Acta Univ. Sapientiae, Informatica, **3,** 2 (2011) 230–268

On Erdős-Gallai and Havel-Hakimi algorithms

Antal Iványi
Department of Computer Algebra,
Eötvös Loránd University, Hungary
email: tony@inf.elte.hu

Loránd Lucz
Department of Computer Algebra,
Eötvös Loránd University, Hungary
email: lorand.lucz@gmail.com

Tamás F. Móri
Department of Probability Theory and Statistics,
Eötvös Loránd University, Hungary
email: moritamas@ludens.elte.hu

Péter Sótér
Department of Computer Algebra,
Eötvös Loránd University, Hungary
email: mapoleon@freemail.hu

Abstract. Havel in 1955 [28], Erdős and Gallai in 1960 [21], Hakimi in 1962 [26], Ruskey, Cohen, Eades and Scott in 1994 [69], Barnes and Savage in 1997 [6], Kohnert in 2004 [49], Tripathi, Venugopalan and West in 2010 [83] proposed a method to decide, whether a sequence of nonnegative integers can be the degree sequence of a simple graph. The running time of their algorithms is $\Omega(n^2)$ in worst case. In this paper we propose a new algorithm called EGL (Erdős-Gallai Linear algorithm), whose worst running time is $\Theta(n)$. As an application of this quick algorithm we computed the number of the different degree sequences of simple graphs for 24, ..., 29 vertices (see [74]).

1 Introduction

In the practice an often appearing problem is the ranking of different objects as hardware or software products, cars, economical decisions, persons etc. A

Computing Classification System 1998: G.2.2. [Graph Theory]: Subtopic - Network problems.
Mathematics Subject Classification 2010: 05C85, 68R10
Key words and phrases: simple graphs, prescribed degree sequences, Erdős-Gallai theorem, Havel-Hakimi theorem, graphical sequences

typical method of the ranking is the pairwise comparison of the objects, assignment of points to the objects and sorting the objects according to the sums of the numbers of the received points.

For example Landau [51] references to biological, Hakimi [26] to chemical, Kim et al. [45], Newman and Barabási [61] to net-centric, Bozóki, Fülöp, Poesz, Kéri, Rónyai and Temesi to economical [1, 10, 11, 42, 80], Liljeros et al. [52] to human applications, while Iványi, Khan, Lucz, Pirzada, Sótér and Zhou [30, 31, 38, 65, 67] write on applications in sports.

From several popular possibilities we follow the terminology and notations used by Paul Erdős and Tibor Gallai [21].

Depending from the rules of the allocation of the points there are many problems. In this paper we deal only with the case when the comparisons have two possible results: either both objects get one point, or both objects get zero points. In this case the results of the comparisons can be represented using simple graphs and the number of points gathered by the given objects are the degrees of the corresponding vertices. The decreasing sequence of the degrees is denoted by $b = (b_1, \dots, b_n)$.

From the popular problems we investigate first of all the question, how can we quickly decide, whether for given b does exist there a simple graph G whose degree sequence is b. In connection with this problem we remark that the main motivation for studying of this problem is the question: what is the complexity of deciding whether a sequence is the score sequence of some football tournament [24, 32, 35, 36, 43, 44, 54].

As a side effect we extended the popular data base *On-line Encyclopedia of Integer Sequences* [72] with the continuation of contained sequences.

In connection with the similar problems we remark, that in the last years a lot of papers and chapters were published on the undirected graphs (for example [8, 9, 12, 16, 19, 29, 37, 41, 55, 68, 81, 83, 84, 85]) and also on directed graphs (for example [7, 11, 14, 23, 24, 30, 31, 33, 38, 45, 48, 50, 57, 58, 63, 65, 64, 66]).

The majority of the investigated algorithms is sequential, but there are parallel results too [2, 18, 20, 36, 60, 62, 77].

Let l, u and m integers ($m \geq 1$ and $u \geq l$). A sequence of integer numbers $b = (b_1, \dots, b_m)$ is called (l, u, m)-*bounded,* if $l \leq b_i \leq u$ for $i = 1, \dots, m$. A (l, u, m)-*bounded* sequence b is called (l, u, m)-*regular,* if $b_m \geq b_{m-1} \geq \dots \geq b_1$. An (l, u, m)-regular sequence is called (l, u, m)-*even,* if the sum of its elements is even. A $(0, n-1, n)$-*regular* sequence b is called n-*graphical,* if there exists a simple graph G whose degree sequence is b. If $l = 0$, $u = n-1$ and $m = n$, then we use the terms n-bounded, n-regular, n-even, and n-

graphical (or simply bounded, regular, even, graphical).

In the following we deal first of all with regular sequences. In our definitions the bounds appear to save the testing algorithms from the checking of such sequences, which are obviously not graphical, therefore these bounds do not mean the restriction of the generality.

The paper consists of nine parts. After the introductory Section 1 in Section 2 we describe the classical algorithms of the testing and reconstruction of degree sequences of simple graphs. Section 3 introduces several linear testing algorithms, then Section 4 summarizes some properties of the approximate algorithms. Section 5 contains the description of new precise algorithms and in Section 6 the running times of the classical testing algorithms are presented. Section 7 contains enumerative results, in Section 8 we report on the application of the new algorithms for the computation of the number of score sequences of simple graphs. Finally Section 9 contains the summary of the results.

Our paper [37] written in Hungarian contains further algorithms and simulation results. [35] contains a short summary on the linear Erdős-Gallai algorithm while in [36] the details of the parallel implementation of enumerating Erdős-Gallai algorithm are presented.

2 Classical precise algorithms

For a given n-regular sequence $b = (b_1, \ldots, b_n)$ the first i elements of the sequence we call *the head* of the sequence belonging to the index i, while the last $n - i$ elements of the sequence we call *the tail* of the sequence belonging to the index i.

2.1 Havel-Hakimi algorithm

The first algorithm for the solution of the testing problem was proposed by Vaclav Havel Czech mathematician [28, 53]. In 1962 Louis Hakimi [26] published independently the same result, therefore the theorem is called today usually as *Havel-Hakimi theorem*, and the method of reconstruction is called *Havel-Hakimi algorithm.*

Theorem 1 (Hakimi [26], Havel [28]). *If* $n \geq 3$*, then the* n*-regular sequence* $b = (b_1, \ldots, b_n)$ *is* n*-graphical if and only if the sequence* $b' = (b_2 - 1, b_3 - 1, \ldots, b_{b_1} - 1, b_{b_1+1} - 1, b_{b_1+2}, \ldots, b_n)$ *is* $(n-1)$*-graphical.*

Proof. See [9, 26, 28, 37]. □

If we write a recursive program based on this theorem, then according to the RAM model of computation its running time will be in worst case $\Omega(n^2)$, since the algorithm decreases the degrees by one, and e.g. if $b = ((n-1)^n)$, then the sum of the elements of b equals to $\Theta(n^2)$. It is worth to remark that the proof of the theorem is constructive, and the algorithm based on the proof not only tests the input in quadratic time, but also construct a corresponding simple graph (of course, only if it there exists).

It is worth to remark that the algorithm was extended to directed graphs in which any pair of the vertices is connected with at least $a \geq 0$ and at most $b \geq a$ edges [30, 31]. The special case $a = b = 1$ was reproved in [23].

In 1965 Hakimi [27] gave a necessary and sufficient condition for *two sequences* $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ to be the in-degree sequences and out-degree sequence of a directed multigraph without loops.

2.2 Erdős-Gallai algorithm

In chronological order the next result is the necessary and sufficient theorem published by Paul Erdős and Tibor Gallai [21].

For an n-regular sequence $b = (b_1, \ldots, b_n)$ let $H_i = b_1 + \cdots + b_i$. For given i the elements $b_1, \ldots, b_i$ are called *the head* of b, belonging to i, while the elements $b_{i+1}, \ldots, b_n$ are called *the tail* of b belonging to i.

When we investigate the realizability of a sequence, a natural observation is that the degree requirement H_i of a head is covered partially with inner and partially with outer degrees (with edges among the vertices of the head, resp. with edges, connecting a vertex of the head and a vertex of the tail). This observation is formalized by the following Erdős-Gallai theorem.

Theorem 2 (Erdős, Gallai [21]) *Let* $n \geq 3$. *The* n*-regular sequence* $b = (b_1, \ldots, b_n)$ *is* n*-graphical if and only if*

$$\sum_{i=1}^{n} b_i \quad \textit{even} \tag{1}$$

and

$$\sum_{i=1}^{j} b_i \leq j(j-1) + \sum_{k=j+1}^{n} \min(j, b_k) \quad (j = 1, \ldots, n-1). \tag{2}$$

Proof. See [9, 15, 21, 70, 83]. □

n	$R(n)$	$E(n)$	$E(n)/R(n)$
1	1	1	1.0000000000000
2	3	2	0.6666666666667
3	10	6	0.6000000000000
4	35	19	0.5428571428571
5	126	66	0.5238095238095
6	462	236	0.5108225108225
7	1716	868	0.5058275058275
8	6435	3235	0.5027195027195
9	24310	12190	0.5014397367339
10	92378	46252	0.5006819805581
11	352716	176484	0.5003572279114
12	1352078	676270	0.5001708481315
13	5200300	2600612	0.5000888410284
14	20058300	10030008	0.5000427753100
15	77558760	38781096	0.5000221251603
16	300540195	150273315	0.5000107057227
17	1166803110	583407990	0.5000055150693
18	4537567650	2268795980	0.5000026787479
19	17672631900	8836340260	0.5000013755733
20	68923264410	34461678394	0.5000006701511
21	269128937220	134564560988	0.5000003432481
22	1052049481860	526024917288	0.5000001676328
23	4116715363800	2058358034616	0.5000000856790
24	16123801841550	8061901596814	0.5000000419280
25	63205303218876	31602652961516	0.5000000213918
26	247959266474052	123979635837176	0.5000000104862
27	973469712824056	486734861612328	0.5000000053420
28	3824345300380220	1912172660219260	0.5000000026224
29	15033633249770520	7516816644943560	0.5000000013342
30	59132290782430712	29566145429994736	0.5000000006558
31	232714176627630544	116357088391374032	0.5000000003333
32	916312070471295267	458156035385917731	0.5000000001640
33	3609714217008132870	1804857108804606630	0.5000000000833
34	14226520737620288370	7113260369393545740	0.5000000000410
35	56093138908331422716	28046569455332514468	0.5000000000208
36	221256270138418389602	110628135071477978626	0.5000000000103
37	873065282167813104916	436532641088444120108	0.5000000000052
38	3446310324346630677300	1723155162182151654600	0.5000000000026

Figure 1: Number of regular and even sequences, and the ratio of these numbers

Although this theorem does not solve the problem of reconstruction of graphical sequences, the systematic application of (2) requires in worst case (for example when the input sequence is graphical) $\Theta(n^2)$ time.

Recently Tripathi and Vijay [83] published a constructive proof of Erdős-Gallai theorem and proved that their construction requires $O(n^3)$ time.

Figure 1 shows the number of n-regular ($R(n)$ and n-even ($E(n)$ sequences and their ratio ($E(n)/R(n)$ for $n = 1, \ldots, 38$. According to (34) the sequence of these ratios tends to $\frac{1}{2}$ as n tends to ∞. According to Figure 1 the convergence is quick: e.g. $E(20)/R(20) = 0.5000006701511$.

The pseudocode of ERDŐS-GALLAI see in [37].

3 Testing algorithms

We are interested in the investigation of football sequences, where often appears the necessity of the testing of degree sequences of simple graphs.

A possible way to decrease the expected testing time is to use quick (linear) filtering algorithms which can state with a high probability, that the given input is not graphical, and so we need the slow precise algorithms only in the remaining cases.

Now we describe a *parity checking*, then a *binomial,* and finally a *headsplitting* filtering algorithm.

3.1 Parity test

Our first test is based on the first necessary condition of Erdős-Gallai theorem. This test is very effective, since according to Figure 1 and Corollary 14 about the half of the regular sequences is odd, and our test establishes in linear time, that these sequences are not graphical.

The following simple algorithm is based on (1).

Input. n: number of the vertices ($n \geq 1$);
$b = (b_1, \ldots, b_n)$: an n-regular sequence.

Output. L: logical variable ($L = $ FALSE shows, that b is not graphical, while the meaning of the value $L = $ TRUE is, that the test *could not decide,* whether b is graphical or not).

Working variable. i: cycle variable;
$H = (H_1, \ldots, H_n)$: H_i is the sum of the first i elements of b.

```
PARITY-TEST(n, b, L)
H_1 = b_1
for i = 2 to n
    H_i = H_{i-1} + b_i
```

```
if H_n odd
    L = FALSE
    return L
L = TRUE
return L
```

The running time of this algorithm is $\Theta(n)$ in all cases. Figure 1 characterizes the efficiency of Parity-test.

(1) is only a necessary condition, therefore Parity-Test is only an approximate (filtering) algorithm.

3.2 Binomial test

Our second test is based on the second necessary condition of Erdős-Gallai theorem. It received the given name since we estimate the number of the inner edges of the head of b using a binomial coefficient. Let $T_i = b_{i+1} + \cdots + b_n$ ($i = 1, \ldots, n$).

Lemma 3 *If* $n \geq 1$ *and* b *is an* n*-graphical sequence, then*

$$H_i \leq i(i-1) + T_i \quad (i = 1, \ldots, n-1). \tag{3}$$

Proof. The left side of (3) represents the degree requirement of the head of b. On the right side of (3) $i(i-1)$ is an upper bound of the inner degree capacity of the head, while T_i is an upper bound of the degree capacity of the tail, belonging to the index i. □

The following program is based on Lemma 3.

Input. n: number of the vertices ($n \geq 1$);
$b = (b_1, \ldots, b_n)$: an n-regular even sequence;
$H = (H_1, \ldots, H_n)$: H_i the sum of the first i elements of b;
H_0: auxiliary variable, helping to compute the elements of H.

Output. L: logical variable ($L =$ False signals, that b is surely not graphical, while $L =$ True shows, that the test *could not decide,* whether b is graphical).

Working variables. i: cycle variable;
$T = (T_1, \ldots, T_n)$: T_i the sum of the last $n - i$ elements of b;
T_0: auxiliary variable, helping to compute the elements of T.

Binomial-Test(n, b, H, L)

```
T_0 = 0
for i = 1 to n - 1
```

```
    T_i = H_n − H_i
    if H_i > i(i − 1) + T_i
        L = FALSE
        return L
L = TRUE
return L
```

The running time of this algorithm is $\Theta(n)$ in worst case, while in best case is only $\Theta(1)$.

According to our simulation experiments BINOMIAL-TEST is an effective filtering test (see Figure 2 and Figure 3).

3.3 Splitting of the head

We can get a better estimation of the inner capacity of the head, than the binomial coefficient gives in (3), if we split the head into two parts. Let $\lfloor i/2 \rfloor = h_i$, p the number of positive elements of b. Then the sequence $(b_1, \ldots, b_{h_i})$ is called *the beginning of the head* belonging to index i and the sequence $(b_{h_i+1}, \ldots, b_i)$ *the end of the head* belonging to index i.

Lemma 4 *If* $n \geq 1$ *and* b *is an* n*-graphical sequence, then*

$$\begin{aligned} H_i \leq \ & \min(\min(H_{h_i}, T_n - T_i, h_i(n-i)) \\ & + \min(H_i - H_{h_i}, T_n - T_i, (i - h_i)(n-i)), T_i) \\ & + \min(h_i(i-h_i) + \binom{h_i}{2} + \binom{i-h_i}{2} \quad (i = 1, \ldots, n), \end{aligned} \tag{4}$$

further

$$\min(H_{h_i}, T_n - T_i, h_i(n-i)) + \min(H_i - H_{h_i}, T_n - T_i, (i-h_i)(n-i)) \leq T_i. \tag{5}$$

Proof. Let G be a simple graph whose degree sequence is b. Then we divide the set of the edges of the head belonging to index i into five subsets: $(S_{i,1})$ contains the edges between the beginning of the head and the tail, $(S_{i,2})$ the edges between the end of the head and the tail, $S_{i,3}$ the edges between the parts of the head, $S_{i,4}$ the edges in the beginning of the head and $S_{i,5}$ the edges in the end of the head. Let us denote the number of edges in these subsets by $X_{i,1}$, ..., $X_{i,5}$.

$X_{i,1}$ is at most the sum H_{h_i} of the elements of the head, at most the sum $T_n - T_i$ of the elements of the tail, and at most the product $h_i(n-i)$ of the elements of the pairs formed from the tail and from the beginning of the head, that is

$$X_{i,1} \leq \min(H_{h_i}, T_n - T_i, h_i(n - i)). \tag{6}$$

A similar train of thought results

$$X_{i,2} \leq \min(H_i - H_{h_i}, T_n - T_i, (i - h_i)(n - i)). \tag{7}$$

$X_{i,3}$ is at most $h_i(i - h_i)$ and at most H_i, implying

$$X_{i,3} \leq \min(h_i(i - h_i), H_i). \tag{8}$$

$X_{i,4}$ is at most $\binom{h_i}{2}$ and at most H_{h_i}, implying

$$X_{i,4} \leq \min(\binom{h_i}{2}, H_{h_i}), \tag{9}$$

while $X_{i,5}$ is at most $\binom{i-h_i}{2}$ and at most $H_i - H_{h_i}$, implying

$$X_{i,5} \leq \binom{i - h_i}{2}. \tag{10}$$

A requirement is also, that the tail can overrun its capacity, that is

$$X_{i,1} + X_{i,2} \leq T_i. \tag{11}$$

Summing of (6), (7), (8), (9), and (10) results

$$H_i \leq X_{i,1} + X_{i,2} + X_{i,3} + 2X_{i,4} + 2X_{i,5}. \tag{12}$$

Substituting of (6), (7), (8), (9), and (10) into (12) results (4), while (11) is equivalent to (5). □

The following algorithm executes the test based on Lemma 4.

Input. n: the number of vertices ($n \geq 1$);
$b = (b_1, \ldots, b_n)$: an n-even sequence, accepted by BINOMIAL-TEST;
$H = (H_1, \ldots, H_n)$: H_i the sum of the first i elements of b;
$T = (T_1, \ldots, T_n)$: T_i the sum of the last $n - i$ elements of b.

Output. L: logical variable ($L =$ FALSE signals,that b is not graphical, while $L =$ TRUE shows, that the test *could nor decide*, whether b is graphical).

Working variables. i: cycle variable;
h: the actual value of h_i;
$X = (X_1, X_2, X_3, X_4, X_5)$: X_j is the value of the actual $X_{i,j}$.

```
Headsplitter-Test(n, b, H, T, L)
for i = 2 to n − 1
    h = ⌊i/2⌋
    X_1 = min(H_h, T_n − T_i, h(n − i))
    X_2 = min(H_i − H_h, T_n − T_i, (i − h)(n − i))
    X_3 = min(h(i − h)
    X_4 = binom(h, 2)
    X_5 = binom(i−h, 2)
    if H_i > X_1 + X_2 + X_3 + 2X_4 + 2X_5 or X_1 + X_2 > T_i
        L = False
        return L
L = True
return L
```

The running time of the algorithm is $\Theta(1)$ in best, and $\Theta(n)$ in worst case.

It is a substantial circumstance that the use of Lemma 3 and Lemma 4 requires only *linear* time (while the earlier two theorems require quadratic time). But these improvements of Erdős-Gallai theorem decrease only the coefficient of the quadratic member in the formula of the running time, the order of growth remains unchanged.

Figure 2 contains the results of the running of Binomial-Test and Headsplitter-Test, further the values $G(n)$ and $\frac{G(n)}{G(n+1)}$ (the computation of the values of the function $G(n)$ will be explained in Section 8).

Figure 3 shows the relative frequency of the zerofree regular, binomial, headsplitted and graphical sequences compared to the number of regular sequences.

3.4 Composite test

Composite-Test uses approximate algorithms in the following order: Parity-Test, Binomial-Test, Positive-Test, Headsplitter-Test.

```
Composite-test(n, b, L)
Parity-Test(n, b, L)
if L == False
    return L
Binomial-Test(n, b, H, L)
if L == False
    return L
Headsplitter-Test(n, b, H, T, L)
```

```
if L == False
    return L
L = True
return L
```

The running time of this composite algorithm is in all cases $\Theta(n)$.

4 Properties of the approximate testing algorithms

We investigate the efficiency of the approximate algorithms testing the regular algorithms. Figure 1 contains the number $R(n)$ of regular, the number $E(n)$ of even, and the number $G(n)$ of graphical sequences for $n = 1, \ldots, 38$.

The relative efficiency of arbitrary testing algorithm A for sequences of given length n we define with the ratio of the number of accepted by A sequences of length n and the number of graphical sequences $G(n)$. This ratio as a function of n will be noted by $X_A(n)$ and called the *error function* of A [34].

We investigate the following approximate algorithms, which are the components of Composite-Test:

1) Parity-Test;
2) Binomial-Test;
3) Headsplitter-Test.

According to (23) there are $R(2) = 3$ 2-regular sequences: $(1,1)$, $(1,0)$ and $(0,0)$. According to (25) among these sequences there are $E(2) = 2$ even sequences. Binomial-Test accepts both even ones, therefore $B(2) = 2$. Both sequences are 2-graphical, therefore $G(2) = 2$ and so the efficiency of Parity-Test (PT) and Binomial-Test (BT) is $X_{PT}(2) = X_{BT}(2) = 2/2 = 1$, in this case both algorithms are optimal.

The number of 3-regular sequences is $R(3) = 10$. From these sequences $(2,2,2)$, $(2,2,0)$, $(2,1,1)$, $(2,0,0)$ $(1,1,0)$ and $(0,0,0)$ are even, so $E(3) = 6$. Binomial-Test excludes the sequences $(2,2,0)$ and $(2,0,0)$, so remains $B(3) = 4$. Since these sequences are 3-graphical, $G(3) = 4$ implies $X_{PT}(3) = \frac{3}{2}$ and $X_{BT}(3) = 1$.

The number of 4-regular sequences equals to $R(4) = 35$. From these sequences 16 is even, and the following 11 are 4-graphical: $(3,3,3,3)$, $(3,3,2,2)$, $(3,2,2,1)$, $(3,1,1,1,)$, $(2,2,2,2)$, $(2,2,2,0)$, $(2,2,1,1)$, $(2,1,1,0)$, $(1,1,1,1)$, $(1,1,0,0)$ and $(0,0,0,0)$. From the 16 even sequences Binomial-Test also excludes the 5 sequences, so $B(4) = G(4) = 11$ and $X_{BT}(4) = 1$.

According to these data in the case of $n \leq 4$ Binomial-Test recognizes all nongraphical sequences. Figure 2 shows, that for $n \leq 5$ we have $B(n) = G(n)$,

n	$B_z(n)$	$F_z(n)$	$G(n)$	$G(n+1)/G(n)$
1	**1**	**0**	**1**	2.000000
2	**2**	**2**	**2**	2.000000
3	**4**	**4**	**4**	2.750000
4	**11**	**11**	**11**	2.818182
5	**31**	**31**	**31**	3.290323
6	103	**102**	**102**	3.352941
7	349	344	**342**	3.546784
8	1256	1230	1213	3.595218
9	4577	4468	4361	3.672552
10	17040	16582	16016	3.705544
11	63944	62070	59348	3.742620
12	242218	234596	222117	3.765200
13	922369	891852	836315	3.786674
14	3530534	3409109	3166852	3.802710
15	13563764	13082900	12042620	3.817067
16	52283429	50380684	45967479	3.828918
17	202075949	194550002	176005709	3.839418
18	782879161	753107537	675759564	3.848517
19	3039168331	2921395019	2600672458	3.856630
20	11819351967	11353359464	10029832754	3.863844
21			38753710486	3.870343
22			149990133774	3.876212
23			581393603996	3.881553
24			2256710139346	3.886431
25			8770547818956	3.890907
26			34125389919850	3.895031
27			132919443189544	3.897978
28			518232001761434	3.898843
29			2022337118015338	

Figure 2: Number of zerofree binomial, zerofree headsplitted and graphical sequences, further the ratio of the numbers of graphical sequences for neighbouring values of n

that is Binomial-Test accepts the same number of sequences as the precise algorithms. If $n > 5$, then the error function of Binomial-Test is increasing: while $X_{\rm BT}(6) = \frac{103}{102}$ (BT accepts one nongraphical sequence), $X_{\rm BT}(7) = \frac{349}{342}$ (BT accepts 7 nongraphical sequences) etc.

Figure 4 presents the average running time of the testing algorithms BT

n	$E_z(n)$	$E_z(n)/R(n)$	$B_z(n)/R(n)$	$F_z(n)/R(n)$	$G(n)/R(n)$
1	0	0.000000	1.000000	1.000000	1.000000
2	1	0.333333	0.666667	0.666667	0.666667
3	2	0.300000	0.400000	0.400000	0.400000
4	9	0.257143	0.314286	0.314286	0.314286
5	28	0.230159	0.246032	0.246031	0.246032
6	110	0.238095	0.222943	0.220779	0.220779
7	396	0.231352	0.203380	0.200466	0.199301
8	1519	0.236053	0.195183	0.191142	0.188500
9	5720	0.235335	0.188276	0.183793	0.179391
10	21942	0.237524	0.184460	0.179502	0.173375
11	83980	0.238098	0.181290	0.175977	0.168260
12	323554	0.239301	0.179145	0.173508	0.164278
13	1248072	0.240000	0.177368	0.171500	0.160821
14	4829708	0.240784	0.176014	0.169960	0.157882
15	18721080	0.241379	0.174884	0.168684	0.155271
16	72714555	0.241946	0.173965	0.167634	0.152950
17	282861360	0.242424	0.173188	0.166738	0.150844
18	1101992870	0.242860	0.172533	0.165972	0.148926
19	4298748300	0.243243	0.171970	0.165306	0.147158
20	16789046494	0.243590	0.171486	0.164725	0.145521
21					0.143997
22					0.142569
23					0.141228
24					0.139961
25					0.138762
26					0.137625
27					0.136542
28					0.135509
29					0.134521

Figure 3: The number of zerofree even sequences, further the ratio of the numbers binomial/regular, headsplitted/regular and graphical/regular sequences

and HT in secundum and in number of operations. The data contain the time and operations necessary for the generation of the sequences too.

n	BT, s	BT, operation	HT, s	HT, operation
1	0	14	0	15
2	0	41	0	43
3	0	180	0	200
4	0	716	0	815
5	0	2 918	0	3 321
6	0	11 918	0	13 675
7	0	48 952	0	56 299
8	0	201 734	0	233 182
9	0	831 374	0	964 121
10	0	3 426 742	0	3 988 542
11	0	14 107 824	0	16 469 036
12	0	58 028 152	0	67 929 342
13	0	238 379 872	0	279 722 127
14	0	978 194 400	1	1 150 355 240
15	2	4 009 507 932	3	4 724 364 716
16	6	16 417 793 698	13	19 379 236 737
17	26	67 160 771 570	51	79 402 358 497
18	106	274 490 902 862	196	324 997 910 595
19	423	1 120 923 466 932	798	1 328 948 863 507
20	1 627	4 573 895 421 484	3 201	5 429 385 115 097

Figure 4: Running time of Binomial-Test (BT) and Headsplitter-Test (HT) in secundum and as the number of operations for $n = 1, \ldots, 20$

5 New precise algorithms

In this section the zerofree algorithms, the shifting Havel-Hakimi, the parity checking Havel-Hakimi, the shortened Erdős-Gallai, the jumping Erdős-Gallai, the linear Erdős-Gallai and the quick Erdős-Gallai algorithms are presented.

5.1 Zerofree algorithms

Since the zeros at the and of the input sequences correspond to isolated vertices, so they have no influence on the quality of the sequence. This observation is exploited in the following assertion, in which p means the number of the positive elements of the input sequence.

Corollary 5 *If* $n \geq 1$*, the* $(b_1, \ldots, b_n)$ n*-regular sequence is* n*-graphical if and only if* $(b_1, \ldots, b_p)$ *is* p*-graphical.*

Proof. If all elements of b are positive (that is $p = n$), then the assertion is equivalent with Erdős-Gallai theorem. If b contains zero element (that is

$p < n$), then the assertion is the consequence of Havel-Hakimi and Erdős-Gallai algorithms, since the zero elements do not help in the pairing of the positive elements, but from the other side they have no own requirement. □

The algorithms based on this corollary are called Havel-Hakimi-Zerofree (HHZ), resp. Erdős-Gallai-Zerofree (EGZ).

5.2 Shifting Havel-Hakimi algorithm

The natural algorithmic equivalent of the original Havel-Hakimi theorem is called Havel-Hakimi Sorting (HHSo), since it requires the sorting of the reduced input sequence in every round.

But it is possible to design such implementation, in which the reduction of the degrees is executed saving the monotonity of the sequence. Then we get Havel-Hakimi-Shifting (HHSh) algorithm.

For the pseudocode of this algorithms see [37].

5.3 Parity checking Havel-Hakimi algorithm

It is an interesting idea the join the application of the conditions of Erdős-Gallai and Havel-Hakimi theorems in such a manner, that we start with the parity checking of the input sequence, and only then use the recursive Havel-Hakimi method.

For the pseudocode of the algorithm Havel-Hakimi-Parity (HHP) see [37].

5.4 Shortened Erdős-Gallai algorithm (EGSh)

In the case of a regular sequence the maximal value of H_i is $n(n-1)$, therefore the inequality (2) certainly holds for $i = n$, therefore it is unnecessary to check.

Even more useful observation is contained in the following assertion due to Tripathi and Vijai.

Lemma 6 (Tripathi, Vijay [82]) *If* $n \geq 1$, *then an* n*-regular sequence* $b = (b_1 \ldots, b_n)$ *is* n*-graphical if and only if*

$$H_n \quad \textit{even} \tag{13}$$

and

$$H_i \leq \min(H_i, i(i-1)) + \sum_{k=i+1}^{n} \min(i, b_k) \quad (i = 1, 2, \ldots, r), \tag{14}$$

where

$$r = \max_{1 \le s \le n} (s \mid s(s-1) < H_s) \tag{15}$$

Proof. If $i(i-1) \ge H_i$, then the left side of (2) is nonpositive, therefore the inequality holds, so the checking of the inequality is nonnecessary. □

The algorithm based on this assertion is called Erdős-Gallai-Shortened. For example if the input sequence is $b = (5^{100})$, then Erdős-Gallai computes the right side of (2) 99 times, while Erdős-Gallai-Shortened only 6 times.

5.5 Jumping Erdős-Gallai algorithm

Contracting the repeated elements a regular sequence $(b_1, \ldots, b_n)$ can be written in the form $(b_{i_1}^{e_1}, \ldots, b_{i_q}^{e_q})$, where $b_{i_1} < \cdots < b_{i_q}$, $e_1, \ \ldots, \ e_q \ge 1$ and $e_1 + \cdots + e_q = n$. Let $g_j = e_1 + \cdots + e_j$ $(j = 1, \ \ldots, \ q)$.

The element b_i is called the *checking points* of the sequence b, if $i = n$ or $1 \le i \le n-1$ és $b_i > b_{i+1}$. Then the checking points are $b_{g_1}, \ldots, b_{g_q}$.

Theorem 7 (Tripathi, Vijay [82]) *An n-regular sequence $b = (b_1, \ldots, b_n)$ is n-graphical if and only if*

$$H_n \quad \textit{even} \tag{16}$$

and

$$H_{g_i} - g_i(g_i - 1) \le \sum_{k=g_i+1}^{n} \min(i, b_k) \quad (i = 1, \ldots, q). \tag{17}$$

Proof. See [82]. □

Later in algorithm Erdős-Gallai-Enumerating we will exploit, that in the inequality (17) g_q is always n, therefore it is enough to check the inequality only up to $i = q - 1$.

The next program implements a quick version of Erdős-Gallai algorithm, exploiting Corollary 5, Lemma 6 and Lemma 7. In this paper we use the pseudocode style proposed in [17].

Input. n: number of vertices $(n \ge 1)$;
$b = (b_1, \ldots, b_n)$: an n-even sequence.

Output. L: logical variable (L = False signalizes, that, b is not graphical, while L = True shows, that b is graphical).

Working variables. i and j: cycle variables;
$H = (H_0, H_1, \ldots, H_n)$: H_i is the sum of the first i elements of b;

C: the degree capacity of the actual tail;
b_{n+1}: auxiliary variable helping to decide, whether b_n is a jumping element.

```
Erdős-Gallai-Jumping(n, b, H, L)
01 H_1 = b_1                                  // lines 01–06: test of parity
02 for i = 2 to n
03     H_i = H_{i-1} + b_i
04 if H_n odd
05     L = False
06     return L
07 b_{n+1} = -1                   // lines 07–20: test of the request of the head
08 i = 1
09 while i ≤ n and i(i - 1) < H_i
10         while b_i == b_{i+1}
11                 i = i + 1
12         C = 0
13         for j = i + 1 to n
14             C = C + min(j, b_j)
15         if H_i > i(i - 1) + C
16             L = False
17             return L
18         i = i + 1
19 L = True
20 return L
```

The running time of EGJ varies between the best $\Theta(1)$ and the worst $\Theta(n^2)$.

5.6 Linear Erdős-Gallai algorithm

Recently we could improve Erdős-Gallai algorithm [35, 37]. The new algorithm Erdős-Gallai-Linear exploits, that b is monotone. It determines the capacities C_i in constant time. The base of the quick computation is the sequence $w(b)$ containing the *weight points* w_i of the elements of the input sequence b.

For given sequence b let $w(b) = (w_1, \ldots, w_{n-1})$, where w_i gives the index of b_k having the maximal index among such elements of b which are greater or equal to i.

Theorem 8 (Iványi, Lucz [35], Iványi, Lucz, Móri, Sótér [37]) *If* $n \geq 1$, *then* *the* n*-regular sequence* $(b_1, \ldots, b_n)$ *is* n*-graphical if and only if*

$$H_n \quad \textit{is even} \tag{18}$$

and if $i > w_i$*, then*

$$H_i \leq i(i-1) + H_n - H_i,$$

further if $i \leq w_i$*, then*

$$H_i \leq i(i-1) + i(w_i - i) + H_n - H_{w_i}.$$

Proof. (18) is the same as (1).

During the testing of the elements of b by Erdős-Gallai-Linear there are two cases:

- if $i > w_i$, then the contribution $C_i = \sum_{k=i+1}^{n} \min(i, b_k)$ of the tail of b equals to $H_n - H_i$, since the contribution c_k of the element b_k is only b_k.
- if $i \leq w_1$, then the contribution of the tail of b consists of contributions of two types: $c_{i+1}, \ldots, c_{w_i}$ are equal to i, while $c_j = b_j$ for $j = w_i + 1, \ldots, n$.

Therefore in the case $n - 1 \geq i > w_i$ we have

$$C_i = H_n - H_i, \tag{19}$$

and in the case $1 \leq i \leq w_i$

$$C_i = i(w_i - i) + H_n - H_{w_i}. \tag{20}$$

□

The following program is based on Theorem 8. It decides on arbitrary n-regular sequence whether it is n-graphical or not.

Input. n: number of vertices $(n \geq 1)$;
$b = (b_1, \ldots, b_n)$: n-regular sequence.

Output. L: logical variable, whose value is True, if the input is graphical, and it is False, if the input is not graphical.

Work variables. i and j: cycle variables;
$H = (H_1, \ldots, H_n)$: H_i is the sum of the first i elements of the tested b;
b_0: auxiliary element of the vector b
$w = (w_1, \ldots, w_{n-1})$: w_i is the weight point of b_i, that is the maximum of the indices of such elements of b, which are not smaller than i;
$H_0 = 0$: help variable to compute the other elements of the sequence H;
$b_0 = n - 1$: help variable to compute the elements of the sequence w.

```
Erdős-Gallai-Linear(n, b, L)
01 H_0 = 0                                     // line 01: initialization
02 for i = 1 to n           // lines 02–03: computation of the elements of H
03     H_i = H_{i−1} + b_i
04 if H_n odd                              // lines 04–06: test of the parity
05    L = False
06    return L
07 b_0 = n − 1                  // line 07: initialization of a working variable
08 for i = 1 to n                     // lines 08–12: computation of the weights
09     if b_i < b_{i−1}
10        for j = b_{i−1} downto b_i + 1
11            w_j = i − 1
12        w_{b_i} = i
13 for j = b_n downto 1                          // lines 13–14: large weights
14     w_j = n
15 for i = 1 to n                      // lines 15–23: test of the elements of b
16     if i ≤ w_i                // lines 16–19: test of indices for large w_i's
17        if H_i > i(i − 1) + i(w_i − i) + H_n − H_{w_i}
18           L = False
19           return L
20     if i > w_i                // lines 20–23: test of indices for small w_i's
21        if H_i > i(i − 1) + H_n − H_i
22           L = False
23           return L
24 L = True              // lines 24–25: the program ends with the value True
25 return L
```

Theorem 9 (Iványi, Lucz [35], Iványi, Lucz, Móri, Sótér [37]) *Algorithm* Erdős-Gallai-Linear *decides in* $\Theta(n)$ *time, whether an* n*-regular sequence* $b = (b_1, \ldots, b_n)$ *is graphical or not.*

Proof. Line 1 requires $O(1)$ time, lines 2–3 $\Theta(n)$ time, lines 4–6 $O(1)$ time, line 07 $O(1)$ time, lines 08–12 $O(1)$ time, lines 13–14 $O(n)$ time, lines 15–23 $O(n)$ time and lines 24–25 $O(1)$ time, therefore the total time requirement of the algorithm is $\Theta(n)$. □

Since in the case of a graphical sequence all elements of the investigated sequence are to be tested, in the case of RAM model of computations [17] Erdős-Gallai-Linear is asymptotically optimal.

6 Running time of the precise testing algorithms

We tested the precise algorithms determining their total running time for all the even sequences. The set of the even sequences is the smallest such set of sequences, whose the cardinality we know exact and explicite formula. The number of n-bounded sequences $K(n)$ is also known, but this function grows too quickly when n grows.

If we would know the average running time of the bounded sequences we would take into account that is is sufficient to weight the running times of the regular sequences with the corresponding frequencies. For example a homogeneous sequence consisting of identical elements would get a unit weight since it corresponds to only one bounded sequence, while a rainbow sequence consisting is n different elements as e.g. the sequence $n, n-1, \dots, 1$ corresponds to $n!$ different bounded sequences and therefore would get a corresponding weight equal to $n!$.

We follow two ways of the decreasing of the running time of the precise algorithms. The first way is the decreasing of the number of the executable operations. The second way is, that we try to use quick (linear time) preprocessing algorithms for the filtering of the sequences in order to decrease of the part of sequences requiring the relative slow precise algorithms.

For the first type of decrease of the expected running time is the shortening of the sequences and the application of the checking points, while for the the second type are examples the completion of HH algorithm with the parity checking or the completion of the EG algorithm with the binomial and headsplitted algorithms.

In this section we investigate the following precise algorithms:

1) Havel-Hakimi-Shorting (HHSo).
2) Havel-Hakimi-Shifting (HHSh).
3) Erdős-Gallai algorithm (EG).
4) Erdős-Gallai-Jumping algorithm (EGJ).
5) Erdős-Gallai-Linear algorithm (EGL).

Figure 5 contains the total number of operations of the algorithms HHSo, HHSh, EG, and EGL required for the testing of all even sequences of length $n = 1, \dots, 15$. The operations necessary to generate the sequences are included.

Comparison of the first two columns shows that algorithm HHSh is much quicker than HHSo, especially if n increases. Comparison of the third and fourth columns shows that we get substantial decrease of the running time if we have to test the input sequence only in the check points. Finally the comparison of the third and fifth columns demonstrates the advantages of a

n	HHSo	HHSh	EG	EGJ	EGL
1	10	15	87	-	-
2	40	61	119	12	37
3	231	236	267	116	148
4	1 170	1 052	946	551	585
5	5 969	4 477	4 000	2 677	2 339
6	31 121	20 153	18 206	12 068	9 539
7	157 345	88 548	82 154	54 184	38 984
8	784 341	393 361	372 363	238 813	160 126
9	3 628 914	1 726 484	1 666 167	1 666 167	656 575
10	17 345 700	7 564 112	7 418 447	4 552 276	2 692 240
11	80 815 538	32 895 244	32 737 155	19 680 986	11 018 710
12	385 546 527	142 460 352	143 621 072	84 608 529	45 049 862
13	1 740 003 588	613 739 913	626 050 861	362 141 061	183 917 288
14	8 066 861 973	2 633 446 908	2 715 026 827	1 543 745 902	750 029 671
15	36 630 285 216	11 254 655 388	11 717 017 238	6 557 902 712	3 055 289 271

Figure 5: Total number of operations as the function of n for precise algorithms HHSo, HHSh, EG, EGJ, and EGL.

n	$E(n)$	$T(n)$, s	$Op(n)$	$T(n)/E(n)/n$, s	$Op(n)/E(n)/n$
2	2	0	37	0	9.25000000000
3	6	0	148	0	8.22222222222
4	19	0	585	0	7.69736842105
5	66	0	2 339	0	7.08787878788
6	236	0	9 539	0	6.73658192090
7	868	0	38 984	0	6.41606319947
8	3 235	0	160 126	0	6.18724884080
9	12 190	0	656 575	0	5.98464132714
10	46 252	0	2 692 240	0	5.82080774885
11	176 484	0	11 018 710	0	5.67587378511
12	676 270	0	45 049 862	0	5.55126675243
13	2 600 612	0	183 917 288	0	5.44005937537
14	10 030 008	1	750 029 671	0.000000007121487	5.34132654018
15	38 781 096	5	3 055 289 271	0.000000008595253	5.25219687963
16	150 273 315	23	12 434 367 770	0.000000009565903	5.17156346504
17	583 407 990	79	50 561 399 261	0.000000007965367	5.09797604337
18	2 268 795 980	297	205 439 740 365	0.00000000727258	5.03056202928

Figure 6: Total and amortized running time of Erdős-Gallai-Linear in secundum, resp. in the number of executed operations

$E(n) - G(n)$	n/i	f_1	f_2	f_3	f_4	f_5	f_6	f_7
2	3	2	0	0	0	0	0	0
8	4	6	2	0	0	0	0	0
35	5	33	2	0	0	0	0	0
134	6	122	12	0	0	0	0	0
526	7	459	65	2	2	0	0	0
2022	8	1709	289	24	0	0	0	0
7829	9	6421	1228	176	4	0	0	0
30236	10	24205	4951	1013	67	0	0	0
115136	11	91786	19603	5126	610	11	0	0
454153	12	349502	76414	23755	4274	208	0	0
1764297	13	1336491	296036	104171	25293	2277	29	0
6863156	14	5128246	1142470	439155	133946	18673	666	0
26738476	15	19739076	4404813	1803496	655291	127116	8603	81

Figure 7: Distribution of the even nongraphical sequences according to the number of tests made by Erdős-Gallai-Jumping to exclude the given sequence for $n = 3, \ldots, 15$

linear algorithm over a quadratic one.

Figure 6 shows the running time of Erdős-Gallai-Linear in secundum and operation, and also the amortized number of operation/even sequence.

The most interesting data of Figure 6 are in the last column: they show that the number of operations/investigated sequence/length of the investigated sequence is monotone decreasing (see [69]).

Figure 7 shows the distribution of the $E(n) - G(n)$ even nongraphical sequences according to the number of tests made by Erdős-Gallai-Jumping to exclude the given sequence for $n = 3, \ldots, 15$ vertices. $f_i(n) = f_i$ gives the frequency of even nongraphical sequences of length n, which requeired exactly i round of the test.

These data show, that the maximal number of tests is about $\frac{n}{2}$ in all lines.

Figure 8 shows the average number of required rounds for the nongraphical, graphical and all even sequences. The data of the column belonging to $G(n)$ are computed using Lemma 17. It is remarkable that the sequences of the coefficients are monotone decreasing in the last three columns.

Figure 9 presents the distribution of the graphical sequences according to their first element. These data help at the design of the algorithm Erdős-Gallai-Enumerating which computes the new values of $G(n)$ (in the slicing of the computations belonging to a given value of n).

n	$E(n)$	$G(n)$	$E(n) - G(n)$	average of $E(n) - G(n)$	average of $G(n)$	average of $E(n)$
3	6	4	2	$0.3333n$	$0.8000n$	$0.6444n$
4	19	11	8	$0.3125n$	$0.5714n$	$0.4661n$
5	66	31	35	$0.2114n$	$0.5555n$	$0.3730n$
6	236	102	134	$0.1967n$	$0.5455n$	$0.3730n$
7	868	342	526	$0.1649n$	$0.5385n$	$0.3475n$
8	3233	1213	2020	$0.1458n$	$0.5333n$	$0.2911n$
9	12190	4363	7829	$0.1337n$	$0.5294n$	$0.2753n$
10	46232	16016	30216	$0.1249n$	$0.5263n$	$0.2700n$
11	174484	59348	115136	$0.1175n$	$0.5238n$	$0.2557n$
12	676270	222117	454153	$0.1085n$	$0.5217n$	$0.2444n$
13	2603612	836313	1767299	$0.1035n$	$0.5200n$	$0.2373n$
14	10030008	3166852	6863156	$0.0960n$	$0.5185n$	$0.2294n$
15	38761096	12042620	26718476	$0.0934n$	$0.5172n$	$0.2251n$

Figure 8: Weighted average number of tests made by Erdős-Gallai-Jumping while investigating the even sequences for $n = 3, \ldots, 15$

n/b_1	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	1	1										
3	1	1	2									
4	1	1	4	4								
5	1	2	7	10	11							
6	1	3	10	22	35	31						
7	1	3	14	34	78	110	102					
8	1	4	18	54	138	267	389	342				
9	1	4	23	74	223	503	968	1352	1213			
10	1	5	28	104	333	866	1927	3496	4895	4361		
11	1	5	34	134	479	1356	3471	7221	12892	17793	16016	
12	1	6	40	176	661	2049	5591	13270	27449	47757	65769	59348

Figure 9: The distribution of the graphical sequences according to b_1 for $n = 1, \ldots, 12$

We see in Figure 9 that from $n = 6$ the multiplicities increase up to $n - 2$, and the last positive value is smaller then the last but one element.

7 Enumerative results

Until now for example Avis and Fukuda [4], Barnes and Savage [5, 6], Burns [13], Erdős and Moser [59], Erdős and Richmond [22], Frank, Savage and Selers [25], Kleitman and Winston [47], Kleitman and Wang [46], Metropolis and Stein [56], Rødseth et al. [68], Ruskey et al. [69], Stanley [78], Simion [71] and Winston and Kleitman [86] published results connected with the enumeration of degree sequences. Results connected with the number of sequences investigated by us can be found in the books of Sloane és Ploffe [76], further Stanley [79] and in the free online database *On-line Encyclopedia of Integer Sequences* [73, 74, 75]

It is easy to show, that if l, u and m are integers, further $u \geq l$, $m \geq 1$, and $l \leq b_i \leq u$ for $i = 1, \ldots, m$, then the number of (l, u, m)-bounded sequences $a = (a_1, \ldots, a_m)$ of integer numbers $K(l, u, m)$ is

$$K(l, u, m) = (u - l + 1)^m. \tag{21}$$

It is known (e.g. see [39, page 65]), that if l, u and m are integers, further $u \geq l$ and $m \geq 1$, and $u \geq b_1 \geq \cdots \geq b_n \geq l$, then the number of (l, u, m)-regular sequences of integer numbers $R(l, u, m)$ is

$$R(l, u, m) = \binom{u - l + m}{m}. \tag{22}$$

The following two special cases of (22) are useful in the design of the algorithm Erdős-Gallai-Enumerating.

If $n \geq 1$ is an integer, then the number of $R(0, n-1, n)$-regular sequences is

$$R(0, n-1, n) = R(n) = \binom{2n-1}{n}. \tag{23}$$

If $n \geq 1$ is an integer, then the number of $R(1, n-1, n)$-regular sequences is

$$R(1, n-1, n) = R_z(n) = \binom{2n-2}{n}. \tag{24}$$

In 1987 Ascher derived the following explicit formula for the number of n-even sequences $E(n)$.

Lemma 10 (Ascher [3], Sloane, Pfoffe [76]) *If*

Lemma 11 *lemma-En* $n \geq 1$, *then the number of* n*-even sequences* $E(n)$ *is*

$$E(n) = \frac{1}{2}\left(\binom{2n-1}{n} + \binom{n-1}{\lfloor n \rfloor}\right). \tag{25}$$

Proof. See [3, 76]. □

At the designing and analysis of the results of the simulation experiments is useful, if we know some features of the functions $R(n)$ and $E(n)$.

Lemma 12 *If* $n \geq 1$, *then*

$$\frac{R(n+2)}{R(n+1)} > \frac{R(n+1)}{R(n)}, \tag{26}$$

$$\lim_{n \to \infty} \frac{R(n+1)}{R(n)} = 4, \tag{27}$$

further

$$\frac{4^n}{\sqrt{4\pi n}}\left(1 - \frac{1}{2n}\right) < R(n) < \frac{4^n}{\sqrt{4\pi n}}\left(1 - \frac{1}{8n+8}\right). \tag{28}$$

Proof. On the base of (23) we have

$$\frac{R(n+2)}{R(n+1)} = \frac{(2n+3)!(n+1)n!}{(n+2)!(n+1)!(2n+1)!} = \frac{4n+6}{n+2} = 4 - \frac{2}{n+2}, \tag{29}$$

from where we get directly (26) and (27). □

Using Lemma 13 we can give the precise asymptotic order of growth of $E(n)$.

Lemma 13 *If* $n \geq 1$, *then*

$$\frac{E(n+2)}{E(n+1)} > \frac{E(n+1)}{E(n)}, \tag{30}$$

$$\lim_{n \to \infty} \frac{E(n+1)}{E(n)} = 4, \tag{31}$$

further

$$\frac{4^n}{\sqrt{\pi n}}(1 - D_3(n)) < E(n) < \frac{4^n}{\sqrt{\pi n}}(1 - D_4(n)), \tag{32}$$

where $D_3(n)$ *and* $D_4(n)$ *are monotone decreasing functions tending to zero.*

Proof. The proof is similar to the proof of Lemma 12. □

Comparison of (23) and Lemma 13 shows, that the order of growth of numbers of even and odd sequences is the same, but there are more even sequences than odd. Figure 1 contains the values of $R(n)$, $E(n)$ and $E(n)/R(n)$ for $n = 1, \ldots, 37$.

As the next assertion and Figure 1 show, the sequence of the ratios $E(n)/R(n)$ is monotone decreasing and tends to $\frac{1}{2}$.

Corollary 14 *If* $n \geq 1$*, then*

$$\frac{E(n+1)}{R(n+1)} < \frac{E(n)}{R(n)} \tag{33}$$

and

$$\lim_{n\to\infty} \frac{E(n)}{R(n)} = \frac{1}{2}. \tag{34}$$

Proof. This assertion is a direct consequence of (23) and (25). □

The expected value of the number of jumping elements has a substantial influence on the running time of algorithms using the jumping elements. Therefore the following two assertions are useful.

The number of different elements in an n-bounded sequence b is called *the rainbow number* of the sequence, and it will be denoted by $r_n(b)$.

Lemma 15 *Let* b *be a random* n*-bounded sequence. Then the expectation and variance of its rainbow number are as follows.*

$$E[r_n(b)] = n\left[1 - \left(1 - \frac{1}{n}\right)^n\right] = n\left(1 - \frac{1}{e}\right) + O(1), \tag{35}$$

$$\begin{aligned} Var[r_n(b)] &= n\left(1 - \frac{1}{n}\right)^n \left[1 - \left(1 - \frac{1}{n}\right)^n\right] \\ &\quad + n(n-1)\left[\left(1 - \frac{2}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n}\right] \\ &= \frac{n}{e}\left(1 - \frac{2}{e}\right) + O(1). \end{aligned} \tag{36}$$

Proof. Let ξ_i denote the indicator of the event that number i is not contained in a random n-bounded sequence. Then the rainbow number of a random sequence is $n - \sum_{i=0}^{n-1} \xi_i$, hence its expectation equals $n - \sum_{i=0}^{n-1} E[\xi_i]$. Clearly,

$$E[\xi_i] = \left(1 - \frac{1}{n}\right)^n \tag{37}$$

holds independently of i, thus

$$E[r_n(b)] = n\left[1 - \left(1 - \frac{1}{n}\right)^n\right]. \tag{38}$$

On the other hand,

$$Var[r_n(b)] = Var\left[\sum_{i=0}^{n-1} \xi_i\right] = \sum_{i=0}^{n-1} Var[\xi_i] + 2 \sum_{0 \le i < j \le n-1} cov[\xi_i, \xi_j]. \tag{39}$$

Here

$$Var[\xi_i] = \left(1 - \frac{1}{n}\right)^n \left[1 - \left(1 - \frac{1}{n}\right)^n\right], \tag{40}$$

and

$$cov[\xi_i, \xi_j] = E[\xi_i \xi_j] - E[\xi_i]E[\xi_j] = \left(1 - \frac{2}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n}, \tag{41}$$

implying (36). □

We remark that this lemma answers a question of Imre Kátai [40] posed in connection with the speed of computers having interleaved memory and with checking algorithms of some puzzles (e.g sudoku).

Lemma 16 *The number of* $(0, n-1, m)$*-regular sequences composed from* k *distinct numbers is*

$$\binom{n}{k}\binom{m-1}{k}, \; k = 1, \ldots, n. \tag{42}$$

In other words, the distribution of the rainbow number $r_n(b)$ *of a random* $(0, n-1, m)$*-regular sequence* b *is hypergeometric with parameters* $n+m-1$*,* n*, and* m*.*

Proof. The k-set of distinct elements of the sequence can be selected from $\{0, 1, \ldots, n-1\}$ in $\binom{n}{k}$ ways. Having this values selected we can tell their multiplicities in $\binom{m-1}{k-1}$ ways. Let us consider the k blocks of identical elements. The first one starts with b_1, and the starting position of the other $k-1$ blocks can be selected in $\binom{m-1}{k-1}$ ways. □

From this the expectation and the variance of a random n-regular sequence follow immediately.

Corollary 17 *Let* b *be a random* n*-regular sequence. Then the expectation and the variance of its rainbow number* $r_n(b)$ *are as follows:*

$$E[r_n(b)] = \frac{n^2}{2n-1} = \frac{n}{2} + \frac{n}{4n-2} = \frac{n}{2} + O(1), \tag{43}$$

$$Var[r_n(b)] = \frac{n^2(n-1)}{2(2n-1)^2} = \frac{n}{8} + \frac{n}{128n^2 - 128n + 32} = \frac{n}{8} + O(1). \tag{44}$$

Lemma 18 *Let* b *be a random* n*-regular sequence. Let us write it in the form* $b = (b_1^{e_1}, \ldots, b_r^{e_r})$*. Then the expected value of the exponents* e_j *is*

$$E[e_j \mid r(b) \geq j] = 4 + o(1). \tag{45}$$

Proof. Let $c(n,j)$ denote the number of n-regular sequences with rainbow number not less than j. By Lemma 16,

$$c(n,j) = \sum_{k=j}^{n} \binom{n}{k}\binom{n-1}{k-1}. \tag{46}$$

Let us turn to the number of n-regular sequences with rainbow number not less than j and $e_j = \ell$. This is equal to the number of $(0, n-1, n-\ell+1)$-regular sequences containing at least j different numbers, that is,

$$\sum_{k=j}^{n} \binom{n}{k}\binom{n-\ell}{k-1}. \tag{47}$$

From this the sum of e_j over all n-regular sequences with $e_j > 0$ is equal to

$$\sum_{\ell=1}^{n-j+1} \ell \sum_{k=j}^{n} \binom{n}{k}\binom{n-\ell}{k-1} = \sum_{k=j} n\binom{n}{k} \sum_{\ell=1}^{n-j+1} \binom{\ell}{1}\binom{n-\ell}{k-1}$$
$$= \sum_{k=j}^{n} \binom{n}{k}\binom{n+1}{k+1} = c(n+1, j+1). \tag{48}$$

This can also be seen in a more direct way. Consider an arbitrary n-regular sequence with at least $j+1$ blocks, then substitute the elements of the $j+1$st block with the number in the jth block (that is, concatenate this two adjacent blocks) and delete one element from the united block; finally, decrease by 1 all elements in the subsequent blocks. In this way one obtains an n-regular sequence with at least j blocks, and it easy to see that every such sequence is obtained exactly e_j times.

Thus the expectation to be computed is just

$$\frac{c(n+1, j+1)}{c(n, j)}. \tag{49}$$

Clearly, $c(n, 1) = R(0, n-1, n) = \binom{2n-1}{n}$, hence

$$c(n, j) = \binom{2n-1}{n} - \sum_{k=1}^{j-1} \binom{n}{k}\binom{n-1}{k-1} = \binom{2n-1}{n} + O\left(n^{2j-3}\right), \tag{50}$$

as $n \to \infty$. This is asymptotically equal to

$$\frac{\binom{2n+1}{n+1}}{\binom{2n-1}{n}} = \frac{4n+2}{n+1} = 4 - \frac{2}{n+1} = 4 + o(1). \tag{51}$$

□

It is interesting to observe that by (43) the average block length in a random n-regular sequence is

$$\frac{1}{r}\sum_{j=1}^{r} e_j = \frac{n}{r(b)} \approx 2 \tag{52}$$

approximately, as $n \to \infty$. This fact could be interpreted as if blocks in the beginning of the sequence were significantly longer. However, fixing $r_n(b) = r$ we find that the lengths of the r blocks are exchangeable random variables with equal expectation n/r. At first sight this two facts seem to be in contradiction. The explanation is that exchangability only holds conditionally. Blocks in the beginning do exist even for smaller rainbow numbers, when the average block length is big, while blocks with large index can only appear when there are many short blocks in the sequence.

The following assertion gives the number of zerofree sequences and the ratio of the numbers of zerofree and regular sequences.

Corollary 19 *The number of the zerofree n-regular sequences $R_z(n)$ is*

$$R_z(n) = \binom{2n-2}{n-1} \tag{53}$$

and

$$\lim_{n\to\infty} \frac{R_z(n)}{R(n)} = \frac{1}{2}. \tag{54}$$

Proof. (53) identical with (22), (54) is a direct consequence of (22) and (23). □

As the experimental data in Figure 3 show, $\frac{E_z(n)}{R(n)} \approx \frac{1}{4}$.

The following lemma allows that the algorithm Erdős-Gallai-Enumerating tests only the zerofree even sequences instead of the even sequences.

Lemma 20 *If* $n \geq 2$, *then the number of the* n*-graphical sequences* $G(n)$ *is*

$$G(n) = G(n-1) + G_z(n). \tag{55}$$

Proof. If an n-graphical sequence b contains at least one zero, that is $b_n = 0$, then $b' = (b_1, \ldots, b_{n-1})$ is $(n-1)$-graphical or not. If $a = (a_1, \ldots, a_{n-1})$ is an $(n-1)$-graphical sequence, then $a' = (a_1, \ldots, a_{n-1}, 0)$ is n-graphical.

The set of the n-graphical sequences S consists of two subsets: set of zerofree sequences S_1 and the set of sequences S_2 containing at least one zero. There is a bijection between the set of the $(n-1)$-graphical sequences and such n-graphical sequences, which contain at least one zero. Therefore $|S| = |S_1| + |S_2| = G_z(n) + G(n-1)$. □

Corollary 21 *If* $n \geq 1$, *then*

$$G(n) = 1 + \sum_{i=2}^{n} G_z(n). \tag{56}$$

Proof. Concrete calculation gives $G(1) = 1$. Then using (55) and induction we get (56). □

A promising direction of researches connected with the characterization of the function $G(n)$ is the decomposition of the even integers into members and the investigation, which decompositions represent a graphical sequence [5, 6, 13]. Using this approach Burns proved the following asymptotic bounds in 2007.

Theorem 22 (Burns [13]) *There exist such positive constants* c *and* C, *that the following bounds of the function* $G(n)$ *is true:*

$$\frac{4^n}{cn} < G(n) < \frac{4^n}{(\log n)^C \sqrt{n}}. \tag{57}$$

Proof. See [13]. □

This result implies that the asymptotic density of the graphical sequences is zero among the even sequences.

Corollary 23 *If $n \geq 1$, then there exists a positive constant C such that*

$$\frac{G(n)}{E(n)} < \frac{1}{(\log_2 n)^C} \tag{58}$$

and

$$\lim_{n \to \infty} \frac{G(n)}{E(n)} = 0. \tag{59}$$

Proof. (58) is a direct consequence of (25) and (58), and (58) implies (59). □

As Figure 1 and Figure 3 show, the convergence of the ratio $G(n)/E(n)$ is relative slow.

8 Number of graphical sequences

Erdős-Gallai-Enumerating algorithm (EGE) [37] generates and tests for given n every zerofree even sequence. Its input is n and output is the number of corresponding zerofree graphical sequences $G_z(n)$.

The algorithm is based on Erdős-Gallai-Linear algorithm. It generates and tests only the zerofree even sequences, that is according to Corollary 5 and Figure 3 about the 25 percent of the n-regular sequences.

EGE tests the input sequences only in the checking points. Corollary 17 shows that about the half of the elements of the input sequences are check points.

Figure 3 contains data showing that EGE investigates even less than the half of the elements of the input sequences.

Important property of EGE is that it solves in $O(1)$ expected time

1. the generation of one input sequence;
2. the updating of the vector H;
3. the updating of the vector of checking points;
4. the updating of the vector of the weight points.

We implemented the parallel version of EGE (EGEP). It was run on about 200 PC's containing about 700 cores. The total running time of EGEP is contained in Figure 10

The pseudocode of the algorithm see in [37]. The amortized running time of this algorithm for a sequence is $\Theta(1)$, so the total running time of the whole program is $O(E(n))$.

n	running time (in days)	number of slices
24	7	415
25	26	415
26	70	435
27	316	435
28	1130	2001
29	6733	15119

Figure 10: The runnng time of EGEP for $n = 24, \ldots, 29$

9 Summary

In Figure 1 the values of $R(n)$ up to $n = 24$ are the elements of the sequence A001700 of OEIS [73], the values of $E(n)$ up to $n = 23$ are the elements of the sequence A005654 [75] of the OEIS, and in Figure 2 the values $G(n)$ are up to $n = 23$ are the elements of sequence A0004251-es [74] of OEIS. The remaining values are new [37, 36].

Figure 2 contains the number of graphical sequences $G(n)$ for $n = 1, \ldots, 29$, and also $G(n+1)/G(n)$ for $n = 1, \ldots, 28$.

The referenced manuscripts, programs and further simulation results can be found at the homepage of the authors, among others at http://compalg.inf.elte.hu/~tony/Kutatas/E

Acknowledgements

The authors thank Zoltán Király (Eötvös Loránd University, Faculty of Science, Dept. of Computer Science) for his advice concerning the weight points, Antal Sándor and his colleagues (Eötvös Loránd University, Faculty of Informatics), further Ádám Mányoki (TFM World Kereskedelmi és Szolgáltató Kft.) for their help in running of our timeconsuming programs and the unknown referee for the useful corrections. The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP 4.2.1/B-09/1/KMR-2010-0003.

References

[1] M. Anholcer, V. Babiy, S. Bozóki, W. W. Koczkodaj, A simplified implementation of the least squares solution for pairwise comparisons matrices. *CEJOR Cent. Eur. J. Oper. Res.* **19,** 4 (2011) 439–444. ⇒231

[2] S. R. Arikati, A. Maheshwari, Realizing degree sequences in parallel. *SIAM J. Discrete Math.* **9,** 2 (1996) 317–338. ⇒231

[3] M. Ascher, Mu torere: an analysis of a Maori game, *Math. Mag.* **60,** 2 (1987) 90–100. ⇒253, 254

[4] D. Avis, K. Fukuda, Reverse search for enumeration, *Discrete Appl. Math.* **2,** 1-3 (1996) 21–46. ⇒253

[5] T. M. Barnes, C. D. Savage, A recurrence for counting graphical partitions, *Electron. J. Combin.* **2** (1995), Research Paper 11, 10 pages (electronic). ⇒253, 259

[6] T. M. Barnes, C. D. Savage, Efficient generation of graphical partitions, *Discrete Appl. Math.* **78,** 1-3 (1997) 17–26. ⇒230, 253, 259

[7] L. B. Beasley, D. E. Brown, K. B. Reid, Extending partial tournaments, *Math. Comput. Modelling* **50,** 1 (2009) 287–291. ⇒231

[8] S. Bereg, H. Ito, Transforming graphs with the same degree sequence, *The Kyoto Int. Conf. on Computational Geometry and Graph Theory* (ed. by H. Ito et al.), LNCS **4535,** Springer-Verlag, Berlin, Heidelberg, 2008. pp. 25–32. ⇒231

[9] N. Bödei, *Degree sequences of graphs* (Hungarian), Mathematical master thesis (supervisor A. Frank), Dept. of Operation Research of Eötvös Loránd University, Budapest, 2010, 43 pages. ⇒231, 232, 233

[10] S. Bozóki S., J. Fülöp, A. Poesz, On pairwise comparison matrices that can be made consistent by the modification of a few elements. *CEJOR Cent. Eur. J. Oper. Res.* **19** (2011) 157–175. ⇒231

[11] Bozóki S., J. Fülöp, L. Rónyai: On optimal completion of incomplete pairwise comparison matrices, *Math. Comput. Modelling* **52** (2010) 318–333. ⇒231

[12] A. R. Brualdi, K. Kiernan, Landau's and Rado's theorems and partial tournaments, *Electron. J. Combin.* **16,** #N2 (2009) 6 pages. ⇒231

[13] J. M. Burns, *The Number of Degree Sequences of Graphs*, PhD Dissertation, MIT, 2007. ⇒253, 259, 260

[14] A. N. Busch, G. Chen, M. S. Jacobson, Transitive partitions in realizations of tournament score sequences, *J. Graph Theory* **64,** 1 (2010), 52–62. ⇒ 231

[15] S. A. Choudum, A simple proof of the Erdős-Gallai theorem on graph sequences, *Bull. Austral. Math. Soc.* **33** (1986) 67–70. ⇒233

[16] J. Cooper, L. Lu, Graphs with asymptotically invariant degree sequences under restriction, *Internet Mathematics* **7,** 1 67–80. ⇒231

[17] T. H. Cormen, Ch. E. Leiserson, R. L. Rivest, C. Stein, *Introduction to Algorithms*, Third edition, The MIT Press/McGraw Hill, Cambridge/New York, 2009. ⇒245, 248

[18] S. De Agostino, R. Petreschi, Parallel recognition algorithms for graphs with restricted neighbourhoods. *Internat. J. Found. Comput. Sci.* **1,** 2 (1990) 123–130. ⇒231

[19] C. I. Del Genio, H. Kim, Z. Toroczkai, K. E. Bassler, Efficient and exact sampling of simple graphs with given arbitrary degree sequence, *PLoS ONE* **5,** 4 (2010) e10012. ⇒231

[20] A. Dessmark, A. Lingas, O. Garrido, On parallel complexity of maximum f-matching and the degree sequence problem. *Mathematical Foundations of Computer Science 1994* (Košice, 1994), LNCS **841,** Springer, Berlin, 1994, 316–325. ⇒231

[21] P. Erdős, T. Gallai, Graphs with prescribed degrees of vertices (Hungarian), *Mat. Lapok* **11** (1960) 264–274. ⇒230, 231, 233

[22] P. Erdős, L. B. Richmond, On graphical partitions, *Combinatorica* **13,** 1 (1993) 57–63. ⇒253

[23] P. L. Erdős, I. Miklós, Z. Toroczkai, A simple Havel-Hakimi type algorithm to realize graphical degree sequences of directed graphs, *Electron. J. Combin.* **17,** 1 (2010) R66 (10 pages). ⇒231, 233

[24] A. Frank, *Connections in Combinatorial Optimization,* Oxford University Press, Oxford, 2011. ⇒231

[25] D. A. Frank, C. D. Savage, J. A. Sellers, On the number of graphical forest partitions, *Ars Combin.* **65** (2002) 33–37. ⇒253

[26] S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a simple graph. *J. SIAM Appl. Math.* **10** (1962) 496–506. ⇒ 230, 231, 232

[27] S. L. Hakimi, On the degrees of the vertices of a graph, *F. Franklin Institute,* **279,** 4 (1965) 290–308. ⇒233

[28] V. Havel, A remark on the existence of finite graphs (Czech), *Časopis Pěst. Mat.* **80** (1955), 477–480. ⇒230, 232

[29] P. Hell, D. Kirkpatrick, Linear-time certifying algorithms for near-graphical sequences, *Discrete Math.* **309,** 18 (2009) 5703–5713. ⇒231

[30] A. Iványi, Reconstruction of complete interval tournaments, *Acta Univ. Sapientiae, Inform.* **1,** 1 (2009) 71–88. ⇒231, 233

[31] A. Iványi, Reconstruction of complete interval tournaments. II, *Acta Univ. Sapientiae, Math.* **2,** 1 (2010) 47–71. ⇒231, 233

[32] A. Iványi, Deciding the validity of the score sequence of a soccer tournament, in: *Open problems of the Egerváry Research Group,* ed. by A. Frank, Budapest, 2011. ⇒231

[33] A. Iványi, Directed graphs with prescribed score sequences, *The 7th Hungarian-Japanese Symposium on Discrete Mathematics and Applications* (Kyoto, May 31–June 3, 2011, ed by S. Iwata), 114–123. ⇒231

[34] A. Iványi, *Memory management,* in: *Algorithms of Informatics* (ed by. A. Iványi), AnTonCom, Budapest, 2011, 797–847. ⇒240

[35] A. Iványi, L. Lucz, Erdős-Gallai test in linear time, *Combinatorica* (submitted). ⇒231, 232, 246, 248

[36] A. Iványi, L. Lucz, Parallel Erdős-Gallai algorithm, *CEJOR Cent. Eur. J. Oper. Res.* (submitted). ⇒231, 232, 261

[37] A. Iványi, L. Lucz, T. F. Móri, P. Sótér, Linear Erdős-Gallai test (Hungarian), *Alk. Mat. Lapok* (submitted). ⇒ 231, 232, 235, 244, 246, 248, 260, 261

[38] A. Iványi, S. Pirzada, Comparison based ranking, in: *Algorithms of Informatics, Vol. 3,* ed. A. Iványi, AnTonCom, Budapest 2011, 1209–1258. ⇒ 231

[39] A. Járai, *Introduction to Mathematics* (Hungarian). ELTE Eötvös Kiadó, Budapest, 2005. ⇒ 253

[40] I. Kátai, Personal communication, Budapest, 2010. ⇒ 256

[41] K. K. Kayibi, M. A. Khan, S. Pirzada, A. Iványi, Random sampling of minimally cyclic digraphs with given imbalance sequence, *Acta Univ. Sapientiae, Math.* (submitted). ⇒ 231

[42] G. Kéri, On qualitatively consistent, transitive and contradictory judgment matrices emerging from multiattribute decision procedures, *CEJOR Cent. Eur. J. Oper. Res.* **19,** 2 (2011) 215–224. ⇒ 231

[43] K. Kern, D. Paulusma, The new FIFA rules are hard: complexity aspects of sport competitions, *Discrete Appl. Math.* **108,** 3 (2001) 317–323. ⇒ 231

[44] K. Kern, D. Paulusma, The computational complexity of the elimination problem in generalized sports competitions, *Discrete Optimization* **1**, 2 (2004) 205–214. ⇒ 231

[45] H. Kim, Z. Toroczkai, I. Miklós, P. L. Erdős, l. A. Székely, Degree-based graph construction, *J. Physics: Math. Theor. A* **42,** 39 (2009) 392–401. ⇒ 231

[46] D. J. Kleitman, D. L. Wang, Algorithms for constructing graphs and digraphs with given valencies and factors, *Discrete Math.* **6** (1973) 79–88. ⇒ 253

[47] D. J. Kleitman, K. J. Winston, Forests and score vectors, *Combinatorica* **1,** 1 (1981) 49–54. ⇒ 253

[48] D. E. Knuth, *The Art of Computer Programming. Volume 4A, Combinatorial Algorithms,* Addison–Wesley, Upper Saddle River, 2011. ⇒ 231

[49] A. Kohnert, Dominance order and graphical partitions, *Elec. J. Comb.* **11,** 1 (2004) No. 4. 17 pp. ⇒230

[50] M. D. LaMar, Algorithms for realizing degree sequences of directed graphs. arXiv-0906:0343ve [math.CO], 7 June 2010. ⇒231

[51] H. G. Landau, On dominance relations and the structure of animal societies. III. The condition for a score sequence, *Bull. Math. Biophys.* **15** (1953) 143–148. ⇒231

[52] F. Liljeros, C. R. Edling, L. Amaral, H. Stanley, Y. Áberg, The web of human sexual contacts, *Nature* **411**, 6840 (2001) 907–908. ⇒231

[53] L. Lovász, *Combinatorial Problems and Exercises* (corrected version of the second edition), AMS Chelsea Publishing, Boston, 2007. ⇒232

[54] L. Lucz, A. Iványi, P. Sótér, S. Pirzada, Testing and enumeration of football sequences, *Abstracts of MaCS 2012*, ed. by Z. Csörnyei (Siófok, February 9–12, 2012). ⇒231

[55] D. Meierling, L. Volkmann, A remark on degree sequences of multigraphs, *Math. Methods Oper. Res.* **69,** 2 (2009) 369–374. ⇒231

[56] N. Metropolis, P. R. Stein, The enumeration of graphical partitions, *European J. Comb.* **1,** 2 (1980) 139–153. ⇒253

[57] I. Miklós, Graphs with prescribed degree sequences (Hungarian), Lecture in Alfréd Rényi Institute of Mathematics, 16 November 2009. ⇒231

[58] I. Miklós, P. L. Erdős, L. Soukup, A remark on degree sequences of multigraphs (submitted). ⇒231

[59] J. W. Moon, *Topics on Tournaments,* Holt, Rinehart, and Winston, New York, 1968. ⇒253

[60] T. V. Narayana, D. H. Bent, Computation of the number of score sequences in round-robin tournaments, *Canad. Math. Bull.* **7,** 1 (1964) 133–136. ⇒231

[61] M. E. J. Newman, A. L. Barabási, *The Structure and Dynamics of Networks*, Princeton University Press, Princeton, NJ. 2006. ⇒231

[62] G. Pécsy, L. Szűcs, Parallel verification and enumeration of tournaments, *Stud. Univ. Babeş-Bolyai, Inform.* **45,** 2 (2000) 11–26. ⇒231

[63] S. Pirzada, *Graph Theory,* Orient Blackswan (to appear). ⇒231

[64] S. Pirzada, A. Iványi, Imbalances in digraphs, *Abstracts of MaCS 2012*, ed. by Z. Csörnyei (Siófok, February 9–12, 2012). ⇒231

[65] S. Pirzada, A. Iványi, M. A. Khan, Score sets and kings, in: *Algorithms of Informatics, Vol. 3,* ed. by A. Iványi. AnTonCom, Budapest 2011, 1410–1450. ⇒231

[66] S. Pirzada, T. A. Naikoo, U. T. Samee, A. Iványi, Imbalances in directed multigraphs, *Acta Univ. Sapientiae, Inform.* **2,** 1 (2010) 47–71. ⇒231

[67] S. Pirzada, G. Zhou, A. Iványi, On k-hypertournament losing scores, *Acta Univ. Sapientiae, Inform.* **2,** 2 (2010) 184–193. ⇒231

[68] Ø J. Rødseth, J. A. Sellers, H. Tverberg, Enumeration of the degree sequences of non-separable graphs and connected graphs, *European J. Comb.* **30,** 5 (2009) 1309–1319. ⇒231, 253

[69] F. Ruskey, R. Cohen, P. Eades, A. Scott, Alley CAT's in search of good homes, *Congr. Numer.* **102** (1994) 97–110. ⇒230, 251, 253

[70] G. Sierksma, H. Hoogeveen, Seven criteria for integer sequences being graphic, *J. Graph Theory* **15,** 2 (1991) 223–231. ⇒233

[71] R. Simion, Convex polytopes and enumeration, *Advances in Applied Math.* **18,** 2 (1996) 149–180. ⇒253

[72] N. J. A. Sloane (Ed.), *Encyclopedia of Integer Sequences,* 2011. ⇒231

[73] N. J. A. Sloane, The number of ways to put $n+1$ indistinguishable balls into $n+1$ distinguishable boxes, in: *The On-line Encyclopedia of the Integer Sequences* (ed. by N. J. A. Sloane), 2011. ⇒253, 261

[74] N. J. A. Sloane, The number of degree-vectors for simple graphs, in: *The On-line Encyclopedia of the Integer Sequences* (ed. by N. J. A. Sloane), 2011. ⇒230, 253, 261

[75] N. J. A. Sloane, The number of bracelets with n red, 1 pink and $n-1$ blue beads, in: *The On-line Encyclopedia of the Integer Sequences* (ed. by N. J. A. Sloane), 2011. ⇒253, 261

[76] N. J. A. Sloane, S. Plouffe, *The Encyclopedia of Integer Sequences,* Academic Press, Waltham, MA, 1995. ⇒253, 254

[77] D. Soroker, *Optimal parallel construction of prescribed tournaments, Discrete Appl. Math.* **29,** 1 (1990) 113–125. ⇒231

[78] R. P. Stanley, A zonotope associated with graphical degree sequence, in: *Applied geometry and discrete mathematics, Festschr. 65th Birthday Victor Klee.* DIMACS Series in Discrete Mathematics and Theoretical Computer Science. **4** (1991) 555-570. ⇒253

[79] R. P. Stanley, *Enumerative Combinatorics*, Cambridge University Press, Cambridge, 1997. ⇒253

[80] J. Temesi, Pairwise comparison matrices and the error-free property of the decision maker, *CEJOR Cent. Eur. J. Oper. Res.* **19,** 2 (2011) 239–249. ⇒231

[81] A. Tripathi, H. Tyagi, A simple criterion on degree sequences of graphs, *Discrete Appl. Math.* **156,** 18 (2008) 3513–3517. ⇒231

[82] A. Tripathi, S. Vijay, A note on a theorem of Erdős & Gallai, *Discrete Math.* **265,** 1-3 (2003) 417–420. ⇒244, 245

[83] A. Tripathi, S. Venugopalanb, D. B. West, A short constructive proof of the Erdös-Gallai characterization of graphic lists, *Discrete Math.* **310,** 4 (2010) 833–834. ⇒230, 231, 233, 235

[84] E. W. Weisstein, *Degree Sequence,* From MathWorld—Wolfram Web Resource, 2011. ⇒231

[85] E. W. Weisstein, *Graphic Sequence,* From MathWorld—Wolfram Web Resource, 2011. ⇒231

[86] K. J. Winston, D. J. Kleitman, On the asymptotic number of tournament score sequences, *J. Combin. Theory Ser. A.* **35** (1983) 208–230. ⇒253

Received: September 30, 2011 • Revised: November 10, 2011