

THE MEDIAN GENOCCHI NUMBERS, Q -ANALOGUES AND CONTINUED FRACTIONS

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ABSTRACT. The goal of this paper is twofold. First, we review the recently developed geometric approach to the combinatorics of the median Genocchi numbers. The Genocchi numbers appear in this context as Euler characteristics of the degenerate flag varieties. Second, we prove that the generating function of the Poincaré polynomials of the degenerate flag varieties can be written as a simple continued fraction. As an application we prove that the Poincaré polynomials coincide with the q -version of the normalized median Genocchi numbers introduced by Han and Zeng.

INTRODUCTION

The Genocchi numbers appear in many different contexts (see e.g. [B], [Du], [DR], [DZ], [DV], [G], [V2]). Probably, the most well-known definition uses the Seidel triangle

							155	155	
						17	17	155	310
				3	3	17	34	138	448
		1	1	3	6	14	48	104	552
1	1	1	2	2	8	8	56	56	608

By definition, the triangle is formed by the numbers $g_{k,n}$ (k is the number of a row counted from bottom to top and n is the number of a column from left to right) with $n = 1, 2, \dots$ and $1 \leq k \leq \frac{n+1}{2}$, subject to the relations $g_{1,1} = 1$ and

$$g_{k,2n} = \sum_{i \geq k} g_{i,2n-1}, \quad g_{k,2n+1} = \sum_{i \leq k} g_{i,2n}.$$

For example, $138 = 56 + 48 + 34$ and $48 = 14 + 17 + 17$. The two sequences of numbers sitting on the edges of the Seidel triangle are called the Genocchi numbers. More precisely, the Genocchi numbers of the first kind are $1, 1, 3, 17, 155, \dots$ and of the second kind are $1, 2, 8, 56, 608, \dots$. The latter numbers are also referred to as the median Genocchi numbers and are denoted by H_{2n-1} . For example, $H_1 = 1$ and $H_7 = 56$. These numbers are known to be divisible by the powers of 2 (see [B], [Du]): $H_{2n+1} \div 2^n$. The ratios are called the normalized median Genocchi numbers and are denoted by h_n . Thus the first values h_0, h_1, h_2, \dots are as follows:

$$1, 1, 2, 7, 38, 295, 3098, \dots$$

It has been shown recently (see [Fe2]) that the numbers h_n are analogues ("degenerations") of the numbers $n!$. More precisely, let \mathcal{F}_n be the variety of flags in an n -dimensional space, i.e. \mathcal{F}_n consists of collections of subspaces $(V_1 \subset V_2 \subset \dots \subset V_{n-1})$ of a given n -dimensional space W such that $\dim V_k = k$. It is well known that the Euler characteristics of \mathcal{F}_n is equal to $n!$. Combinatorially, the number $n!$ appears in this context as the number of sequences $(I_1 \subset I_2 \subset \dots \subset I_{n-1})$ of subsets of $\{1, \dots, n\}$ such that $\#I_k = k$. The varieties \mathcal{F}_n have natural degenerations \mathcal{F}_n^a , called the degenerate flag varieties (see [Fe1],[Fe2],[FF], [FFL]). In order to define \mathcal{F}_n^a we fix a basis w_1, \dots, w_n of W and the projection maps $pr_k : W \rightarrow W$ mapping w_k to 0 and w_i with $i \neq k$ to w_i . The degenerate flag varieties consist of collections $(V_1, V_2, \dots, V_{n-1})$ of subspaces of W such that $\dim V_k = k$ and $pr_{k+1}V_k \subset V_{k+1}$. It turns out that the Euler characteristic of \mathcal{F}_n^a is equal to the normalized median Genocchi number:

$$(0.1) \quad \chi(\mathcal{F}_n^a) = h_n.$$

Combinatorially this means that the number of sequences $(I_1, I_2, \dots, I_{n-1})$ of subsets of $\{1, \dots, n\}$ such that $\#I_k = k$ and $I_k \subset I_{k+1} \cup \{k+1\}$ is equal to h_n .

In this paper we review (following [Fe2] and [CFR]) the applications of the observation (0.1) to the combinatorics of h_n . We give several new combinatorial objects counted by the normalized median Genocchi numbers. As an application the formula for the numbers h_n is derived in terms of binomial coefficients. Using (0.1) we introduce natural q -analogues $h_n(q)$ as Poincaré polynomials of the degenerate flag varieties. We note that the degenerate flag varieties are singular, but share the following important property with their classical analogues: the varieties \mathcal{F}_n^a can be decomposed into a disjoint union of complex (even-dimensional real) affine cells. Therefore the Poincaré polynomials $P_{\mathcal{F}_n^a}(t)$ are functions of $q = t^2$ (odd powers do not show up). Hence we define

$$h_n(q) = P_{\mathcal{F}_n^a}(q^{1/2}).$$

Obviously, one has $h_n(1) = h_n$. We note also that the degree of $h_n(q)$ is equal to $n(n-1)/2$, since the complex dimension of \mathcal{F}_n^a is $n(n-1)/2$. In the paper we recall two formulas for the polynomials $h_n(q)$: one uses certain statistic on the set of Dellac configurations (see [De],[Fe2]) and another is obtained via the geometric arguments (see [CFR]). Various q -analogues of the Genocchi numbers appear in the literature (see e.g. [HZ1], [HZ2], [ZZ]). In particular, in [HZ1] Han and Zeng used the q -analogues to give a third proof of the Barsky theorem ([B], [Du]).

Our new result is the continued fraction presentation of the generating function of the polynomials $h_n(q)$. Namely, it is convenient to introduce the "reversed" polynomials $\tilde{h}_n(q) = q^{n(n-1)/2}h_n(q^{-1})$. Then we have

Theorem 0.1.

$$\sum_{n \geq 0} \tilde{h}_n(q) s^n = \frac{1}{1 - \frac{s}{1 - \frac{qs}{1 - \frac{\binom{3}{2}_q s}{1 - \frac{q \binom{3}{2}_q s}{1 - \frac{\binom{4}{2}_q s}{1 - \frac{q \binom{4}{2}_q s}{1 - \dots}}}}}}$$

Using this formula, we prove that $\tilde{h}_n(q)$ coincide with the q version of the normalized median Genocchi numbers introduced by Han and Zeng in [HZ1], [HZ2]. We also show that the Viennot formula (see [V1], [V3], [Du], [DZ]) for the generating function of the median Genocchi numbers H_{2n-1} can be derived by specialization at $q = 1$.

Our paper is organized as follows.

In Section 1 we give several definitions of the normalized median Genocchi numbers.

In Section 2 we give two formulas for the polynomials $h_n(q)$.

In Section 3 we obtain the continued fraction presentation for the generating function of $\tilde{h}_n(q)$.

1. COMBINATORICS OF THE NORMALIZED MEDIAN GENOCCHI NUMBERS

The normalized median Genocchi numbers h_n , $n = 0, 1, 2, \dots$ form a sequence which starts with 1, 1, 2, 7, 38, 295. These numbers enjoy many definitions (see [B], [Du], [De], [G], [K], [Sl], [Fe2]). We recall some of them now.

1.1. **The Seidel triangle.** The Seidel triangle [Se] is formed by the numbers $g_{k,n}$ with $n = 1, 2, \dots$ and $1 \leq k \leq \frac{n+1}{2}$, subject to the relations $g_{1,1} = 1$ and

$$g_{k,2n} = \sum_{i \geq k} g_{i,2n-1}, \quad g_{k,2n+1} = \sum_{i \leq k} g_{i,2n}.$$

The numbers $g_{n,2n-1}$ are called the Genocchi numbers of the first kind and the numbers $H_{2n-1} = g_{1,2n}$ are called the Genocchi numbers of the second kind (or the median Genocchi numbers). Barsky [B] and then Dumont [Du] proved that the number H_{2n+1} is divisible by 2^n . The normalized median Genocchi numbers h_n are defined as the corresponding ratios: $h_n = H_{2n+1}/2^n$.

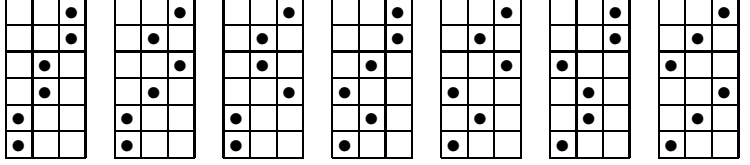
1.2. **Dellac's configurations.** The earliest definition was given by Dellac in [De]. Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l, j) , where $l = 1, \dots, n$ is the number of a column

and $j = 1, \dots, 2n$ is the number of a row. A Dellac configuration D is a subset of boxes, subject to the following conditions:

- each column contains exactly two boxes from D ,
- each row contains exactly one box from D ,
- if the (l, j) -th box is in D , then $l \leq j \leq n + l$.

Let DC_n be the set of such configurations. Then the number of elements in DC_n is equal to h_n .

We list all Dellac's configurations for $n = 3$. We specify boxes in a configuration by putting fat dots inside.



1.3. Permutations. In [K] Kreweras suggested another description of the numbers h_n . Namely, a permutation $\sigma \in S_{2n+2}$ is called a normalized Dumont permutation of the second kind if the following conditions are satisfied:

- $\sigma(k) < k$ if k is even,
- $\sigma(k) > k$ if k is odd,
- $\sigma^{-1}(2k) < \sigma^{-1}(2k+1)$ for $k = 1, \dots, n$.

According to Kreweras, the number of such permutations is equal to h_n (see also [Fe2], Proposition 3.3).

1.4. A la $n!$. Let $\mathbf{I} = (I_1, I_2, \dots, I_{n-1})$ be a sequence of subsets of the set $\{1, \dots, n\}$. We call such a sequence admissible if $\#I_l = l$ for all l and

$$(1.1) \quad I_l \subset I_{l+1} \cup \{l+1\}, \quad l = 1, \dots, n-2.$$

Then the number of admissible sequences is equal to the normalized median Genocchi number h_n .

Remark 1.1. If we replace (1.1) with $I_l \subset I_{l+1}$, then the number of admissible sequences will obviously become $n!$.

Admissible sequences can be visualized as follows. Consider an oriented graph Γ with the set of vertices (l, j) , $1 \leq l \leq n-1$, $1 \leq j \leq n$. Two vertices (l_1, j_1) and (l_2, j_2) are connected by an arrow $(l_1, j_1) \rightarrow (l_2, j_2)$ if and only if $j_1 = j_2$, $l_2 = l_1 + 1$, $l_2 \neq j_2$ (note that in [CFR] this graph appeared as a

coefficient quiver). For $n = 5$ the graph is as follows:

$$\begin{aligned}
(1, 5) &\rightarrow (2, 5) \rightarrow (3, 5) \rightarrow (4, 5) \\
(1, 4) &\rightarrow (2, 4) \rightarrow (3, 4) \quad (4, 4) \\
(1, 3) &\rightarrow (2, 3) \quad (3, 3) \rightarrow (4, 3) \\
(1, 2) &\quad (2, 2) \rightarrow (3, 2) \rightarrow (4, 2) \\
(1, 1) &\rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (4, 1)
\end{aligned}$$

To a collection $\mathbf{I} = (I_1, \dots, I_{n-1})$ with $\#I_l = l$ for all l we associate a subset $S_{\mathbf{I}}$ of vertices of Γ by the formula

$$S_{\mathbf{I}} = \{(l, j) : j \in I_l\}.$$

Then \mathbf{I} is admissible if and only if $S_{\mathbf{I}}$ is closed in Γ , i.e. if $p \in S_{\mathbf{I}}$ and $p \rightarrow q$ is an arrow in Γ then $q \in S_{\mathbf{I}}$.

1.5. Euler characteristic. The admissible sequences label the cells in the degenerate flag varieties \mathcal{F}_n^a (see [Fe2]). Recall that \mathcal{F}_n^a consists of sequences (V_1, \dots, V_{n-1}) of the subspaces of a given n -dimensional space W subject to the conditions given below. Let w_1, \dots, w_n be a basis of W and let $pr_k : W \rightarrow W$ be the projection along w_k to $\text{span}(w_i : i \neq k)$: $pr_k(\sum_{i=1}^n c_i w_i) = \sum_{i \neq k} c_i w_i$.

Definition 1.2. A collection (V_1, \dots, V_{n-1}) of subspaces $V_k \subset W$ belongs to \mathcal{F}_n^a if and only if $\dim V_k = k$ for all k and

$$pr_{k+1}V_k \subset V_{k+1}, \quad k = 1, \dots, n-2.$$

We recall (see e.g. [Fu]) that the classical flag varieties \mathcal{F}_n consist of collections $(V_1 \subset \dots \subset V_{n-1})$ of sequentially embedded subspaces of W with $\dim V_k = k$ for all k . These varieties are acted upon by a torus $T = (\mathbb{C}^*)^n$. The action is induced from the natural action of T on W :

$$(a_1, \dots, a_n) \cdot (c_1 w_1 + \dots + c_n w_n) = a_1 c_1 w_1 + \dots + a_n c_n w_n.$$

The varieties \mathcal{F}_n enjoy several important properties:

- (i) \mathcal{F}_n can be decomposed into a disjoint union of complex (even-dimensional real) cells. Each cell contains exactly one T -fixed point.
- (ii) The T -fixed points on \mathcal{F}_n are labeled by permutations from S_n . The fixed point $p(\sigma)$ attached to $\sigma \in S_n$ is given by

$$p(\sigma) = (\text{span}(w_{\sigma(1)}), \text{span}(w_{\sigma(1)}, w_{\sigma(2)}), \dots, \text{span}(w_{\sigma(1)}, \dots, w_{\sigma(n-1)})).$$

- (iii) The (real) dimension of the cell containing $p(\sigma)$ is equal to $2l(\sigma)$ (twice length of the permutation).

In particular, the Euler characteristic of \mathcal{F}_n is equal to $n!$. The degenerate flag varieties \mathcal{F}_n^a share property (i). However, the labeling set for the T -fixed points is different: the number of torus fixed points in \mathcal{F}_n^a is equal to the

number of admissible sequences. Namely, an admissible sequence \mathbf{I} defines a point

$$(1.2) \quad p(\mathbf{I}) = (V_1, \dots, V_{n-1}), \quad V_k = \text{span}(w_i : i \in I_k).$$

Any such point belongs to \mathcal{F}_n^a , is T -fixed and all T -fixed points in \mathcal{F}_n^a are of this form.

Corollary 1.3. *The Euler characteristic of \mathcal{F}_n^a is equal to h_n :*

$$\chi(\mathcal{F}_n^a) = h_n.$$

1.6. Two triangles. We add one more combinatorial description of the numbers h_n obtained in [CFR] using the representation theory of quivers.

Proposition 1.4. *The normalized median Genocchi number h_{n+1} is equal to the number of pairs of collections of non-negative integers $(r_{i,j}), (m_{i,j})$, $1 \leq i \leq j \leq n$ subject to the following conditions for all $k = 1, \dots, n$:*

$$\sum_{i=k}^n r_{k,i} \leq 1, \quad \sum_{j=1}^k m_{j,k} \leq 1, \quad \sum_{i \leq k \leq j} r_{i,j} = \sum_{i \leq k \leq j} m_{i,j}.$$

1.7. Explicit formula. The following explicit formula for the numbers h_n is available (see [CFR])

$$(1.3) \quad h_n = \sum_{f_0, \dots, f_n \geq 0} \prod_{k=1}^{n-1} \binom{1+f_{k-1}}{f_k} \prod_{k=1}^n \binom{1+f_{k+1}}{f_k}$$

with $f_0 = f_n = 0$. This formula can be rewritten as a weighted sum of Motzkin paths. Namely, let M_n be the set of Motzkin paths starting at $(0,0)$ and ending at $(n,0)$. For a Motzkin path $\mathbf{f} = (f_0, \dots, f_n) \in M_n$ with $f_0 = f_n = 0$ let $l(\mathbf{f})$ be the number of "rises" ($f_{i+1} = f_i + 1$) plus the number of "falls" ($f_{i+1} = f_i - 1$). Then we obtain

$$(1.4) \quad h_n = \sum_{\mathbf{f} \in M_n} \frac{\prod_{k=0}^n (1+f_k)^2}{2^{l(\mathbf{f})}}.$$

1.8. The Viennot formula. In section 3 we use the q -version of the formula (1.4) to derive the continued fraction presentation of the generating function of q -Genocchi numbers. We close this section with the continued fraction form of the generating function of the (non-normalized) median Genocchi numbers H_{2n-1} (see subsection 1.1) due to Viennot ([V1], [V3], [Du], [DZ]):

$$1 + \sum_{n=1}^{\infty} H_{2n-1} x^n = \frac{1}{1 - \frac{1^2 x}{1 - \frac{1^2 x}{1 - \frac{2^2 x}{1 - \frac{2^2 x}{\dots}}}}}$$

2. Q-VERSIONS

Several q -versions of the median Genocchi numbers can be found in the literature (see [HZ1], [HZ1], [ZZ]). We briefly recall the definition of the Han-Zeng polynomials below. In our approach the normalized median Genocchi numbers appear as the Euler characteristics of the degenerate flag varieties. Thus we obtain a natural q -analogue defined by the Poincaré polynomials of \mathcal{F}_n^a . We give combinatorial description as well as an explicit formula for these polynomials below.

2.1. The Han-Zeng polynomials. Consider the polynomials $C_n(x, q)$ in two variables defined by $C_1(x, q) = 1$ and

$$C_n(x, q) = (1 + qx) \frac{(1 + qx)C_{n-1}(1 + qx, q) - xC_{n-1}(x, q)}{1 + qx - x}, \quad n \geq 2.$$

Define

$$\bar{c}_n(q) = \frac{C_n(1, q)}{(1 + q)^{n-1}}, \quad n \geq 1,$$

Han and Zeng proved that these are polynomials satisfying $\bar{c}_n(1) = h_{n-1}$ (see formula (17) in [HZ1]). Hence $\bar{c}_n(q)$ can be viewed as q -analogues of the normalized median Genocchi numbers.

2.2. Statistic on the Dellac configurations. For a Dellac configuration $D \in DC_n$ (see subsection 1.2) we define the length $l(D)$ of D as the number of pairs $(l_1, j_1), (l_2, j_2)$ such that the boxes (l_1, j_1) and (l_2, j_2) are both in D and $l_1 < l_2, j_1 > j_2$. This definition resembles the definition of the length of a permutation. We note that in the classical case the complex dimension of the cell attached to a permutation σ in a flag variety is equal to the number of pairs $j_1 < j_2$ such that $\sigma(j_1) > \sigma(j_2)$, which equals to the length of σ (see property (iii) of \mathcal{F}_n).

Proposition 2.1. *The real dimension of the cell in \mathcal{F}_n^a containing a point $p(\mathbf{I})$ (1.2) is equal to $2l(D)$. Thus the Poincaré polynomial $h_n(t) = P_{\mathcal{F}_n^a}(t)$ is given by $h_n(t) = \sum_{D \in DC_n} t^{2l(D)}$.*

Since the polynomials $h_n(t)$ do not contain odd powers of t , it is convenient to introduce a new variable $q = t^2$ and define $h_n(q) = P_{\mathcal{F}_n^a}(q^{1/2})$. The first four polynomials $h_n(q)$ are as follows:

$$\begin{aligned} h_1(q) &= 1, & h_2(q) &= 1 + q, \\ h_3(q) &= 1 + 2q + 3q^2 + q^3, \\ h_4(q) &= 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6. \end{aligned}$$

In general, the degree of $h_n(q)$ is equal to $n(n-1)/2$. Obviously, $h_n(1) = h_n$.

2.3. Explicit formula. Let $m \geq n \geq 0$. Then the q -binomial (Gaussian) coefficient $\binom{m}{n}_q$ is defined as

$$\binom{m}{n}_q = \frac{m_q!}{n_q!(m-n)_q!}, \quad m_q! = \prod_{i=1}^m \frac{1-q^i}{1-q}.$$

The following formula is obtained in [CFR] using the geometry of quiver Grassmannians.

Proposition 2.2. *The Poincaré polynomial of the degenerate flag variety \mathcal{F}_n^a is equal to*

$$(2.1) \quad \sum_{f_1, \dots, f_{n-1} \geq 0} q^{\sum_{k=1}^{n-1} (k-f_k)(1-f_k+f_{k+1})} \prod_{k=1}^{n-1} \binom{1+f_{k-1}}{f_k}_q \prod_{k=1}^{n-1} \binom{1+f_{k+1}}{f_k}_q,$$

(we assume $f_0 = f_n = 0$).

Such kind of formulas are usually referred to as fermionic: these are sums of products of q -binomial coefficients multiplied by certain powers of q . Geometrically, formula (2.1) appears as follows. The varieties \mathcal{F}_n^a can be cut into disjoint pieces, such that each piece is fibered over a product of several Grassmannians with fibers being affine spaces. Since the Poincaré polynomial of a Grassmannian is given by a q -binomial coefficient, we arrive at the formula as above.

3. GENERATING FUNCTION AND CONTINUED FRACTION

Our goal in this section is to give an explicit continued fraction form of the generating function of the Poincaré polynomials $h_n(q)$ and to prove that they coincide with the q -versions of the normalized median Genocchi numbers defined in [HZ1], [HZ2].

We note that formula (2.1) can be seen as a sum over the set M_n of Motzkin paths $\mathbf{f} = (f_0, f_1, \dots, f_n)$ starting at $(0, f_0) = (0, 0)$ and ending at $(n, f_n) = (n, 0)$.

We first recall the formalism of the weighted generating functions of Motzkin paths due to Flajolet (see [Fl]). Let α_n, β_n and $\gamma_n, n \geq 0$ be sequences of complex numbers called weights. For a nonnegative integer k we define $w(k, k) = \gamma_k, w(k, k+1) = \alpha_k$ and $w(k, k-1) = \beta_{k-1}$ (if $k \geq 1$). We denote by α_\bullet the whole collection $(\alpha_k)_{k=0}^\infty$ and similarly for β_\bullet and γ_\bullet . The weighted generating function of Motzkin paths is given by the formula

$$F(s; \alpha_\bullet, \beta_\bullet, \gamma_\bullet) = \sum_{n \geq 0} s^n \sum_{\mathbf{f} \in M_n} \prod_{k=0}^{n-1} w(f_k, f_{k+1}).$$

The following result is due to Flajolet [Fl].

Theorem 3.1. *The weighted generating sum of the Motzkin paths is given by the continued fraction*

$$F(s; \alpha_\bullet, \beta_\bullet, \gamma_\bullet) = \frac{1}{1 - \gamma_0 s - \frac{\alpha_0 \beta_0 s^2}{1 - \gamma_1 s - \frac{\alpha_1 \beta_1 s^2}{1 - \gamma_2 s - \dots}}}$$

Let us apply this formalism to our situation. Formula (2.1) can be rewritten as follows.

(3.1)

$$h_n(q) = q^{n(n-1)/2} \sum_{\mathbf{f} \in M_n} q^{\sum_{k=1}^{n-1} f_k(f_k - f_{k+1} - 2)} \prod_{k=1}^{n-1} \binom{1 + f_{k-1}}{f_k}_q \prod_{k=1}^{n-1} \binom{1 + f_{k+1}}{f_k}_q.$$

We introduce three sequences of weights

$$\alpha_m(q) = q^{-3m} \binom{m+2}{2}_q, \quad \beta_m(q) = q^{-m-1} \binom{m+2}{2}_q, \quad \gamma_m(q) = q^{-2m} \binom{m+1}{1}_q$$

and define $w(f_k, f_{k+1})$ using these weights. Then formula (3.1) implies the following lemma.

Lemma 3.2.

$$(3.2) \quad q^{-n(n-1)/2} h_n(q) = \sum_{\mathbf{f} \in M_n} \prod_{k=0}^{n-1} w(f_k, f_{k+1}).$$

In order to use the Flajolet theorem we need to get rid of the factor $q^{n(n-1)/2}$ in (3.2). We introduce the notation

$$\tilde{h}_n(q) = q^{n(n-1)/2} h_n(q^{-1})$$

(note that the degree of $h_n(q)$ is exactly $n(n-1)/2$). Let $\tilde{h}(q, s) = \sum_{n \geq 0} \tilde{h}_n(q) s^n$. We note that

$$\gamma_m(q) = \binom{m+1}{1}_{q^{-1}}^2, \quad \alpha_m(q) \beta_m(q) = q^{-1} \binom{m+2}{2}_{q^{-1}}^2.$$

Using Theorem 3.1 we arrive at the following theorem.

Theorem 3.3. *The generating function $\tilde{h}(q, s)$ can be written as follows*

$$(3.3) \quad \tilde{h}(q, s) = \frac{1}{1 - s - \frac{q s^2}{1 - \binom{2}{1}_q^2 s - \frac{q \binom{3}{2}_q^2 s^2}{1 - \binom{3}{1}_q^2 s - \frac{q \binom{4}{2}_q^2 s^2}{1 - \binom{4}{1}_q^2 s - \dots}}}}$$

Proof. Follows from Lemma 3.2 and the Flajolet theorem. \square

Corollary 3.4.

$$(3.4) \quad \tilde{h}(q, s) = \frac{1}{1 - \frac{s}{1 - \frac{qs}{1 - \frac{\binom{3}{2}_q s}{1 - \frac{q \binom{3}{2}_q s}{1 - \frac{\binom{4}{2}_q s}{1 - \frac{q \binom{4}{2}_q s}{1 - \dots}}}}}}$$

Proof. Recall the following formula (see [DZ], Lemma 2)

$$\frac{c_0}{1 - c_1 s - \frac{c_1 c_2 s^2}{1 - (c_2 + c_3)s - \frac{c_3 c_4 s^2}{1 - (c_4 + c_5)s - \dots}}} = \frac{c_0}{1 - \frac{c_1 s}{1 - \frac{c_2 s}{1 - \frac{c_3 s}{1 - \dots}}}}$$

Now our corollary follows from Theorem 3.3. \square

Specializing at $q = 1$ we arrive at formula (1.4) for the normalized median Genocchi numbers.

Corollary 3.5. *The number h_n is equal to the weighted sum over the set M_n of Motzkin paths $\sum_{\mathbf{f} \in M_n} \prod_{k=0}^{n-1} w(f_k, f_{k+1})$ with the weights $w(\cdot, \cdot)$ defined by*

$$\alpha_m = (m+1)(m+2)/2 = \beta_m, \quad \gamma_m = (m+1)^2.$$

The generating function $\sum_{n \geq 0} h_n s^n$ is given by the continued fraction

$$(3.5) \quad \frac{1}{1 - \frac{s}{1 - \frac{s}{1 - \frac{3s}{1 - \frac{3s}{1 - \frac{6s}{1 - \frac{6s}{1 - \frac{10s}{1 - \dots}}}}}}}}$$

Corollary 3.6. $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$.

Proof. Formula (18) in [HZ1] gives a continued fraction form of the generating function of the polynomials $\bar{c}_n(q)$. Comparing this formula with (3.4)

and using the equations

$$\begin{aligned} \binom{2n}{2}_q &= (1 + q^2 + q^4 + \cdots + q^{2n-2})(1 + q + q^2 + \cdots + q^{2n-2}), \\ \binom{2n+1}{2}_q &= (1 + q^2 + q^4 + \cdots + q^{2n-2})(1 + q + q^2 + \cdots + q^{2n}), \end{aligned}$$

we obtain $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$. \square

We also derive the Viennot formula for the generating function of the (non-normalized) median Genocchi numbers H_{2n+1} (see [Du], [DZ], [V1], [V3]):

Corollary 3.7.

$$1 + \sum_{n=1}^{\infty} H_{2n-1} s^n = \frac{1}{1 - \frac{1^2 s}{1 - \frac{2^2 s}{1 - \frac{2^2 s}{\dots}}}}$$

Proof. Recall the formula $H_{2n+1} = 2^n h_n$. Therefore, one has

$$1 + \sum_{n=1}^{\infty} H_{2n-1} s^n = 1 + \sum_{n=0}^{\infty} h_n 2^n s^{n+1} = 1 + s \sum_{n=0}^{\infty} h_n (2s)^n.$$

Specializing (3.3) at $q = 1$, we obtain that the generating function for the (non-normalized) median Genocchi numbers is given by

$$1 + \frac{s}{1 - 2s - \frac{4s^2}{1 - 2 \cdot 4s - \frac{4 \cdot 3^2 s^2}{1 - 2 \cdot 9s - \frac{4 \cdot 6^2 s^2}{1 - 2 \cdot 16s - \frac{4 \cdot 10^2 s^2}{1 - 2 \cdot 25s - \dots}}}}}$$

Finally, we use the following formula from [DZ], Lemma 2:

$$c_0 + \frac{c_0 c_1 s}{1 - (c_1 + c_2)s - \frac{c_2 c_3 s^2}{1 - (c_3 + c_4)s - \frac{c_4 c_5 s^2}{1 - \dots}}} = \frac{c_0}{1 - \frac{c_1 s}{1 - \frac{c_2 s}{1 - \frac{c_3 s}{1 - \dots}}}}$$

\square

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