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A combinatorial interpretation of the Catalan transform of the Catalan numbers

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

Abstract

The Catalan transform of a sequence $(a_n)_{n\geq 0}$ is the sequence $(b_n)_{n\geq 0}$ with $b_n = \sum_{k=0}^{n} \frac{k}{2n-k}\binom{2n-k}{n-k} a_k$. Here we show that the Catalan transform of the Catalan numbers has a simple interpretation: it counts functions $f : [1, n] \to [1, n]$ satisfying the condition that, for all $i < j$, $f(j) - (j - i)$ is not in the interval $[1, f(i) - 1]$.

1 Introduction

The Catalan transform [1] of a sequence $(a_n)_{n\geq 0}$ is the sequence $(b_n)_{n\geq 0}$ with $b_n = \sum_{k=0}^{n} \frac{k}{2n-k}\binom{2n-k}{n-k} a_k$, where $\frac{k}{2n-k}\binom{2n-k}{n-k}$ is interpreted as 1 if $n = k = 0$. The Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The purpose of this note is to establish the following combinatorial interpretation.

Theorem 1. *The Catalan transform $(b_n)_{n\geq 0}$ of the Catalan numbers $(C_n)_{n\geq 0}$ counts functions $f : [1, n] \to [1, n]$ satisfying the condition that, for all $i < j$, $f(j) - (j - i)$ is not in the interval $[1, f(i) - 1]$.*

We will represent functions $f : [1, n] \to [1, n]$ as sequences $(u_i)_{i=1}^{n}$ where $u_i = f(i)$ and, hence, $1 \leq u_i \leq n$ for all i.

In Section 2 we review the Catalan numbers, and in Section 3 we establish some preliminary results about sequences $(u_i)_{i=1}^{n}$ of positive integers that satisfy the key condition

$$\boxed{u_j - (j - i) \notin [1, u_i - 1]} \qquad 1 \leq i < j \leq n \qquad (1)$$

of Theorem 1. In Section 4 we prove the main result and Section 5 presents an extension.

2 Generalized Catalan numbers

The generalized Catalan number $C_n^{(k)}$ is defined by $C_n^{(k)} = \frac{k+1}{2n+k+1}\binom{2n+k+1}{n}$ with $C_n^{(-1)} := 1$ if $n = 0$ and $:= 0$ if $n \geq 1$. Taking $k = 0$ gives the ordinary Catalan number $C_n := C_n^{(0)} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$. It is well known that C_n counts Dyck n-paths [2]. A Dyck n-path has n upsteps and n downsteps and its semilength is n. It is also known [3] that the sequence $\left(C_n^{(k)}\right)_{n\geq 0}$ is the k-fold convolution of $(C_n)_{n\geq 0}$ [3]. It follows, as is well known, that $C_n^{(k)}$ is the number of Dyck $(k+n)$-paths that start with at least k upsteps—discard the first k upsteps and then consider the decomposition into Dyck paths induced by the last upstep at level i, $i = k, k-1, \ldots, 1$.

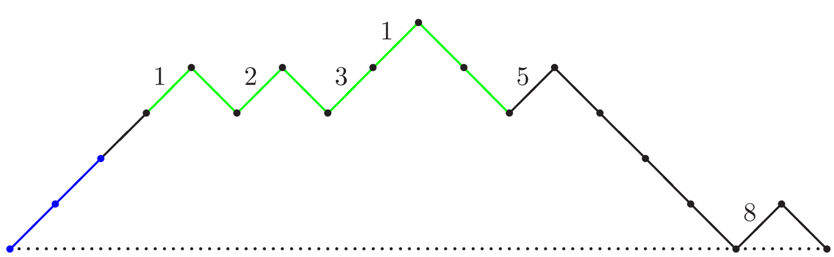
We will use an equivalent formulation of the Catalan transform obtained by reversing the order of summation in the expression for b_n. Thus

$$b_n = \sum_{k=0}^{n} C_k^{(n-1-k)} a_{n-k}.$$

3 Preliminary results

Proposition 2. *Fix a nonnegative integer k. Let $\mathcal{B}_n^{(k)}$ denote the set of sequences $(v_1, \ldots, v_n)$ satisfying $1 \leq v_i \leq i + k$ and condition (1). Then $|\mathcal{B}_n^{(k)}| = C_n^{(k)}$.*

Proof. As noted in Section 2, $C_n^{(k)}$ is the number of Dyck $(k+n)$-paths that start with at least k upsteps. Given such a path, define $v_i = 1 +$ semilength of the longest Dyck subpath immediately preceding the $(i+k)$-th upstep, $1 \leq i \leq n$. This is a bijection to $\mathcal{B}_n^{(k)}$. For example, with $k = 2$ and $n = 6$, the mandatory k upsteps in the Dyck path shown are in blue and the $(i+k)$-th upstep is labeled with the corresponding v_i.



The key observation to show this bijection works is that if an upstep in a Dyck path is immediately preceded by a nonempty Dyck subpath and the steps of the maximal such subpath are colored green, as for the $(6+k)$-th upstep in the example, then the Dyck subpath associated with each green upstep is a subpath of the green path. □

In particular, for $k=0$ we have

Proposition 3. *Let $\mathcal{B}_n = \mathcal{B}_n^{(0)}$ denote the set of sequences $(v_1,\dots,v_n)$ satisfying $1 \le v_i \le i$ and condition (1). Then $|\,\mathcal{B}_n\,| = C_n$.* □

Remark. The map "reverse and decrement each entry by 1" is a bijection from $\mathcal{B}_n$ to the inversion codes for 231-avoiding permutations of $[n]$. (The inversion code for a permutation $(p_1,\dots,p_n)$ of $[n]$ is the sequence $(u_1,\dots,u_n)$ where u_i is the number of $j \in [i+1,n]$ with $p_i > p_j$.)

Proposition 4. *For $s \ge 0$, let $\mathcal{D}_n^{(s)}$ denote the set of sequences $(u_1,\dots,u_n)$ satisfying $u_i \ge 1$ for all i, $u_n = n+s$, and condition (1). Then $|\,\mathcal{D}_n^{(s)}| = C_{n-1}^{(s)}$.*

Proof. Suppose $(u_1,\dots,u_n) \in \mathcal{D}_n^{(s)}$. Take $i \in [n-1]$. If $u_i \ge 2$, condition (1) applied with $j=n$ implies $s+i \le 0$ or $s+i \ge u_i$. The first inequality cannot hold, so $u_i \le s+i$. But the latter inequality obviously also holds when $u_i = 1$. Hence, $(u_1,\dots,u_{n-1}) \in \mathcal{B}_{n-1}^{(s)}$. Apply Proposition 2. □

Proposition 5. *For $k \ge 0$, let $\mathcal{F}_n^{(k)}$ denote the set of sequences $(u_1,\dots,u_n)$ satisfying $1 \le u_i \le n+k+1$ for all i, $u_n > n$, and condition (1). Then $|\,\mathcal{F}_n^{(k)}| = C_n^{(k)}$.*

Proof. Clearly, $\{(u_1,\dots,u_n) \in \mathcal{F}_n^{(k)} : \; u_n = n+s\} \subseteq \mathcal{D}_n^{(s)}$ for $1 \le s \le k+1$. On the other hand, for $(u_1,\dots,u_n) \in \mathcal{D}_n^{(s)}$, condition (1) implies $u_i \le s+i \le k+1+i \le n+k+1$. Hence, the reverse inclusion also holds and $\{(u_1,\dots,u_n) \in \mathcal{F}_n^{(k)} : \; u_n = n+s\} = \mathcal{D}_n^{(s)}$. The result now follows from Proposition 4 and the identity

$$\sum_{s=1}^{k+1} C_{n-1}^{(s)} = C_n^{(k)},$$

which can be established, for example, by counting nonnegative paths of $k+n$ upsteps and n downsteps by s = number of steps weakly after the last downstep.

4 Main result

Let $\mathcal{A}_n$ denote the set of sequences $\mathbf{u} = (u_1,\dots,u_n)$ satisfying $1 \le u_i \le n$ for all i, and condition (1). The main result, Theorem 1, can now be stated as

Theorem 6. $|\,\mathcal{A}_n\,| = \sum_{k=0}^{n-1} C_k^{(n-1-k)} C_{n-k}$ *for $n \ge 1$.*

Proof. Define the statistic X on $\mathbf{u} \in \mathcal{A}_n$ by $X =$ largest $i \in [n]$ such that $u_i > i$, with $X = 0$ if there is no such i. Set $\mathcal{A}_{n,k} = \{\mathbf{u} \in \mathcal{A}_n \,:\, X(\mathbf{u}) = k\}$, $0 \le k \le n-1$. We claim $|\,\mathcal{A}_{n,k}\,| = C_k^{(n-1-k)} C_{n-k}$, from which the Theorem follows. To see the claim, we have $\mathcal{A}_{n,0} = \mathcal{B}_n$ and Proposition (3) says $|\,\mathcal{B}_n\,| = C_n$. So the claim holds for $k = 0$. Now suppose $k \ge 1$ and $(u_1, \ldots, u_n) \in \mathcal{A}_{n,k}$. We have $u_k > k$ and, applying condition (1) with $i = k$ and $j \in [k+1, n]$, we have

$$u_j - (j-k) \notin [1, u_k - 1].$$

Hence, $u_j - (j-k) \le 0$ or $u_j - (j-k) \ge u_k$, that is, $u_j \le j - k$ or $u_j \ge j + u_k - k > j$. We can't have $u_j > j$, by the maximality in the definition of $X(\mathbf{u}) = k$, and so $u_j \le j - k$ for $j \in [k+1, n]$. Now $(v_1, \ldots, v_{n-k}) := (u_{k+1}, \ldots, u_n)$ inherits condition (1) and $v_i \le i$ for $1 \le i \le n-k$. Thus $(u_{k+1}, \ldots, u_n) \in \mathcal{B}_{n-k}$.

Also, for $1 \le i \le k$, we have $u_i \le n = k + (n-k-1) + 1$ and $u_k > k$, in other words, $(u_1, \ldots, u_k) \in \mathcal{F}_k^{(n-k-1)}$. The map $(u_1, \ldots, u_n) \to \{(u_1, \ldots, u_k), (u_{k+1}, \ldots, u_n)\}$ is in fact a bijection from $\mathcal{A}_{n,k}$ to the Cartesian product $\mathcal{F}_k^{(n-k-1)} \times \mathcal{B}_{n-k}$. The claim now follows from the counting results of Propositions 3 and 5. □

5 Extensions

Similar methods establish a combinatorial interpretation for the k-shifted Catalan sequence $(0, 0, \ldots, 0, 1, 1, 2, 5, 14, \ldots)$ with k initial 0's.

Theorem 7. *Let $(b_n)_{n \ge 0}$ be the Catalan transform of the k-shifted Catalan sequence. Then $b_n = 0$ for $n \le k-1$, $b_k = 1$ and, for $n \ge k+1$, b_n is the number of sequences $(u_1, \ldots, u_{n-k})$ satisfying $1 \le u_i \le n$ and condition (1).*

Stefan Forcey [4] gives a combinatorial interpretation for the Catalan transform of the 1-shifted Catalan sequence (A121988 in The On-Line Encyclopedia of Integer Sequences [5]): b_n is the number of vertices of the n-th multiplihedron.

References

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