes use of techniques similar to what was used in [21].

A basis of functions on S_n is given by the matrix elements of irreducible representations. Start with an element in the group algebra of $(S_n \times S_n)$ labelled by representations and states in the representations.

$$\tilde{\mathcal{O}}^{R_1,R_2}_{i_1j_1;i_2j_2} = \sum_{\sigma_1,\sigma_2\in S_n} D^{R_1}_{i_1j_1}(\sigma_1)D^{R_2}_{i_2j_2}(\sigma_2)\sigma_1 \otimes \sigma_2 \qquad \text{(B.1)}$$

We can make it invariant under left action of S_n and under right action of $H_1 \times H_2$ by taking

$$\mathcal{O}^{R_1,R_2}_{i_1j_1;i_2j_2} = \frac{1}{n!|H_1||H_2|} \sum_{\alpha\in S_n} \sum_{\beta_1\in H_1} \sum_{\beta_2\in H_2} (\alpha\otimes\alpha)\tilde{\mathcal{O}}^{R_1,R_2}_{i_1,j_1;i_2j_2} \quad \beta_1\otimes\beta_2 \tag{B.2}$$

Some basic manipulations lead to

$$\begin{aligned}
\mathcal{O}^{R_1,R_2}_{i_1j_1;i_2j_2} &= \frac{1}{n!|H_1||H_2|} \sum_{\sigma_1,\sigma_2\in S_n} \sum_{\alpha\in S_n} \sum_{\beta_1\in H_1} \sum_{\beta_2\in H_2} D^{R_1}_{i_1j_1}(\alpha\sigma_1\beta_1) D^{R_2}_{i_2j_2}(\alpha\sigma_2\beta_2) \quad \sigma_1\otimes\sigma_2 \\
&= \frac{1}{n!|H_1||H_2|} \sum_{\sigma_1,\sigma_2\in S_n} \sum_{\alpha\in S_n} \sum_{\beta_1\in H_1} \sum_{\beta_2\in H_2} \sigma_1\otimes\sigma_2 D^{R_1}_{i_1k_1}(\alpha) D^{R_1}_{k_1l_1}(\sigma_1) D^{R_1}_{l_1j_1}(\beta_1) D^{R_2}_{i_2k_2}(\alpha) D^{R_2}_{k_2l_2}(\sigma_2) D^{R_2}_{l_2j_2}(\beta_2) \\
&= \sum_{\sigma_1,\sigma_2} \sum_{\mu_1,\mu_2} \sigma_1\otimes\sigma_2 D^{R_1}_{k_1l_1}(\sigma_1) D^{R_2}_{k_2l_2}(\sigma_2) \\
&\qquad \delta_{R_1,R_2}\delta_{i_1,i_2}\delta_{k_1,k_2} B^{R_1,\mu_1}_{l_1}(\mathbf{1}_{H_1}) B^{R_1,\mu_1}_{j_1}(\mathbf{1}_{H_1}) B^{R_2,\mu_2}_{l_2}(\mathbf{1}_{H_2}) B^{R_2,\mu_2}_{j_2}(\mathbf{1}_{H_2})
\end{aligned} \tag{B.3}$$

The label μ_1 runs over the multiplicity with which the trivial irrep. of H_1 appears in the decomposition of the S_n irrep. R with respect to the subgroup. Thus

$$\begin{aligned}
\mathcal{O}^{R_1,R_2}_{i_1j_1;i_2j_2} &= \delta_{R_1R_2} \sum_{\sigma_1,\sigma_2} \delta_{i_1i_2}\sigma_1\otimes\sigma_2 D^{R_1}_{l_1l_2}(\sigma_1^{-1}\sigma_2)\delta_{R_1,R_2} B^{R_1,\mu_1}_{l_1}(\mathbf{1}_{H_1}) B^{R_1,\mu_1}_{j_1}(\mathbf{1}_{H_1}) B^{R_2,\mu_2}_{l_2}(\mathbf{1}_{H_2}) B^{R_2,\mu_2}_{j_2}(\mathbf{1}_{H_2}) \\
&= \delta_{R_1R_2} \sum_{\sigma,\sigma_2} \delta_{i_1i_2}(\sigma_2\sigma^{-1}\otimes\sigma_2)\delta_{R_1,R_2} D^{R_1}_{l_1l_2}(\sigma) B^{R_1,\mu_1}_{l_1}(\mathbf{1}_{H_1}) B^{R_1,\mu_1}_{j_1}(\mathbf{1}_{H_1}) B^{R_2,\mu_2}_{l_2}(\mathbf{1}_{H_2}) B^{R_2,\mu_2}_{j_2}(\mathbf{1}_{H_2})
\end{aligned} \tag{B.4}$$

The sum over σ_2 is trivial, so it suffices to consider

$$\mathcal{O}^{R}_{j_1,j_2} = \sum_{\sigma\in S_n} \sigma \quad D^{R}_{l_1l_2}(\sigma) B^{R,\mu_1}_{l_1}(\mathbf{1}_{H_1}) B^{R,\mu_1}_{j_1}(\mathbf{1}_{H_1}) B^{R,\mu_2}_{l_2}(\mathbf{1}_{H_2}) B^{R,\mu_2}_{j_2}(\mathbf{1}_{H_2}) \tag{B.5}$$

From this $\mathcal{O}^{R_1,R_2}_{i_1j_1;i_2j_2}$ is reconstructed by using $\delta_{R_1,R_2}\delta_{i_1i_2}$.

Now define an element in the group algebra of S_n labeled by multiplicities (ν_1,ν_2) of the identity irrep of (H_1,H_2) appearing in a decomposition of irrep R of S_n. Using orthogonality of the branching coefficients we have

$$\begin{aligned}
\mathcal{O}^{R}_{\nu_1,\nu_2} &\equiv \sum_{j_1,j_2} B^{R,\nu_1}_{j_1}(\mathbf{1}_{H_1}) B^{R,\nu_2}_{j_2}(\mathbf{1}_{H_1}) \mathcal{O}^{R}_{j_1,j_2} \\
&= \sum_{\sigma} \sigma D^{R}_{l_1l_2}(\sigma) B^{R,\nu_1}_{l_1}(\mathbf{1}_{H_1}) B^{R,\nu_2}_{l_2}(\mathbf{1}_{H_2})
\end{aligned} \tag{B.6}$$

So we see that a complete set of functions on the coset

$$S_n \setminus (S_n \times S_n)/(H_1 \times H_2) \tag{B.7}$$

can be labelled by the representations R of S_n which contain the trivial of H_1 and of H_2.

We have a complete set of elements in the group algebra of $(S_n \times S_n)$, which are invariant under the left action of S_n and the right action of $H_1 \times H_2$.

This can be viewed as a derivation of the formula for Feynman graph counting in terms of multiplicities given in formulas (7.4) and (5.25).

B.1 QED counting in terms of representation theory

To count the number of QED vacuum graphs with $2v$ vertices we need to evaluate

$$\begin{aligned}
F_{QED}(2v) \quad &= \frac{1}{2^v v!} \sum_{\sigma \in S_v[S_2]} \sum_{\gamma \in S_{2v}} \delta(\sigma\gamma\sigma\gamma^{-1}\sigma) \\
&= \frac{1}{(2v)!}\frac{1}{2^v v!} \sum_{R \vdash 2v} \sum_{\sigma \in S_v[S_2]} \sum_{\gamma \in S_{2v}} d_R \chi_R(\sigma\gamma\sigma^{-1}\gamma^{-1}) \\
&= \frac{1}{(2v)!}\frac{1}{2^v v!} \sum_{R \vdash 2v} \sum_{\sigma \in S_v[S_2]} \sum_{\gamma \in S_{2v}} d_R D^R_{ij}(\sigma) D^R_{jk}(\gamma) D^R_{kl}(\sigma^{-1}) D^R_{li}(\gamma^{-1}) \\
&= \frac{1}{2^v v!} \sum_{R \vdash 2v} \sum_{\sigma \in S_v[S_2]} \chi_R(\sigma)\chi_R(\sigma)
\end{aligned} \tag{B.8}$$

We have used the orthogonality of matrix elements

$$\sum_{\gamma \in S_{2v}} D^R_{jk}(\gamma) D^R_{li}(\gamma^{-1}) = \frac{(2v)!}{d_R} \delta_{kl}\delta_{ji} \tag{B.9}$$

The product of characters can be be expanded using the Clebsch-Gordan (inner-product) multiplicities. $C(R,R,\Lambda)$ is the number of times the representation Λ of S_{2v} appears in the tensor product $R \times R$, when this is decomposed in terms of the diagonal S_{2v}. Thus

$$\begin{aligned}
F_{QED}(2v) \quad &= \frac{1}{2^v v!} \sum_{R \vdash 2v} \sum_{\sigma \in S_v[S_2]} C(R,R,\Lambda)\chi_\Lambda(\sigma) \\
&= \sum_{R \vdash 2v} C(R,R,\Lambda) \mathcal{M}^{\Lambda}_{\mathbf{1}_{S_v[S_2]}}
\end{aligned} \tag{B.10}$$

The multiplicity $\mathcal{M}^{\Lambda}_{\mathbf{1}_{S_v[S_2]}}$ is the number of times the identity representation of $S_v[S_2]$ appears when the representation Λ of S_{2v} is decomposed into the irreducible representations of the subgroup $S_v[S_2]$.

Similar manipulations in the case of Furry QED leads to expressions involving, not Clebsch-Gordan multiplicities, but Clebsch-Gordan coefficients, along with the matrix elements for $\sum_{\tau even} \tau \otimes \tau$ in $R \otimes R$.

C Feynman graphs with GAP

We have explained different points of view on the enumeration of Feynman graphs, perhaps the most intuitive and useful is that Feynman graph is an orbit of the vertex symmetry group acting on the Wick contractions. For the case of ϕ^4 theory, we have the wreath product $S_v[S_4]$ acting on the conjugacy class of permutations in S_{4v} consisting of 2-cycles, which we denoted $[2^{2v}]$. Calculations with this formulation are easy to implement directly in the GAP software for group theoretic computations [22].

C.1 Vacuum graphs of ϕ^4

We illustrate with the sequence $1, 3, 7, 20, 56...$ of vacuum Feynman graphs in ϕ^4 theory. The count of 3 for $v = 2$ vertices can be obtained from GAP using the commands.

```
gap> C := ConjugacyClass( SymmetricGroup(8), ( 1,2) (3,4) (5,6) ( 7,8) ) ;;
gap> G := WreathProduct ( SymmetricGroup(4), SymmetricGroup(2) ) ;;
gap> Length(OrbitsDomain ( G , C ) ) ;
[ 3
```

This directly implements the formulation of vacuum Feynman graphs as orbits of $S_v[S_4]$ on the conjugacy class $[2^{2v}]$ described in section 4. The numbers $7, 20$ can also be recovered without much trouble with this method. For the purposes of just getting the number of vacuum graphs, this method is an overkill. Better methods with cycle indices are described in Section 5. We found the use of GAP to be very useful as a tool to check in formulating the correct group theoretical formulations of various Feynman graph counting problems. The command OrbitsDomain (G , C) actually gives a nested list of lists of pairings. The list runs over inequivalent Feynman graphs. For each Feynman graph, there is a list of different Wick contractions which lead to the given graph. Hence these orbits encode not just the number of Feynman graphs, but the Feynman graphs themselves.

C.2 ϕ^4 with external edges

It is a small generalization to consider graphs with external edges. For ϕ^4 theory graphs with E external edges we act with the wreath product $S_v[S_4]$ on the conjugacy class $[2^{2v+\frac{E}{2}}]$ of S_{4v+2E}. The relevant sequence for $E = 2$ is $1, 2, 7, 23, 85, 340, ...$ The count of 7 for $v = 2$ and $E = 2$ can be obtained from GAP as follows

```
gap> C := ConjugacyClass( SymmetricGroup(10), ( 1,2) (3,4) (5,6) (7,8) (9,10) ) ;;
gap> G := WreathProduct ( SymmetricGroup(4), SymmetricGroup(2) ) ;;
gap> Length(OrbitsDomain ( G , C ) ) ;
[ 7
```

Comments made above are again applicable: there are better methods to count these graphs which use cycle indices.

C.3 Symmetry factor

The symmetry factor of a given Feynman graph can be constructed using GAP. The following commands compute the symmetry factor of the diagram in Figure 22.

```
gap> H1 := WreathProduct( SymmetricGroup(4) , SymmetricGroup(3) ) ;
gap> Sig1 := (1,10) ( 4,11) (3,8) (2,5) (6,9) ( 7,12) ;
gap> for g in H1 do
         if OnPoints( Sig1 , g ) := Sig1 then
        Countsym := Countsym +1 ; fi ; od ; Countsym ;
[ 48
```

Simple modifications of these commands can construct the group, study its subgroups etc. , but these will have no immediate interest for us.

C.4 Action of S_v on a set

An alternative way to code the graphs is to label the vertices $\{1, \cdots, v\}$. Consider lists of $2v$ unordered pairs.

$$\{(i_1, j_i), (i_2, j_2)...(i_{2v}, j_{2v})\} \tag{C.1}$$

Put the constraint that

$$\sum_{k=1}^{2v} \delta(i, i_k) + \delta(i, j_k) = 4 \tag{C.2}$$

for all i from $1, ..v$. This imposes the condition that all vertices are 4-valent.

There is an action of S_v on this set. We are interested in the orbits. This can be programmed in GAP and gives an alternative method to get the sequence of vacuum Feynman graphs in ϕ^4 theory.

The method of using S_v is more efficient in generating the whole set of Feynman graphs with GAP, than the $S_v[S_4]$ method. But it does not automatically have the edge symmetries, unlike the $S_v[S_4]$ approach. It only gives the automorphism G_V discussed in Appendix A.

D Integer sequences

In this Appendix we collect the numerical results we have obtained by counting Feynman graphs.

- The number of vacuum graphs in ϕ^4 theory, with v vertices, starting with $v = 1$, is given by the sequence

$$1, 3, 7, 20, 56, 187, 654, 2705, 12587, 67902, 417065, 2897432, 22382255, 189930004, 1750561160, .. \tag{D.1}$$

  This sequence is listed in "The On-Line Encyclopedia of Integer Sequences" [23], where it is described as the sequence of 4-regular multi-graphs (loops allowed). This sequence is derived using (5.21) or (6.4).

- The number of connected vacuum graphs in ϕ^4 theory with v vertices, starting with $v = 1$, is given by the sequence

$$1,\ 2,\ 4,\ 10,\ 28,\ 97,\ 359,\ 1635,\ 8296,\ 48432,\ 316520,\ 2305104, 18428254, ... \tag{D.2}$$

  This sequence was generated using (6.5).

- The number of graphs in ϕ^4 theory with $E = 2$ external legs, with v vertices, starting with $v = 0$, is given by the sequence

$$1,\ 2,\ 7,\ 23,\ 85,\ 340,\ 1517,\ 7489,\ 41276,\ 252410,\ 1706071,\ 12660012, 102447112, ... \tag{D.3}$$

  This sequence is generated by the formula (6.8) with $E = 2$.

- The number of connected graphs in ϕ^4 theory with $E = 2$ external legs, with v vertices, starting with $v = 0$, is given by the sequence

$$1,\ 1,\ 3,\ 10,\ 39,\ 174,\ 853,\ 4632,\ 27607,\ 180148,\ 1281437,\ 9896652, 82610706, ... \tag{D.4}$$

  This sequence is generated by the formula (6.9). These graphs have been tabulated in [56]. Comparing the Table II of [56], we find a match between the number of diagrams at first and second order in perturbation theory. At third order in perturbation theory we have 10 graphs compared to the 8 graphs listed in [56]. The two graphs not listed in [56] are obtained from graph # 5.3 by swapping the position of the closed loop and setting sun, and from # 7.2 by swapping the order of the double bubble and the bubble. At fourth order in perturbation theory, there are 30 graphs listed in [56] compared to our count of 39. However, there are 9 graphs that are obtained from graphs #11.2, #11.4, #12.3, #14.2, #14.3, #15.1, #13.3, #15.5 and #17.2, in much the same way that we described for third order in perturbation theory.

- The number of graphs in ϕ^4 theory with $E = 4$ external legs, with v vertices, starting with $v = 0$, is given by the sequence

$$3,\ 10,\ 44,\ 190,\ 889,\ 4490,\ 24736,\ 148722,\ 976427,\ 6980529,\ 54151689,\ 453922676, ... \tag{D.5}$$

  This sequence is generated by the formula (6.8) with $E = 4$.

- The number of vacuum graphs for ϕ^3 theory, with $2v$ vertices, starting from $v = 1$

$$2,\ 8,\ 31,\ 140,\ 722,\ 4439,\ 32654,\ 289519, ... \tag{D.6}$$

  This sequence was generated using (6.12).

- The number of connected vacuum graphs in ϕ^3 theory, with $2v$ vertices, starting from $v = 1$

$$2,\ 5,\ 17,\ 71,\ 388,\ 2592,\ 21096,\ 204638, ... \tag{D.7}$$

  This sequence was generated using (6.5).

- The number of graphs in ϕ^3 theory with $E = 2$ external legs, with $2v$ vertices starting from $v = 1$, is

$$5,\ 30,\ 186,\ 1276,\ 9828,\ 86279,\ 866474,\ 9924846, ... \tag{D.8}$$

  This sequence was generated using (6.15) with $E = 2$.

- The number of QED/Yukawa vacuum graphs with $2v$ vertices, which equals the total number of ribbon graphs with $2v$ edges is given by

$$2,\ 8,\ 34,\ 182,\ 1300,\ 12634,\ 153590,\ 2230979,\ 37250144, ... \tag{D.9}$$

  This sequence was generated using (8.27).

- The number of connected QED/Yukawa vacuum graphs with $2v$ vertices, starting from $v = 1$, is given by

$$2,\ 5,\ 20,\ 107,\ 870,\ 9436,\ 122832,\ 1863350,\ 32019816, ... \tag{D.10}$$

  This sequence was generated using (6.5).

- Vacuum graphs in QED after implementing the constraint due to Furry's theorem.

$$1,\ 4,\ 12,\ 57,\ 321,\ 2816,\ 31092,\ 423947,\ldots \tag{D.11}$$

This sequence was generated using (9.13).

- Connected Vacuum graphs in QED after implementing the constraint due to Furry's theorem.

$$1,\ 3,\ 8,\ 39,\ 240,\ 2332,\ 27196,\ 382,\ 802,\ldots \tag{D.12}$$

This sequence was generated using (6.5). The $1, 3, 8$ agree with the results of [43].

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