A permutation pattern that illustrates the strong law of small numbers

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Abstract

We obtain an explicit formula for the number of permutations of $[n]$ that avoid the barred pattern $\bar{1}43\bar{5}2$. A curious feature of its counting sequence, $1, 1, 2, 5, 14, 43, 145, 538, 2194, \ldots$, is that the displayed terms agree with A122993 in the On-Line Encyclopedia of Integer Sequences, but the two sequences diverge thereafter.

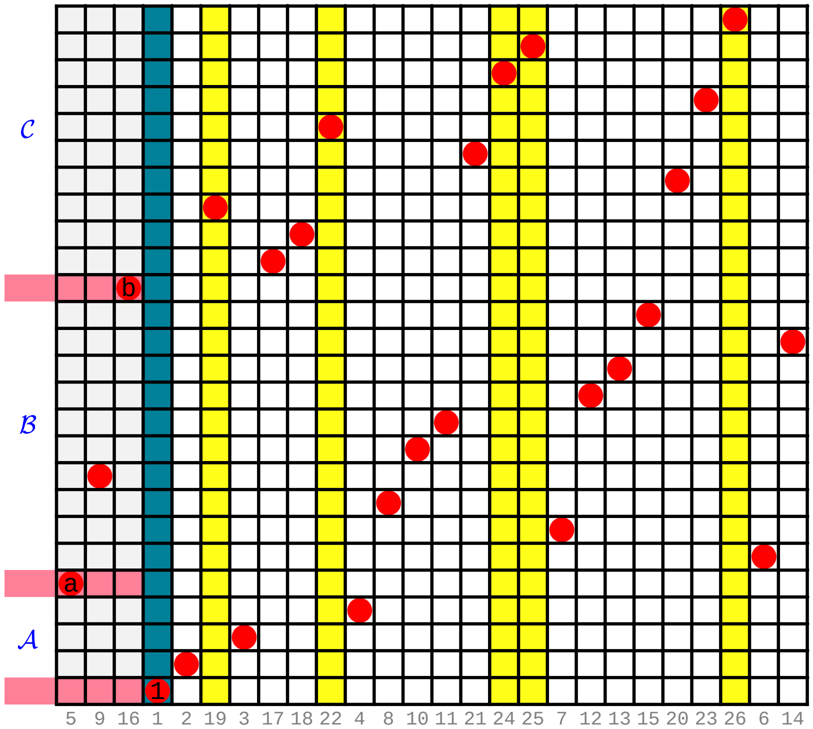
1 Introduction

A permutation π avoids the barred pattern $\bar{1}43\bar{5}2$ if each instance of a not-necessarily-consecutive 432 pattern in π is part of a 14352 pattern in π, and similarly for other barred patterns. This paper is one of a series of notes counting permutations avoiding a 5 letter pattern with 2 bars that do not yield to Lara Pudwell's method of Enumeration Schemes [3]. The question of whether there may be an automated method to fill in these and other gaps in Pudwell's enumeration remains open. Here we treat the pattern $\bar{1}43\bar{5}2$. A curious feature of the counting sequence is that it agrees through the $n = 8$ term with sequence A122993 in the On-Line Encyclopedia of Integer Sequences [4], an instance of the Strong Law of Small Numbers [5, 6].

Our method is to identify the structure of a $\bar{1}43\bar{5}2$-avoider. This permits a direct count as a 5-summation formula according to five statistics of the permutation, four of which are the first entry a, the immediate predecessor of 1 denoted b, the position of 1 denoted j, and the number of left to right maxima that occur after 1 denoted k. One of these sums can be evaluated, leading to a faster formula.

2 $\bar{1}43\bar{5}2$-Avoiders

A "typical" $\bar{1}43\bar{5}2$-avoider is illustrated in Figure 1 in matrix form. It has first entry $a = 5$, 1 is in position $j = 4$, the immediate predecessor of 1 is $b = 16$, and there are $k = 5$ left to right maxima that occur after 1. Here $j \geq 3$ so that $1, a, b$ are all distinct. The special cases $j = 1$ or 2 are treated later. There is a vertical blue line through the bullet representing the entry 1, and yellow vertical lines through the left to right maxima that occur after 1. These k yellow lines divide the the part of the matrix to the right of the blue line into $k + 1$ vertical strips (in white, one of which is vacuous in Figure 1).



Permutation as matrix diagram

Figure 1

Furthermore, horizontal lines through 1, a and b determine three horizontal strips indexed by $\mathcal{A} = [2, a-1]$, $\mathcal{B} = [a+1, b-1]$, $\mathcal{C} = [b+1, n]$. There are $j-3$ bullets to the left of the blue line in strip $\mathcal{B}$ and none in $\mathcal{A}$ or $\mathcal{C}$. Hence, to the right of the blue line there are $A := a-2$ bullets in strip $\mathcal{A}$, $B := |\,\mathcal{B}\,| - (j-3) = b-a-j+2$ bullets in strip $\mathcal{B}$, and $C := n-b$ bullets in strip $\mathcal{C}$. The following properties of a $\bar{1}43\bar{5}2$-avoider are evident in the illustration and easily proved from the definition.

- The entries to the left of 1 are increasing, else together with 1, there is a 432 pattern with no available 1. Equivalently, the bullets in the gray vertical strip on the left are rising.
- The entries $2, 3, \ldots, a-1$ occur in that order, else together with b, there is a 432 pattern with no available 1. Equivalently, the bullets in horizontal strip $\mathcal{A}$ are rising.

- Entries in the interval (a, b) lie either to the left of 1 or to the right of $a-1$, else together with b, there is a 432 pattern with no available 1. Equivalently, all bullets in horizontal strip $\mathcal{B}$ to the right of 1 are also to the right of $a-1$.
- Every descent initiator after 1 is a left to right maximum, else together with a left to right maximum to its left (there is one), we have a 432 pattern with no available 5. Equivalently, the bullets in each vertical white strip $\mathcal{A}$ are rising.

Conversely, when the position j of 1 is ≥ 3, one can check that a permutation with these properties is $\bar{1}43\bar{5}2$-avoiding.

To count permutations with these four properties, let $i \in [1, k+1]$ denote the left to right position of the first white strip containing an entry in (a, b), that is, containing a bullet in horizontal strip $\mathcal{B}$ (when there is one).

The subpermutation of entries in $\mathcal{C} = [b+1, n]$, when split at its left to right maxima, forms a partition in a canonical form: in each block, the largest entry occurs first and the rest of the block is increasing, and the blocks are ordered by increasing first entry. This yields $\left\{ {C \atop k} \right\}$ choices to determine the relative positions of entries in $\mathcal{C}$.

Next, choose $j-3$ elements from $[a+1, b-1]$ to precede 1—$\binom{b-a-1}{j-3}$ choices. The entries in $\mathcal{B}$ following 1 must be distributed into boxes (white strips) labeled $i, i+1, \ldots, k+1$ in such a way that box i is nonempty—$(k-i+2)^B - (k-i+1)^B$ choices when $B > 0$. The bullets for entries in $\mathcal{A}$ must be distributed into boxes $1, 2, \ldots, i$—$\binom{A+i-1}{i-1}$ choices when $B > 0$. In case $B = 0$, we merely distribute the bullets for entries in $\mathcal{A}$ into $k+1$ boxes—$\binom{A+k}{k}$ choices.

Recalling that $A = a-2,\ B = b-a-j+2,\ C = n-b$, the contribution of the case $j \geq 3$ to the desired count is now seen to be

$$\sum_{a=2}^{n-1} \sum_{b=a+1}^{n} \sum_{j=3}^{b-a+1} \sum_{k=1}^{n-b} \sum_{i=1}^{k+1} \left\{ {n-b \atop k} \right\} \binom{b-a-1}{j-3} \left((k-i+2)^{b-a-j+2} - (k-i+1)^{b-a-j+2} \right) \times \binom{a+i-3}{i-1} + \sum_{a=2}^{n-1} \sum_{b=a+1}^{n} \left\{ {n-b \atop k} \right\} \binom{a+i-3}{i-1} \quad (1)$$

When $j = 1$, the map "delete first entry" is a bijection to $43\bar{5}2$-avoiding permutations of size $n-1$, counted by the Bell number B_{n-1} [7]. When $j = 2$, we have $a = b$ in Figure 1, and the count reduces to $\sum_{a=2}^{n} \sum_{k=0}^{n-a} \binom{k+a-2}{a-2} \left\{ {n-a \atop k} \right\}$ where $\left\{ {0 \atop 0} \right\} := 1$.

The sum over j in (1) can be evaluated using the binomial theorem, and putting it all together we have, after minor simplifications, the following result.

Theorem. *For $n \geq 2$, the number of permutations of $[n]$ avoiding the barred pattern $\bar{1}43\bar{5}2$ is*

$$B_{n-1} + 1 + 2^{n-2} - n + \sum_{a=0}^{n-3} \sum_{b=0}^{a-1} \sum_{k=0}^{a-b} \left(\sum_{i=0}^{k} \binom{n-4-a+k-i}{k-i} (i+2)^b - \binom{n-3-a+k}{k} \right) \left\{ {a-b \atop k} \right\} + \sum_{a=0}^{n-2} \sum_{k=0}^{n-2-a} \binom{k+a+1}{k+1} \left\{ {n-2-a \atop k} \right\}.$$

The first few terms of the counting sequence, starting at $n = 1$, are 1, 2, 5, 14, 43, 145, 538, 2194, 9790, 47491, 248706.

References

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