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Adam Adamandy Kochański's approximations of π: reconstruction of the algorithm

Henryk Fukś

Department of Mathematics, Brock University
St. Catharines, Ontario L2S 3A1, Canada
Toronto, Ontario M5T 3J1, Canada
Email: hfuks@brocku.ca

Abstract

In his 1685 paper "Observationes cyclometricae" published in Acta Eruditorum, Adam Adamandy Kochański presented an approximate ruler-and-compass construction for rectification of the circle. It is not generally known that the first part of this paper included an interesting sequence of rational approximations of π. Kochański gave only a partial explanation of the algorithm used to produce these approximations, while promising to publish details at a later time, which has never happened. We reconstruct the complete algorithm and discuss some of its properties. We also argue that Kochański was very close to discovery of continued fractions and convergents of π.

1. Introduction

Adam Adamandy Kochański SJ (1631–1700) was a Polish Jesuit mathematician, inventor, and polymath. His interest were very diverse, including problems of geometry, mechanics, and astronomy, design and construction of mechanical clocks, *perpetuum mobile* and mechanical computers, as well as many other topics. He published relatively little, and most of his mathematical works appeared in *Acta Eruditorum* between 1682 and 1696. He left a reach correspondence, however, which currently consists of 163 surviving letters [1]. These letters include correspondence with Gottfried Leibniz, Athanasius Kircher SJ, Johannes Hevelius, Gottfried Kirch, and many other luminaries of the 17-th century, giving a rich record of Kochański's activities and a vivid description of the intellectual life of the period. Recently published comprehensive monograph [2] gives detailed account of his life and work, and includes extensive bibliography of the relevant literature.

Among his mathematical works, the most interesting and well-known is his paper on the rectification of the circle and approximations of π, published in 1685 in *Acta Eruditorum* under the title

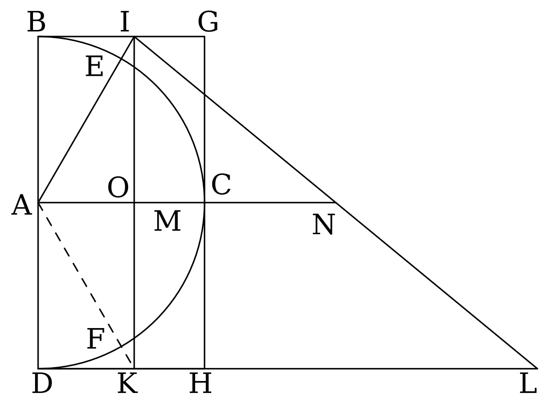

Figure 1: Kochański's construction of approximation of π.

Observationes cyclometricae ad facilitandam praxin accomodatae [3]. Annotated English translation of *Observationes* with parallel Latin version has been made available online by the author [4].

The paper has three distinct parts, the first one giving a sequence of rational approximations of π. This will be the main subject of this note, so more about this will follow in the next section.

The second part of *Observationes* is the one which is most often commented and quoted. There the author proposes an approximate solution of the problem of the rectification of the circle, giving an elegant and simple construction of a linear segment whose length approximates π. Figure 1 shows this construction exacly as Kochański had it the original paper. We start with drawing a semi-circle of radius AB centered at A, inscribed in a rectangle $BGHD$. Than we draw a line AI such that $\measuredangle IAC = 60^\circ$. When the lower side of the rectangle is extended so that HL is equal to the diameter of the circle, and a line is drawn from I to L, one can easily show that

$$|IL| = \frac{1}{3}\sqrt{120 - 18\sqrt{3}} = 3.1415333\ldots, \tag{1}$$

which agrees with π in the first four digits after the decimal point. This compass and ruler construction is often referred to as *Kochański's construction*.

The third part of [3] gives yet another approximation of π, this time expressing it as a sum of multiples and fractional parts of $1/32$,

$$\frac{96}{32} + \frac{4}{32} + \frac{1}{2} \cdot \frac{1}{32} + \frac{1}{32 \cdot 32} = \frac{3217}{1024} = 3.1416015625.$$

It is fair to say that if the rectification of the circle reported in *Observationes* received a lot of attention from both contemporaries of Kochański and historians of mathematics [5, 6, 7], then the first part of his paper has mostly been forgotten. In what follows we will show that this is perhaps unjustly so, as it includes some intriguing sequences of fractions approximating π, origin of which has not been explained by commentators of Kochański's work.

2. Sequence of rational approximations of π

In the table on p. 395 of [3] (also p. 2 of [4]), Kochański gives the following sequence of pairs of lower and upper rational approximants of π:

$$\left\{\frac{25}{8},\frac{22}{7}\right\},\left\{\frac{333}{106},\frac{355}{113}\right\},\left\{\frac{1667438}{530762},\frac{1667793}{530875}\right\},\left\{\frac{9252915567}{2945294501},\frac{9254583360}{2945825376}\right\},$$
$$\left\{\frac{136727214560643}{43521624105025},\frac{136736469144003}{43524569930401}\right\}. \tag{2}$$

As mentioned in the original paper, two of these fractions can be further reduced, $\frac{1667438}{530762}=\frac{833719}{265381}$, and $\frac{9254583360}{2945825376}=\frac{96401910}{30685681}$, while all the others are already written in their lowest terms. He then partially describes the algorithm generating these approximants, which could be explained using modern terminology and notation as follows. Let us denote the first element of the n-th pair (lower approximant) by P_n/Q_n, and the second element (upper approximant) by R_n/S_n. The approximants are then generated by recurrence equations

$$Q_{n+1} = S_n x_n + 1, \tag{3}$$
$$P_{n+1} = R_n x_n + 3, \tag{4}$$
$$S_{n+1} = S_n(x_n+1)+1, \tag{5}$$
$$R_{n+1} = R_n(x_n+1)+3, \tag{6}$$

where $R_0 = 22$, $S_0 = 7$. In these formulae x_n is a sequence of numbers which Kochański calls *genitores*, giving the first four values of x_n:

$$15, 4697, 5548, 14774. \tag{7}$$

Unfortunately, he does not explain how he obtained these numbers. He only makes the following remark regarding them:

> Methodicam prædictorum Numerorum Synthesin in *Cogitatis, & Inventis Polymathematicis*, quæ, si DEUS vitam prorogaverit, utilitati publicæ destinavi, plenius exponam;[1]

In spite of this declaration, *Cogitata & Inventa Polymathematica* have never appeared in print. It is possible that some explanation could have been found in unpublished manuscripts of Kochański, but unfortunately all his personal papers gathered by the National Library in Warsaw perished during the Warsaw Uprising in 1944. To the knowledge of the author, nobody has ever attempted to find the algorithm for generating the sequence of *genitores*. In the most comprehensive analysis of Kochański's mathematical works published up to date [8], Z. Pawlikowska did not offer any explanation either.

In the subsequent section, we attempt to reproduce the most likely method by which Kochański could have obtained the sequence of *genitores*, and, consequently, a sequence of approximants of π converging to π. We will also explain why he gave only first four terms of the sequence.

[1] I will explain the method of generating the aforementioned numbers more completely in *Cogitata & Inventa Polymathematica*, which work, if God prolongs my life, I have decided to put out for public benefit. (transl. H.F.)

3. Construction of genitores

It seems plausible that the starting point for Kochański's considerations was Archimedes approximation of π by $22/7$ and the result of Metius

$$\frac{333}{106} < \pi < \frac{355}{113}. \tag{8}$$

It is also likely that Kochański then noticed that Metius' result can be obtained from Archimedes' approximation by writing

$$\frac{333}{106} = \frac{22 \cdot 15 + 3}{7 \cdot 15 + 1}. \tag{9}$$

Where is the factor 15 coming from? The key observation here is that 15 is the "optimal" factor, in the sense that it is the largest integer value of x for which $\frac{22 \cdot x+3}{7 \cdot x+1}$ remains smaller than π.

The next most likely step in Kochański's reasoning was the observation that the upper approximant can be obtained by incrementing 15 to 16,

$$\frac{355}{113} = \frac{22 \cdot 16 + 3}{7 \cdot 16 + 1}. \tag{10}$$

Repeating this procedure for $\frac{355}{113}$ produces another pair of approximants,

$$\frac{355 \cdot 4697 + 3}{113 \cdot 4697 + 1} = \frac{1667438}{530762}, \tag{11}$$

$$\frac{355 \cdot 4698 + 3}{113 \cdot 4698 + 1} = \frac{1667793}{530875}, \tag{12}$$

where 4697 is again the largest integer x for which $\frac{355 \cdot x+3}{113 \cdot x+1} < \pi$. Recursive application of the above process produces the desired sequence of pairs given in eq. (2), and the values of x thus obtained are precisely what Kochański calls *genitores*.

What remains to be done is proving that the above algorithm indeed produces a sequence of lower and upper approximants of π, and that these converge to π in the limit of $n \to \infty$.

4. Kochański approximants

We will present the problem in a general setting. In what follows, α will denote a positive irrational number which we want to approximate by rational fractions.

Suppose that we have a pair of positive integers R and S such that their ratio is close to α but exceeds α, $R/S > \alpha$. Together with $\lfloor \alpha \rfloor$, we then have two rational bounds on α,

$$\frac{\lfloor \alpha \rfloor}{1} < \alpha < \frac{R}{S}. \tag{13}$$

Suppose now that we want to improve these bounds. As we will shortly see, this can be achieved by considering fractions which have the form

$$\frac{Rx + \lfloor \alpha \rfloor}{Sx + 1}, \tag{14}$$

where x is some positive integer. Before we go on, let us first note that the function $f(x) = \frac{Rx+\lfloor\alpha\rfloor}{Sx+1}$, treated as a function of real x has positive derivative everywhere except at $x = -1/S$, where it is undefined, and that there exists x where $f(x) = \alpha$, given by $x = (\alpha - \lfloor\alpha\rfloor)/(R - \alpha S)$.

Definition 1 *Let α be a positive irrational number, and let R and S be positive integers such that $\frac{R}{S} > \alpha$.* Genitor *of R, S with respect to α will be defined as*

$$g_\alpha(R,S) = \left\lfloor \frac{\alpha - \lfloor\alpha\rfloor}{R - \alpha S} \right\rfloor. \tag{15}$$

Let us note that if $g_\alpha(R,S)$ is positive, then it is the largest positive integer x such that $\frac{Rx+\lfloor\alpha\rfloor}{Sx+1} < \alpha$, i.e.,

$$g_\alpha(R,S) = \max\{x \in \mathbb{N} : \frac{Rx + \lfloor\alpha\rfloor}{Sx+1} < \alpha\}. \tag{16}$$

In this notation, the four *genitores* given in the paper can thus be written as $g_\pi(22,7) = 15$, $g_\pi(355,113) = 4697$, $g_\pi(1667793,530875) = 5548$, and $g_\pi(9254583360,2945825376) = 14774$.

Using the concept of *genitores*, we can now tighten the bounds given in eq. (13).

Proposition 1 *For any $\alpha \in \mathbb{IQ}^+$ and $R, S \in \mathbb{Q}^+$, if $\frac{R}{S} > \alpha$ and if the genitor $g_\alpha(R,S)$ is positive, then*

$$\lfloor\alpha\rfloor < \frac{Rg_\alpha(R,S) + \lfloor\alpha\rfloor}{Sg_\alpha(R,S)+1} < \alpha < \frac{R(g_\alpha(R,S)+1) + \lfloor\alpha\rfloor}{S(g_\alpha(R,S)+1)+1} < \frac{R}{S}. \tag{17}$$

The second and third inequality is a simple consequence of the definition of $g_\alpha(R,S)$ and eq. (16). The first one can be demonstrated as follows. Since $R/S > \alpha$, then $R/S > \lfloor\alpha\rfloor$, and therefore $Rg_\alpha(R,S) > \lfloor\alpha\rfloor Sg_\alpha(R,S)$. Now

$$Rg_\alpha(R,S) + \lfloor\alpha\rfloor > \lfloor\alpha\rfloor Sg_\alpha(R,S) + \lfloor\alpha\rfloor,$$

and

$$\frac{Rg_\alpha(R,S) + \lfloor\alpha\rfloor}{Sg_\alpha(R,S)+1} > \lfloor\alpha\rfloor,$$

as required. To show the last inequality let us note that

$$\frac{R(g_\alpha(R,S)+1) + \lfloor\alpha\rfloor}{S(g_\alpha(R,S)+1)+1} - \frac{R}{S} = \frac{\lfloor\alpha\rfloor S - R}{S(S(g_\alpha(R,S)+1)+1)}.$$

Since $R/S > \lfloor\alpha\rfloor$, the numerator is negative, and the last inequality of (17) follows. □

The above proposition gives us a method to tighten the bounds of (13), and the next logical step is to apply this proposition recursively.

Definition 2 *Let $\alpha \in \mathbb{IQ}^+$ and let R_0, S_0 be positive integers such that $R_0/S_0 > \alpha$ and $g_\alpha(R_0,S_0) > 0$.* Kochański approximants *of α starting from R_0, S_0 are sequences of rational numbers $\{P_n/Q_n\}_{n=1}^\infty$ and $\{R_n/S_n\}_{n=0}^\infty$ defined recursively for $n \in \mathbb{N} \cup \{0\}$ by*

$$\begin{aligned} P_{n+1} &= R_n x_n + \lfloor\alpha\rfloor, \\ Q_{n+1} &= S_n x_n + 1, \\ R_{n+1} &= R_n(x_n+1) + \lfloor\alpha\rfloor, \\ S_{n+1} &= S_n(x_n+1) + 1, \end{aligned} \tag{18}$$

where $x_n = g_\alpha(R_n, S_n)$. *Elements of the sequence* $\{P_n/Q_n\}_{n=1}^{\infty}$ *will be called* lower approximants, *and element of the sequence* $\{R_n/S_n\}_{n=0}^{\infty}$ – upper approximants.

Note that

$$P_n = R_n - R_{n-1}, \tag{19}$$
$$Q_n = S_n - S_{n-1}, \tag{20}$$

therefore it is sufficient to consider sequences of R_n and S_n only, as these two sequences uniquely define both upper and lower approximants.

Proposition 2 *Kochański approximants have the following properties:*

(i) x_n *is non-decreasing sequence of positive numbers,*

(ii) $\lfloor \alpha \rfloor < \frac{P_n}{Q_n} < \alpha < \frac{R_n}{S_n} < \frac{R_0}{S_0}$ *for all* $n \geq 1$,

(iii) $\frac{R_n}{S_n}$ *is decreasing,*

(iv) $\frac{P_n}{Q_n}$ *is increasing,*

(v) $\lim_{n\to\infty} \frac{R_n}{S_n} = \lim_{n\to\infty} \frac{P_n}{Q_n} = \alpha.$

For (i), because of the definition of $x_n = g_\alpha(R_n, S_n)$ shown in eq. (15), we need to demonstrate that $R_n - \alpha S_n$ is non-increasing. To do this, let us check the sign of

$$\begin{aligned} R_n - \alpha S_n - (R_{n+1} - \alpha S_{n+1}) &= R_n - \alpha S_n - (R_n(x_n+1) + \lfloor \alpha \rfloor) \\ +\pi(S_n(x_n+1)+1) &= \alpha - \lfloor \alpha \rfloor - (R_n - \alpha S_n)x_n \\ &= \alpha - \lfloor \alpha \rfloor - (R_n - \alpha S_n)\left\lfloor \frac{\alpha - \lfloor \alpha \rfloor}{R_n - \alpha S_n} \right\rfloor. \end{aligned}$$

The last expression, by the definition of the floor operator, must be non-negative, thus $R_n - \pi S_n$ is non-increasing, and x_n is non-decreasing as a result. Now, since the definition of Kochański approximants requires that x_0 is positive, all other x_n must be positive too.

Property (ii) is just a consequence of the Proposition 1, which becomes clear once we note that

$$\frac{R_n}{S_n} = \frac{R_{n-1}(x_{n-1}+1)+3}{S_{n-1}(x_{n-1}+1)+1},$$

and

$$\frac{P_n}{Q_n} = \frac{R_{n-1}x_{n-1}+3}{S_{n-1}x_{n-1}+1},$$

where $x_{n-1} = g_\alpha(R_{n-1}, S_{n-1})$.

To show (iii), let us compute the difference between two consecutive terms of the sequence R_n/S_n,

$$\frac{R_n}{S_n}-\frac{R_{n-1}}{S_{n-1}}=\frac{R_{n-1}y_{n-1}+\lfloor\alpha\rfloor}{S_{n-1}y_{n-1}+1}-\frac{R_{n-1}}{S_{n-1}},$$

where we defined $y_{n-1}=g_\alpha(R_{n-1},S_{n-1})+1$. This yields

$$\begin{aligned}\frac{R_n}{S_n}-\frac{R_{n-1}}{S_{n-1}}&=\frac{(R_{n-1}y_{n-1}+\lfloor\alpha\rfloor)S_{n-1}-R_{n-1}(S_{n-1}y_{n-1}+1)}{S_{n-1}(S_{n-1}y_{n-1}+1)}\\&=\frac{\lfloor\alpha\rfloor S_{n-1}-R_{n-1}}{S_{n-1}(S_{n-1}y_{n-1}+1)}<0,\end{aligned}$$

because, by (ii), $R_{n-1}/S_{n-1}>\lfloor\alpha\rfloor$. The sequence R_n/S_n is thus decreasing. Proof of (iv) is similar and will not be presented here.

Le us now note that R_n/S_n is bounded from below by α and decreasing, thus it must have a limit. Similarly, $\frac{P_n}{Q_n}=\frac{R_n-R_{n-1}}{S_n-S_{n-1}}$ is bounded from above by α and increasing, so again it must have a limit. To demonstrate (v), it is therefore sufficient to show that limits of $\frac{R_n}{S_n}$ and $\frac{R_n-R_{n-1}}{S_n-S_{n-1}}$ are the same, or, what is equivalent, that

$$\lim_{n\to\infty}\left(\frac{R_n}{S_n}-\frac{R_n-R_{n-1}}{S_n-S_{n-1}}\right)=0. \tag{21}$$

We start by defining $\gamma_n=R_n/S_n-P_n/Q_n$ and observing that

$$\gamma_n=\frac{R_n}{S_n}-\frac{R_n-R_{n-1}}{S_n-S_{n-1}}=\frac{R_{n-1}S_n-R_nS_{n-1}}{S_n(S_n-S_{n-1})}.$$

By substituting $R_n=R_{n-1}(x_{n-1}+1)+\lfloor\alpha\rfloor$ and $S_n=S_{n-1}(x_{n-1}+1)+1$, one obtains after simplification

$$\gamma_n=\frac{R_{n-1}-\lfloor\alpha\rfloor S_{n-1}}{S_n(S_n-S_{n-1})}=\frac{\frac{R_{n-1}}{S_{n-1}}-\lfloor\alpha\rfloor}{S_n(\frac{S_n}{S_{n-1}}-1)}. \tag{22}$$

Since R_n/S_n is decreasing, and starts from R_0/S_0, we can write

$$\gamma_n<\frac{\frac{R_0}{S_0}-\lfloor\alpha\rfloor}{S_n(\frac{S_n}{S_{n-1}}-1)}=\frac{\frac{R_0}{S_0}-\lfloor\alpha\rfloor}{S_n(x_{n-1}+\frac{1}{S_{n-1}})}, \tag{23}$$

where we used the fact that $S_n=S_{n-1}(x_{n-1}+1)+1$ and where $x_{n-1}=g_\alpha(R_{n-1},S_{n-1})$. Since x_n is non-decreasing, and S_n increases with n, we conclude that $\gamma_n\to 0$ ad $n\to\infty$, as required. □

Let us remark here that x_n is indeed only non-decreasing, and it is possible for two consecutive values of x_n to be the same. For example, for $\alpha=\sqrt{2}$ and $R_0/S_0=3/2$, we obtain

$$\{x_n\}_{n=0}^{\infty}=2,4,4,15,17,77,101,119,\dots,$$

where $x_1=x_2$.

5. Initial values

One last thing to explain is the choice of the starting values R_0, S_0. Definition 2 requires that the *genitor* of these initial values is positive, so how can we choose R_0, S_0 to ensure this? We start by noticing that the second pair of Kochański's approximants (Metius' fractions $\frac{333}{106}$, $\frac{355}{113}$) are known to appear in the sequence of convergents of the continuous fraction representation of π. As we shall see, this is not just a coincidence.

Let us first recall two basic properties of continuous fraction expansion of a positive irrational number α,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}.$$

By convergent p_n/q_n we will mean a fraction (written in its lowest terms) obtained by truncation of the above infinite continued fraction after a_n. The first property we need is the recursive algorithm for generating convergents and values of a_n.

Proposition 3 *Consecutive convergents p_n/q_n of α can be obtained by applying the recursive formula*

$$a_{n+1} = \left\lfloor \frac{\alpha q_{n-1} - p_{n-1}}{p_n - \alpha q_n} \right\rfloor, \tag{24}$$

$$p_{n+1} = p_n a_{n+1} + p_{n-1}, \tag{25}$$

$$q_{n+1} = q_n a_{n+1} + q_{n-1}, \tag{26}$$

with initial conditions $a_0 = \lfloor \alpha \rfloor$, $a_1 = \lfloor \frac{1}{\alpha - a_0} \rfloor$, $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$, $q_1 = a_1$.

For example, for $\alpha = \pi$ we obtain

$$\left\{ \frac{p_n}{q_n} \right\}_{n=0}^{\infty} = \left\{ \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}, \ldots \right\} \tag{27}$$

Convergents are know to be the best rational approximations of irrational numbers, which can formally be stated as follows.

Proposition 4 *If p_n/q_n is a convergent for an irrational number α and p/q is an arbitrary fraction with $q < q_{n+1}$, then*

$$|q_n \alpha - p_n| < |qa - p| \tag{28}$$

Elementary proofs of both of the above propositions can be found in [9]. We also need to recall that convergents p_n/q_n are alternatively above and below α, so that for odd n we always have $p_n/q_n >$

α, and for even n, $p_n/q_n < \alpha$. Suppose that we now take some odd convergent p_{2k+1}/q_{2k+1}, and further set $p = \lfloor\alpha\rfloor$, $q = 1$. Inequality (28) then becomes

$$p_{2k+1} - \alpha q_{2k+1} < \alpha - \lfloor\alpha\rfloor, \tag{29}$$

and hence

$$\frac{\alpha - \lfloor\alpha\rfloor}{p_{2k+1} - \alpha q_{2k+1}} > 1. \tag{30}$$

This, by the definition of the *genitor* given in eq. (15), yields $g_\alpha(p_{2k+1}, q_{2k+1}) > 0$, leading to the following corollary.

Corollary 1 *If p_{2k+1}/q_{2k+1} is an odd convergent of α, then $g_\alpha(p_{2k+1}, q_{2k+1}) > 0$, and $R_0 = p_{2k+1}$, $S_0 = q_{2k+1}$ can be used as initial values in the construction of Kochański's approximants. In particular, one can generate Kochański's approximants starting from the first convergent of α, by taking $R_0 = a_0a_1 + 1$, $S_0 = a_1$, where $a_0 = \lfloor\alpha\rfloor$, $a_1 = \lfloor 1/(\alpha - a_0)\rfloor$.*

Note that Kochański in his paper indeed started from the first convergent of π, by taking $R_0/S_0 = 22/7$. Obviously, if one starts from the first convergent $R_0 = p_1$, $S_0 = q_1$, then the first lower approximant will be the second convergent, $P_1 = p_2$, $Q_1 = q_2$, and indeed in Kochański's case $P_1/Q_1 = p_2/q_2 = 333/106$. Other approximants do not have to be convergents, and they normally aren't, although convergents may occasionally appear in the sequence of lower or upper approximants. For example, in the case of $\alpha = \pi$, $R_1/S_1 = p_3/q_3 = 355/113$ and $P_2/Q_2 = p_8/q_8 = 833719/265381$.

We should also add here that the choice of the first convergent as the starting point is the most natural one. Among all pairs R_0, S_0 where $S_0 < 106$, the only cases for which $g_\pi(R_0, S_0) > 0$ are $R = 22k$, $R = 7k$, where $k \in \{1, 2, \ldots, 15\}$. If one wants to obtain fractions expressed by as small integers as possible, then taking $k = 1$ is an obvious choice.

6. Concluding remarks

We have reconstructed the algorithm for construction of rationals approximating π used in [3], and we have demonstrated that it can be generalized to produce approximants of arbitrary irrational number α. Under a suitable choice of initial values, approximants converge to α.

Using these results, we can generate more terms of the sequence of *genitores* for $\alpha = \pi$, $R_0/S_0 = 22/7$, going beyond first four terms found in Kochański's paper:

$$\{x_n\}_{n=0}^{\infty} = \{g_\pi(R_n, S_n)\}_{n=0}^{\infty} = \{15, 4697, 5548, 14774, 33696, 61072, 111231, 115985, 173819, 563316, 606004, \ldots\}. \tag{31}$$

We propose to call this sequence *Kochański sequence*. It has been submitted to the Online Encyclopedia of Integer Sequences as A191642 [10], and its entry in the Encyclopedia includes Maple code for generating its consecutive terms.

Knowing that $x_n = \left\lfloor \frac{\alpha - \lfloor\alpha\rfloor}{R_n - \alpha S_n} \right\rfloor$, we can also understand why only four terms of the sequence are given in the paper. In order to compute x_n, one needs to know π with sufficient accuracy. For example, 20 digits after the decimal point are needed in order to compute x_0 to x_3. Kochański was familiar with the work of Ludolph van Ceulen, who computed 35 digits of π, and this was more than enough to compute x_4. Nevertheless, Kochański in his paper performed all computations keeping track of "only" 25 digits, and this was falling just one digit short of the precision needed to compute x_4.

It is also interesting to notice that the recurrence equations in Definition 2 strongly resemble recurrence equations for convergents p_n/q_n in Proposition 3. Kochański was always adding 3 and 1 to the numerator and denominator in his approximants, because, as remarked earlier, he noticed that

$$\frac{22\cdot 15+3}{7\cdot 15+1}=\frac{333}{106}, \quad \frac{22\cdot 16+3}{7\cdot 16+1}=\frac{355}{113}. \tag{32}$$

He apparently failed to notice that

$$\frac{333\cdot 1+22}{106\cdot 1+7}=\frac{355}{113}, \tag{33}$$

that is, instead of finding the largest x for which $(22x+3)/(7x+1)<\pi$, one can take the last two approximants, $22/7$ and $333/106$, and then find the largest x such that $(333x+22)/(106x+7)>\pi$. If he had done this he would have discovered convergents and continued fractions. His genitores would then be a_n values in the continued fraction expansion of π. In the meanwhile, continued fractions and convergents had to wait until 1695 when John Wallis laid the groundwork for their theory in his book *Opera Mathematica* [11].

One little puzzling detail remains, however. If we look at the Definition 2, we notice that the sequence of lower approximants P_n/Q_n starts from $n=1$, not from $n=0$, as is the case for the upper approximants R_n/S_n. Indeed, P_0, Q_0 are not needed to start the recursion. Nevertheless, in the table of approximants given in [3], in the second row there is a pair of values corresponding to $n=0$, namely $P_0/Q_0=25/8$ (in the first row of the table he also gives the obvious bounds $3<\pi<4$). These numbers are not needed in any subsequent calculation, and Kochański does not explain where do they come from. One can only speculate that perhaps he wanted the table to appear "symmetric", thus he entered some arbitrary fraction approximating π from below as P_0/Q_0.

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