Crossing on hyperbolic lattices

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We divide the circular boundary of a hyperbolic lattice into four equal intervals, and study the probability of a percolation crossing between an opposite pair, as a function of the bond occupation probability p. We consider the $\{7,3\}$ (heptagonal), enhanced or extended binary tree (EBT), the EBT-dual, and $\{5,5\}$ (pentagonal) lattices. We find that the crossing probability increases gradually from zero to one as p increases from the lower p_l to the upper p_u critical values. We find bounds and estimates for the values of p_l and p_u for these lattices, and identify the self-duality point p^* corresponding to where the crossing probability equals 1/2. Comparison is made with recent numerical and theoretical results.

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I. INTRODUCTION

Hyperbolic lattices represent curved surfaces in a space that is effectively of infinite dimensions. While long of interest to mathematicians [1], and even artists [2], such lattices have only relatively recently been studied in statistical physics, where many problems [3–13] including percolation [14–27] have been examined. Investigation has also been carried out on closely-related hierarchical lattices [28–31]. The study of hyperbolic lattices helps in the understanding of how geometry affects the behavior of systems. There are also physical systems that show negative curvature on the nanoscale [32]. Networks are not confined to a physical dimensionality and can show hyperbolic behavior, and the current strong interest in network physics is another motivation to study these systems.

Connectivity in networks is described by percolation, which has been studied for a wide variety of systems for well over 50 years [33]. For most systems in percolation, there is typically a bond or site occupation probability p, such that when p is less than a threshold value p_c, all connected components are finite, while above p_c there is an infinite network which connects every region of the system.

For hyperbolic lattices, however, percolation shows a more complicated behavior, with two distinct transitions, as described below. Various numerical and theoretical methods have been developed to study these transitions, and in particular to find the threshold values [14–27].

In this paper, we investigate percolation on hyperbolic lattices by considering the crossing probability, a technique which has not been applied to this system before. The lattices we study are some that have been considered by others, so that a comparison of the values for the transition points can be made. We also study (for the first time numerically) percolation on the pentagonal

lattice, which has some interesting self-dual features. In general, the determination of the transition points allows one to compare the different numerical methods and to test theoretical predictions and bounds. These threshold points will be useful for future studies of the nature of the transition behavior, such as the determination of critical exponents.
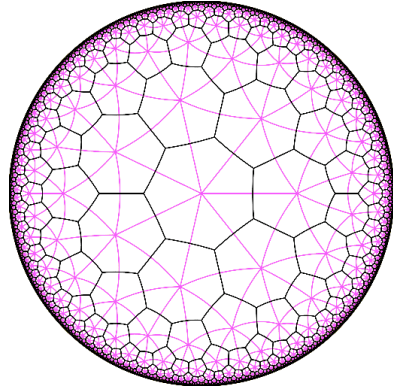

FIG. 1. (color online) Heptangonal lattice $\{7,3\}$ (black or dark) and the dual lattice $\{3,7\}$ (magenta or light).

Common examples of hyperbolic lattices are those composed of identical polygons of n sides, m of which meet at a vertex. These lattices can be characterized by the Schläfli symbol $\{n, m\}$, corresponding to the Grünbaum-Shepard [34] notation (n^m). Thus, $\{6, 3\}$ is a regular planar hexagonal (honeycomb) lattice, while $\{7, 3\}$ is the heptagonal hyperbolic lattice shown in Fig. 1. The dual to the heptagonal lattice is the $\{3, 7\}$ lattice, also shown in Fig. 1. The self-dual hyperbolic $\{5, 5\}$ is shown in Fig. 9.

Recently, another type of hyperbolic lattice has been introduced, the Enhanced or Extended Binary Tree (EBT) [14, 18]), which is made by adding transverse bonds to the Bethe lattice. The EBT, which is simpler to represent and code on a computer, has been studied extensively for percolation [18–20, 22–24]. The EBT and its dual are shown in Fig. 2.

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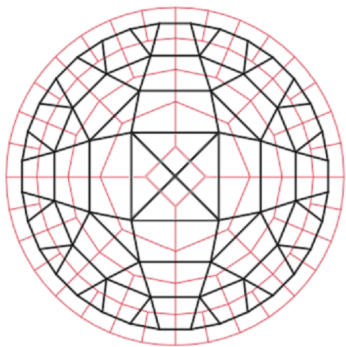* ghbright@umich.edu
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FIG. 2. (color online) The Enhanced Binary Tree (EBT) lattice (black or dark) and EBT-dual lattice (red or light). The EBT is made by joining four trees together. We joined four bonds at the center rather than having two join there as in [24].

The general picture that has emerged for percolation on hyperbolic lattices [14, 15, 17, 18] is that there are two thresholds p_l and p_u, and the behavior is continuous between them. For $0 < p < p_l$, there are no "infinite" (large) clusters connecting the central area to the boundary sites, for the intermediate region $p_l < p < p_u$ there are many infinite clusters touching the boundary, and for $p_u < p < 1$, there is exactly one infinite cluster. These three regions persist in the limit that the system size goes to infinity. This behavior is in contrast to ordinary percolation, in which there is no intermediate region (in an infinite system) and the crossing behavior is discontinuous.

Here we study percolation on hyperbolic lattices by examining a suitably defined crossing probability $R(p)$ as a function of the bond occupation probability p. The crossing probability is often studied in ordinary percolation to locate the threshold and to investigate the critical scaling behavior [35–37]. There, the crossing probability (the probability of a continuous path from one opposite side to another) becomes steeper as the size of the system is increased, with a slope proportional to $L^{1/\nu}$, where ν is the correlation-length exponent, equal to $4/3$ in two dimensions. This behavior defines the transition point uniquely in the limit of $L \to \infty$ [33]. Furthermore, when the system boundary is symmetric such as a perfect square, the crossing probability between opposite sides is exactly $1/2$ (because the dual lattice percolates if the original lattice does not) [38, 39]. A crossing probability of $1/2$ applies to a disk also, with the circumference divided into four equal intervals. In this paper, we set up a similar crossing problem for hyperbolic system by dividing the boundary of a finite system into four equal-size intervals (or as equal as possible), and study the probability of crossing between an opposite pair of these intervals. We consider the heptagonal, EBT, EBT-dual, and pentagonal hyperbolic lattices, and investigate how the resulting crossing probability behaves with p. We also discuss how the value

of $p = p^*$ that corresponds to a crossing probability of exactly $1/2$ relates to the two transition points p_l and p_u.

In the following sections we discuss the methods (II), the results (III), and the conclusions (IV).

II.   METHOD

We begin by generating a hyperbolic lattice to a fixed number of generations or levels, so that the outside is essentially circular as in Fig. 1. Practically, it is only feasible to generate a relatively small number of levels (up to $10 - 15$) because of the exponential growth in the number of lattice sites with level.

For the heptagonal lattice with an open heptagon centered at the origin, we can derive an expression for the number of sites $N(l)$ as a function of level l as follows: Let a_l equal the number of new sites which connect to the next generation, and b_l equal the number of new sites which connect to the previous generation. Inspection of the Fig. 1 shows that we have the relations

$$\begin{aligned} a_l &= 3a_{l-1} + b_l \\ b_l &= a_{l-1} \end{aligned} \tag{1}$$

Thus, the total number of sites up to level l is equal to $\sum_{l'=1}^{l}(a_l + b_l)$. By means of generating function techniques, we find the explicit relation

$$N(l) = 7\left[\left(\frac{3+\sqrt{5}}{2}\right)^l + \left(\frac{3-\sqrt{5}}{2}\right)^l - 2\right] \tag{2}$$

which yields $N(l) = 7, 35, 112, 315, 847, 2240, 5887, 15\,435, 40\,432, 105\,875\ldots$ for $l = 1, 2, \ldots, 10, \ldots$. In Ref. [17], the corresponding formula for $N(l)$ with a vertex rather than an open heptagon at the center of the system is given. For large l, $N(l)$ grows exponentially as $\sim 7[(3 + \sqrt{5})/2]^l$. These $N(l)$ are related to other mathematical quantities, such as the number of fixed points of period l in iterations of Arnold's cat map at its hyperbolic fixed point, multiplied by 7 [40].

For the pentagonal lattice, we find

$$\begin{aligned} a_l &= 5a_{l-1} + 3b_l \\ b_l &= 3a_{l-1} + 2b_l \end{aligned} \tag{3}$$

which yields

$$N(l) = \left[\left(\frac{7+3\sqrt{5}}{2}\right)^l + \left(\frac{7-3\sqrt{5}}{2}\right)^l - 2\right] \tag{4}$$

and equals $5, 45, 320, 2205, 15\,125, 103\,680, 710\,645, 4\,870\,845\ldots$ for $l = 1, 2, \ldots, 8, \ldots$. Here the $N(l)$ are related to the Fibonacci numbers $F(l)$ by $N(l) = 5 \cdot F(2l)^2$.

For the EBT, we consider a geometry with four trees meeting at the origin as shown in Fig. 2, so that it is easy to divide the boundary into four equal intervals.

The number of sites grows as $N(l) = 2^{l+2} - 3$. For the EBT-dual, we have $N(l) = \cdot 2^{l+2} - 4$.

We applied the algorithm of [41, 42] to find the crossing probability on these four lattices. This algorithm allows one to find an estimate of $R(p)$ for all values of p in a single sweep of the lattice; averaging over many sweeps yields an accurate estimate of $R(p)$. The connections between points in a cluster are represented by a tree structure, bonds are added one at a time, and clusters are joined together by means of a union-find operation. The algorithm yields the crossing probability R_n as a function of the number of occupied bonds n added to the system, corresponding to a fixed-n or canonical ensemble. To get the grand canonical result $R(p)$ corresponding to a fixed probability p, one must convolve R_n with the binomial distribution:

$$R(p) = \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} R_n \qquad (5)$$

where N is the total number of bonds in the system. For large N, and the binomial distribution becomes very sharp, and for many problems it is not necessary to carry out this convolution, but instead use just the value at the maximum of the distribution $n = Np$, so that $R(p) \approx R_{Np}$. However, for smaller systems it is necessary to use this convolution to get accurate results.

III. RESULTS

We carried out simulations for each of the four lattices, recording $R(p)$ at 500 values of p. Below we describe the results for each lattice.

A. Heptagonal {7,3} lattice

We considered the heptagonal {7, 3} lattice up to level $l = 10$ with $N(l) = 105\,875$ total sites. Fig. 3 shows the resulting $R(p)$ as a function of p for levels $5, \ldots, 10$. We find a gradual transition of $R(p)$ from 0 to 1 as the p increases, as is typical for finite systems for ordinary percolation. However, here the width of the transition region is more spread out and, more significantly, the width limits to a non-zero value as $l \to \infty$. Equivalently the slope at the inflection point limits to a finite value as $l \to \infty$. In Fig. 4 we plot the maximum slope as a function of $N(l)^{-0.7}$ where $N(l)$ is given by (2), and find an extrapolation to a maximum value of ≈ 6.12. The exponent -0.7 was chosen empirically to get a fairly straight line; different choices of the exponent do not change the intercept significantly and especially do not change the conclusion that the slope limits to a finite value as $N \to \infty$.

The close-up in Fig. 3 shows that the curves do not quite cross at a single point, but the crossing point changes with N. For a perfectly self-dual system in which
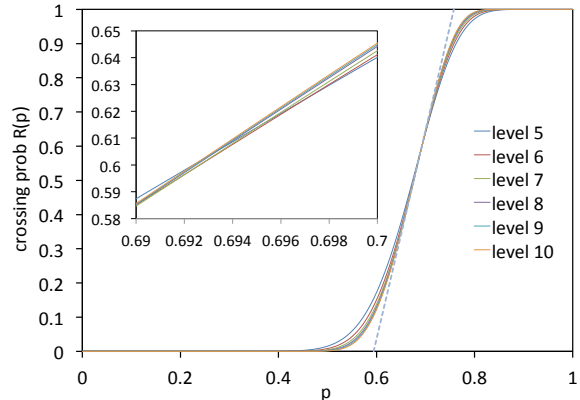

FIG. 3. (color online) Curves of crossing probability for the heptagonal lattice, convoluted with (5), for various levels l; the slope increases as l increases. The dashed line passes through the inflection point, and its intercepts with the lines at $R = 0$ and $R = 1$ gives our estimates of p_l^B and p_u^B. Inset: close-up near crossing point.
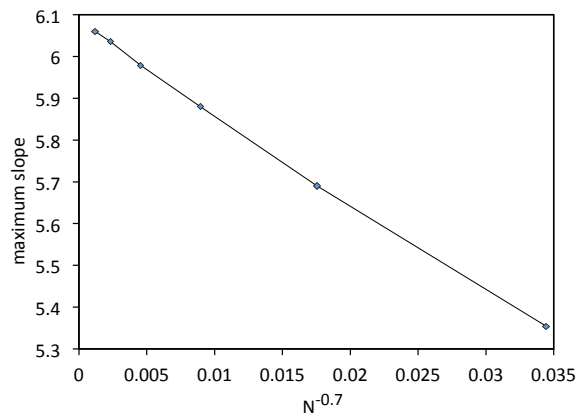

FIG. 4. Maximum of the slope of $R(p)$ *versus* $N^{-0.7}$ for the heptagonal lattice, where N is given by (2).

the dual lattice is identical to the original lattice, such as bond percolation on a square lattice and square boundary in ordinary percolation, the curves cross at a single point corresponding to $R = 1/2$ and $p = 1/2$, but because this system is not self-dual, one would not expect the crossing to be at $(1/2, 1/2)$ here.

We define the duality point $p^*(l)$ as the value of p where $R(p) = 1/2$. We call this the duality point because on a truly dual lattice the crossing probability should also be $1/2$ (although below we see that there are differences in the center that limit the extent that one can make a completely self-dual system). We find $p^*(\infty) \approx 0.6759$ by extrapolating to $l = \infty$ as shown in Fig. 5. Here we
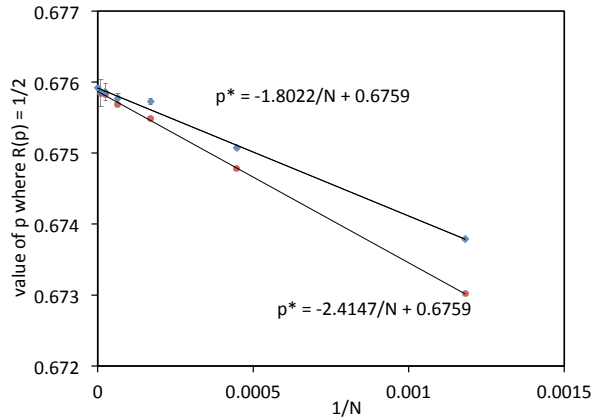
FIG. 5. (color online) Dual point $p^*(l)$ versus $1/N(l)$, where $N(l)$ is the number of sites on the lattices of levels $l = 5, \ldots, 10$ for the heptagonal lattice as given by (2). We show results for the raw (canonical) (upper points) and convoluted (grand canonical) (lower points) data; both extrapolate to the same value, $p \approx 0.6759$, as $L \to \infty$.

observed a scaling of order $1/N(l)$.

The transition points for the heptagonal lattice were found by Baek et al. [17] to be $p_l \approx 0.53$ and $p_u \approx 0.72$, and on the dual lattice $\{3,7\}$ they found $p_l \approx 0.20$ and $p_u \approx 0.37$. These four values are evidently not completely consistent because one should have, for any lattice and its dual,

$$
\begin{aligned}
p_l + p_u^{\mathrm{dual}} &= 1 \\
p_u + p_l^{\mathrm{dual}} &= 1
\end{aligned}
\tag{6}
$$

One would expect that for $p > p_u$, $R(p) = 1$, and for $p < p_l$, $R(p) = 0$. However, how $R(p)$ approaches those values from the region $p_l < p < p_u$ is not clear. It appears from our data the approach is tangential (with slope zero), and therefore it is rather hard to identify the transition points accurately. We can, however, find bounds to that behavior by drawing a tangent line from the inflection point. Drawing a line through the inflection point in Fig. 3 with the maximum slope ≈ 6.12, and the intercepts for $R(p) = 0$ and $R(p) = 1$ give us the rather crude bounds $p_l < p_l^B = 0.594$ and $p_u > p_u^B = 0.758$.

A more precise method to get bounds or estimates for the transition point is to look at the values of p where $R(p) = \epsilon$ and $R(p) = 1 - \epsilon$, where we chose $\epsilon = 10^{-5}$, and then extrapolating to $L = \infty$. Fig. 6 shows that these estimates appear to scale as $1/l$, and extrapolating the points to $l \to \infty$ gives the values of p_l^e and p_u^e listed in Table I. In principle, the smaller value of ϵ the better, but noise in the data and precision of the numbers in our output files limited how small we could make ϵ. The fixed number of points that we recorded (500) also limited the final precision of the thresholds. Varying the value of ϵ by an order of magnitude in each direction and finding

the extrapolated thresholds, we estimate that the overall error in our threshold estimates is about ± 0.01.
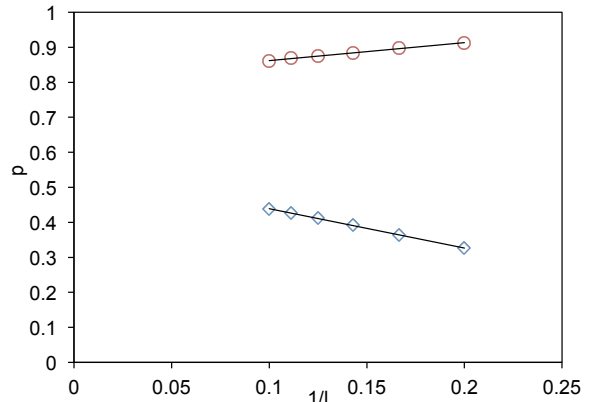

FIG. 6. Values of p where $R(p) = 1 - 10^{-5}$ (upper data points) and $R(p) = 10^{-5}$ (lower data points), plotted as a function $1/l$, for the $\{7,3\}$ heptagonal lattice. The linear extrapolation to $l \to \infty$ gives our estimates $p_u^e = 0.810$ and $p_l^e = 0.551$. Extrapolations for the other lattices show similar linear behavior and the values for p_l^e and p_u^e are given in Table I.

B. EBT and EBT-dual lattices

We simulated the EBT lattice to the level of 15, and the EBT-dual lattice to the level of 10. Figs. 7 and 8 show the resulting crossing probability distribution for these two lattices. For the EBT, the maximum slope converges to ≈ 6.79. Its duality point is at $p^* \approx 0.4299$, yielding the bounds $p_l^B \approx 0.356$ and $p_u^B \approx 0.503$. The EBT-dual's crossing probability distribution curve also converges to a maximum slope ≈ 6.83, the duality point of which is at $p^* \approx 0.5698$, yielding the bounds $p_l^B = 0.497$ and $p_u^B = 0.643$. These bounds satisfy the expected duality (6) within errors. The estimates are also found to the scale as $1/l$ and the resulting values p_l^e and p_u^e given in Table I. These estimates do not satisfy the duality relations (6) very precisely, reflecting rather large error bars in their values.

C. Pentagonal lattice

We also considered the pentagonal $\{5,5\}$ lattice, which is shown in Fig. 9. This lattice is interesting because it is self-dual in an infinite system. For the systems of a finite number of levels, it is not precisely self-dual because the center is different: on what we call the pentagonal lattice, there is a pentagon at the center, while for the pentagonal-dual, there is a vertex at the center (see
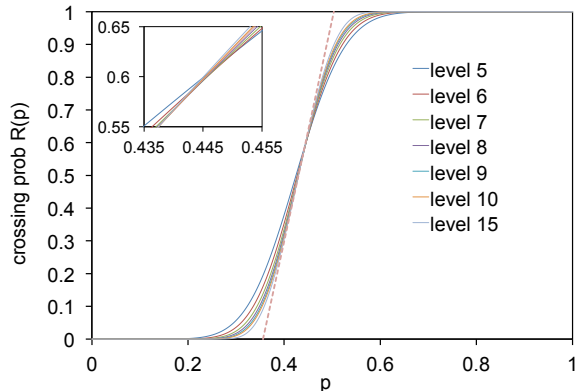
FIG. 7. (color online) The crossing probability R as a function of p for the EBT lattice, for systems of $5 - 15$ levels. The dashed line passes through the inflection point.
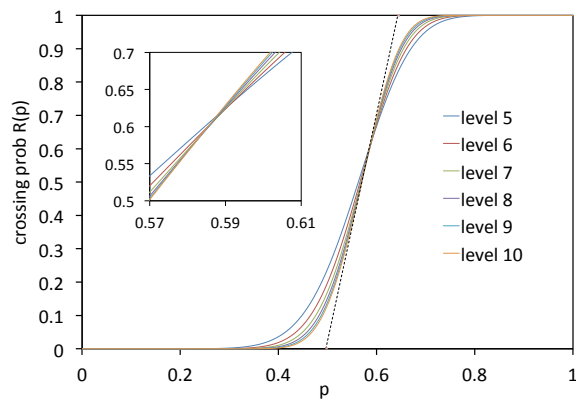

FIG. 8. (color online) The crossing probability R as a function of p for the EBT-dual lattice, for systems of $5 - 10$ levels. The dashed line passes through the inflection point.
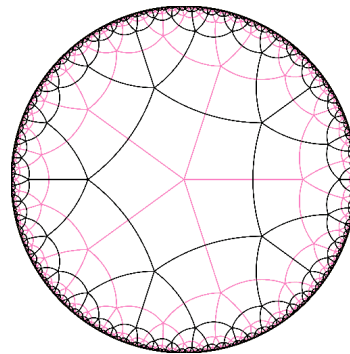

FIG. 9. (color online) Pentagonal (black or dark) and dual pentagonal (red or light), both $\{5, 5\}$.
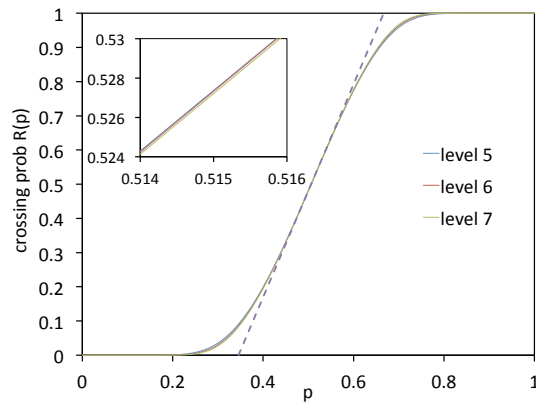

FIG. 10. (color online) The crossing probability R as a function of p for the $\{5,5\}$ or pentagonal lattice, for systems of 5, 6, and 7 levels. The curves are nearly indistinguishable on this plot. The dashed line passes through the inflection point.

and p_u^B fall well within these values, and our estimates p_l^e and p_u^e are close to the bound $1/4$ and (by duality) $3/4$, respectively. This bound follows from approximating the hyperbolic lattice as a tree (Bethe lattice) of coordination number 5.

Lattice	p^*	max. slope	p_l^B	p_u^B	p_l^e	p_u^e
$\{7,3\}$	0.6759	6.12	0.594	0.758	0.551	0.810
EBT	0.4299	6.79	0.356	0.503	0.306	0.564
EBT-dual	0.5698	6.83	0.497	0.643	0.452	0.699
$\{5,5\}$	0.506	3.12	0.346	0.666	0.263	0.749

TABLE I. Values of the dual point p^* where $R(p) = 1/2$, the slope at that point, our bounds p_l^B and p_u^B for the various lattices we studied, such that $p_l < p_l^B$, and $p_u > p_u^B$, and our extrapolated estimates of the transition points p_l^e and p_u^e. In general, the numbers are expected to be accurate to about ± 1 in the last digit shown, except the estimates p_l^e and p_u^e, which are expected to be accurate to about ± 10 in the last digits.

Fig 10). We find that $p^* = 0.506 \pm 0.001$, so it is not exactly at 0.5 as one might expect from duality. Evidently, the central region plays an important role and a significant fraction of the percolating clusters connecting opposite sides pass through it, making the pentagonal and pentagonal-dual lattices slightly different with respect to the crossing problem we consider.

The slope of its crossing probability curve converges to a maximum value ≈ 3.12, with bounds $p_l^B \approx 0.346$ and $p_u^B \approx 0.666$, which indicates the distribution is nearly symmetric.

Recently, Delfosse and Zémor [43] have shown that, for any self-dual hyperbolic lattice $\{m, m\}$, $1/(m-1) \leq p_l \leq 2/m$, so that for $m = 5$, $1/4 \leq p_l \leq 2/5$. Our bounds p_l^B

Lattice	p_l	p_u	Ref.
{7,3}	0.53	0.72	[17]
{3,7}	0.20	0.37	[17]
EBT	0.304(1)	0.564(1)	[18]
"		0.48	[17]
"	$(\sqrt{13}-3)/2 \approx 0.3028$	0.5	[23]
EBT-dual	0.436(1)	0.696(1)	[18]
{5,5}	$0.25 \le p_l \le 0.4$		[43]

TABLE II. Previous values of the transition points.

IV. CONCLUSIONS

In summary, we have the following results and conclusions:

(i) The crossing probability approaches a continuous S-shaped curve with a finite maximum slope at the inflection point as $l \to \infty$.

(ii) By drawing a tangent line through the inflection point and finding its intercept with $R(p) = 0$ and $R(p) = 1$, we find the bounds p_l^B and p_u^B for the transition points p_l and p_u listed in Table I. Also, by extrapolating where $R(p) = \epsilon$ and $R(p) = 1 - \epsilon$ to $L \to \infty$, we find the estimates p_l^e and p_u^e, also listed in Table I. In comparison, previously measured and predicted values of p_l and p_u (determined through other methods) are listed in Table II.

(iii) For the {7,3} lattice, the reported value $p_u = 0.72$ [17] is inconsistent with our lower bound $p_u^B = 0.758$ and estimate 0.810. However, those authors' value for $p_l = 0.20$ on the dual lattice {3,7} is consistent with this bound, by (6).

(iv) For the EBT lattice, our bound $p_u^B = 0.503$ and especially our estimate $p_u^e = 0.564$ are inconsistent with the prediction $p_u = 1/2$ [23]. Our results for p_u are however consistent with the measurement of p_u by [18]. For p_l, our estimate 0.306 is in substantial agreement with the results of both Refs. [17] and [18].

(v) For the EBT-dual lattice, our bounds and estimates for the transition points agree with those of [18] within expected errors.

(vi) For the {5,5} lattice, we report measurements of the thresholds for the first time, and our estimates are close to the theoretical bounds $p_l = 1/4$ and $p_u = 3/4$,

which follow by assuming a tree structure [43].

(vii) We determine the point p^* where $R(p^*) = 1/2$ for all four lattices we consider, and find that the behavior of $R(p)$ is nearly symmetric about that point. For the {5,5} lattice, $p^* \approx 0.506$, slightly larger than the value 0.5 one might expect from self-duality. We believe the deviation from 0.5 is due to the non-equivalent configurations at the center for the lattice and its dual, implying that the two finite systems are not exactly dual to each other.

(viii) We have found a variety of finite-size scaling relations, as given in Figs. 4, 5, and 6. Another scaling relation that exists in the literature is that of Nogawa and Hasegawa [18], who find that the mass of the root cluster in the EBT scales as $N^{\psi(p)}$ in the intermediate region, where ψ is a function of p. Clearly it would be desirable to have a general scaling theory that combines all of these finite-size scaling relations. This is an interesting problem for future study.

(ix) Another area for future study is site percolation on hyperbolic lattices. For site percolation on fully triangulated lattices in ordinary two-dimensional space, $p_c = 1/2$. For fully triangulated hyperbolic lattices, such as the {3,7} lattice, we guess that the behavior of $R(p)$ will be precisely symmetric about $p = 1/2$, because of its self-matching property. Also, because bond percolation on a given lattice is equivalent to site percolation on its covering lattice (or line graph), the results here for bond percolation on the {7,3} lattice can be mapped to site percolation on its covering lattice, which is an interesting kind of hyperbolic kagomé lattice. Covering lattices of the other lattices we considered here contain crossing bonds.

Note added in proof: Just recently, Baek [44] argued that the conjectured result $p_u = 0.5$ for the EBT in Ref. [23] should be replaced by a lower bound of about 0.55, which is consistent with our results here.

V. ACKNOWLEDGMENTS

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Notes added after publication: This paper has been published in Physical Review E **85**, 051141 (May 29, 2012). We add the following update and correction:

1. Shortly after our paper was published, Delfosse and Zémor [45] posted a paper in which the rigorous upper bound for p_l for the {5,5} lattice is reduced from 0.40 to 0.38, closer to our approximate bound $p_l^B = 0.346$. They also give upper bounds for self-dual lattices $\{m, m\}$ for larger values of m.

2. A correction in the printed version: In section III A, third paragraph, the second sentence should read: "We call this the duality point because on a truly dual lattice the *crossing* probability should also be $1/2$...." We have changed "occupation" to "crossing" in the present version.

[45] Nicolas Delfosse and Gilles Zémor, "Upper Bounds on the Rate of Low Density Stabilizer Codes for the Quantum Erasure Channel," arXiv:1205.7036 (May 31, 2012).