# Generalized Stieltjes transforms: basic aspects

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**Abstract.** The paper surveys the basic properties of generalized Stieltjes functions including some new ones. We introduce the notion of the exact Stieltjes order and give a criterion of exactness, simple sufficient conditions and some prototypical examples. The paper includes an appendix, where we define the left sided Riemann-Liouville and the right sided Kober-Erdelyi fractional integrals of measures supported on half axis and give inversion formulas for them.

Keywords: *generalized Stieltjes transform, generalized Stieltjes function, fractional integral, fractional derivative, exact Stieltjes order*

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**1. Introduction.** The generalized Stieltjes transform of a non-negative measure $\mu$ supported on $[0,\infty)$ is defined by

$$\int\limits_{[0,\infty)} \frac{\mu(du)}{(u+z)^{\alpha}},$$

where $\alpha > 0$ and we always choose the branch of the power function which is positive on the positive half-axis. The measure is assumed to produce a convergent integral for each $z \in \mathbb{C}\backslash(-\infty,0]$ thus generating a function holomorphic in $\mathbb{C}\backslash(-\infty,-r]$, where $r = \inf\{x: x \in \mathbf{supp}(\mu)\}$. Functions representable by the above integral plus a non-negative constant are known as generalized Stieltjes functions [30, 32], [35, Section 8], [34, Chapter VIII].

The case $\alpha = 1$ has been thoroughly studied by many authors beginning with the classical work of Stieltjes [31] followed by Krein (see [18] and references therein), Widder [35, 34], Hirsch [11], Berg [3] and many others. Among the most important tools facilitating such study are the complex inversion formula due to Stieltjes [31, 34], the complex variable characterization found by Krein (see Theorem 11 below) and the real inversion formulas by Widder [35, 34]. When the measure $\mu$ has compact support, the Stieltjes functions are also known as Markov functions studied by Chebyshev and Markov in connection with continued fractions. The deep connection of the Stieltjes and Markov functions with continued fractions and Padé approximation is investigated in the monographs [2, 8]. See also the survey paper [10]. Connection with Bernstein functions and various other similar classes can be found in the carefully written recent monograph [28].

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For general $\alpha > 0$ much less is known. A complex inversion formula in this case has been found by Sumner [32] and later rediscovered by Schwarz [29]. It was amended by several other complex inversion formulas by Byrne and Love in [6]. These authors also found several real inversion formulas in [20, 21]. A simple real variable characterization has been discovered recently by Sokal [30] generalizing the corresponding result of Widder. The asymptotic expansion of generalized Stieltjes transforms of slowly decaying functions has been studied by López and Ferreira in [19], factorization as iterative Laplace transforms - by Yürekli [36], closely related classes on half plane have been investigated by Jerbashian [13]. An interesting connection to entire functions has been discovered in a recent work of Pedersen [23].

In this paper we collect a number of facts about generalized Stieltjes functions. Most of them are scattered in the literature, those we could not find are furnished with detailed proofs. Some of them may be new. In the last section we introduce the notion of the exact Stieltjes order, which is the "natural" exponent defined for each generalized Stieltjes function. We give a criterion of exactness and its two practical corollaries. We also added an appendix, where we define the left sided Riemann-Liouville and the right sided Kober-Erdelyi fractional integrals of measures supported on $[0,\infty]$ (the one point compactification of $[0,\infty)$ - see details below) and give inversion formulas for them. Our study of the generalized Stieltjes transforms carried out in this paper has been largely motivated by the applications to the theory of hypergeometric functions presented in our forthcoming work [14].

**2. Definition and real variable properties.** Define $S_\alpha$, $\alpha > 0$, to be the class of functions representable by the integral

$$f(z) = \int\limits_{(0,\infty)} \frac{\mu_\alpha(du)}{(u+z)^\alpha} + \mu_\infty + \frac{\mu_0}{z^\alpha}, \tag{1}$$

where $0 \le \mu_0, \mu_\infty < \infty$ and $\mu_\alpha$ runs over the set of non-negative measures supported on $[0,\infty)$ such that

$$\int\limits_{[0,\infty)} \frac{\mu_\alpha(dt)}{(1+t)^\alpha} < \infty. \tag{2}$$

This condition guarantees the finiteness of the integral (1) for all $z \in \mathbb{C}\setminus(-\infty,0]$. The classical Stieltjes cone $\mathcal{S}$ corresponds to the case $\alpha = 1$, i.e. $\mathcal{S} = S_1$, see [3, formula (1)], [30, formula (2)] or [28, formula (2.1)].

If we define $[0,\infty]$ to be the one point compactification of $[0,\infty)$ formula (1) may be rewritten as

$$f(z) = \int\limits_{[0,\infty]} \left(\frac{u+1}{u+z}\right)^\alpha \tilde{\mu}_\alpha(du), \tag{3}$$

where

$$\tilde{\mu}_\alpha = \frac{\mu_\alpha(du)}{(1+u)^\alpha} + \mu_0\delta_0 + \mu_\infty\delta_\infty,$$

is a finite measure on the compact interval $[0,\infty]$. Here $\delta_a$ stands for the Dirac measure with mass 1 concentrated in $a$. This explains the notation $\mu_\infty$ for the non-negative constant in

(1). The set of measures supported on $[0,\infty]$ and satisfying (2) will be denoted by $\mathcal{M}_\alpha$. The majority of references on the classical and generalized Stieltjes functions use formula (1) (or its particular case $\alpha=1$) to define them. See, for instance, [3, 11, 6, 20, 21, 28, 30, 34, 36]. However, the literature on Padé approximation frequently defines the Stieltjes functions by $\alpha=1$ case of the following formula

$$f(z)=\int\limits_{(0,\infty)}\frac{\rho_\alpha(dt)}{(1+tz)^\alpha}+\rho_0+\frac{\rho_\infty}{z^\alpha}. \tag{4}$$

Define the map $N_\alpha$ on $\mathcal{M}_\alpha$ by

$$[N_\alpha\mu](A):=\int\limits_{1/A}t^{-\alpha}\mu(dt)=\int\limits_{A}u^\alpha\mu^*(du)\text{ for each Borel set }A\subset(0,\infty), \tag{5}$$

where $\mu^*$ is the image measure of $\mu$ under the map $t\to 1/t$ and by definition $[N_\alpha\mu](\{\infty\})=\mu(\{0\})$, $[N_\alpha\mu](\{0\})=\mu(\{\infty\})$. If (1) and (4) represent the same function, the change of variable $t=1/u$ in (1) shows that $\rho_\alpha=N_\alpha\mu_\alpha$. By the same change of variable applied to (2) we see that

$$\int\limits_{[0,\infty)}\frac{\rho_\alpha(dt)}{(1+t)^\alpha}<\infty, \tag{6}$$

so that $N_\alpha$ maps $\mathcal{M}_\alpha$ into itself. Moreover, $N_\alpha$ is easily seen to be an involution on $\mathcal{M}_\alpha$: $N_\alpha N_\alpha\mu=\mu$ for each $\mu\in\mathcal{M}_\alpha$.

Definition (4) is a natural extension of the definition of Stieltjes functions used in [2, formula(5.1)] and [10, formula (1)]. In some situations this representation leads to simpler expressions for hypergeometric functions. We will work with both representations (1) and (4). If we define the finite measure

$$\tilde{\rho}_\alpha=\frac{\rho_\alpha(du)}{(1+u)^\alpha}+\rho_0\delta_0+\rho_\infty\delta_\infty$$

on the compact interval $[0,\infty]$, then $\tilde{\rho}_\alpha$ and $\tilde{\mu}_\alpha$ from (3) are related by $\tilde{\rho}_\alpha(A)=\tilde{\mu}_\alpha(1/A)$ for each Borel set $A\subset[0,\infty]$.

We will denote by $F_\mu:[0,\infty)\to[0,\infty)$ the left-continuous distribution function of the measure $\mu\in\mathcal{M}_\alpha$: $F_\mu(x)=\mu([0,x))$ normalized by $F_\mu(0)=0$. The distribution function $F_\mu$ defines the measure $\mu$ uniquely except for a possible atom at infinity which must be specified separately. According to [4, section 1.8] or [27, Problem 7.9(iii)] every measure $\mu\in\mathcal{M}_\alpha$ is generated by such non-decreasing left-continuous function satisfying

$$\int\limits_{[0,\infty)}\frac{dF_\mu(t)}{(1+t)^\alpha}<\infty,$$

and a non-negative constant $\mu_\infty:=\mu(\{\infty\})$. Here the integral is understood as Lebesgue-Stieltjes integral [25, Section 20.3] or [4, 2.12(vi)].

The Stieltjes cone $\mathcal{S}$ possesses a number of nice stability properties which can be found in [3]. The majority of these properties do not carry over to $S_\alpha$, $\alpha\neq 1$. On the other hand, here we have some new effects related to transition from $S_\alpha$ to $S_\beta$, $\beta\neq\alpha$ and certain stability properties of the class $S_\infty:=\cup_{\alpha>0}S_\alpha$. Below we list the basic facts about these classes.

**Theorem 1** *If $f \in S_\alpha$ then $g(z) := z^{-\alpha} f(1/z)$ also belongs to $S_\alpha$ and their representing measures are related by $\mu_g = N_\alpha \mu_f$.*

**Proof.** Indeed, using definition (4) we get:

$$z^{-\alpha} f(1/z) = z^{-\alpha} \int_{(0,\infty)} \frac{z^\alpha \rho_\alpha(dt)}{(z+t)^\alpha} + z^{-\alpha}\rho_0 + \rho_\infty = \int_{(0,\infty)} \frac{\rho_\alpha(dt)}{(z+t)^\alpha} + z^{-\alpha}\rho_0 + \rho_\infty$$

which is precisely the representation (1). Since $\rho_\alpha \in \mathcal{M}_\alpha$ is arbitrary, the claim follows. $\square$

The next important result is due to Sokal [30]:

**Theorem 2** (Sokal, [30]) *A function $f$ defined on $(0,\infty)$ has holomorphic extension $f \in S_\alpha$ if and only if*

$$F^\alpha_{n,k}(x) := (-1)^n D^k (x^{n+k+\alpha-1} D^n f(x)) \geq 0 \tag{7}$$

*for all integers $n, k \geq 0$ and all $x > 0$. Here $D = d/dx$.*

**Remark 1.** Differentiating (1) under the integral sign or writing

$$(-1)^n D^k (x^{n+k+\alpha} D^n(-f'(x)) = (-1)^{(n+1)} D^k (x^{(n+1)+k+\alpha-1} D^{(n+1)} f(x)), \tag{8}$$

we see that $f \in S_\alpha$ implies $-f' \in S_{\alpha+1}$, so that $(-1)^n f^{(n)} \in S_\infty$ for all integers $n \geq 0$ once $f \in S_\infty$.

**Theorem 3** *If $\alpha < \beta$ then $S_\alpha \subset S_\beta$. Moreover, each $f \in S_\alpha$ defined by (1) can be written as*

$$f(z) = \int_{(0,\infty)} \frac{\mu_\beta(dy)}{(y+z)^\beta} + \mu_\infty, \tag{9}$$

*where $\mu_\beta \in \mathcal{M}_\beta$ and*

$$\mu_\beta(dy) = \frac{\Gamma(\beta) dy}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_{[0,y)} \frac{\mu_\alpha(du)}{(y-u)^{\alpha+1-\beta}}. \tag{10}$$

*Conversely, given $\mu_\beta \in \mathcal{M}_\beta$ which represents $f \in S_\alpha$ we can recover $\mu_\alpha$ from*

$$F_{\mu_\alpha}(y) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta+n+1)} \left(\frac{d}{dy}\right)^n \int_{[0,y)} \frac{\mu_\beta(du)}{(y-u)^{\beta-\alpha-n}} \tag{11}$$

*with $n = [\beta - \alpha]$ and $\mu_\alpha(\{\infty\}) = \mu_\beta(\{\infty\})$.*

**Remark 2.** Formula (10) generalizes [34, Chapter VIII, Corollary 3a.1, p.330] which in our notation connects the measures $\mu_1$ and $\mu_2$. The transformation $\mu_\alpha \to \mu_\beta$ defined in (10) is the left-sided Riemann-Liouville fractional integral. Its precise definition and inversion are investigated in the Appendix.

**Proof.** Since

$$\frac{1}{(u+z)^\alpha} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_{(0,\infty)} \frac{t^{\beta-\alpha-1} dt}{(z+u+t)^\beta}, \tag{12}$$

we have

$$\int\limits_{[0,\infty)} \frac{\mu_\alpha(du)}{(u+z)^\alpha} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int\limits_{[0,\infty)} \mu_\alpha(du) \int\limits_{(0,\infty)} \frac{t^{\beta-\alpha-1}dt}{(z+u+t)^\beta}\Bigg|_{y=u+t}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int\limits_{[0,\infty)} \mu_\alpha(du) \int\limits_{(u,\infty)} \frac{(y-u)^{\beta-\alpha-1}dy}{(z+y)^\beta}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int\limits_{(0,\infty)} \frac{dy}{(z+y)^\beta} \int\limits_{[0,y)} (y-u)^{\beta-\alpha-1}\mu_\alpha(du) = \int\limits_{[0,\infty)} \frac{\mu_\beta(dy)}{(z+y)^\beta}, \quad (13)$$

where $\mu_\beta(dy)$ is defined by (10) and we can use Tonelli's theorem [25, Chapter 20, Corollary 7] or [4, Theorem 3.4.5] to justify the interchange of integrations. Also setting $z=1$ we see by Tonelli's theorem and condition (2) that the measure $\mu_\beta$ belongs to $\mathcal{M}_\beta$. According to Remark A1 in the Appendix the measure $\mu_\beta$ has no atom at zero. This allows us to remove zero from the domain of integration in (9). This proves (9) and the inclusion $S_\alpha \subset S_\beta$. The inversion formula (11) is Theorem A1 in the Appendix. □

**Theorem 4** *Each $f \in S_\alpha$ defined by (4) can also be written as*

$$f(z) = \int\limits_{(0,\infty)} \frac{\rho_\beta(dx)}{(1+xz)^\beta} + \rho_0,$$

*where $\rho_\beta \in \mathcal{M}_\beta$ and*

$$\rho_\beta(dx) = \frac{\Gamma(\beta)x^{\alpha-1}dx}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\Bigg\{ \int\limits_{(x,\infty)} \frac{u^{1-\beta}\rho_\alpha(du)}{(u-x)^{\alpha-\beta+1}} + \rho_\infty \Bigg\}. \quad (14)$$

*Conversely, given $\rho_\beta \in \mathcal{M}_\beta$ which represents $f \in S_\alpha$ we can recover $\rho_\alpha$ from*

$$F_{\rho_\alpha}(y) = \rho_0 + \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta+n+1)}\times$$
$$\Bigg\{\alpha \int\limits_{(0,y)} x^{\alpha-1}dx \left(-x^2\frac{d}{dx}\right)^n x^{\beta-\alpha-n}\int\limits_{(x,\infty)} \frac{\rho_\beta(ds)}{s^{\alpha+n}(s-x)^{\beta-\alpha-n}} - y^\alpha\left(-y^2\frac{d}{dy}\right)^n y^{\beta-\alpha-n}\int\limits_{(y,\infty)} \frac{\rho_\beta(ds)}{s^{\alpha+n}(s-y)^{\beta-\alpha-n}}\Bigg\} \quad (15)$$

*with $n=[\beta-\alpha]$ and $\rho_\alpha(\{\infty\}) = [\Gamma(\beta-\alpha)\Gamma(\alpha+1)/\Gamma(\beta)] \lim_{y\to\infty} y^{-\alpha}F_{\rho_\beta}(y)$.*

**Remark 3.** The transformation $\rho_\alpha \to \rho_\beta$ defined in (14) is the right-sided Kober-Erdelyi fractional integral. Its precise definition and inversion are investigated in the Appendix.

**Proof.** To demonstrate (14) we employ the connection formula $\rho_\beta^*(dy) = y^{-\beta}\mu_\beta(dy)$, where $\rho_\beta^*$ is the image of $\rho_\beta$ under $y \to y^{-1}$. We have ($A = \Gamma(\beta)/[\Gamma(\alpha)\Gamma(\beta-\alpha)]$ for brevity)

$$y^\beta\rho_\beta^*(dy) = \mu_\beta(dy) = Ady\int\limits_{(0,y)} \frac{\mu_\alpha(du)}{(y-u)^{\alpha+1-\beta}} + \mu_0 Ay^{\beta-\alpha-1}dy$$

$$= Ady\int\limits_{(1/y,\infty)} \frac{t^{\alpha+1-\beta}\mu_\alpha^*(dt)}{(yt-1)^{\alpha+1-\beta}} + A\rho_\infty y^{\beta-\alpha-1}dy = Ay^{\beta-\alpha-1}dy\int\limits_{(1/y,\infty)} \frac{t^{1-\beta}\rho_\alpha(dt)}{(t-1/y)^{\alpha+1-\beta}} + A\rho_\infty y^{\beta-\alpha-1}dy.$$

Dividing by $y^\beta$ and making substitution $y = 1/x$, $dy = -dx/x^2$ we arrive at (14). The inversion formula (15) is Theorem A2 in the Appendix. $\square$

The following result is also due to Sokal [30, formulas (10a) and (10b)].

**Theorem 5** $\bigcap_{\alpha>0} S_\alpha = \{\text{non-negative constants}\}$.

This result suggests the following definition:

$$S_0 := \{\text{non-negative constants}\}.$$

Recall that a function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic if $f$ has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \geq 0$, for all $x > 0$ and $n = 0, 1, \ldots$ We denote the set of completely monotonic functions by $\mathcal{CM}$. The following result was pointed out to us by Christian Berg and is also hinted at in [30].

**Theorem 6** $\overline{S_\infty} = \mathcal{CM}$, *where the closure is taken with respect to pointwise convergence on* $(0,\infty)$.

**Proof.** The inclusion $\overline{S_\infty} \subset \mathcal{CM}$ follows from the fact that each $f \in \bigcup_{\alpha>0} S_\alpha$ is completely monotonic combined with closedness of $\mathcal{CM}$ under pointwise convergence [28, Corollary 1.6]. To prove the reverse inclusion recall that according to Bernstein's theorem (see [34, Chapter IV, Theorem 12b] or [28, Theorem 1.4]) each completely monotonic function is the Laplace transform of a nonnegative measure:

$$f(x) = \mathcal{L}(\sigma; x) := \int\limits_{[0,\infty)} e^{-xt}\sigma(dt).$$

Since $f(x)$ is non-increasing the sequence $f(1/n)$ is non-decreasing. Hence, two cases are possible: 1) $f(1/n)$ is bounded by a constant or 2) $\lim_{n\to\infty} f(1/n) = +\infty$. Define $a_n := \sqrt{n}$ in the first case and $a_n := f(1/n)$ in the second. Define the following sequence of functions:

$$f_n(x) = \int\limits_{[0,\infty)} \left(1 + \frac{xt}{a_n^2}\right)^{-a_n^2} e^{-\frac{t}{n}}\sigma(dt).$$

Obviously,

$$f_n(x) \leq \int\limits_{[0,\infty)} e^{-\frac{t}{n}}\sigma(dt) = f(1/n),$$

so that the integral defining $f_n(x)$ exists. Moreover,

$$f_n(x) = \int\limits_{[0,\infty)} \frac{\sigma_n(du)}{(1+xu)^{a_n^2}},$$

where $\sigma_n(du) = a_n^2 e^{\frac{-a_n^2 u}{n}}\sigma(du)$. Clearly, $f_n \in \bigcup_{\alpha>0} S_\alpha$. Next, we have

$$f_n(x) - f(x) = \int\limits_{[0,\infty)} \left(\left(1 + \frac{xt}{a_n^2}\right)^{-a_n^2} - e^{-xt} + e^{-xt} - e^{-(x-1/n)t}\right) e^{-\frac{t}{n}}\sigma(dt)$$

$$f_n(x)-f(x)=\int\limits_{[0,\infty)}\left(\left(1+\frac{xt}{a_n^2}\right)^{-a_n^2}-e^{-xt}\right)e^{-\frac{t}{n}}\sigma(dt)+f(x+1/n)-f(x). \qquad (16)$$

It is verified by straightforward calculus that the maximum of

$$\psi(t):=\left|\left(1+\frac{xt}{a_n^2}\right)^{-a_n^2}-e^{-xt}\right|$$

is attained at the point $t^*$ satisfying

$$\left(1+\frac{xt}{a_n^2}\right)^{-a_n^2-1}=e^{-xt}.$$

Hence,

$$\max_t|\psi(t)|=\frac{xt^*e^{-xt^*}}{a_n^2}\le\frac{e^{-1}}{a_n^2}.$$

It follows from (16) that

$$|f_n(x)-f(x)|\le\frac{e^{-1}}{a_n^2}f(1/n)+|f(x+1/n)-f(x)|.$$

Due to the definition of $a_n$ and continuity of $f(x)$ we conclude that $\lim_{n\to\infty}f_n(x)=f(x)$ for each $x>0$. $\square$

**Theorem 7** *If $f\in S_\alpha$ and $g\in S_\beta$ then $fg\in S_{\alpha+\beta}$.*

**Proof.** See [12, Chapter VII, paragraph 7.4].
**Remark 4.** Theorems 3 and 7 show that the union $S_\infty$ is a cone with multiplication: if $f,g\in S_\infty$ then $af+bg\in S_\infty$, for all $a,b\ge 0$ and $fg\in S_\infty$.

**Theorem 8** *Each*

$$f(z)=\int\limits_{[0,\infty)}\frac{\mu_\alpha(du)}{(u+z)^\alpha}+\mu_\infty\in S_\alpha$$

*can be represented in the form*

$$f(z)=\frac{1}{\Gamma(\alpha)}\mathcal{L}(u^{\alpha-1}\mathcal{L}(\mu_\alpha;u)du;z)+\mu_\infty, \qquad (17)$$

*where $\mathcal{L}$ denotes the Laplace transform.*

**Proof.** Write

$$\frac{1}{(u+z)^\alpha}=\frac{1}{\Gamma(\alpha)}\int\limits_{[0,\infty)}e^{-(u+z)t}t^{\alpha-1}dt$$

and apply Tonelli's theorem to show that the iterated integral in (17) exists and is equal to $f(z)$. $\square$
**Remark 5.** Formula (17) has been found in [36] for absolutely continuous measures.

Introduce the standard notation for the right-sided Riemann-Liouville fractional integral [15, section 2.2], [26, §5] and the right-sided Caputo fractional derivative [15, section 2.4], [24, section 2.4.1]:

$$I_\lambda^- f := \frac{1}{\Gamma(\lambda)} \int_x^\infty \frac{f(t)dt}{(t-x)^{1-\lambda}},$$

$${}^C D_\lambda^- f = (I_\lambda^-)^{-1} f = \frac{(-1)^n}{\Gamma(n-\lambda)} \int_x^\infty \frac{f^{(n)}(t)dt}{(t-x)^{1+\lambda-n}}, \quad n = [\lambda]+1.$$

**Theorem 9** *For a fixed non-negative measure $\mu \in \mathcal{M}_\alpha$ and $\beta > \alpha > 0$ denote*

$$f_\alpha(z) = \int_{[0,\infty)} \frac{\mu(du)}{(u+z)^\alpha} + \mu_\infty, \tag{18}$$

*and*

$$f_\beta(z) = \int_{[0,\infty)} \frac{\mu(du)}{(u+z)^\beta} + \mu_\infty. \tag{19}$$

*Then for all $x > 0$*

$$f_\alpha(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_x^\infty \frac{(f_\beta(t)-\mu_\infty)dt}{(t-x)^{1+\alpha-\beta}} + \mu_\infty = \frac{\Gamma(\beta)}{\Gamma(\alpha)} I_{\beta-\alpha}^-[f_\beta(t)-\mu_\infty](x) + \mu_\infty \tag{20}$$

*and*

$$f_\beta(x) = \frac{\Gamma(\alpha)(-1)^n}{\Gamma(\beta)\Gamma(n-\beta+\alpha)} \int_x^\infty \frac{f_\alpha^{(n)}(t)dt}{(t-x)^{1+\beta-\alpha-n}} + \mu_\infty = \frac{\Gamma(\alpha)}{\Gamma(\beta)} {}^C D_{\beta-\alpha}^-[f_\alpha(t)](x) + \mu_\infty, \tag{21}$$

*where $n = [\beta-\alpha]+1$.*

**Proof.** To prove (20) substitute (19) for $f_\beta$ into (20) and exchange the order of integration which is legitimate by Tonelli's theorem again. Conditions $\beta > \alpha > 0$ guarantee the existence of the inner integral. Formula (21) is one of several inversion formulas for the Riemann-Liouville integral in (20) which is applicable since $f_\alpha$ is infinitely differentiable (see, for instance, [15, section 2.4], [24, section 2.4.1]). To demonstrate its validity differentiate under integral sign ($n \geq 1$)

$$f_\alpha^{(n)}(t) = (-1)^n (\alpha)_n \int_{[0,\infty)} \frac{\mu(du)}{(u+t)^{\alpha+n}},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$, substitute into (21) and exchange the order of integrations. $\square$

**Remark 6.** The Riemann-Liouville fractional derivative cannot be used in (21) since, in general, the resulting integral would diverge.

**Theorem 10** *The class $S_\alpha$, $\alpha > 1$ is closed under pointwise limits: if $\{f_n\}_{n=1}^\infty \subset S_\alpha$ and if the limit $\lim\limits_{n\to\infty} f_n(x) = f(x)$ exists for all $x > 0$ then $f \in S_\alpha$.*

**Proof.** A proof for $\alpha = 1$ can be found in [11, Proposition 1] or by a different argument in [28, Theorem 2.2(iii)]. The latter carries over *mutatis mutandis* to all $\alpha > 0$. $\square$

**3. Complex variable properties.** Clearly, $f$ is holomorphic in $\mathbb{C}\setminus(-\infty,-r]$, where $r=\inf\{x:\ x\in\mathbf{supp}(\mu_\alpha)\}$ and $f(\overline{z})=\overline{f(z)}$. In particular, if $0\notin\mathbf{supp}(\mu_\alpha)$ (or, equivalently, $\mathbf{supp}(\rho_\alpha)$ is bounded) then the function $f$ can be represented by the power series

$$f(z)=\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha)_k}{k!}\rho_k(\alpha)z^k,$$

convergent in the disk $|z|<1/R$. Here $R=\sup\{x:\ x\in\mathbf{supp}(\rho_\alpha)\}$, $(\alpha)_k=\Gamma(\alpha+k)/\Gamma(\alpha)$, and

$$\rho_k(\alpha)=\int\limits_{[0,R]}t^k d\rho_\alpha(t)<\infty,\quad k=0,1,\ldots \tag{22}$$

are the moments of the measure $\rho_\alpha$ which are finite due to (6).

For functions belonging to $S_1$ Krein [18, Appendix, Theorem 4] found the following celebrated characterization.

**Theorem 11** *A function $f$ holomorphic in the cut plane $\mathbb{C}\setminus(-\infty,0]$ belongs to $S_1$ iff $f(x)\geq 0$ for $x>0$ and $\Im f(z)\leq 0$ for $\Im z>0$.*

Because of the special role played by $S_1$ (including a number of stability properties [3], connection to continued fractions [8] and Padé approximation [2]) it is interesting to relate the functions from $S_\alpha$ to $S_1$. One way of doing this is provided by Theorem 9 (it works if $\mu\in\mathcal{M}_1$), another approach is presented in the following two theorems.

**Theorem 12** *Suppose $f\in S_\alpha$, $0<\alpha\leq 1$. Then $f^{1/\alpha}\in S_1$.*

**Proof.** For $\Im z>0$ and $t>0$ we have ($\arg(z)$ denotes the principal value of the argument of $z$):

$$-\pi\alpha<\arg(z+t)^{-\alpha}<0\quad\Rightarrow\quad-\pi\alpha<\arg(f(z))\leq 0\quad\Rightarrow\quad\Im(f(z)^{1/\alpha})\leq 0.$$

Since $f^{1/\alpha}(z)$ is holomorphic in $\mathbb{C}\setminus(-\infty,0]$ and non-negative for $z>0$ we get the conclusion by Krein's theorem 11. $\square$

**Theorem 13** *Suppose $f\in S_\alpha$, $\alpha\geq 1$. Then $g(z):=f(z^{1/\alpha})\in S_1$.*

**Proof.** Indeed $g(z)$ is holomorphic in $\mathbb{C}\setminus(-\infty,0]$ and non-negative for $x>0$. Next for $\Im z>0$ and $t>0$ we have ($\arg(z)$ denotes the principal value of the argument of $z$):

$$0<\arg(z^{1/\alpha})<\pi/\alpha\quad\Rightarrow\quad 0<\arg(z^{1/\alpha}+t)<\pi/\alpha,\quad\Rightarrow\quad\Im(z^{1/\alpha}+t)^{-\alpha}<0$$

Integrating the last expression with respect to non-negative measure preserves the lower half plane, so that the proof is completed by Krein's theorem 11. $\square$

**Remark 7.** For $\alpha>1$ the mapping $S_\alpha\to S_1$ defined by $f(z)\to g(z):=f(z^{1/\alpha})$ is clearly not surjective as can be seen immediately by taking $g(z)=1/(1+z)\in S_1$, $f(z)=g(z^\alpha)\notin S_\alpha$ (since it is not holomorphic in the upper half-plane). The conditions (a) $f(z)$ is holomorphic in the cut plane $\mathbb{C}\setminus(-\infty,0]$, (b) $f(x)\geq 0$ for $x>0$ and (c) $\Im f(z)\leq 0$ for $0<\arg(z)<\pi/\alpha$

are necessary for $f$ to belong to $S_\alpha$. Unfortunately, these conditions are not sufficient as the following example shows:

$$f(z) = \frac{1}{(z+1)^2} - \frac{1}{2(z+2)^2}.$$

Indeed, a straightforward computation yields ($z = x + iy$):

$$\Im f(z) = -\frac{y(30 + 87x + 96x^2 + 50x^3 + 12x^4 + x^5 + 12y^2 + 22xy^2 + 12x^2y^2 + 2x^3y^2 + xy^4)}{(1 + 2x + x^2 + y^2)^2(4 + 4x + x^2 + y^2)^2}.$$

Hence, for $\{z \in \mathbb{C} : \ \Re z > 0, \Im z > 0\}$ we get $\Im f(z) < 0$ and $f(x) \geq 0$ for $x > 0$. However, $f \notin S_2$ since the representing measure is signed. In fact, $f \in S_3$ since

$$f(z) = \int_1^\infty \frac{d\nu(t)}{(z+t)^3}, \ \text{where } d\nu(t) = \begin{cases} 2dt, \ t \in (1,2], \\ dt, \ t \in (2,\infty). \end{cases}$$

We thank Alex Gomilko for this example. It provides a partial answer to the question asked by Sokal at the end of [30].

**Theorem 14** *Each $f \in S_\alpha$ satisfies*

$$|f(z)| \leq A \left| \frac{z-1}{\Im(z)} \right|^\alpha + \mu_\infty, \tag{23}$$

*where $A = \int_{[0,\infty)} \mu_\alpha(du)/(u+1)^\alpha < \infty$, and $\Re(z) \leq 0$.*

**Proof.** We have

$$|f(z)| = \left| \int_{[0,\infty)} \frac{\mu_\alpha(du)}{(u+z)^\alpha} + \mu_\infty \right| = \left| \int_{[0,\infty)} \frac{\mu_\alpha(du)}{(u+1)^\alpha} \left( \frac{u+1}{u+z} \right)^\alpha + \mu_\infty \right|$$
$$\leq A \max_{u \geq 0} \left| \frac{u+1}{u+z} \right|^\alpha + \mu_\infty = A[\psi(z)]^{\alpha/2} + \mu_\infty,$$

with $A$ given above and

$$\psi(z) = \max_{u \geq 0} \left| \frac{u+1}{u+z} \right|^2 = \left| \frac{z-1}{\Im(z)} \right|^2.$$

The last equality is true for $\Re(z) \leq 0$ by standard calculus. $\square$

**Remark 8.** The estimate (23) may seem very weak because the right hand side does not tend to zero as $|z| \to \infty$ while $f(z) \to 0$. However, the decrease of $f$ may be arbitrarily slow so that it is difficult to expect a better estimate valid for the whole class $S_\alpha$.

**4. Exact Stieltjes order.** We will say that $f$ is of exact Stieltjes order $\alpha^*$ if $f \in S_\infty$ and

$$\alpha^*[f] = \inf\{\alpha : \ f \in S_\alpha\}. \tag{24}$$

**Theorem 15** *If $f$ is of exact Stieltjes order $\alpha^*$ then $f \in S_{\alpha^*}$. In other words the infimum in (24) is always attained.*

**Proof.** Since $f \in S_\infty$ it is infinitely differentiable so that $F^\alpha_{n,k}(x)$ in (7) is well defined and continuous in $\alpha$ for each fixed $x > 0$ for all $\alpha > \alpha^*$. Passing to the limit $\alpha \to \alpha^*$ we verify that (7) is true for $\alpha = \alpha^*$. Hence, by Sokal's theorem 2 $f \in S_{\alpha^*}$. $\square$

**Theorem 16** *Suppose $f \in S_\beta$. Then $\beta > \alpha^*[f]$ iff the function*

$$\Phi(y) = \int\limits_{(0,y)} \frac{\mu_\beta(du)}{(y-u)^\varepsilon} \tag{25}$$

*is non-decreasing on $(0, \infty)$ for some $\varepsilon \in (0, \min\{\beta, 1\})$.*

**Proof.** Assume $\Phi(y)$ is non-decreasing. We need to show that $\beta$ is not exact. According to Definition 1 in the Appendix and Theorem A1

$$I^+_{1-\varepsilon}\mu_\beta = \frac{\Phi(y)dy}{\Gamma(1-\varepsilon)} + \mu_\infty \delta_\infty \in \mathcal{M}_{\beta+1-\varepsilon} = \mathcal{M}_{\alpha+1},$$

where $\alpha = \beta - \varepsilon$ and $\mu_\infty = \mu_\beta(\{\infty\})$. Hence,

$$\int\limits_{[0,\infty)} \frac{\Phi(y)dy}{(1+y)^{\alpha+1}} < \infty.$$

This implies that

$$\int\limits_{[0,\infty)} \frac{d\Phi(y)}{(1+y)^{\alpha}} < \infty. \tag{26}$$

Indeed, for each $t > 0$ integration by parts,

$$\left.\frac{\Phi(y)}{\alpha(1+y)^\alpha}\right|_0^t = \frac{1}{\alpha} \int\limits_{[0,t)} \frac{d\Phi(y)}{(1+y)^\alpha} - \int\limits_{[0,t)} \frac{\Phi(y)dy}{(1+y)^{\alpha+1}},$$

shows that $\lim_{t\to\infty} \Phi(t)(1+t)^{-\alpha}$ exists (finite or infinite). This limit must be zero since otherwise $\Phi(y)(1+y)^{-\alpha} > C > 0$ for all $y > M$ and

$$\int\limits_{[0,\infty)} \frac{\Phi(y)dy}{(1+y)^{\alpha+1}} > C \int\limits_{[M,\infty)} \frac{dy}{1+y} = \infty.$$

Hence, $\lim_{t\to\infty} \Phi(t)(1+t)^{-\alpha} = 0$ and the above integration by parts proves (26). It follows that the measure $\mu_\alpha$ whose distribution function is equal to $A\Phi(y)$ and whose atom at infinity is equal to $\mu_\infty$ belongs to $\mathcal{M}_\alpha$. Here $A = \Gamma(\alpha)/[\Gamma(\beta)\Gamma(\alpha - \beta + 1)]$. Consider the function

$$g(z) = \int\limits_{[0,\infty)} \frac{\mu_\alpha(du)}{(u+z)^\alpha} + \mu_\infty.$$

By Theorem 3 we have $g \in S_\beta$ and

$$g(z) = \int\limits_{[0,\infty)} \frac{\tilde{\mu}_\beta(du)}{(u+z)^\beta} + \mu_\infty,$$

where $\tilde{\mu}_\beta(du)$ is given by (10). But then $\mu_\alpha$ and $\tilde{\mu}_\beta$ are related by (11) which coincides with (25) times $A$ (note that $n = 0$ in (11) because $\beta - \alpha = \varepsilon < 1$). This proves that $\tilde{\mu}_\beta = \mu_\beta$ so that $f(z) = g(z) \in S_\alpha$, $\alpha < \beta$.

Conversely, if $\beta$ is not exact choose $\varepsilon \in (0, \beta - \alpha^*)$, $\varepsilon < 1$. We have $f \in S_{\beta-\varepsilon}$. According to Theorem 3 the function $A\Phi(y)$, where $\Phi$ is defined in (25) equals the distribution function of the measure $\mu_{\beta-\varepsilon}$ and so is non-decreasing. $\square$

**Corollary 1** *Suppose $f \in S_\beta$, $\varepsilon \in (0, \min\{\beta, 1\})$ and the following limit exists:*

$$\lim_{y\to+\infty} \frac{\Phi(2y)}{\Phi(y)} = A,$$

*where $\Phi$ is defined by (25). If $A < 1$ then $\beta$ is the exact Stieltjes order of $f$.*

**Proof.** Clearly, the condition $A < 1$ implies that $\Phi(y)$ cannot be non-decreasing so that by Theorem 16 $\beta$ must be exact. $\square$

**Corollary 2** *Suppose $f \in S_\beta$ and the support of the measure $\mu_\beta$ is compact. Then $\beta$ is the exact Stieltjes order of $f$.*

**Proof.** Indeed, for all $y > B$, $B := \sup\{x : \ x \in \mathbf{supp}(d\mu)\}$, the function $\Phi(y)$ is strictly decreasing for each $\varepsilon \in (0, \min\{\beta, 1\})$, so that by Theorem 16 $\beta$ must be exact. $\square$

Consider three prototypical examples.

**Example 1**. Find the exact Stieltjes order of ($\alpha > 1$)

$$f(z) = \int\limits_0^1 \frac{dt}{(z+t)^\alpha} = \frac{1}{\alpha-1}\left(\frac{1}{z^{\alpha-1}} - \frac{1}{(1+z)^{\alpha-1}}\right).$$

Method I: by corollary 2 $\alpha^* = \alpha$ since $\mathbf{supp}(d\mu)$ is compact.

Method II: by theorem 16 compute $\Phi(y)$. Let $I(A)$ be the indicator function of a set $A$. We have ($0 < \varepsilon < 1$)

$$\Phi(y) = \int\limits_0^y \frac{I([0,1])du}{(y-u)^\varepsilon} = \begin{cases} y^{1-\varepsilon}/(1-\varepsilon), & 0 < y \le 1, \\ [y^{1-\varepsilon} - (y-1)^{1-\varepsilon}]/(1-\varepsilon), & y > 1. \end{cases}$$

It is straightforward to check that this function is decreasing for $y > 1$ so that again $\alpha^* = \alpha$.

**Example 2.** Find the exact Stieltjes order of ($\alpha > 1$)

$$f(z) = \int\limits_1^\infty \frac{dt}{(z+t)^\alpha}.$$

By theorem 16 compute

$$\Phi(t) = \int_0^t \frac{I([1,\infty))du}{(t-u)^\varepsilon} = \begin{cases} 0, & 0 < t \le 1 \\ (t-1)^{1-\varepsilon}/(1-\varepsilon), & t > 1 \end{cases}$$

This function is non-decreasing on $[0,\infty)$, so that $\alpha^* < \alpha$. To find the exact order we compute

$$f(z) = \frac{1}{(\alpha-1)(1+z)^{\alpha-1}} = \int_0^\infty \frac{d\nu(t)}{(z+t)^{\alpha-1}},$$

where the measure $d\nu(t)$ is concentrated at one point $t = 1$ with $\nu(\{1\}) = (\alpha-1)^{-1}$ so that by Corollary 2 $\alpha^* = \alpha - 1$.

**Example 3.** According to Euler's integral representation [1, Theorem 2.2.1] the Gauss hypergeometric functions ${}_2F_1$ can be written as

$$f(z) :={}_2 F_1(a,b;c;-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{u^{b-1}(1-u)^{c-b-1}}{(1+zu)^a} du, \quad c > b > 0.$$

Assume that $0 < a \le b$. By the above formula $f \in S_a$. Change of variable $u = 1/t$ yields

$${}_2F_1(a,b;c;-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty \frac{\mu(t)dt}{(z+t)^a},$$

where

$$\mu(t) = t^{a-c}(t-1)^{c-b-1}, \quad t > 1.$$

We aim to show that $a$ is the exact Stieltjes order of $f$. Computation gives

$$\Phi(y) := \int_1^y \frac{\mu(t)dt}{(y-t)^\varepsilon} = A(y-1)^{c-b-\varepsilon}{}_2F_1(c-a, c-b; c-b-\varepsilon+1; -(y-1)),$$

where $A = \Gamma(c-b)\Gamma(1-\varepsilon)/[\Gamma(c-b-\varepsilon+1)]$ for any given $\varepsilon \in (0, \min\{1, c-b\})$. Using the asymptotic formula [1, formula (2.3.12)] for ${}_2F_1$ as $y \to \infty$, we get

$$\Phi(y) = A(y-1)^{-\varepsilon} \left(D + o(1)\right),$$

where $D$ is a constant. This implies that $\lim_{y\to\infty}[\Phi(2y)/\Phi(y)] = 2^{-\varepsilon} < 1$ so that by Corollary 1 the exact order is equal to $a$.

In our forthcoming paper [14] we will investigate the exact Stieltjes order of the generalized hypergeometric function ${}_{q+1}F_q$.

**Remark 9.** According to Theorem 10 the class $S_\alpha$ is closed under pointwise limits. It is then reasonable to ask whether the exact Stieltjes order is also preserved by such limits. The following example shows that the answer is negative in general. Consider the sequence of functions ($\alpha > 1$)

$$f_m(z) = \int_1^m \frac{dt}{(z+t)^\alpha} = \frac{1}{(\alpha-1)(z+1)^{\alpha-1}} - \frac{1}{(\alpha-1)(z+m)^{\alpha-1}}, \quad m = 2, 3, \ldots$$

According to Corollary 2 each $f_m$ has exact order $\alpha$ while pointwise

$$\lim_{m\to\infty} f_m(z) = \frac{1}{(\alpha-1)(z+1)^{\alpha-1}} \text{ for each } z \in \mathbb{C}\backslash(-\infty, 0].$$

According, to Example 2 above, the limit function is of exact order $\alpha - 1$.
**Remark 10.** Theorem 9 shows that if $\alpha$ is the exact Stieltjes order of $f$ then its fractional derivative of order $\gamma$ will have the exact order $\alpha + \gamma$ while its fractional integral of order $\gamma$ will have the exact order $\alpha - \gamma$ provided that $\alpha > \gamma$ and $\mu \in \mathcal{M}_{\alpha-\gamma}$.

# APPENDIX
# Inversion formulas for some fractional integrals of measures on half-axis

In this appendix we prove some facts about fractional integrals and derivatives of Borel measures supported on $\mathbb{R}^+$. Fractional calculus is certainly a classical subject with a number of great monographs available today, including [9, 15, 16, 22, 24, 26]. Fractional integrals and derivatives have been studied in (weighted) spaces of integrable functions, in Hölder classes, in spaces of generalized functions, in the complex plane, for functions of several variables and in various other contexts. We could not find, however, a good reference for fractional integrodifferentiation of measures. One reason could be is that the definition of the fractional integral of a measure is fairly straightforward. The inversion problem, nonetheless, i.e. the definition of a fractional derivative, might not be so trivial. Here we prove two inversion theorems required in the study of generalized Stieltjes transforms. They might be useful in some other contexts as well, for instance in connection with completely monotonic functions of positive order, see [17].

Let us remind the reader that $\mathcal{M}_\alpha$ is the positive cone comprising non-negative Borel measures supported on $[0,\infty]$ and satisfying (2). Clearly, $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$ if $\alpha < \beta$. There is a natural involution $N_\alpha$ defined on $\mathcal{M}_\alpha$ by (5). For a measure $\mu \in \mathcal{M}_\alpha$ we denote by $F_\mu$ its left-continuous distribution function normalized by $F_\mu(0) = 0$. The distribution function $F_\mu$ defines the measure $\mu$ uniquely except for a possible atom at infinity which must be specified separately.

**Definition A1.** Let $\mu \in \mathcal{M}_\alpha$. The measure $\nu := I_\eta^+ \mu$ is called the left-sided Riemann-Liouville fractional integral of $\mu$ of order $\eta > 0$ if

$$\nu(B) := \frac{1}{\Gamma(\eta)} \int\limits_{B\setminus\{\infty\}} dy \int\limits_{[0,y)} \frac{\mu(du)}{(y-u)^{1-\eta}} + \mu(B \cap \{\infty\}) \ \text{ for each Borel set } B \subset [0,\infty]. \quad (27)$$

**Remark A1.** Formula (27) is certainly a straightforward generalization of the left-sided Riemann-Liouville fractional integral as given in [15, (2.2.1),(2.2.2)] and [26, Chapter 2(5.1),(5.3)]. One can check that $\nu(\{0\}) = 0$ regardless of $\mu(\{0\})$ by computing the limit $\lim_{y\to 0} F_\nu(y) = 0$.

**Definition A2.** Let $\mu \in \mathcal{M}_\alpha$. The measure $\tau := K_{\alpha,\eta}^- \mu$ is called the right-sided Kober-Erdelyi fractional integral of $\mu$ of order $\eta > 0$ if

$$\tau(B) := \mu(B \cap \{0\}) + \frac{1}{\Gamma(\eta)} \int\limits_{B\setminus\{0\}} y^{\alpha-1} dy \left\{ \int\limits_{(y,\infty)} \frac{\mu(du)}{u^{\eta+\alpha-1}(u-y)^{1-\eta}} + \mu_\infty \right\} \quad (28)$$

for each Borel set $B \subset [0,\infty]$.

**Remark A2.** Formula (28) is certainly a straightforward generalization of the right-sided Kober-Erdelyi fractional integral as given in [15, (2.6.8)] and [26, Chapter 4, (18.6)] with a slight change of notation. Condition (2) ensures that the right hand side of (28) exists. By definition the measure $\tau$ has no atom at infinity.

**Theorem A1** *If $\mu \in \mathcal{M}_\alpha$ then $\nu = I_\eta^+ \mu \in \mathcal{M}_{\alpha+\eta}$ and given $\nu$ we can recover $\mu$ from*

$$F_\mu(y) = \frac{1}{\Gamma(1+n-\eta)} \left(\frac{d}{dy}\right)^n \int\limits_{[0,y)} \frac{\nu(du)}{(y-u)^{\eta-n}}, \tag{29}$$

*where $n = [\eta]$ and $\mu(\{\infty\}) = \nu(\{\infty\})$.*

**Proof.** To show that $\nu \in \mathcal{M}_{\alpha+\eta}$ compute

$$\int\limits_{[0,\infty)} \frac{\nu(dt)}{(1+t)^{\alpha+\eta}} = \frac{1}{\Gamma(\eta)} \int\limits_{[0,\infty)} \frac{dt}{(1+t)^{\alpha+\eta}} \int\limits_{[0,t)} \frac{\mu(du)}{(t-u)^{1-\eta}}$$
$$= \frac{1}{\Gamma(\eta)} \int\limits_{[0,\infty)} \mu(du) \underbrace{\int\limits_{(u,\infty)} \frac{dt}{(t-u)^{1-\eta}(1+t)^{\alpha+\eta}}}_{=(1+u)^{-\alpha}B(\eta,\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\eta)} \int\limits_{[0,\infty)} \frac{\mu(du)}{(1+u)^\alpha} < \infty$$

according to (2). Here

$$B(\eta,\alpha) = \frac{\Gamma(\alpha)\Gamma(\eta)}{\Gamma(\alpha+\eta)}$$

is Euler's beta function. The interchange of the order of integrations is legitimate by Tonelli's theorem [27, Theorem 13.8], [25, Chapter 20, Corollary 7] or [4, Theorem 3.4.5]. The proof of (29) is a paraphrase of the standard proof of the inversion formula for the Riemann-Liouville fractional integral [26, Theorem 2.4] except that we recover the distribution function of the measure $\mu$ so that we differentiate one time less the standard and we employ Tonelli's theorem to justify the interchange of integrations. Substitute (27) into the integral on the right hand side of (29) (recall that $n = [\eta]$):

$$\int\limits_{[0,y)} \frac{\nu(du)}{(y-u)^{\eta-n}}$$
$$= \frac{1}{\Gamma(\eta)} \int\limits_{[0,y)} \frac{du}{(y-u)^{\eta-n}} \int\limits_{[0,u)} \frac{\mu(dt)}{(u-t)^{1-\eta}} = \frac{1}{\Gamma(\eta)} \int\limits_{[0,y)} \mu(dt) \int\limits_{(t,yh)} \frac{du}{(u-t)^{1-\eta}(y-u)^{\eta-n}}$$
$$= \frac{\Gamma(1-\eta+n)}{n!} \int\limits_{[0,y)} (y-t)^n \mu(dt) = \Gamma(1-\eta+n) \int\limits_{[0,y)} dt_1 \int\limits_{[0,t_1)} dt_2 \cdots \int\limits_{[0,t_{n-1})} F_\mu(t_n) dt_n.$$

Since $F_\mu(t_n)$ is non-decreasing it is locally Lebesgue integrable which implies that the function on the right belongs to $AC^n[0,R]$ (the function and $n-1$ its derivative are absolutely continuous) for any $R > 0$. Hence, we can recover $F_\mu$ by $n$-fold differentiation. □

**Theorem A2** *If $\mu \in \mathcal{M}_\alpha$ then $\tau := K_{\alpha,\eta}^-(\mu) \in \mathcal{M}_{\alpha+\eta}$ and given $\tau$ we can recover $\mu$ from*

$$F_\mu(y) = \tau(\{0\}) + \frac{1}{\Gamma(n+1-\eta)} \times$$
$$\left\{\alpha \int\limits_{(0,y)} x^{\alpha-1} dx \left(-x^2 \frac{d}{dx}\right)^n x^{\eta-n} \int\limits_{(x,\infty)} \frac{\tau(ds)}{s^{\alpha+n}(s-x)^{\eta-n}} - y^\alpha \left(-y^2 \frac{d}{dy}\right)^n y^{\eta-n} \int\limits_{(y,\infty)} \frac{\tau(ds)}{s^{\alpha+n}(s-y)^{\eta-n}}\right\}, \tag{30}$$

*where* $n=[\eta]$ *and* $\mu_\infty = \alpha\Gamma(\eta)\lim_{y\to\infty} y^{-\alpha}F_\tau(y)$.

**Proof.** To show that $\tau\in\mathcal{M}_{\alpha+\eta}$ compute

$$\int\limits_{(0,\infty)} \frac{\tau(dt)}{(1+t)^{\alpha+\eta}} = \frac{1}{\Gamma(\eta)}\int\limits_{[0,\infty)} \frac{t^{\alpha-1}dt}{(1+t)^{\alpha+\eta}}\int\limits_{(t,\infty)} \frac{\mu(du)}{u^{\eta+\alpha-1}(u-t)^{1-\eta}}$$
$$= \frac{1}{\Gamma(\eta)}\int\limits_{(0,\infty)} u^{1-\eta-\alpha}\mu(du)\underbrace{\int\limits_{(0,u)} \frac{t^{\alpha-1}dt}{(1+t)^{\alpha+\eta}(u-t)^{1-\eta}}}_{=u^{\eta+\alpha-1}(1+u)^{-\alpha}B(\alpha,\eta)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\eta)}\int\limits_{(0,\infty)} \frac{\mu(du)}{(1+u)^{\alpha}} < \infty$$

The interchange of integrations is again justified by Tonelli's theorem. To prove formula (30) assume for the moment that $\mu_\infty=0$ and substitute (28) into (30):

$$\alpha\int\limits_{(0,y)} x^{\alpha-1}dx\left(-x^2\frac{d}{dx}\right)^n x^{\eta-n}\int\limits_{(x,\infty)}\frac{\tau(ds)}{s^{\alpha+n}(s-x)^{\eta-n}} - y^{\alpha}\left(-y^2\frac{d}{dy}\right)^n y^{\eta-n}\int\limits_{(y,\infty)}\frac{\tau(ds)}{s^{\alpha+n}(s-y)^{\eta-n}}$$
$$= \frac{\alpha}{\Gamma(\eta)}\int\limits_{(0,y)} x^{\alpha-1}dx\left(-x^2\frac{d}{dx}\right)^n x^{\eta-n}\int\limits_{(x,\infty)}\frac{s^{\alpha-1}ds}{s^{\alpha+n}(s-x)^{\eta-n}}\int\limits_{(s,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}(u-s)^{1-\eta}}$$
$$- \frac{y^{\alpha}}{\Gamma(\eta)}\left(-y^2\frac{d}{dy}\right)^n y^{\eta-n}\int\limits_{(y,\infty)}\frac{s^{\alpha-1}ds}{s^{\alpha+n}(s-y)^{\eta-n}}\int\limits_{(s,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}(u-s)^{1-\eta}}$$
$$= \frac{\alpha}{\Gamma(\eta)}\int\limits_{(0,y)} x^{\alpha-1}dx\left\{\left(\frac{d}{dt}\right)^n (1/t)^{\eta-n}\int\limits_{(1/t,\infty)}\frac{ds}{s^{n+1}(s-1/t)^{\eta-n}}\int\limits_{(s,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}(u-s)^{1-\eta}}\right\}_{|\,t=1/x}$$
$$- \frac{y^{\alpha}}{\Gamma(\eta)}\left\{\left(\frac{d}{dt}\right)^n (1/t)^{\eta-n}\int\limits_{(1/t,\infty)}\frac{ds}{s^{n+1}(s-1/t)^{\eta-n}}\int\limits_{(s,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}(u-s)^{1-\eta}}\right\}_{|\,t=1/y}, \quad (31)$$

where we have used the formula

$$\left(-y^2\frac{d}{dy}\right)^n\varphi(y) = \left\{\left(\frac{d}{dt}\right)^n\varphi(1/t)\right\}_{|\,t=1/y}.$$

Further, exchange of the order of integrations justified by Tonelli's theorem yields:

$$\int\limits_{(1/t,\infty)}\frac{ds}{s^{n+1}(s-1/t)^{\eta-n}}\int\limits_{(s,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}(u-s)^{1-\eta}}$$
$$= \int\limits_{(1/t,\infty)}\frac{\mu(du)}{u^{\eta+\alpha-1}}\underbrace{\int\limits_{(1/t,u)}\frac{ds}{s^{n+1}(s-1/t)^{\eta-n}(u-s)^{1-\eta}}}_{=t^{\eta-n}u^{\eta-n-1}(ut-1)^n B(\eta,1-\eta+n)} = B(\eta,1-\eta+n)t^{\eta-n}\int\limits_{(1/t,\infty)}\frac{\mu(du)}{u^{n+\alpha}}(ut-1)^n.$$

We will show now that

$$\left(\frac{d}{dt}\right)^n\int\limits_{(1/t,\infty)}\frac{\mu(du)}{u^{n+\alpha}}(ut-1)^n = n!\int\limits_{(1/t,\infty)} u^{-\alpha}\mu(du).$$

Denote by $\mu^*$ the image of the measure $\mu$ under the mapping $\lambda(u) = 1/u$, so that for each Borel set $A \subset (0,\infty)$ we have $\mu^*(A) := \mu(\lambda^{-1}(A))$. Then

$$\left(\frac{d}{dt}\right)^n \int\limits_{(1/t,\infty)} \frac{\mu(du)}{u^{n+\alpha}}(ut-1)^n = \left(\frac{d}{dt}\right)^n \int\limits_{(0,t)} s^{n+\alpha}(t/s-1)^n\mu^*(ds) = \left(\frac{d}{dt}\right)^n \int\limits_{(0,t)}(t-s)^n s^\alpha\mu^*(ds)$$
$$= n!\left(\frac{d}{dt}\right)^n \int\limits_{(0,t)} dt_1 \int\limits_{(0,t_1)} dt_2 \cdots \int\limits_{(0,t_n)} s^\alpha\mu^*(ds) = n! \int\limits_{(0,t)} s^\alpha\mu^*(ds) = n! \int\limits_{(1/t,\infty)} u^{-\alpha}\mu(du)$$

Finally, by integration by parts for Lebesgue-Stieltjes integral (see [7, Theorem 6.2.2]) we obtain:

$$\alpha\int\limits_{(0,y)} x^{\alpha-1}dx\left(-x^2\frac{d}{dx}\right)^n x^{\eta-n}\int\limits_{(x,\infty)}\frac{\tau(ds)}{s^{\alpha+n}(s-x)^{\eta-n}} - y^\alpha\left(-y^2\frac{d}{dy}\right)^n y^{\eta-n}\int\limits_{(y,\infty)}\frac{\tau(ds)}{s^{\alpha+n}(s-y)^{\eta-n}}$$
$$= \Gamma(1-\eta+n)\Bigg(\alpha\int\limits_{(0,y)} x^{\alpha-1}dx\int\limits_{(x,\infty)} u^{-\alpha}\mu(du) - y^\alpha\int\limits_{(y,\infty)} u^{-\alpha}\mu(du)\Bigg)$$
$$= \Gamma(1-\eta+n)\Bigg(x^\alpha\int\limits_{(x,\infty)} u^{-\alpha}\mu(du)\Bigg|_0^y + \int\limits_{(0,y)}\mu(dx) - y^\alpha\int\limits_{(y,\infty)} u^{-\alpha}\mu(du)\Bigg)$$
$$= \Gamma(1-\eta+n)\int\limits_{(0,y)}\mu(dx) = \Gamma(1-\eta+n)(F_\mu(y)-F_\mu(0+)) = \Gamma(1-\eta+n)(F_\mu(y)-\mu(\{0\})).$$

Since $\mu(\{0\}) = \tau(\{0\})$ by (28) this proves formula (30). While integrating by parts we have also used the following limit:

$$\lim_{x\to 0} x^\alpha\int\limits_{(x,\infty)} u^{-\alpha}\mu(du) = \lim_{x\to 0} x^\alpha\int\limits_{(x,1)} u^{-\alpha}dF_\mu(u) = \lim_{x\to 0} x^\alpha\Bigg(u^{-\alpha}F_\mu(u)\Bigg|_x^1 + \alpha\int\limits_{(x,1)} u^{-\alpha-1}F_\mu(u)du\Bigg)$$
$$= \lim_{x\to 0}\Bigg(\alpha x^\alpha\int\limits_{(x,1)} u^{-\alpha-1}F_\mu(u)du - F_\mu(x)\Bigg) = \lim_{x\to 0}\frac{-x^{-\alpha-1}F_\mu(x)}{(1/\alpha)(-\alpha)x^{-\alpha-1}} - F_\mu(0+) = 0.$$

Here the first equality is due to (2), the second is integration by parts and the preultimate is L'Hôpital's rule applied if $\int_{(x,1)} u^{-\alpha-1}F_\mu(u)du$ is unbounded.

If $\mu_\infty \neq 0$ we need to add the following term to the third line of (31):

$$\frac{\alpha\mu_\infty}{\Gamma(\eta)}\int\limits_{(0,y)} x^{\alpha-1}dx\left(-x^2\frac{d}{dx}\right)^n x^{\eta-n}\underbrace{\int\limits_{(x,\infty)}\frac{s^{\alpha-1}ds}{s^{\alpha+n}(s-x)^{\eta-n}}}_{=B(\eta,1-\eta+n)x^{-\eta}} - \frac{\mu_\infty y^\alpha}{\Gamma(\eta)}\left(-y^2\frac{d}{dy}\right)^n y^{\eta-n}\overbrace{\int\limits_{(y,\infty)}\frac{s^{\alpha-1}ds}{s^{\alpha+n}(s-y)^{\eta-n}}}^{=B(\eta,1-\eta+n)y^{-\eta}}$$
$$= \frac{\alpha\mu_\infty}{\Gamma(\eta)}\int\limits_{(0,y)} x^{\alpha-1}dx\left(-x^2\frac{d}{dx}\right)^n x^{-n} - \frac{\mu_\infty y^\alpha}{\Gamma(\eta)}\left(-y^2\frac{d}{dy}\right)^n y^{-n} = \frac{\alpha\mu_\infty n!}{\Gamma(\eta)}\int\limits_{(0,y)} x^{\alpha-1}dx - \frac{\mu_\infty n! y^\alpha}{\Gamma(\eta)} = 0,$$

where
$$\left(-x^2\frac{d}{dx}\right)^n x^{-n} = \left\{\left(\frac{d}{dt}\right)^n t^n\right\}_{|\,t=1/x} = n!.$$
This shows that (31) is still valid for $\mu_\infty \neq 0$.

Finally, to recover the atom at infinity $\mu_\infty$ we compute

$$\lim_{x\to\infty} x^{-\alpha}F_\tau(x) = \lim_{x\to\infty}[\mu(\{0\})x^{-\alpha}] + \frac{1}{\Gamma(\eta)}\lim_{x\to\infty} x^{-\alpha}\int\limits_{(0,x)} y^{\alpha-1}dy \int\limits_{(y,\infty)} \frac{\mu(du)}{u^{\eta+\alpha-1}(u-y)^{1-\eta}}$$
$$+\frac{\mu_\infty}{\Gamma(\eta)}\lim_{x\to\infty} x^{-\alpha}\int\limits_{(0,x)} y^{\alpha-1}dy = \frac{\mu_\infty}{\alpha\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\lim_{x\to\infty} x^{-\alpha}\int\limits_{(0,x)} y^{\alpha-1}dy \int\limits_{(y,\infty)} \frac{\mu(du)}{u^{\eta+\alpha-1}(u-y)^{1-\eta}}.$$

In order to show that the last limit is zero we interchange the order of integrations (justified again by Tonelli's theorem):

$$\int\limits_{(0,x)} y^{\alpha-1}dy \int\limits_{(y,\infty)} \frac{\mu(du)}{u^{\eta+\alpha-1}(u-y)^{1-\eta}} = \int\limits_{(0,x)} u^{1-\eta-\alpha}\mu(du)\overbrace{\int\limits_{(0,u)} (u-y)^{\eta-1}y^{\alpha-1}dy}^{=B(\alpha,\eta)u^{\alpha+\eta-1}}$$
$$+\int\limits_{[x,\infty)} u^{1-\eta-\alpha}\mu(du)\underbrace{\int\limits_{(0,x)} (u-y)^{\eta-1}y^{\alpha-1}dy}_{=(1/\alpha)x^\alpha u^{\eta-1}{}_2F_1(\alpha,1-\eta;1+\alpha;x/u)} = F_\mu(x) - \mu(\{0\})$$
$$+\frac{1}{\alpha}x^\alpha \int\limits_{[x,\infty)} u^{-\alpha}{}_2F_1(\alpha,1-\eta;1+\alpha;x/u)\mu(du),$$

where ${}_2F_1$ is the Gauss hypergeometric function [1, Chapter 2]. Hence we need to prove that:
$$\lim_{x\to\infty} x^{-\alpha}F_\mu(x) = 0$$
and
$$\lim_{x\to\infty}\int\limits_{[x,\infty)} u^{-\alpha}{}_2F_1(\alpha,1-\eta;1+\alpha;x/u)\mu(du) = 0.$$

Both equalities follow from (2): the first was proved by Widder [34, Corollary 3a.3], the second follows from the fact that ${}_2F_1(a,b;c;x)$ is bounded on $[0,1]$ if $c > a+b$ by the Gauss formula [1, Theorem 2.2.2]. $\square$

**Remark A3.** The operator $(-x^2D_x)^n$ encountered in (30) can be expanded as follows

$$(-x^2D_x)^n f = \sum_{m=1}^{n} a(n,m)x^{n+m}f^{(m)}(x),$$

where the numbers
$$a(n,m) = (-1)^n\frac{n!}{m!}\binom{n-1}{m-1}$$

are known as Lah numbers [33, A008297] satisfying

$$a(n+1,m) = (n+m)a(n,m) + a(n,m-1).$$

Applying $-x^2 D_x$ to the above expansion we see the same recurrence which given the same initial values furnishes a proof of the above expansion.

**Remark A4.** Another way to obtain a representation for $\mu$ via $\tau$ for $\eta > 1$ is the following. Denote $\tilde{\mu} = K^-_{\alpha,1}(\mu)$, i.e. according to (28)

$$\tilde{\mu}(dx) = \mu(\{0\})\delta_0 + x^{\alpha-1}dx \int\limits_{(x,\infty)} u^{-\alpha}\mu(du) = \mu(\{0\})\delta_0 + \phi_{\tilde{\mu}}(x)dx,$$

where the last equality is the definition of $\phi_{\tilde{\mu}}(x)$. It is easy to verify that this formula is inverted as follows

$$F_\mu(y-0) - F_\mu(0+) = \alpha \int\limits_{(0,y)} \tilde{\mu}(dx) - y\phi_{\tilde{\mu}}(y).$$

If $\eta > 1$ we have according to (28) for $\tilde{\mu} \in \mathcal{M}_{\alpha+1}$ and the semigroup property of Kober-Erdeliy operator [15, (2.6.24)]

$$\begin{aligned}\tau(dx) &:= K^-_{\alpha,\eta}(\mu) = K^-_{\alpha+1,\eta-1}(\tilde{\mu}) \\ &= \mu(\{0\})\delta_0 + \frac{x^\alpha dx}{\Gamma(\eta-1)} \int\limits_{(x,\infty)} \frac{\phi_{\tilde{\mu}}(u)du}{u^{\alpha+\eta-1}(u-x)^{2-\eta}} = \mu(\{0\})\delta_0 + \phi_\tau(x)dx. \quad (32)\end{aligned}$$

Here we have the standard Riemann-Liouvile fractional integral of the function $u^{1-\alpha-\eta}\phi_{\tilde{\mu}}(u)$. We cannot, however, use the Riemann-Liouvile fractional derivative to invert the above formula, since, in general the integral in its definition will diverge. Instead, we can employ Caputo's fractional derivative [15, section 2.4] to invert (32):

$$u^{1-\alpha-\eta}\phi_{\tilde{\mu}}(u) = {}^C\!D^{\eta-1}_{-}[x^{-\alpha}\phi_\tau(x)](u) = \frac{(-1)^n}{\Gamma(n-\eta+1)} \int\limits_{(u,\infty)} \frac{[x^{-\alpha}\phi_\tau(x)]^{(n)}dx}{(x-u)^{\eta-n}},$$

where $n = [\eta]$ and

$$\tilde{\mu}(du) = \mu(\{0\})\delta_0 + \frac{(-1)^n u^{\alpha+\eta-1}du}{\Gamma(n-\eta+1)} \int\limits_{(u,\infty)} \frac{[x^{-\alpha}\phi_\tau(x)]^{(n)}dx}{(x-u)^{\eta-n}},$$

so that

$$\begin{aligned}F_\mu(y) = \mu(\{0\})\delta_0 + \frac{(-1)^n\alpha}{\Gamma(n-\eta+1)} \int\limits_{(0,y)} u^{\alpha+\eta-1}du \int\limits_{(u,\infty)} \frac{[x^{-\alpha}\phi_\tau(x)]^{(n)}}{(x-u)^{\eta-n}}dx \\ - \frac{(-1)^n y^{\alpha+\eta}}{\Gamma(n-\eta+1)} \int\limits_{(y,\infty)} \frac{[x^{-\alpha}\phi_\tau(x)]^{(n)}}{(x-y)^{\eta-n}}dx.\end{aligned}$$

This formula is, however, less general then (30) since it requires that $\phi_\tau(x) \in AC^n(0,\infty)$ which is not guaranteed by $\mu \in \mathcal{M}_\alpha$. For $0 < \eta < 1$ both formulas take the same form.

The relation between $I^+_\eta$ and $K^-_{\alpha,\eta}$ is revealed in the following theorem.

**Theorem A3** *Suppose $\mu \in \mathcal{M}_\alpha$. Then $N_{\alpha+\eta} I^+_\eta \mu = K^-_{\alpha,\eta} N_\alpha \mu$.*

**Proof.** Indeed, if $f$ is given by (1) and $\nu = I^+_\eta \mu$ then by Theorem 3

$$f(z) - \mu_\infty = \int\limits_{[0,\infty)} \frac{\mu(du)}{(u+z)^\alpha} = \frac{\Gamma(\alpha+\eta)}{\Gamma(\alpha)} \int\limits_{(0,\infty)} \frac{\nu(du)}{(u+z)^{\alpha+\eta}} = \frac{\Gamma(\alpha+\eta)}{\Gamma(\alpha)} \int\limits_{(0,\infty)} \frac{\tau_1(dt)}{(1+tz)^{\alpha+\eta}},$$

where $\tau_1 = N_{\alpha+\eta}\nu = N_{\alpha+\eta} I^+_\eta \mu$. On the other hand, if $\rho = N_\alpha \mu$ then according to the comment after formula (5) and by Theorem 4 we have

$$f(z) - \mu_\infty = \int\limits_{(0,\infty)} \frac{\rho(dt)}{(1+tz)^\alpha} + \frac{\mu_0}{z^\alpha} = \frac{\Gamma(\alpha+\eta)}{\Gamma(\alpha)} \int\limits_{(0,\infty)} \frac{\tau_2(dt)}{(1+tz)^{\alpha+\eta}},$$

where $\tau_2 = K^-_{\alpha,\eta}\rho = K^-_{\alpha,\eta} N_\alpha \mu$. Comparing these two formulas we conclude that $\tau_1 = \tau_2$ due to uniqueness of the representing measure. $\square$

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