Enumeration of permutations by number of alternating runs*

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Abstract

Let $R(n,k)$ denote the number of permutations of $\{1,2,\ldots,n\}$ with k alternating runs. We find a grammatical description of the numbers $R(n,k)$ and then present several convolution formulas involving the generating function for the numbers $R(n,k)$.

Keywords: Alternating runs; Longest alternating subsequences; Context-free grammars

1 Introduction

Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathcal{S}_n$. We say that π changes direction at position i if either $\pi(i-1)<\pi(i)>\pi(i+1)$, or $\pi(i-1)>\pi(i)<\pi(i+1)$, where $i\in\{2,3,\ldots,n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Let $R(n,k)$ denote the number of permutations in $\mathcal{S}_n$ with k alternating runs. There is a large literature devoted to the numbers $R(n,k)$. The reader is referred to [2, 3, 14, 17] for recent results on this subject.

André [1] was the first to study the alternating runs of permutations and he obtained the following recurrence

$$R(n,k)=kR(n-1,k)+2R(n-1,k-1)+(n-k)R(n-1,k-2) \tag{1}$$

for $n,k\geq 1$, where $R(1,0)=1$ and $R(1,k)=0$ for $k\geq 1$. For $n\geq 1$, we define $R_n(x)=\sum_{k=1}^{n-1}R(n,k)x^k$. Then the recurrence (1) induce

$$R_{n+2}(x)=x(nx+2)R_{n+1}(x)+x\left(1-x^2\right)R'_{n+1}(x), \tag{2}$$

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with initial value $R_1(x) = 1$. The first few terms of $R_n(x)$'s are given as follows:

$$\begin{aligned}
R_2(x) &= 2x,\\
R_3(x) &= 2x + 4x^2,\\
R_4(x) &= 2x + 12x^2 + 10x^3,\\
R_5(x) &= 2x + 28x^2 + 58x^3 + 32x^4.
\end{aligned}$$

The *Eulerian number* $\left\langle {n \atop k} \right\rangle$ enumerates the number of permutations in $\mathcal{S}_n$ with k descents (i.e., $1 \leq i < n, \pi(i) > \pi(i+1)$). Let $A_n(x) = x\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$ be the *Eulerian polynomials*. The polynomial $R_n(x)$ is closely related to $A_n(x)$:

$$R_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right), \quad w = \sqrt{\frac{1-x}{1+x}},$$

which was first established by David and Barton [8, 157-162] and then stated more concisely by Knuth [12, p. 605]. In a series of papers [4, 5, 6], Carlitz studied the generating functions for the numbers $R(n,k)$. In particular, Carlitz [4] proved that

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2. \tag{3}$$

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. An *interior peak* in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. Let $\mathrm{pk}\,(\pi)$ denote *the number of interior peaks* in π. An *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\mathrm{pk}^l\,(\pi)$ denote *the number of left peaks* in π. For example, the permutation $\pi = 21435$ has $\mathrm{pk}\,(\pi) = 1$ and $\mathrm{pk}^l\,(\pi) = 2$. Let $W(n,k)$ denote the number of permutations in $\mathcal{S}_n$ with k interior peaks, and let $\widetilde{W}(n,k)$ denote the number of permutations in $\mathcal{S}_n$ with k left peaks. For $n \geq 1$, we define

$$W_n(x) = \sum_{k\geq 0} W(n,k)x^k \quad \text{and} \quad \widetilde{W}_n(x) = \sum_{k\geq 0}\widetilde{W}(n,k)x^k.$$

It is well known that the polynomials $W_n(x)$ and $\widetilde{W}_n(x)$ respectively satisfy

$$W_{n+1}(x) = (nx - x + 2)W_n(x) + 2x(1-x)W_n'(x),$$

with initial values $W_1(x) = 1, W_2(x) = 2$ and $W_3(x) = 4 + 2x$, and

$$\widetilde{W}_{n+1}(x)(x) = (nx+1)\widetilde{W}_n(x) + 2x(1-x)\widetilde{W}_n'(x)(x),$$

with initial values $\widetilde{W}_0(x) = \widetilde{W}_1(x) = 1, \widetilde{W}_2(x) = 1+x$ and $\widetilde{W}_3(x) = 1+5x$ (see [16, A008303,A008971]).

For $n \geq 0$, we define

$$P_n(\tan\theta) = \frac{d^n}{d\theta^n}\tan\theta.$$

It is clear that $P_0(x) = x$ and $P_{n+1}(x) = (1+x^2)P_n'(x)$ (see [11]). The following result was obtained in [13, 14].

Theorem 1. *For $n \geq 2$, we have*

$$W_n(x) = \frac{1}{x}(x-1)^{\frac{n+1}{2}} P_n\left(\frac{1}{\sqrt{x-1}}\right),$$

and

$$R_n(x) = \left(\frac{x+1}{2}\right)^{n-1} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}(n+1)} P_n\left(\sqrt{\frac{x+1}{x-1}}\right),$$

So the following corollary is immediate.

Corollary 2. *For $n \geq 2$, we have*

$$R_n(x) = \frac{x(1+x)^{n-2}}{2^{n-2}} W_n\left(\frac{2x}{1+x}\right). \tag{4}$$

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. An *alternating subsequence* of length k is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).$$

Let $\text{as}\,(\pi)$ denote the length of the longest alternating subsequence of π. For $n \geq 1$, we define

$$a_k(n) = \#\{\pi \in \mathcal{S}_n : \text{as}\,(\pi) = k\}.$$

Recently, Stanley [17] initiated a study of the distribution of the length of the longest alternating subsequences of π. Put

$$F_k(x) = \sum_{n\geq 0} a_k(n)\frac{x^n}{n!} \quad \text{and} \quad A(x,t) = \sum_{k\geq 0} F_k(x)t^k.$$

Stanley [17, Theorem 2.3] showed that

$$A(x,t) = (1-t)\frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}},$$

where $\rho = \sqrt{1-t^2}$. Let $T_n(x) = \sum_{k=1}^n a_k(n)x^k$. Bóna [17, Section 4] observed an important identity:

$$T_n(x) = \frac{1}{2}(1+x)R_n(x) \quad \text{for} \quad n \geq 2. \tag{5}$$

Then using (2), the polynomials $T_n(x)$ satisfy the recurrence

$$T_{n+1}(x) = x(nx+1)T_n(x) + x\left(1-x^2\right)T_n'(x).$$

with initial values $T_0(x) = 1$ and $T_1(x) = x$. Therefore,

$$a_k(n) = ka_k(n-1) + a_{k-1}(n-1) + (n-k+1)a_{k-2}(n-1), \tag{6}$$

with initial values $a_0(0) = a_1(1) = 1$ and $a_k(0) = a_0(n) = 0$ for $n, k \geq 1$. Combining (4) and (5), we immediately get

$$T_n(x) = \frac{x(1+x)^{n-1}}{2^{n-1}} W_n\left(\frac{2x}{1+x}\right) \quad \text{for} \quad n \geq 1.$$

In this paper we further explore connections between the polynomials $R_n(x)$ and $T_n(x)$. In Section 2, we present a description of the numbers $R(n,k)$ and $T(n,k)$ using the notion of context-free grammars. As applications, we obtain several convolution formulas involving the polynomials $R_n(x)$ and $T_n(x)$.

2 Context-free grammars

The grammatical method was introduced by Chen [7] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. Following Chen [7], a *context-free grammar* G over A is defined as a set of substitution rules replacing a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. For any formal functions u and v, we have

$$D(u+v)=D(u)+D(v), \quad D(uv)=D(u)v+uD(v) \quad \textit{and} \quad D(f(u))=\frac{\partial f(u)}{\partial u}D(u),$$

where $f(x)$ is a analytic function. Using Leibniz's formula, we have

$$D^n(uv)=\sum_{k=0}^{n}\binom{n}{k}D^k(u)D^{n-k}(v). \tag{7}$$

It is well know that many combinatorial objects permit a description using the notion of context-free grammars. For example, Schett [15] considered the grammar

$$G=\{x\rightarrow yz, y\rightarrow xz, z\rightarrow xy\},$$

and established a relationship between the expansion of $D^n(x)$ and the Jacobi elliptic functions. Dumont [9] established the connections between Schett's grammar and permutations. In [10], Dumont considered chains of general substitution rules on words. Let us now recall two results on context-free grammars.

Proposition 3 ([10, Section 2.1])**.** *If* $G=\{x\rightarrow xy, y\rightarrow xy\}$, *then*

$$D^n(x)=x\sum_{k=0}^{n-1}\left\langle {n \atop k}\right\rangle x^k y^{n-k} \quad \textit{for} \quad n\geq 1.$$

Proposition 4 ([13])**.** *If* $G=\{y\rightarrow yz, z\rightarrow y^2\}$, *then*

$$D^n(y)=\sum_{k=0}^{\lfloor n/2\rfloor}\widetilde{W}(n,k)y^{2k+1}z^{n-2k} \quad \textit{for} \quad n\geq 0,$$

and

$$D^n(z)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}W(n,k)y^{2k+2}z^{n-2k-1} \quad \textit{for} \quad n\geq 1.$$

As a conjunction of Proposition 3 and Proposition 4, it is natural to consider the context-free grammar

$$G=\{x\rightarrow xy, y\rightarrow yz, z\rightarrow y^2\}. \tag{8}$$

Thus $D(x)=xy, D(y)=yz$ and $D(z)=y^2$. In the following discussion we will consider the grammar (8). For convenience, we will always assume that $n\geq 1$. The main result of this paper is the following.

Theorem 5. *If $G = \{x \to xy, y \to yz, z \to y^2\}$, then*

$$D^n(x^2) = x^2 \sum_{k=1}^{n} R(n+1,k) y^k z^{n-k}, \tag{9}$$

and

$$D^n(x) = x \sum_{k=1}^{n} a_k(n) y^k z^{n-k}. \tag{10}$$

Proof. Note that $D(x^2) = 2x^2y$ and $D^2(x^2) = 2x^2yz + 4x^2y^2$. For $n \geq 1$, we define

$$D^n(x^2) = x^2 \sum_{k=1}^{n} M(n+1,k) y^k z^{n-k} \tag{11}$$

Then $M(2,1) = R(2,1) = 2, M(3,1) = R(3,1) = 2$ and $M(3,2) = R(3,2) = 4$. From (11), we get

$$\begin{aligned} D^{n+1}(x^2) &= D(D^n(x^2)) \\ &= x^2 \sum_{k=1}^{n} kM(n+1,k) y^k z^{n-k+1} + 2x^2 \sum_{k=1}^{n} M(n+1,k) y^{k+1} z^{n-k} \\ &+ x^2 \sum_{k=1}^{n} (n-k) M(n+1,k) y^{k+2} z^{n-k-1}. \end{aligned}$$

Thus

$$M(n+2,k) = kM(n+1,k) + 2M(n+1,k-1) + (n-k+2)M(n+1,k-2).$$

Comparing with (1), we see that the coefficients $M(n,k)$ satisfy the same recurrence relation and initial conditions as $R(n,k)$, so they agree. Using (6), the formula (10) can be proved in a similar way and we omit the proof for brevity. □

Note that $D(xy) = D(xz) = xyz + xy^2$. Hence $D^n(xy) = D^n(xz)$. Moreover,

$$D^{n+1}(x^2) = D^n(2x^2y).$$

Then from (7), we obtain

$$D^n(x^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x),$$

$$D^{n+1}(x) = D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(y),$$

$$D^n(xz) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(z),$$

$$D^n(2x^2y) = 2D^n(x^2y) = 2\sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(xy) = 2\sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k+1}(x),$$

$$D^n(2x^2y) = 2\sum_{k=0}^{n} \binom{n}{k} D^k(x^2) D^{n-k}(y) = 2x^2 D^n(y) + 2\sum_{k=1}^{n} \binom{n}{k} D^{k-1}(2x^2y) D^{n-k}(y).$$

Thus we can immediately use Proposition 4 and Theorem 5 to get several convolution formulas.

Corollary 6. *For $n \geq 1$, we have*

$$R_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) T_{n-k}(x), \tag{12}$$

$$R_{n+2}(x) = 2 \sum_{k=0}^{n} \binom{n}{k} T_k(x) T_{n-k+1}(x),$$

$$R_{n+2}(x) = 2x\widetilde{W}_n(x^2) + 2x \sum_{k=1}^{n} \binom{n}{k} R_{k+1}(x) \widetilde{W}_{n-k}(x^2),$$

$$T_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} T_k(x) \widetilde{W}_{n-k}(x^2),$$

$$T_{n+1}(x) = T_n(x) + x^2 \sum_{k=0}^{n-1} \binom{n}{k} T_k(x) W_{n-k}(x^2).$$

Combining (3) and (12), it is easy to verify that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} a_k(n) x^{n-k} = -\sqrt{\frac{1-x}{1+x}} \left(\frac{\sqrt{1-x^2} + \sin(z\sqrt{1-x^2})}{x - \cos(z\sqrt{1-x^2})} \right).$$

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