# ROBIN'S THEOREM, PRIMES, AND A NEW ELEMENTARY REFORMULATION OF THE RIEMANN HYPOTHESIS 

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Let Abstract
Let

$$
G(n)=\frac{\sigma(n)}{n \log \log n} \quad(n>1)
$$

where $\sigma(n)$ is the sum of the divisors of $n$. We prove that the Riemann Hypothesis is true if and only if 4 is the only composite number $N$ satisfying

$$
G(N) \geq \max (G(N / p), G(a N))
$$

for all prime factors $p$ of $N$ and each positive integer $a$. The proof uses Robin's and Gronwall's theorems on $G(n)$. An alternate proof of one step depends on two properties of superabundant numbers proved using Alaoglu and Erdős's results.

## 1. Introduction

The sum-of-divisors function $\sigma$ is defined by

$$
\sigma(n):=\sum_{d \mid n} d
$$

For example, $\sigma(4)=7$ and $\sigma(p n)=(p+1) \sigma(n)$, if $p$ is a prime not dividing $n$.
In 1913, the Swedish mathematician Thomas Gronwall [5] found the maximal order of $\sigma$.

Theorem 1 (Gronwall). The function

$$
G(n):=\frac{\sigma(n)}{n \log \log n} \quad(n>1)
$$

satisfies

$$
\limsup _{n \rightarrow \infty} G(n)=e^{\gamma}=1.78107 \ldots,
$$

where $\gamma$ is the Euler-Mascheroni constant.
Here $\gamma$ is defined as the limit

$$
\gamma:=\lim _{n \rightarrow \infty}\left(H_{n}-\log n\right)=0.57721 \ldots,
$$

where $H_{n}$ denotes the $n t h$ harmonic number

$$
H_{n}:=\sum_{j=1}^{n} \frac{1}{j}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} .
$$

Gronwall's proof uses Mertens's theorem [6, Theorem 429], which says that if $p$ denotes a prime, then

$$
\lim _{x \rightarrow \infty} \frac{1}{\log x} \prod_{p \leq x}\left(1-\frac{1}{p}\right)^{-1}=e^{\gamma}
$$

Since $\sigma(n)>n$ for all $n>1$, Gronwall's theorem "shows that the order of $\sigma(n)$ is always 'very nearly' $n$ " (Hardy and Wright [6, p. 350]).

In 1915, the Indian mathematical genius Srinivasa Ramanujan proved an asymptotic inequality for Gronwall's function $G$, assuming the Riemann Hypothesis (RH). (Ramanujan's result was not published until much later [9] for the interesting reasons, see [9, pp. 119-121] and [7, pp. 537-538].)

Theorem 2 (Ramanujan). If the Riemann Hypothesis is true, then

$$
G(n)<e^{\gamma} \quad(n \gg 1) .
$$

Here $n \gg 1$ means for all sufficiently large $n$.
In 1984, the French mathematician Guy Robin [10] proved that a stronger statement about the function $G$ is equivalent to the RH .

Theorem 3 (Robin). The Riemann Hypothesis is true if and only if

$$
\begin{equation*}
G(n)<e^{\gamma} \quad(n>5040) . \tag{1}
\end{equation*}
$$

| $r$ | SA | Factorization | $\sigma(r)$ | $\sigma(r) / r$ | $G(r)$ | $p(r)$ | $G(11 r)$ |
| ---: | ---: | :--- | ---: | :---: | ---: | :---: | :---: |
| 3 |  | 3 | 4 | 1.333 | 14.177 |  | 1.161 |
| 4 | $\checkmark$ | $2^{2}$ | 7 | 1.750 | 5.357 |  | 1.434 |
| 5 |  | 5 | 6 | 1.200 | 2.521 |  | 0.943 |
| 6 | $\checkmark$ | $2 \cdot 3$ | 12 | 2.000 | 3.429 | 2 | 1.522 |
| 8 |  | $2^{3}$ | 15 | 1.875 | 2.561 | 2 | 1.364 |
| 9 |  | $3^{2}$ | 13 | 1.444 | 1.834 | 3 | 1.033 |
| 10 |  | $2 \cdot 5$ | 18 | 1.800 | 2.158 | 2 | 1.268 |
| 12 | $\checkmark$ | $2^{2} \cdot 3$ | 28 | 2.333 | 2.563 | 2 | 1.605 |
| 16 |  | $2^{4}$ | 31 | 1.937 | 1.899 | 2 | 1.286 |
| 18 |  | $2 \cdot 3^{2}$ | 39 | 2.166 | 2.041 | 3 | 1.419 |
| 20 |  | $2^{2} \cdot 5$ | 42 | 2.100 | 1.913 | 5 | 1.359 |
| 24 | $\checkmark$ | $2^{3} \cdot 3$ | 60 | 2.500 | 2.162 | 3 | 1.587 |
| 30 |  | $2 \cdot 3 \cdot 5$ | 72 | 2.400 | 1.960 | 3 | 1.489 |
| 36 | $\checkmark$ | $2^{2} \cdot 3^{2}$ | 91 | 2.527 | 1.980 | 2 | 1.541 |
| 48 | $\checkmark$ | $2^{4} \cdot 3$ | 124 | 2.583 | 1.908 | 3 | 1.535 |
| 60 | $\checkmark$ | $2^{2} \cdot 3 \cdot 5$ | 168 | 2.800 | 1.986 | 5 | 1.632 |
| 72 |  | $2^{3} \cdot 3^{2}$ | 195 | 2.708 | 1.863 | 3 | 1.556 |
| 84 |  | $2^{2} \cdot 3 \cdot 7$ | 224 | 2.666 | 1.791 | 7 | 1.514 |
| 120 | $\checkmark$ | $2^{3} \cdot 3 \cdot 5$ | 360 | 3.000 | 1.915 | 2 | 1.659 |
| 180 | $\checkmark$ | $2^{2} \cdot 3^{2} \cdot 5$ | 546 | 3.033 | 1.841 | 5 | 1.632 |
| 240 | $\checkmark$ | $2^{4} \cdot 3 \cdot 5$ | 744 | 3.100 | 1.822 | 5 | 1.638 |
| 360 | $\checkmark$ | $2^{3} \cdot 3^{2} \cdot 5$ | 1170 | 3.250 | 1.833 | 5 | 1.676 |
| 720 | $\checkmark$ | $2^{4} \cdot 3^{2} \cdot 5$ | 2418 | 3.358 | 1.782 | 3 | 1.669 |
| 840 | $\checkmark$ | $2^{3} \cdot 3 \cdot 5 \cdot 7$ | 2880 | 3.428 | 1.797 | 7 | 1.691 |
| 2520 | $\checkmark$ | $2^{3} \cdot 3^{2} \cdot 5 \cdot 7$ | 9360 | 3.714 | 1.804 | 7 | 1.742 |
| 5040 | $\checkmark$ | $2^{4} \cdot 3^{2} \cdot 5 \cdot 7$ | 19344 | 3.838 | 1.790 | 2 | 1.751 |

Table 1: The set $R$ of all known numbers $r$ for which $G(r) \geq e^{\gamma}$. (Section 1 defines SA, $\sigma(r)$, and $G(r)$; Section 2 defines $p(r)$.)

The condition (1) is called Robin's inequality. Table 1 gives the known numbers $r$ for which the reverse inequality $G(r) \geq e^{\gamma}$ holds, together with the value of $G(r)$ (truncated).

Robin's statement is elementary, and his theorem is beautiful and elegant, and is certainly quite an achievement.

In 10 Robin also proved, unconditionally, that

$$
\begin{equation*}
G(n)<e^{\gamma}+\frac{0.6483}{(\log \log n)^{2}} \quad(n>1) \tag{2}
\end{equation*}
$$

This refines the inequality $\lim \sup _{n \rightarrow \infty} G(n) \leq e^{\gamma}$ from Gronwall's theorem.
In 2002, the American mathematician Jeffrey Lagarias [7] used Robin's theorem to give another elementary reformulation of the RH.

Theorem 4 (Lagarias). The Riemann Hypothesis is true if and only if

$$
\sigma(n)<H_{n}+\exp \left(H_{n}\right) \log \left(H_{n}\right) \quad(n>1)
$$

Lagarias's theorem is also a beautiful, elegant, and remarkable achievement. It improves upon Robin's statement in that it does not require the condition $n>5040$, which appears arbitrary. It also differs from Robin's statement in that it relies explicitly on the harmonic numbers $H_{n}$ rather than on the constant $\gamma$.

Lagarias [8] also proved, unconditionally, that

$$
\sigma(n)<H_{n}+2 \exp \left(H_{n}\right) \log \left(H_{n}\right) \quad(n>1)
$$

The present note uses Robin's results to derive another reformulation of the RH. Before stating it, we give a definition and an example.

Definition 1. A positive integer $N$ is extraordinary if $N$ is composite and satisfies
(i). $G(N) \geq G(N / p)$ for all prime factors $p$ of $N$, and
(ii). $G(N) \geq G(a N)$ for all multiples $a N$ of $N$.

The smallest extraordinary number is $N=4$. To show this, we first compute $G(4)=5.357 \ldots$ Then as $G(2)<0$, condition (i) holds, and since Robin's unconditional bound (2) implies

$$
G(n)<e^{\gamma}+\frac{0.6483}{(\log \log 5)^{2}}=4.643 \ldots<G(4) \quad(n \geq 5)
$$

condition (ii) holds a fortiori.
No other extraordinary number is known, for a good reason.
Theorem 5. The Riemann Hypothesis is true if and only if 4 is the only extraordinary number.

This statement is elementary and involves prime numbers (via the definition of an extraordinary number) but not the constant $\gamma$ or the harmonic numbers $H_{n}$, which are difficult to calculate and work with for large values of $n$. On the other hand, to disprove the RH using Robin's or Lagarias's statement would require only a calculation on a certain number $n$, while using ours would require a proof for a certain number $N$.

Here is a near miss. One can check that the number

$$
\begin{equation*}
\nu:=183783600=2^{4} \cdot 3^{3} \cdot 5^{2} \cdot 7 \cdot 11 \cdot 13 \cdot 17 \tag{3}
\end{equation*}
$$

satisfies condition (i), that is, $G(\nu) \geq G(\nu / p)$ for $p=2,3,5,7,11,13,17$. However, $\nu$ is not extraordinary, because $G(\nu)<G(19 \nu)$. Thus 183783600 is not quite a counterexample to the RH!

In [9, Section 59] Ramanujan introduced the notion of a "generalized highly composite number." The terminology was changed to "superabundant number" by the Canadian-American mathematician Leonidas Alaoglu and the Hungarian mathematician Paul Erdős [2].

Definition 2 (Ramanujan and Alaoglu-Erdős). A positive integer $s$ is superabundant (SA) if

$$
\frac{\sigma(n)}{n}<\frac{\sigma(s)}{s} \quad(0<n<s)
$$

For example, the numbers 1,2 , and 4 are SA, but 3 is not SA, because

$$
\frac{\sigma(1)}{1}=1<\frac{\sigma(3)}{3}=\frac{4}{3}<\frac{\sigma(2)}{2}=\frac{3}{2}<\frac{\sigma(4)}{4}=\frac{7}{4} .
$$

For lists of SA numbers, see the links at [11, Sequence A004394] and the last table in [2]. The known SA numbers $s$ for which $G(s) \geq e^{\gamma}$ are indicated in the "SA" column of Table 1 Properties of SA numbers are given in [2, 3, 7, 9, Proposition 1 , and Section 4

As $\sigma(n) / n=G(n) \log \log n$, Gronwall's theorem yields $\lim \sup _{n \rightarrow \infty} \sigma(n) / n=\infty$, implying there exist infinitely many $S A$ numbers.

Let us compare Definition 2 with condition (i) in Definition 1 If $n<s$, then $\sigma(n) / n<\sigma(s) / s$ is a weaker inequality than $G(n)<G(s)$. On the other hand, condition (i) only requires $G(n) \leq G(N)$ for factors $n=N / p$, while Definition 2 requires $\sigma(n) / n<\sigma(s) / s$ for all $n<s$. In particular, the near miss (3) is the smallest $S A$ number greater than 4 that satisfies (i). For more on (i), see Section 5 ,

Amir Akbary and Zachary Friggstad [1] observed that, "If there is any counterexample to Robin's inequality, then the least such counterexample is a superabundant number." Combined with Robin's theorem, their result implies the RH is true if and only if $G(s)<e^{\gamma}$ for all $S A$ numbers $s>5040$.

Here is an analog for extraordinary numbers of Akbary and Friggstad's observation on SA numbers.

Corollary 1. If there is any counterexample to Robin's inequality, then the maximum $M:=\max \{G(n): n>5040\}$ exists and the least number $N>5040$ with $G(N)=M$ is extraordinary.

Using Gronwall's theorem and results of Alaoglu and Erdős, we prove two properties of SA numbers.

Proposition 1. Let $S$ denote the set of superabundant numbers.
SA1. We have

$$
\limsup _{s \in S} G(s)=e^{\gamma} .
$$

SA2. For any fixed positive integer $n_{0}$, every sufficiently large number $s \in S$ is a multiple of $n_{0}$.

The rest of the paper is organized as follows. The next section contains three lemmas about the function $G$; an alternate proof of the first uses Proposition 1 The lemmas are used in the proof of Theorem 5and Corollary [1, which is in Section 3. Proposition 1 is proved in Section 4 . Section 5 gives some first results about numbers satisfying condition (i) in Definition 1 .

We intend to return to the last subject in another paper [4, in which we will also study numbers satisfying condition (ii).

## 2. Three lemmas on the function $G$

The proof of Theorem 5 requires three lemmas. Their proofs are unconditional.
The first lemma generalizes Gronwall's theorem (the case $n_{0}=1$ ).
Lemma 1. If $n_{0}$ is any fixed positive integer, then $\underset{a \rightarrow \infty}{\limsup } G\left(a n_{0}\right)=e^{\gamma}$.
We give two proofs.
Proof 1. Theorem $\square$ implies $\limsup _{a \rightarrow \infty} G\left(a n_{0}\right) \leq e^{\gamma}$. The reverse inequality can be proved by adapting that part of the proof of Theorem [ in [6, Section 22.9]. Details are omitted.

Proof 2. The lemma follows immediately from Proposition 1 .
The remaining two lemmas give properties of the set $R$ of all known numbers $r$ for which $G(r) \geq e^{\gamma}$.

Lemma 2. Let $R$ denote the set

$$
\begin{aligned}
& \qquad \text { R:=\{rડ5040:G(r)} \begin{aligned}
& \text { If } r \in R\left.e^{\gamma}\right\} . \\
& \text { and } r>5 \text {, then } G(r)<G(r / p) \text {, for some prime factor } p \text { of } r .
\end{aligned}
\end{aligned}
$$

Proof. The numbers $r \in R$ and the values $G(r)$ are computed in Table 1 Assuming $G(r)<G(r / p)$ for some prime factor $p$ of $r$, denote the smallest such prime by

$$
p(r):=\min \{\text { prime } p \mid r: G(r / p)>G(r)\}
$$

Whenever $5<r \in R$, a value of $p(r)$ is exhibited in the " $p(r)$ " column of Table 1 . This proves the lemma.

Lemma 3. If $r \in R$ and $p \geq 11$ is prime, then $G(p r)<e^{\gamma}$.
Proof. Note that if $p>q$ are odd primes not dividing a number $n$, then

$$
G(p n)=\frac{\sigma(p n)}{p n \log \log p n}=\frac{p+1}{p} \frac{\sigma(n)}{n \log \log p n}<\frac{q+1}{q} \frac{\sigma(n)}{n \log \log q n}=G(q n) .
$$

Also, Table 1 shows that no prime $p \geq 11$ divides any number $r \in R$, and that $G(11 r)<1.76$ for all $r \in R$. As $1.76<e^{\gamma}$, we obtain $G(p r) \leq G(11 r)<e^{\gamma}$.

Note that the inequality $G(p n)<G(q n)$ and its proof remain valid for all primes $p>q$ not dividing $n$, if $n>1$, since then $\log \log q n \neq \log \log 2<0$ when $q=2$.

## 3. Proof of Theorem 5 and Corollary 1

We can now prove that our statement is equivalent to the RH.
Proof of Theorem 5 and Corollary 1. Assume $N \neq 4$ is an extraordinary number. Then condition (ii) and Lemma 1 imply $G(N) \geq e^{\gamma}$. Thus if $N \leq 5040$, then $N \in R$, but now since $N \neq 4$ is composite we have $N>5$, and Lemma 2 contradicts condition (i). Hence $N>5040$, and by Theorem 3 the RH is false.

Conversely, suppose the RH is false. Then from Theorems 1 and 3 we infer that the maximum

$$
\begin{equation*}
M:=\max \{G(n): n>5040\} \tag{4}
\end{equation*}
$$

exists and that $M \geq e^{\gamma}$. Set

$$
\begin{equation*}
N:=\min \{n>5040: G(n)=M\} \tag{5}
\end{equation*}
$$

and note that $G(N)=M \geq e^{\gamma}$. We show that $N$ is an extraordinary number.
First of all, $N$ is composite, because if $N$ is prime, then $\sigma(N)=1+N$ and $N>5040$ imply $G(N)<5041 /(5040 \log \log 5040)=0.46672 \ldots$, contradicting $G(N) \geq e^{\gamma}$.

Since (4) and (5) imply $G(N) \geq G(n)$ for all $n \geq N$, condition (ii) holds. To see that (i) also holds, let prime $p$ divide $N$ and set $r:=N / p$. In the case $r>5040$, as $r<N$ the minimality of $N$ implies $G(N)>G(r)$. Now consider the case $r \leq 5040$.

By computation, $G(n)<e^{\gamma}$ if $5041 \leq n \leq 35280$, so that $N>35280=7 \cdot 5040$ and hence $p \geq 11$. Now if $G(r) \geq e^{\gamma}$, implying $r \in R$, then Lemma 3 yields $e^{\gamma}>G(p r)=G(N)$, contradicting $G(N) \geq e^{\gamma}$. Hence $G(r)<e^{\gamma} \leq G(N)$. Thus in both cases $G(N)>G(r)=G(N / p)$, and so (i) holds. Therefore, $N \neq 4$ is extraordinary. This proves both the theorem and the corollary.

Remark 1. The proof shows that Theorem 5 and Corollary 1 remain valid if we replace the inequality in Definition (i) with the strict inequality $G(N)>G(N / p)$.

## 4. Proof of Proposition 1

We prove the two parts of Proposition 1 separately.
Proof of $S A 1$. It suffices to construct a sequence $s_{1}, s_{2}, \ldots \rightarrow \infty$ with $s_{k} \in S$ and $\limsup \operatorname{sum}_{k \rightarrow \infty} G\left(s_{k}\right) \geq e^{\gamma}$. By Theorem [1, there exist positive integers $\nu_{1}<\nu_{2}<\cdots$ with $\lim _{k \rightarrow \infty} G\left(\nu_{k}\right)=e^{\gamma}$. If $\nu_{k} \in S$, set $s_{k}:=\nu_{k}$. Now assume $\nu_{k} \notin S$, and set $s_{k}:=\max \left\{s \in S: s<\nu_{k}\right\}$. Then $\left\{s_{k}+1, s_{k}+2, \ldots, \nu_{k}\right\} \cap S=\emptyset$, and we deduce that there exists a number $r_{k} \leq s_{k}$ with $\sigma\left(r_{k}\right) / r_{k} \geq \sigma\left(\nu_{k}\right) / \nu_{k}$. As $s_{k} \in S$, we obtain $\sigma\left(s_{k}\right) / s_{k} \geq \sigma\left(\nu_{k}\right) / \nu_{k}$, implying $G\left(s_{k}\right)>G\left(\nu_{k}\right)$. Now since $\lim _{k \rightarrow \infty} \nu_{k}=\infty$ and $\# S=\infty$ imply $\lim _{k \rightarrow \infty} s_{k}=\infty$, we get $\lim \sup _{k \rightarrow \infty} G\left(s_{k}\right) \geq e^{\gamma}$, as desired.

Proof of SA2. We use the following three properties of a number $s \in S$, proved by Alaoglu and Erdős [2].

AE1. The exponents in the prime factorization of $s$ are non-increasing, that is, $s=2^{k_{2}} \cdot 3^{k_{3}} \cdot 5^{k_{5}} \cdots p^{k_{p}}$ with $k_{2} \geq k_{3} \geq k_{5} \geq \cdots \geq k_{p}$.
AE2. If $q<r$ are prime factors of $s$, then $\left|\left\lfloor k_{q} \frac{\log q}{\log r}\right\rfloor-k_{r}\right| \leq 1$.
AE 3 . If $q$ is any prime factor of $s$, then $q^{k_{q}}<2^{k_{2}+2}$.

To prove SA2, fix an integer $n_{0}>1$. Let $K$ denote the largest exponent in the prime factorization of $n_{0}$, and set $P:=P\left(n_{0}\right)$, where $P(n)$ denotes the largest prime factor of $n$. As $n_{0}$ divides $(2 \cdot 3 \cdot 5 \cdots P)^{K}$, by AE1 it suffices to show that the set

$$
F:=\left\{s \in S: s \text { is not divisible by } P^{K}\right\}=\left\{s \in S: 0 \leq k_{P}=k_{P}(s)<K\right\}
$$

is finite.
From AE2 with $q=2$ and $r=P$, we infer that $k_{2}=k_{2}(s)$ is bounded, say $k_{2}(s)<B$, for all $s \in F$. Now if $q$ is any prime factor of $s$, then AE1 implies $k_{q}=k_{q}(s)<B$, and AE3 implies $q^{k_{q}}<2^{B+2}$. The latter with $q=P(s)$ forces $P(s)<2^{B+2}$. Therefore, $s<\left(2^{B+2}!\right)^{B}$ for all $s \in F$, and so $F$ is a finite set.

Remark 2. We outline another proof of SA2. Observe first that, if $p^{k+1}$ does not divide $n$, then (compare the proof of [6, Theorem 329])

$$
\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)}\left(1-\frac{1}{p^{k+1}}\right)
$$

where $\varphi(n)$ is Euler's totient function. Together with the classical result [6, Theorem 328]

$$
\limsup _{n \rightarrow \infty} \frac{n}{\varphi(n) \log \log n}=e^{\gamma}
$$

this implies that there exists $\epsilon=\epsilon\left(n_{0}\right)>0$ such that, if $n \gg 1$ is not multiple of $n_{0}$, then

$$
G(n) \leq e^{\gamma}-\epsilon,
$$

so that, by SA1, $n$ cannot be SA.

## 5. GA numbers of the first kind

Let us say that a positive integer $n$ is a GA number of the first kind (GA1 number) if $n$ is composite and satisfies condition (i) in Definition with $N$ replaced by $n$, that is, $G(n) \geq G(n / p)$ for all primes $p$ dividing $n$. For example, 4 is GA1, as are all other extraordinary numbers, if any. Also, the near miss 183783600 is a GA number of the first kind. By Lemma 2 if $4 \neq r \in R$, then $r$ is not a GA1 number.

Writing $p^{k} \| n$ when $p^{k} \mid n$ but $p^{k+1} \nmid n$, we have the following criterion for GA1 numbers.

Proposition 2. A composite number $n$ is a $G A$ number of the first kind if and only if prime $p \mid n$ implies

$$
\frac{\log \log n}{\log \log \frac{n}{p}} \leq \frac{p^{k+1}-1}{p^{k+1}-p} \quad\left(p^{k} \| n\right)
$$

Proof. This follows easily from the definitions of GA1 and $G(n)$ and the formulas

$$
\sigma(n)=\prod_{p^{k} \| n}\left(1+p+p^{2}+\cdots+p^{k}\right)=\prod_{p^{k} \| n} \frac{p^{k+1}-1}{p-1}
$$

The next two propositions determine all GA1 numbers with exactly two prime factors.

Proposition 3. Let $p$ be a prime. Then $2 p$ is a $G A$ number of the first kind if and only if $p=2$ or $p>5$.

Proof. As $G(2)<0<G(2 p)$, the number $2 p$ is GA1 if and only if $G(2 p) \geq G(p)$. Thus $2 p$ is GA1 for $p=2$, but, by computation, not for $p=3$ and 5 . If $p>5$, then since $3 \log \log x>2 \log \log 2 x$ for $x \geq 7$, we have

$$
\frac{G(2 p)}{G(p)}=\frac{\sigma(2 p)}{2 p \log \log 2 p} \div \frac{\sigma(p)}{p \log \log p}=\frac{3(p+1)}{2 p \log \log 2 p} \cdot \frac{p \log \log p}{p+1}=\frac{3 \log \log p}{2 \log \log 2 p}>1
$$

Thus $2 p$ is GA1 for $p=7,11,13, \ldots$ This proves the proposition.
Proposition 4. Let $p \geq q$ be odd primes. Then $p q$ is not a GA1 number.
Proof. As $(x+1) \log \log y<x \log \log x y$ when $x \geq y \geq 3$, it follows that if $p>q \geq 3$ are primes, then

$$
\frac{G(p q)}{G(q)}=\frac{(p+1)(q+1)}{p q \log \log p q} \div \frac{q+1}{q \log \log q}=\frac{(p+1) \log \log q}{p \log \log p q}<1
$$

and if $p \geq 3$ is prime, then

$$
\frac{G\left(p^{2}\right)}{G(p)}=\frac{p^{2}+p+1}{p^{2} \log \log p^{2}} \div \frac{p+1}{p \log \log p}=\frac{\left(p^{2}+p+1\right) \log \log p}{\left(p^{2}+p\right) \log \log p^{2}}<1
$$

Hence $p q$ is not GA1 for odd primes $p \geq q$.

## 6. Concluding remarks

Our reformulation of the RH, like Lagarias's, is attractive because the constant $e^{\gamma}$ does not appear. Also, there is an elegant symmetry to the pair of conditions (i) and (ii): the value of the function $G$ at the number $N$ is not less than its values at the quotients $N / p$ and at the multiples $a N$. The statement reformulates the Riemann Hypothesis in purely elementary terms of divisors, prime factors, multiples, and logarithms.

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