The Number of $\overline{31}542$-Avoiding Permutations

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Abstract

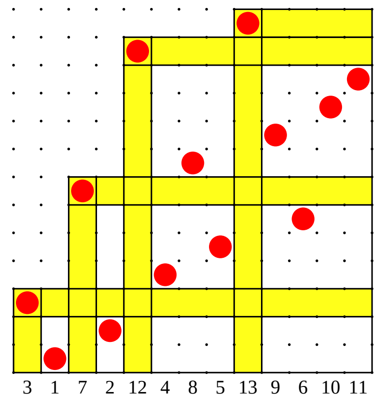
We confirm a conjecture of Lara Pudwell and show that permutations of $[n]$ that avoid the barred pattern $\overline{31}542$ are counted by OEIS sequence A047970. In fact, we show bijectively that the number of $\overline{31}542$ avoiders of length n with $j+k$ left-to-right maxima, of which j initiate a descent in the permutation and k do not, is $\binom{n}{k} j! \left\{ {n-j-k \atop j} \right\}$, where $\left\{ {n \atop j} \right\}$ is the Stirling partition number.

1 Introduction

A permutation π avoids the barred pattern $\overline{31}542$ if each instance of a not-necessarily-consecutive 542 pattern in π is part of a 31542 pattern in π. Lara Pudwell [1] introduces an automated method to produce "prefix enumeration schemes" for permutations avoiding a set of one or more barred patterns. Such a scheme, if found, translates to a recurrence to count the corresponding avoiders. For single patterns of length 5 with 2 bars, she notes a success rate of 136/172 (79.1%). The title pattern is among the failures but Pudwell observes [2] that the counting sequence for this pattern appears to be A047970 in The On-Line Encyclopedia of Integer Sequences [3]. Here we confirm this conjecture. The definition of A047970 is $a(n) = \sum_{i=0}^{n} \left((i+1)^{n-i} - i^{n-i} \right)$, which is equivalent to $a(n) = \sum_{j,k\ge 0} \binom{n}{k} j! \left\{ {n-j-k \atop j} \right\}$, where $\left\{ {n \atop j} \right\}$ is the Stirling partition number. (This equivalence is not entirely obvious and a proof is outlined in the Appendix.) Our result is the following.

Theorem 1. *The number of* $\overline{31}542$ *avoiders of length* n *with* $j+k$ *left-to-right maxima, of which* j *initiate a descent in the permutation and* k *do not, is* $\binom{n}{k} j! \left\{ {n-j-k \atop j} \right\}$.

Section 2 explains the $\binom{n}{k}$ factor, and reduces the problem to the case $k=0$. Section 3 looks at the structure of the avoiders with $k=0$ and presents a bijection to complete the proof. To standardize a permutation means to replace its smallest entry by 1, next smallest by 2, and so on.



Permutation as matrix diagram

Figure 1

2 Reduction

We begin with a reduction of the problem. Deleting the left-to-right maxima that do not initiate a descent and standardizing the remaining permutation does not affect the avoider status because a left-to-right maximum can serve only as the 5 of a 542 pattern or as the 3 in the 31 of a 31542 pattern, and each deleted left-to-right maximum is immediately followed (in the original permutation) by another left-to-right maximum or ends the path and so "it won't be missed". Furthermore, the original permutation can be recovered from knowledge of the deleted entries and the standardized permutation. These observations account for the $\binom{n}{k}$ factor (choose k entries from $[n]$ to serve as the left-to-right maxima that do not initiate a descent), and it means that Theorem 1 will follow from

Theorem 2. *The number of* $\overline{31}542$*-avoiding permutations with* j *left-to-right maxima, all of which initiate a descent, is* $j!\left\{ {n-j \atop j} \right\}$.

3 Proof of Theorem 2

To characterize $\overline{31}542$-avoiders in which each left-to-right maximum initiates a descent, it is convenient to represent a permutation π as the usual matrix diagram, as in Figure 1, with a bullet in the (a, b) position (measuring from lower left) if and only if $\pi(a) = b$.

The left-to-right maxima determine the upper left staircase boundary and the yellow lines through the left-to-right maxima delineate a collection of $1 + 2 + \ldots + j$ cells (the white rectangles), where j is the number of left-to-right maxima. Some cells may be empty (contain no bullets) or even vacuous (contain no area). The cells in turn split into horizontal strips and also into vertical strips: the i-th horizontal (resp. vertical) strip contains $j + 1 - i$ (resp. i) cells. The condition that each left-to-right maximum initiates a descent means that no vertical strip consists entirely of empty cells.

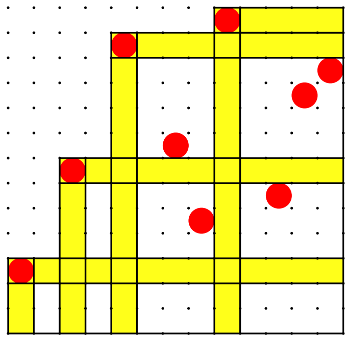


Figure 2a

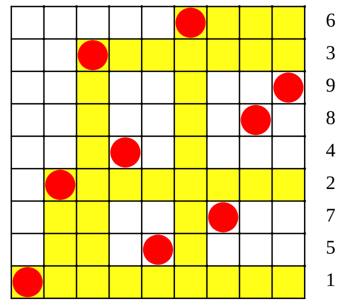


Figure 2b

Proposition 3. *Suppose each left-to-right maximum in a permutation π initiates a descent. Then π is $\overline{31}542$-avoiding if and only if the bullets in each horizontal strip are rising from left to right.*

Proof. Two falling bullets correspond to an inversion ba in the permutation. If they lie in the same horizontal strip, let c denote the left-to-right maximum associated with this strip. Then cba is a 542 with no available 3 let alone 31.

On the other hand, neither the 4 nor 2 of a 542 pattern can be a left-to-right maximum and, if the rising bullet condition holds, the 4 and 2 lie in *different* horizontal strips. The left-to-right maximum m associated with the horizontal strip containing the 2 serves as the 3. The vertical strip associated with m is nonempty (since all vertical strips are nonempty) and any bullet in it serves as the 1. $\square$

Let $\mathcal{T}_j = [1] \times [2] \times \ldots \times [j]$ (which may conveniently be referred to as the set of inversion codes for permutations of $[j]$), and let $\mathcal{P}_{n,j}$ denote the set of partitions of $[n]$ into j blocks in a canonical form: the smallest entry of each block is last, the remaining entries increase left to right, and the blocks are arranged in order of increasing smallest entry. For example, $1\,/\,5\,7\,2\,/\,4\,8\,9\,3\,/\,6 \in \mathcal{P}_{9,4}$. Clearly, $|\,\mathcal{T}_j\,| = j!$ and $|\,\mathcal{P}_{n,j}\,| = \left\{ {n \atop j} \right\}$.

The *successors* in π are the entries immediately following the left-to-right maxima, that is, the terminators of the descents which the left-to-right maxima initiate. Here is a bijection from the set $\mathcal{A}_{n,j}$ of $\overline{31}542$-avoiding permutations of $[n]$ with j left-to-right maxima, all of which initiate descents, to $\mathcal{T}_j \times \mathcal{P}_{n-j,\,j}$.

Given $\pi \in \mathcal{A}_{n,j}$, draw its diagram. Record the horizontal strips (numbered bottom to top) of the successors in π. In the example of Figure 1, the successors are $(1, 2, 4, 9)$ and their horizontal strips are $(1, 1, 2, 3)$. This is the desired inversion code.

Erase the bullets of the successors (Figure 2a), "prettify" the diagram (Figure 2b), record the position of each bullet, bottom to top, among all the bullets and place a divider to the right of each right-to-left minimum. This is the desired partition in $\mathcal{P}_{n-j,\,j}$ and it is already in canonical form. The example yields $1\,/\,5\,7\,2\,/\,4\,8\,9\,3\,/\,6$.

The map is reversible because the diagram of Figure 2b can be recovered from the partition, the inversion code determines the cell of each missing bullet and, within the cell, the bullet's horizontal position is the extreme left (since it lies immediately after a left-to-right maximum) and its vertical position is determined by the rising bullet condition.

4 Appendix

Proposition 4.

$$\sum_{i=0}^{n}\left((i+1)^{n-i}-i^{n-i}\right)=\sum_{j,k\geq 0}\binom{n}{k}j!\left\{ {n-j-k \atop j} \right\}. \tag{1}$$

Proof. Let $A_n=\sum_{i=0}^{n} i^{n-i}$. Then $\sum_{i=0}^{n}(i+1)^{n-i}=A_{n+1}$, and the left hand side of (1) is $A_{n+1}-A_n$. The binomial theorem says $A_{n+1}=\sum_{i=0}^{n}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}i^{\ell}$, leading to

$$A_{n+1}-A_n=\sum_{\ell=0}^{n}\sum_{i=0}^{n-\ell}\left(\binom{n-i}{\ell}-\binom{n-i-1}{\ell}\right)i^{\ell}=\sum_{\ell=0}^{n}\sum_{i=0}^{n-\ell}\binom{n-i-1}{\ell-1}i^{\ell}.$$

On the other hand, the right side of (1) is

$$\begin{aligned}\sum_{j,k\geq 0}\binom{n}{k}j!\left\{ {n-j-k \atop j} \right\} &\overset{\text{(i)}}{=} \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{i=0}^{j}\binom{n}{k}(-1)^{j-i}\binom{j}{i}i^{n-j-k}\\ &\overset{\text{(ii)}}{=} \sum_{\ell=0}^{n}\sum_{i=0}^{\ell}i^{n-\ell}\sum_{k=0}^{\ell-i}\binom{n}{k}(-1)^{\ell-i-k}\binom{\ell-k}{i}\\ &\overset{\text{(iii)}}{=} \sum_{\ell=0}^{n}\sum_{i=0}^{\ell}i^{n-\ell}\binom{n-i-1}{n-\ell-1}\\ &\overset{\text{(iv)}}{=} \sum_{\ell=0}^{n}\sum_{i=0}^{n-\ell}i^{\ell}\binom{n-i-1}{\ell-1},\end{aligned}$$

agreeing with the left side.

Notes on equalities:

(i) Expand Stirling partition number.

(ii) Change summation index from j to ℓ with $\ell=j+k$.

(iii) Evaluate inner sum; this is the essential content of identity (5.25) in [4]. The identity (5.25) in [4] ostensibly has 4 parameters, but a change of summation index from k to $j=k-n$ in (5.25) leaves only 3 independent parameters, as here.

(iv) Reverse order of outer summation.

References

[1] Lara Pudwell, Enumeration Schemes for Permutations Avoiding Barred Patterns, *Electronic J. Combinatorics*, **17** (1) (2010), R29.

[2] Lara Pudwell, Comment on sequence A047970 in The On-Line Encyclopedia of Integer Sequences.

[3] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

[4] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics* (2nd edition), Addison-Wesley, 1994.