The Number of $\bar{2}413\bar{5}$-Avoiding Permutations

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Abstract

We answer a question of R. J. Mathar and confirm that the counting sequence for $\bar{2}413\bar{5}$-avoiding permutations is the Invert transform of the Bell numbers. The proof relies on a simple decomposition of these permutations and the known fact that $\bar{2}413$-avoiding permutations are counted by the Bell numbers.

1 Introduction

A permutation π avoids the barred pattern $\bar{2}413\bar{5}$ if each instance of a not-necessarily-consecutive 413 pattern in π is part of a 24135 pattern in π, and similarly for other barred patterns. Lara Pudwell [1] presents a general approach to counting permutations avoiding a given 5-letter barred pattern that often produces a recurrence relation but does not do so in this case. R. J. Mathar observed [2] that the first few terms of the counting sequence for $\bar{2}413\bar{5}$-avoiding permutations agree with those of the Invert transform of the Bell numbers—Invert$(1, 1, 2, 5, 15, 52, \ldots) = (1, 2, 5, 14, 43, 144, \ldots)$—and asked if the two sequences coincide. We will show that the answer is yes. Section 2 reviews terminology. Section 3 presents a decomposition for $\bar{2}413\bar{5}$-avoiding permutations in terms of $\bar{2}413$-avoiding permutations, yielding a bijection that proves the result.

2 Review of terminology

The Invert transform of a sequence $(a_n)_{n\geq 1}$ is $(b_n)_{n\geq 1}$ defined by

$$1 + \sum_{n\geq 1} b_n x^n = \frac{1}{1 - \sum_{n\geq 1} a_n x^n},$$

and has the following combinatorial interpretation [3, 4]. If the counting sequence by size of a class of combinatorial structures, say A-structures, is $(a_n)_{n\geq 1}$, then b_n is the number of lists (of unspecified length) of A-structures whose total size is n.

For any barred pattern ρ, we use $S_n(\rho)$ for the set of ρ-avoiding permutations of $[n]$. A permutation is *standard* if its support set is an initial segment of the positive integers (or empty). To *standardize* a permutation means to replace its smallest entry by 1, next smallest by 2, and so on. We use stand(π) for the result of standardizing π.

3 A decomposition and bijection

We begin with two observations about a $\bar{2}41 3\bar{5}$-avoiding permutation π. The entries *after* n in π must decrease, else n would start a 413 pattern with no available "5". If entries $c > a$ occur in that order *before* n, then all elements of the interval $[a, c]$ must occur before n, else an element b of (a, c) would occur after n and cab is a 413 pattern, again with no available "5". From these observations, it follows that π has the form

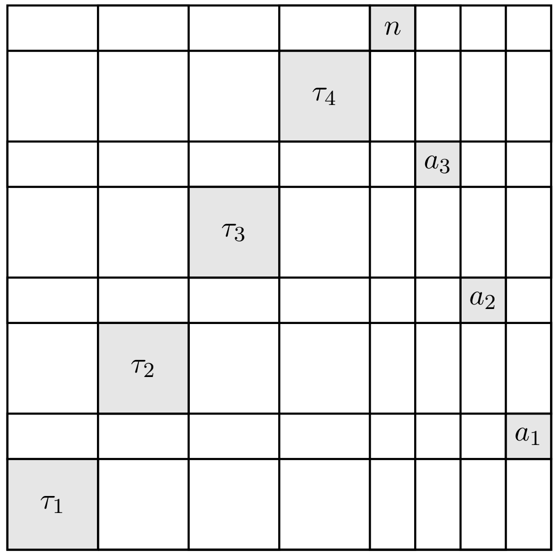
$$\tau_1 \tau_2 \ \ldots \tau_r \, n \, a_{r-1} \, a_{r-2} \, \ldots \, a_1$$

where

- each τ_i is a subpermutation, possibly empty, with support an interval of integers,
- each a_i is a single entry in π and is the only integer lying (in value) between the support intervals of τ_i and τ_{i+1}, $1 \le i \le r-1$,
- $\tau_1 < \tau_2 < \ldots < \tau_r$ in the sense that each entry of τ_i is less than each entry of τ_{i+1}, $i = 1, 2, \ldots, r-1$,
- $a_{r-1} > a_{r-2} > \ldots > a_1$,
- each τ_i is $\bar{2}413$-avoiding.

Conversely, any permutation π with a decomposition $\tau_1 \tau_2 \ \ldots \tau_r \, n \, a_{r-1} \, a_{r-2} \, \ldots \, a_1$ satisfying these conditions is $\bar{2}413\bar{5}$-avoiding.

It is helpful to interpret the bulleted conditions in the setting of a modified permutation matrix where, for a permutation π, the entry in the (i, j) cell, measuring from the southwest corner, is $\pi(i)$ if $j = \pi(i)$ and 0 otherwise. We see that a permutation π has a decomposition that meets all the bulleted conditions if and only if its matrix has the form pictured schematically below for $r = 4$ with each τ_i a $\bar{2}413$ avoider, each a_i an entry of π, and 0's in all unshaded regions.



a $\overline{2}413\overline{5}$ avoider as a permutation matrix

With $a_r := n$, the map $\pi \to \big(\text{stand}(\tau_1 a_1), \ldots, \text{stand}(\tau_r a_r)\big)$ is a bijection from $S_n(\overline{2}413\overline{5})$ to lists $(\sigma_1, \ldots, \sigma_r)$ with $r \geq 1$ where each σ_i is an *end-max avoider*—a standard $\overline{2}413\overline{5}$-avoiding permutation that ends at its maximum entry—and the total length of the σ_i's is n. Clearly, the number of end-max avoiders of length k is the number of $\overline{2}413$-avoiding permutations of length $k-1$ and it is known [5] that $|\, S_{k-1}(\overline{2}413)\,| = B_{k-1}$, the Bell number.

Set $a_n = B_{n-1}$, so that a_n is the number of end-max avoiders of length (size) n. Then the Invert transform $(b_n)_{n\geq 1}$ of $(a_n)_{n\geq 1}$ is the number of lists of of end-max avoiders of total size n, which the bijection above shows is $|\, S_n(\overline{2}413\overline{5})\,|$. Hence, the counting sequence for $S_n(\overline{2}413\overline{5})$ is the Invert transform of $(a_n)_{n\geq 1} = (1, 1, 2, 5, 15, \ldots)$, the full sequence of Bell numbers.

References

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