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CONGRUENCES FOR FRANEL NUMBERS

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Abstract. The Franel numbers given by $f_n = \sum_{k=0}^{n}\binom{n}{k}^3$ $(n = 0, 1, 2, \dots)$ play important roles in both combinatorics and number theory. In this paper we initiate the systematic investigation of fundamental congruences for Franel numbers. Let $p > 3$ be a prime. We mainly establish the following congruences:

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\quad \sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k \equiv 0 \pmod{p^2},\quad \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k \equiv 0 \pmod{p}.$$

We also pose several conjectural congruences.

1. Introduction

It is well known that

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}\ (n = 0, 1, 2, \dots)$$

and central binomial coefficients play important roles in mathematics. A famous theorem of J. Wolstenholme [W] asserts that

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3} \quad \text{for any prime } p > 3.$$

The reader may consult [S11a], [S11b], [ST1] and [ST2] for recent work on congruences involving central binomial coefficients.

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In 1895 J. Franel [F] noted that the numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots) \tag{1.1}$$

(cf. [Sl, A000172]) satisfy the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n(n+1)+2) f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

Such numbers are now called Franel numbers. For combinatorial interpretations of Franel numbers and Barrucand's identity

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{with } g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}, \tag{1.2}$$

the reader may consult D. Callan [C]. Recall that Apéry numbers given by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2 \ (n = 0, 1, 2, \ldots)$$

were introduced by Apéry [Ap] (see also [Po]) in his famous proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$, and they can be expressed in terms of Franel numbers as follows:

$$A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k \tag{1.3}$$

(see V. Strehl [St92]).

The Franel numbers are also related to the theory of modular forms. Let η be the Dedkind eta function given by

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

with $\mathrm{Im}(\tau) > 0$. It is known that

$$\sum_{n=0}^{\infty} f_n \left(\frac{\eta(\tau)^3 \eta(6\tau)^9}{\eta(2\tau)^3 \eta(3\tau)^9} \right)^n = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2 \eta(6\tau)^3}$$

for any complex number τ with $\mathrm{Im}(\tau) > 0$. (See, e.g., D. Zagier [Z].)

In this paper we study congruences for Franel numbers systematically. As usual, for an odd prime p and integer a, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a)$ stands for the Fermat quotient $(a^{p-1} - 1)/p$ if $p \nmid a$.

Now we state our main results.

Theorem 1.1. *Let $p>3$ be a prime. For any p-adic integer r we have*

$$\sum_{k=0}^{p-1}(-1)^k\binom{k+r}{k}f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\binom{k+r}{k}^2\pmod{p^2}.\tag{1.4}$$

In particular,

$$\sum_{k=0}^{p-1}(-1)^kf_k\equiv\left(\frac{p}{3}\right)\pmod{p^2},\tag{1.5}$$

$$\sum_{k=0}^{p-1}(-1)^kkf_k\equiv-\frac{2}{3}\left(\frac{p}{3}\right)\pmod{p^2},\tag{1.6}$$

$$\sum_{k=0}^{p-1}(-1)^kk^2f_k\equiv\frac{10}{27}\left(\frac{p}{3}\right)\pmod{p^2},\tag{1.7}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\pmod{p^2}.\tag{1.8}$$

We also have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k\equiv0\pmod{p^2},\tag{1.9}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k\equiv0\pmod{p},\tag{1.10}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_{k-1}\equiv3q_p(2)+3p\,q_p(2)^2\pmod{p^2},\tag{1.11}$$

and

$$\sum_{k=0}^{p-1}(3k+1)\frac{f_k}{8^k}\equiv p^2-2p^3q_p(2)+4p^4q_p(2)^2\pmod{p^5}.\tag{1.12}$$

Remark 1.1. Fix a prime $p>3$. As $f_k\equiv(-8)^kf_{p-1-k}\pmod{p}$ for all $k=0,\ldots,p-1$ by [JV, Lemma 2.6], (1.11) implies that

$$\sum_{k=1}^{p-1}\frac{f_k}{k8^k}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_{p-1-k}=\sum_{k=1}^{p-1}\frac{(-1)^{p-k}}{p-k}f_{k-1}\equiv3q_p(2)\pmod{p}.$$

Concerning (1.8) the author [S11b, Conj. 5.2(ii)] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 3\ \&\ p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv2\pmod 3.\end{cases}$$

We also have many such conjectures for $\sum_{k=0}^{p-1}\binom{2k}{k}f_k/m^k$ mod p^2. (1.10) can be extended as

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)}\equiv0\pmod p,\tag{1.13}$$

where r is any positive integer and $f_k^{(r)}:=\sum_{j=0}^k\binom{k}{j}^r$. Note that $f_k^{(2)}=\binom{2k}{k}$ and $\sum_{k=1}^{p-1}\binom{2k}{k}/k\equiv0\pmod{p^2}$ by [ST1].

Let $p>3$ be a prime. Similar to (1.5)-(1.7), we are also able to show that

$$\sum_{k=0}^{p-1}(-1)^kk^3f_k\equiv-\frac{10}{81}\left(\frac{p}{3}\right)\pmod{p^2}\ \text{and}\ \sum_{k=0}^{p-1}(-1)^kk^4f_k\equiv-\frac{14}{3^5}\left(\frac{p}{3}\right)\pmod{p^2}.$$

In general, for any positive integer r there should be an odd integer a_r such that

$$\sum_{k=0}^{p-1}(-1)^kk^rf_k\equiv\frac{2a_r}{3^{2r-1}}\left(\frac{p}{3}\right)\pmod{p^2}.$$

For example, if $p>5$ then

$$\sum_{k=0}^{p-1}(-1)^kk^5f_k\equiv\frac{322}{3^7}\left(\frac{p}{3}\right)\pmod{p^2},\quad\sum_{k=0}^{p-1}(-1)^kk^6f_k\equiv-\frac{2030}{3^9}\left(\frac{p}{3}\right)\pmod{p^2}.$$

The Apéry polynomials introduced in [S11c] are those

$$A_n(x)=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2x^k\quad(n=0,1,2,\ldots).$$

Here we define

$$g_n(x)=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}x^k$$

for all $n=0,1,2,\ldots$. Note that $g_n(1)=g_n$.

Theorem 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} g_k(x) \equiv p \sum_{k=0}^{p-1} \frac{x^k}{2k+1} \pmod{p^2}. \tag{1.14}$$

Consequently,

$$\sum_{k=1}^{p-1} g_k \equiv 0 \pmod{p^2}, \tag{1.15}$$

$$\sum_{k=0}^{p-1} g_k(-1) \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \tag{1.16}$$

$$\sum_{k=0}^{p-1} g_k(-3) \equiv \left(\frac{p}{3}\right) \pmod{p^2}. \tag{1.17}$$

We also have

$$\sum_{k=1}^{p-1} \frac{g_k(x)}{k} \equiv 0 \pmod{p}, \tag{1.18}$$

$$\sum_{k=1}^{p-1} \frac{g_{k-1}}{k} \equiv -\left(\frac{p}{3}\right) 2q_p(3) \pmod{p}, \tag{1.19}$$

$$\sum_{k=1}^{p-1} k g_k \equiv -\frac{3}{4} \pmod{p^2}, \tag{1.20}$$

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k^2} \equiv 0 \pmod{p} \quad \text{if } p > 5. \tag{1.21}$$

Moreover,

$$\frac{1}{3n^2} \sum_{k=0}^{n-1} (4k+3) g_k = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 C_k \tag{1.22}$$

for all $n = 1, 2, 3, \ldots$, where C_k denotes the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$.

Remark 1.2. Let $p > 3$ be a prime. By [JV, Lemma 2.7], $g_k \equiv (\frac{p}{3}) 9^k g_{p-1-k} \pmod{p}$ for all $k = 0, \ldots, p-1$. So (1.17) implies that

$$\sum_{k=1}^{p-1} \frac{g_k}{k 9^k} \equiv \left(\frac{p}{3}\right) \sum_{k=1}^{p-1} \frac{g_{p-1-k}}{k} = \left(\frac{p}{3}\right) \sum_{k=1}^{p-1} \frac{g_{k-1}}{p-k} \equiv 2q_p(3) \pmod{p}.$$

We also introduce the polynomials

$$f_n(x)=\sum_{k=0}^n\binom nk^2\binom{2k}nx^k=\sum_{k=0}^n\binom nk\binom k{n-k}\binom{2k}kx^k\quad(n=0,1,2,\ldots)$$

which play a central role in our proof of Theorem 1.1.

We are going to show Theorem 1.1 in the next section and investigate in Section 3 connections among the polynomials $A_n(x)$, $f_n(x)$ and $g_n(x)$. In Section 4 we will prove Theorem 1.2. In Section 5 we shall raise some conjectures for further research.

2. Proof of Theorem 1.1

Lemma 2.1. *For any nonnegative integer n, the integer $f_n(1)$ coincides with the Franel number f_n.*

Remark 2.1. This is a known result due to V. Strehl [St94].

Lemma 2.2. *For any nonnegative integer k we have*

$$\sum_{l=k}^{2k}(-1)^l\binom lk\binom k{l-k}\binom{x+l}l=\binom{x+k}k^2.$$

Proof. Observe that

$$\begin{aligned}&\sum_{l=k}^{2k}(-1)^l\binom lk\binom k{l-k}\binom{x+l}l\\=&\sum_{l=k}^{2k}\binom lk\binom k{l-k}\binom{-x-1}l\\=&\binom{-x-1}k\sum_{l=k}^{2k}\binom{-x-1-k}{l-k}\binom k{l-k}\\=&\binom{-x-1}k\sum_{j=0}^{k}\binom{-x-1-k}{j}\binom k{k-j}=\binom{-x-1}k^2=\binom{x+k}k^2\end{aligned}$$

with the help of the Chu-Vandemonde identity (cf. (3.1) of [G, p.22]). We are done. □

Lemma 2.3. *For each positive integer m we have*

$$\sum_{k=0}^{n-1}P_m(k)\binom{2k}k=n^m\binom{2n}n\quad\text{for all } n=1,2,3,\ldots,$$

where

$$P_m(x) := 2(2x+1)(x+1)^{m-1} - x^m.$$

Proof. The desired result follows immediately by induction on n. □

Remark 2.2. The author thanks Prof. Qing-Hu Hou at Nankai Univ. for his comments on the author's original version of Lemma 2.3.

Lemma 2.4. *Let $p > 3$ be a prime.*
(i) ([ST, (1.6)]) *We have*

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

(ii) ([S11c]) We have

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p \pmod{p^4}.$$

Lemma 2.5. *Let m be a positive integer. For $n = 0, 1, \ldots, m$ we have*

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{m-k} = \frac{m-n}{m}\binom{x-1}{n}\binom{-x}{m-n}.$$

Remark 2.3. This is a known result due to E. S. Andersen, see, e.g., (3.14) of [G, p. 23].

Lemma 2.6 (Sun [S11b, Lemma 2.1])**.** *Let p be an odd prime. For any $k = 1, \ldots, p-1$ we have*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p\rfloor - 1}2p \pmod{p^2}.$$

Recall that harmonic numbers are given by

$$H_n = \sum_{0<k\leqslant n}\frac{1}{k} \quad (n = 0, 1, 2, \ldots).$$

In general. for any positive integer m, harmonic numbers of order m are defined by

$$H_n^{(m)} := \sum_{0<k\leqslant n}\frac{1}{k^m} \quad (n = 0, 1, 2, \ldots).$$

Let $p>3$ be a prime. In 1862 J. Wolstenholme [W] proved that

$$H_{p-1}\equiv 0 \pmod{p^2}\ \text{ and }\ H_{p-1}^{(2)}\equiv 0 \pmod{p}.$$

Note that

$$H_{(p-1)/2}^{(2)}=\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\frac{1}{2}H_{p-1}^{(2)}\equiv 0 \pmod{p}.$$

In 1938 E. Lehmer [L] showed that

$$H_{(p-1)/2}\equiv -2q_p(2)+p\,q_p(2)^2 \pmod{p^2}. \tag{2.1}$$

These fundamental congruences are quite useful. Recently the author [S12a] proved some further congruences involving harmonic numbers.

Lemma 2.7. *Let $p>3$ be a prime. Then*

$$f_{p-1}\equiv 1+3p\,q_p(2)+3p^2q_p(2)^2 \pmod{p^3}. \tag{2.2}$$

Proof. For any $k=1,\ldots,p-1$, we obviously have

$$\begin{aligned}(-1)^k\binom{p-1}{k}&=\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\\ &\equiv 1-pH_k+\frac{p^2}{2}\sum_{1\leqslant i<j\leqslant k}\frac{2}{ij}=1-pH_k+\frac{p^2}{2}\left(H_k^2-H_k^{(2)}\right) \pmod{p^3}.\end{aligned}$$

Thus

$$\begin{aligned}f_{p-1}-1=&\sum_{k=1}^{p-1}\binom{p-1}{k}^3\equiv\sum_{k=1}^{p-1}(-1)^k\left(1-pH_k+\frac{p^2}{2}\left(H_k^2-H_k^{(2)}\right)\right)^3\\ =&-3p\sum_{k=1}^{p-1}(-1)^kH_k+\frac{9}{2}p^2\sum_{k=1}^{p-1}(-1)^kH_k^2-\frac{3}{2}p^2\sum_{k=1}^{p-1}(-1)^kH_k^{(2)} \pmod{p^3}.\end{aligned}$$

Clearly

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^kH_k=&\sum_{k=1}^{p-1}\sum_{j=1}^{k}\frac{(-1)^k}{j}=\sum_{j=1}^{p-1}\frac{\sum_{k=j}^{p-1}(-1)^k}{j}=\sum_{\substack{j=1\\ 2\mid j}}^{p-1}\frac{1}{j}\\ =&\frac{1}{2}H_{(p-1)/2}\equiv -q_p(2)+\frac{p}{2}q_p(2)^2 \pmod{p^2}\ \ (\text{by (2.1)})\end{aligned}$$

and

$$\sum_{k=1}^{p-1}(-1)^k H_k^{(2)} = \sum_{j=1}^{p-1}\frac{\sum_{k=j}^{p-1}(-1)^k}{j^2} = \sum_{i=1}^{(p-1)/2}\frac{1}{(2i)^2} \equiv 0 \pmod p.$$

Observe that

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^k H_k^2 =& \sum_{k=1}^{p-1}(-1)^{p-k}H_{p-k}^2 = \sum_{k=1}^{p-1}(-1)^{k-1}\left(H_{p-1} - \sum_{0<j<k}\frac{1}{p-j}\right)^2 \\ \equiv& -\sum_{k=1}^{p-1}(-1)^k\left(H_k - \frac{1}{k}\right)^2 \\ \equiv& -\sum_{k=1}^{p-1}(-1)^k H_k^2 + 2\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k - \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2} \pmod p.\end{aligned}$$

Clearly,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2} \equiv \sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^2} = \sum_{j=1}^{(p-1)/2}\frac{2}{(2j)^2} \equiv 0 \pmod p,$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k = \sum_{\substack{k=1\\2\mid k}}^{p-1}\frac{H_k}{k} - \sum_{\substack{k=1\\2\nmid k}}^{p-1}\frac{H_k}{k} \equiv \frac{q_p(2)^2}{2} - \left(-\frac{q_p(2)^2}{2}\right) \pmod p$$

by [S12a, Lemma 2.3]. Therefore

$$\sum_{k=1}^{p-1}(-1)^k H_k^2 \equiv \sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k \equiv q_p(2)^2 \pmod p.$$

Combining the above, we finally obtain

$$f_{p-1} - 1 \equiv -3p\left(-q_p(2) + \frac{p}{2}q_p(2)^2\right) + \frac{9}{2}p^2 q_p(2)^2 \pmod{p^3}$$

and hence (2.2) holds. □

Proof of Theorem 1.1. (i) Let r be any p-adic integer. Observe that

$$\begin{aligned}\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x) =& \sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k \\ =& \sum_{k=0}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{\min\{2k,p-1\}}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l}.\end{aligned}$$

If $(p-1)/2<k\leqslant p-1$ and $k\leqslant l\leqslant 2k$, then

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod p \quad\text{and}\quad \binom{l}{k}=\frac{l!}{k!(l-k)!}\equiv 0 \pmod p.$$

Therefore

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{2k}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l} \pmod{p^2}.$$

Applying Lemma 2.2 we obtain

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k\binom{k+r}{k}^2 \pmod{p^2}. \tag{2.3}$$

In the case $x=1$ this yields (1.4).

Taking $r=0$ in (1.4) and applying Lemma 2.4(i), we immediately get (1.5).

By (2.3) with $r=0,1$,

$$\begin{aligned}&\sum_{k=0}^{p-1}(3(k+1)-1)(-1)^kf_k(x)\\ \equiv&\sum_{k=0}^{p-1}\binom{2k}{k}x^k\left(3(k+1)^2-1\right)=\sum_{k=0}^{p-1}P_2(k)\binom{2k}{k}x^k \pmod{p^2}\end{aligned}$$

where $P_2(x)=2(2x+1)(x+1)-x^2=3x^2+6x+2$. Thus, with the help of Lemma 2.3, we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^kf_k\equiv 0 \pmod{p^2} \tag{2.4}$$

and hence (1.6) holds in view of (1.5).

Taking $r=2$ in (2.3) we get

$$2\sum_{k=0}^{p-1}(k^2+3k+2)(-1)^kf_k(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k((k+1)(k+2))^2 \pmod{p^2}.$$

In view of (2.4), this yields

$$2\sum_{k=0}^{p-1}(-1)^kk^2f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}(k^2+3k+2)^2 \pmod{p^2}.$$

Note that

$$27(k^2+3k+2)^2 = 9P_4(k)+12P_3(k)+23P_2(k)+20$$

where $P_m(x)$ is given by Lemma 2.3. Therefore, with the help of Lemma 2.3 and Lemma 2.4(i), we have

$$54\sum_{k=0}^{p-1}(-1)^k k^2 f_k \equiv \sum_{k=0}^{p-1}(9P_4(k)+12P_3(k)+23P_2(k)+20)\binom{2k}{k} \equiv 20\left(\frac{p}{3}\right) \pmod{p^2}$$

and hence (1.7) follows.

Putting $r=-1/2$ in (2.3) and noting that

$$(-1)^k\binom{k-1/2}{k} = \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \quad \text{for } k=0,1,2,\ldots,$$

we then obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k(x)}{(-4)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k \pmod{p^2}. \tag{2.5}$$

In the case $x=1$ this gives (1.8).

(ii) Now we prove (1.9). Observe that

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k = \sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}x^k\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}.$$

If $1 \leqslant k \leqslant (p-1)/2$, then

$$\begin{aligned}\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} &= \sum_{l=k}^{2k}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}\\ &= \sum_{j=0}^{k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}\\ &= (-1)^k\sum_{j=0}^{k}\binom{-k}{j}\binom{k}{k-j} = (-1)^k\binom{0}{k} = 0\end{aligned}$$

by the Chu-Vandermonde identity. If $(p+1)/2 \leqslant k \leqslant p-1$, then

$$\begin{aligned}\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} &= \sum_{j=0}^{p-1-k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}\\ &= (-1)^k\sum_{j=0}^{p-1-k}\binom{-k}{j}\binom{k}{k-j}\end{aligned}$$

and hence applying Lemma 2.5 we get

$$\begin{aligned}&\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}\\ =&(-1)^k\frac{k-(p-1-k)}{k}\binom{-k-1}{p-1-k}\binom{k}{k-(p-1-k)}\\ =&(-1)^{p-1}\left(\frac{p-k}{k}\right)^2\binom{p-1}{k-1}\binom{k}{p-k}\\ \equiv&(-1)^{k-1}\binom{k}{p-k}=\binom{p-2k-1}{p-k}\\ \equiv&\binom{2(p-k)-1}{p-k-1}=\frac{1}{2}\binom{2(p-k)}{p-k}\pmod p.\end{aligned}$$

Note that $\binom{2k}{k}\equiv 0\pmod p$ for $k=(p+1)/2,\ldots,p-1$. So we have

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}f_l(x)\equiv\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{k}x^k\frac{\binom{2(p-k)}{p-k}}{2}\equiv p\sum_{k=(p+1)/2}^{p-1}\frac{x^k}{k^2}\pmod{p^2}\quad(2.6)$$

with the help of Lemma 2.6. Clearly

$$2\sum_{k=(p+1)/2}^{p-1}\frac{1}{k^2}\equiv\sum_{k=(p+1)/2}^{p-1}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p$$

since $\sum_{k=1}^{p-1}1/(2k)^2\equiv\sum_{k=1}^{p-1}1/k^2\pmod p$. Therefore (1.9) is valid.

Instead of proving (1.10) we show its extension (1.13). Clearly,

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}=\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^{kr}}{k^{r-1}}+\frac{(-1)^{(p-k)r}}{(p-k)^{r-1}}\right)\equiv 0\pmod p.$$

Thus

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)}\equiv&\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}\sum_{k=1}^{l}\binom{l}{k}^r=\sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{l=k}^{p-1}(-1)^{lr}\binom{l-1}{k-1}^{r-1}\binom{l}{k}\\ =&\sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{j=0}^{p-1-k}(-1)^{(k+j)r}\binom{k+j-1}{j}^{r-1}\binom{k+j}{j}\\ =&\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-1-k}\binom{-k}{j}^{r-1}\binom{-k-1}{j}\\ \equiv&\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-k-1}\binom{p-k}{j}^{r-1}\binom{p-k-1}{j}\pmod p.\end{aligned}$$

For any positive integer n, we have

$$f_n^{(r)} = \sum_{k=0}^{n}\left(\frac{k}{n}+\frac{n-k}{n}\right)\binom{n}{k}^r = 2\sum_{k=0}^{n}\frac{n-k}{n}\binom{n}{k}^r = 2\sum_{k=0}^{n-1}\binom{n}{k}^{r-1}\binom{n-1}{k}.$$

Therefore,

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} \equiv & \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\cdot\frac{f_{p-k}}{2} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{(p-k)r}f_k}{(p-k)^{r-1}}\\ \equiv & -\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)} \pmod p\end{aligned}$$

and hence (1.10) follows.

(iii) Next we show (1.11). Note that

$$\begin{aligned}\frac{1}{p}\sum_{n=0}^{p-1}(2n+1)A_n =&\frac{1}{p}\sum_{n=0}^{p-1}(2n+1)\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}f_k\\ =&\frac{1}{p}\sum_{k=0}^{p-1}\binom{2k}{k}f_k\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}f_k\frac{p-k}{k+1}\binom{p+k}{2k} = \sum_{k=0}^{p-1}\frac{pf_k}{k+1}\binom{p-1}{k}\binom{p+k}{k}\\ =&f_{p-1}\binom{2p-1}{p-1}+\sum_{k=0}^{p-2}\frac{pf_k}{k+1}\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\\ \equiv&f_{p-1}+p\sum_{k=0}^{p-2}\frac{(-1)^k f_k}{k+1} \pmod{p^3}.\end{aligned}$$

Combining this with Lemma 2.4(ii) and (2.2) we obtain

$$\sum_{k=0}^{p-2}\frac{(-1)^{k-1}f_k}{k+1} \equiv \frac{f_{p-1}-1}{p} \equiv 3q_p(2)+3p\,q_p(2)^2 \pmod{p^2},$$

which is equivalent to (1.11).

(iv) Finally we prove (1.12). For any positive integer n we have the identity

$$\begin{aligned}\sum_{k=0}^{n-1}(3k+1)f_k 8^{n-1-k} =&n^2\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\binom{n-1}{k}^2-\binom{n-1}{k}\binom{n}{k+1}+\binom{n}{k+1}^2\right)\\ =&n^2\sum_{k=0}^{n-1}\binom{n-1}{k}^3\left(1-\frac{n}{k+1}+\frac{n^2}{(k+1)^2}\right),\end{aligned}$$

which can be easily proved via the Zeilberger algorithm (cf. [PWZ]) since both sides satisfy the same recurrence relation with respect to n.

Putting $n=p$ in the above identity, we get

$$\begin{aligned}&\frac{1}{p^2}\sum_{k=0}^{p-1}(3k+1)\frac{f_k}{8^k}\\&=(1+p\,q_p(2))^{-3}\left(f_{p-1}-1-\sum_{k=0}^{p-2}\binom{p-1}{k}^3\frac{p}{k+1}+1+\sum_{k=0}^{p-2}\binom{p-1}{k}^3\frac{p^2}{(k+1)^2}\right)\\&\equiv(1-p\,q_p(2)+p^2q_p(2)^2)^3\left(f_{p-1}-p\sum_{k=0}^{p-2}\frac{(-1)^k(1-pH_k)^3}{k+1}+p^2\sum_{k=0}^{p-2}\frac{(-1)^k}{(k+1)^2}\right)\\&\equiv(1-3p\,q_p(2)+6p^2q_p(2)^2)\left(f_{p-1}-p\sum_{k=0}^{p-2}\frac{(-1)^k(1-3pH_k)}{k+1}\right)\pmod{p^3}\end{aligned}$$

since

$$\sum_{k=0}^{p-2}\frac{(-1)^{k+1}}{(k+1)^2}=\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^k}{k^2}+\frac{(-1)^{p-k}}{(p-k)^2}\right)\equiv 0\pmod p$$

In view of the proof of Lemma 2.7,

$$\begin{aligned}\sum_{k=0}^{p-2}\frac{(-1)^{k+1}(1-3pH_k)}{k+1}&=\sum_{k=1}^{p-1}\frac{(-1)^k}{k}(1-3pH_{k-1})\\&=\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}-H_{p-1}-3p\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k+3p\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\\&\equiv H_{(p-1)/2}-3p\,q_p(2)^2\pmod{p^2}\end{aligned}$$

Thus, with the helps of (2.1) and (2.2) we deduce from the above our desired congruence (1.12).

So far we have completed the proof of Theorem 1.1. □

Remark 2.3. Let $p>3$ be a prime. By (2.3) in the case $r=0$, we have

$$\sum_{k=0}^{p-1}(-1)^kf_k(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k\pmod{p^2}.$$

Note that

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}x^k&\equiv\sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}(-4x)^k\\&\equiv\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(-4x)^k=(1-4x)^{(p-1)/2}\pmod p.\end{aligned}$$

Thus

$$\sum_{k=0}^{p-1}(-1)^k f_k(x) \equiv \left(\frac{1-4x}{p}\right) \pmod p \quad \text{for all } x\in\mathbb{Z}. \tag{2.7}$$

In view of (2.5) we also have many conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)/m^k \bmod p^2$ related to the representation of p or $4p$ by certain binary quadratic form.

3. Relations among $A_n(x), f_n(x)$ and $g_n(x)$

Lemma 3.1. *For any nonnegative integers m and n we have the combinatorial identity*

$$\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n} = \binom{x}{m}\binom{y}{n}. \tag{3.1}$$

Remark 3.1. (3.1) is due to Nanjundiah, see, e.g., (4.17) of [G, p. 53].

Our following theorem presents the polynomial forms of some known identities.

Theorem 3.1. *Let n be any nonnegative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}f_n(x) = g_n(x), \quad f_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}g_k(x), \tag{3.2}$$

and

$$A_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}f_k(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x). \tag{3.3}$$

Proof. By the binomial inversion formula, the two identities in (3.2) are equivalent. Observe that

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}f_l(x) &= \sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\sum_{l=k}^{n}\binom{n-k}{n-l}\binom{k}{l-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\binom{n}{n-k} = g_n(x)\end{aligned}$$

with the help of the Chu-Vandermonde identity. Thus (3.2) holds.

Next we show (3.3). Clearly

$$\begin{aligned}\sum_{l=0}^n\binom{n}{l}\binom{n+l}{l}f_l(x)=&\sum_{l=0}^n\binom{n}{l}\binom{n+l}{l}\sum_{k=0}^l\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k\\=&\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}x^k\sum_{l=k}^n\binom{n-k}{l-k}\binom{k}{l-k}\binom{n+l}{n}\\=&\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}x^k\sum_{j=0}^k\binom{n-k}{j}\binom{k}{k-j}\binom{n+k+j}{n}\\=&\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}x^k\binom{n+k}{n-k}\binom{n+k}{k}\quad\text{(by Lemma 2.1)}.\end{aligned}$$

This proves the first identity in (3.3). Observe that

$$\begin{aligned}&\sum_{l=0}^n\binom{n}{l}\binom{n+l}{l}(-1)^l g_l(x)\\=&\sum_{l=0}^n\binom{n}{l}\binom{-n-1}{l}\sum_{k=0}^l\binom{l}{k}f_k(x)\\=&\sum_{k=0}^n\binom{n}{k}f_k(x)\sum_{l=k}^n\binom{n-k}{n-l}\binom{-n-1}{l}\\=&\sum_{k=0}^n\binom{n}{k}f_k(x)\binom{-k-1}{n}\quad\text{(by the Chu-Vandermonde identity)}\end{aligned}$$

and hence the second identity of (3.3) follows.

The proof of Theorem 3.1 is now complete. □

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$ we set

$$[n]_q=\frac{1-q^n}{1-q}=\sum_{0\leqslant k<n}q^k,$$

this is the usual q-analogue of n. For any $n,k\in\mathbb{N}$, if $k\leqslant n$ then we call

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{\prod_{0<r\leqslant n}[r]_q}{(\prod_{0<s\leqslant k}[s]_q)(\prod_{0<t\leqslant n-k}[t]_q)}$$

a *q-binomial coefficient*; if $k>n$ then we let $\begin{bmatrix}n\\k\end{bmatrix}_q=0$. Obviously we have $\lim_{q\to1}\begin{bmatrix}n\\k\end{bmatrix}_q=\binom{n}{k}$. It is easy to see that

$$\begin{bmatrix}n\\k\end{bmatrix}_q=q^k\begin{bmatrix}n-1\\k\end{bmatrix}_q+\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\qquad\text{for all }k,n=1,2,3,\ldots.$$

By this recursion, each q-binomial coefficient is a polynomial in q with integer coefficients.

For $n \in \mathbb{N}$ we define

$$A_n(x;q) := \sum_{k=0}^{n} q^{2n(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_q^2 x^k$$

and

$$g_n(x;q) := \sum_{k=0}^{n} q^{2n(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2k \\ k \end{bmatrix}_q x^k.$$

Clearly

$$\lim_{q\to 1} A_n(x;q) = A_n(x) \quad \text{and} \quad \lim_{q\to 1} g_n(x;q) = g_n(x).$$

Those identities in Theorem 3.1 have their q-analogues. For example, the following theorem gives a q-analogue of the identity

$$A_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} g_k(x).$$

Theorem 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$A_n(x;q) = \sum_{k=0}^{n} (-1)^{n-k} q^{(n-k)(5n+3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q g_k(x;q). \tag{3.4}$$

Proof. Let $j \in \{0, \ldots, n\}$. By the q-Chu-Vandermonde identity (see, e.g., Ex. 4(b) of [AAR, p. 542]),

$$\sum_{k=j}^{n} q^{(k-j)^2} \begin{bmatrix} -n-1-j \\ k-j \end{bmatrix}_q \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q = \begin{bmatrix} -2j-1 \\ n-j \end{bmatrix}_q.$$

This, together with

$$\begin{bmatrix} -n-1 \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} -n-1 \\ j \end{bmatrix}_q \begin{bmatrix} -n-1-j \\ k-j \end{bmatrix}_q,$$

yields that

$$\sum_{k=j}^{n} q^{(k-j)^2} \begin{bmatrix} -n-1 \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q = \begin{bmatrix} -n-1 \\ j \end{bmatrix}_q \begin{bmatrix} -2j-1 \\ n-j \end{bmatrix}_q.$$

It is easy to see that

$$\begin{bmatrix} -m-1 \\ k \end{bmatrix}_q = (-1)^k q^{-km-k(k+1)/2} \begin{bmatrix} m+k \\ k \end{bmatrix}_q.$$

So we are led to the identity

$$\sum_{k=j}^{n}(-1)^{n-k}q^{\binom{n-k+1}{2}+2j(n-k)}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}n-j\\k-j\end{bmatrix}_q=\begin{bmatrix}n+j\\j\end{bmatrix}_q\begin{bmatrix}n+j\\2j\end{bmatrix}_q. \quad (3.5)$$

Since

$$\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q=\begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}n-j\\k-j\end{bmatrix}_q \text{ and } \begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}n+j\\j\end{bmatrix}_q=\begin{bmatrix}n+j\\2j\end{bmatrix}_q\begin{bmatrix}2j\\j\end{bmatrix}_q,$$

Multiplying both sides of (3.5) by $\begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}2j\\j\end{bmatrix}_q x^j$ we get

$$\sum_{k=j}^{n}(-1)^{n-k}q^{\binom{n-k+1}{2}+2j(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q^2\begin{bmatrix}2j\\j\end{bmatrix}_q x^j=\begin{bmatrix}n\\j\end{bmatrix}_q^2\begin{bmatrix}n+j\\j\end{bmatrix}_q^2 x^j.$$

In view of the last identity we can easily deduce the desired (3.4). □

Theorem 3.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}A_k(x)\equiv p\sum_{k=0}^{p-1}\frac{(-1)^k f_k(x)}{2k+1}\pmod{p^2} \quad (3.6)$$

and

$$\sum_{k=0}^{p-1}(-1)^k A_k(x)\equiv p\sum_{k=0}^{p-1}\frac{g_k(x)}{2k+1}\pmod{p^2}. \quad (3.7)$$

Proof. Observe that

$$\begin{aligned}\sum_{l=0}^{p-1}A_l(x)&=\sum_{l=0}^{p-1}\sum_{k=0}^{l}\binom{k+l}{2k}\binom{2k}{k}f_k(x)=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\sum_{l=k}^{p-1}\binom{k+l}{2k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\frac{p}{(2k+1)!}\prod_{0<j\leqslant k}(p^2-j^2)\\&\equiv\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\frac{p}{2k+1}(-1)^k\pmod{p^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{l=0}^{p-1}(-1)^l A_l(x)&=\sum_{l=0}^{p-1}\sum_{k=0}^{l}\binom{k+l}{2k}\binom{2k}{k}(-1)^k g_k(x)\\&=\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k g_k(x)\binom{p+k}{2k+1}\\&\equiv\sum_{k=0}^{p-1}\binom{2k}{k}g_k(x)\frac{p}{2k+1}\pmod{p^2}.\end{aligned}$$

This concludes the proof of Theorem 3.3. □

Remark 3.2. In [S11c] the author investigated $\sum_{k=0}^{p-1}(\pm1)^k A_k(x) \bmod p^2$ (where p is an odd prime) and made some conjectures.

Theorem 3.4. *Let n be any positive integer. Then*

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}(-1)^{n-k}(6k^3+9k^2+5k+1)A_k(x)\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(3k+2-3n^2)f_k(x),\end{aligned} \tag{3.8}$$

and also

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}(-1)^{n-k}P(k)A_k(x)\\&=-\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(9n^4-2n^2(9k+11)+18k^2+31k+14)f_k(x).\end{aligned} \tag{3.9}$$

where

$$P(x)=18x^5+45x^4+46x^3+24x^2+7x+1. \tag{3.10}$$

Proof. In view of (3.3), we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}(-1)^{n-k}(6k^3+9k^2+5k+1)A_k(x)\\&=\frac{(-1)^n}{n}\sum_{k=0}^{n-1}(-1)^k(6k^3+9k^2+5k+1)\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}f_j(x)\\&=\frac{(-1)^n}{n}\sum_{j=0}^{n-1}\binom{2j}{j}f_j(x)\sum_{k=j}^{n-1}(-1)^k(6k^3+9k^2+5k+1)\binom{k+j}{2j}\\&=\frac{(-1)^n}{n}\sum_{j=0}^{n-1}\binom{2j}{j}f_j(x)(-1)^{n-1}(n-j)(3n^2-3j-2)\binom{n+j}{2j}\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(3k+2-3n^2)f_k(x).\end{aligned}$$

This proves (3.8). Similarly,

$$\begin{aligned}&\frac{1}{n}\sum_{m=0}^{n-1}(-1)^{n-m}P(m)A_m(x)\\=&\frac{(-1)^n}{n}\sum_{m=0}^{n-1}(-1)^mP(m)\sum_{k=0}^{m}\binom{k+m}{2k}\binom{2k}{k}f_k(x)\\=&\frac{(-1)^n}{n}\sum_{k=0}^{n-1}\binom{2k}{k}f_k(x)\sum_{m=k}^{n-1}(-1)^mP(m)\binom{m+k}{2k}\\=&\frac{(-1)^n}{n}\sum_{k=0}^{n-1}\binom{2k}{k}f_k(x)(-1)^{n-1}(n-k)\\&\times(9n^4-2n^2(9k+11)+18k^2+31k+14)\binom{n+k}{2k}\\=&-\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(9n^4-2n^2(9k+11)+18k^2+31k+14)f_k(x).\end{aligned}$$

So (3.9) holds. □

The author [S11c] conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kA_k\equiv p\left(\frac{p}{3}\right)\pmod{p^3},\tag{3.11}$$

and this has been confirmed by Guo and Zeng [GZ].

Corollary 3.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(2k+1)^3(-1)^kA_k\equiv-\frac{p}{3}\left(\frac{p}{3}\right)\pmod{p^3}\tag{3.12}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^5(-1)^kA_k\equiv-\frac{13}{27}p\left(\frac{p}{3}\right)\pmod{p^3}.\tag{3.13}$$

Proof. Clearly

$$3(2k+1)^3=4(6k^3+9k^2+5k+1)-(2k+1)$$

and

$$9(2k+1)^5+2(2k+1)^3+5(2k+1)=16(18k^5+45k^4+46k^3+24k^2+7k+1).$$

Combining these with (3.11), it suffices to show that

$$\sum_{k=0}^{p-1}(-1)^k(6k^3+9k^2+5k+1)A_k \equiv 0 \pmod{p^2} \tag{3.14}$$

and

$$\sum_{k=0}^{p-1}(-1)^k P(k)A_k \equiv 0 \pmod{p^2}, \tag{3.15}$$

where $P(x)$ is given by (3.10).

Taking $n=p$ in (3.8) we get

$$\begin{aligned}&\frac{1}{p}\sum_{k=0}^{p-1}(-1)^{k-1}(6k^3+9k^2+5k+1)A_k\\ =&\sum_{k=0}^{p-1}(3k+2-3p^2)f_k\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\\ \equiv&\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k \equiv 0 \pmod{p^2}\end{aligned}$$

with the help of (1.5) and (1.6). Similarly, (3.9) with $n=p$ yields (3.15) since

$$\sum_{k=0}^{p-1}(18k^2+31k+14)(-1)^k f_k \equiv 0 \pmod{p^2}.$$

by (1.5)-(1.7). We are done.

Remark 3.3. Let $p>3$ be a prime. We can also prove that

$$\sum_{k=0}^{p-1}(2k+1)^7(-1)^k A_k \equiv \frac{5}{9}p\left(\frac{p}{3}\right) \pmod{p^3}. \tag{3.16}$$

In general, for each $r=0,1,2,\ldots$ there is a p-adic integer c_r only depending on r such that

$$\sum_{k=0}^{p-1}(2k+1)^{2r+1}(-1)^k A_k \equiv c_r p\left(\frac{p}{3}\right) \pmod{p^3}.$$

4. Proof of Theorem 1.2

Lemma 4.1. *For any positive integer n, we have*

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k g_k(x). \tag{4.1}$$

For any odd prime p and integer x, we have

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(x)\equiv\sum_{k=0}^{p-1}g_k(x) \pmod{p^2}. \tag{4.2}$$

Proof. Let n be any positive integer. In view of (1.15),

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)A_m(x)=&\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}(-1)^{m-k}g_k(x)\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k g_k(x)\sum_{m=k}^{n-1}(-1)^m(2m+1)\binom{m+k}{2k}\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k g_k(x)(-1)^{n-1}(n-k)\binom{n+k}{2k}\\ =&(-1)^{n-1}n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k g_k(x).\end{aligned}$$

This proves (4.1).

Now let p be an odd prime and let $x\in\mathbb{Z}$. As

$$\binom{p-1}{k}\binom{p+k}{k}=\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\equiv(-1)^k \pmod{p^2}$$

for every $k=0,\ldots,p-1$, (4.2) follows from (4.1) with $n=p$. □

Lemma 4.2. *Let $p>3$ be a prime. Then*

$$g_{p-1}\equiv\left(\frac{p}{3}\right)(1+2p\,q_p(3)) \pmod{p^2}. \tag{4.3}$$

Proof. For $k=0,\ldots,p-1$, clearly

$$\binom{p-1}{k}^2=\prod_{0<j\leqslant k}\left(1-\frac{p}{j}\right)^2\equiv\prod_{0<j\leqslant k}\left(1-\frac{2p}{j}\right)=(-1)^k\binom{2p-1}{k} \pmod{p^2}.$$

Thus, with the help of [S12b] we obtain

$$g_{p-1}\equiv\sum_{k=0}^{p-1}\binom{2p-1}{k}(-1)^k\binom{2k}{k}\equiv\left(\frac{p}{3}\right)(2\times 3^{p-1}-1) \pmod{p^2}.$$

and hence (4.3) holds. □

Lemma 4.3. *For any odd prime p, we have*

$$p\sum_{k=0}^{p-1}\frac{(-3)^k}{2k+1}\equiv\left(\frac{p}{3}\right)\pmod{p^2}.\tag{4.4}$$

Proof. Clearly (4.4) holds for $p=3$. Below we assume $p>3$. Observe that

$$\begin{aligned}\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-3)^k}{2k+1}=&\sum_{k=1}^{(p-1)/2}\left(\frac{(-3)^{(p-1)/2-k}}{2((p-1)/2-k)+1}+\frac{(-3)^{(p-1)/2+k}}{2((p-1)/2+k)+1}\right)\\ \equiv&\left(\frac{-3}{p}\right)\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{(-3)^k}{k}-\frac{1}{3}\cdot\frac{(-3)^{p-k}}{p-k}\right)\\ =&\frac{1}{2}\left(\frac{p}{3}\right)\left(\frac{4}{3}\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k}-\frac{1}{3}\sum_{k=1}^{p-1}\frac{(-3)^k}{k}\right)\\ =&-2\left(\frac{p}{3}\right)\sum_{k=1}^{(p-1)/2}\frac{(-3)^{k-1}}{k}+\frac{1}{2}\left(\frac{p}{3}\right)\sum_{k=1}^{p-1}\frac{(-3)^{k-1}}{k}\pmod{p^2}.\end{aligned}$$

Since

$$\frac{1}{p}\binom{p}{k}=\frac{1}{k}\binom{p-1}{k-1}\equiv\frac{(-1)^{k-1}}{k}\pmod{p}\quad\text{for }k=1,\ldots,p-1,$$

we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-3)^{k-1}}{k}\equiv&\frac{1}{3p}\sum_{k=1}^{p-1}\binom{p}{k}3^k=\frac{4^p-1-3^p}{3p}=4(2^{p-1}+1)\frac{2^{p-1}-1}{3p}-\frac{3^{p-1}-1}{p}\\ \equiv&\frac{8}{3}q_p(2)-q_p(3)\pmod{p}.\end{aligned}$$

Note also that

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-3)^{k-1}}{k}=&\sum_{k=1}^{(p-1)/2}\int_0^1(-3x)^{k-1}dx=\int_0^1\frac{1-(-3x)^{(p-1)/2}}{1+3x}dx\\ =&\int_0^1\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}(-1-3x)^{k-1}dx\\ =&\sum_{k=1}^{p-1}\binom{(p-1)/2}{k}\frac{(-1-3x)^k}{-3k}\bigg|_{x=0}^1\\ \equiv&\sum_{k=1}^{p-1}\binom{-1/2}{k}\frac{(-1)^k-(-4)^k}{3k}=\frac{1}{3}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}-\frac{1}{3}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\\ \equiv&\frac{2}{3}q_p(2)\pmod{p}\end{aligned}$$

since

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv 2q_p(2) \pmod p\quad\text{and}\quad \sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod{p^2}$$

by [ST1, (1.12) and (1.20)]. Thus, in view of the above, we get

$$\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-3)^k}{2k+1}\equiv -2\left(\frac p3\right)\frac23 q_p(2)+\frac12\left(\frac p3\right)\left(\frac83 q_p(2)-q_p(3)\right)=-\frac{q_p(3)}{2}\pmod p.$$

It follows that

$$\begin{aligned}p\sum_{k=0}^{p-1}\frac{(-3)^k}{2k+1}\equiv&(-3)^{(p-1)/2}-\frac{3^{p-1}-1}{2}\\ =&(-3)^{(p-1)/2}-\frac{(-3)^{(p-1)/2}+(\frac{-3}{p})}{2}\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)\\ \equiv&(-3)^{(p-1)/2}-\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)=\left(\frac p3\right)\pmod{p^2}.\end{aligned}$$

We are done. □

Proof of Theorem 1.2. (i) By [S11c, (1.5)],

$$\begin{aligned}&\frac1p\sum_{k=0}^{p-1}(2k+1)A_k(x)\\ =&\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}\binom{p+k}{2k+1}\binom{2k}{k}x^k\\ =&\sum_{k=0}^{p-1}\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\times\frac{p}{(2k+1)!}\prod_{0<j\leqslant k}(p^2-j^2)\times\binom{2k}{k}x^k\\ \equiv&\sum_{k=0}^{p-1}\frac{p(k!)^2}{(2k+1)!}\binom{2k}{k}x^k=p\sum_{k=0}^{p-1}\frac{x^k}{2k+1}\pmod{p^2}.\end{aligned}$$

Combining this with (4.2) we immediately get (1.14).

Since

$$p\sum_{k=0}^{p-1}\frac{1}{2k+1}=1+p\sum_{k=0}^{(p-3)/2}\left(\frac{1}{2k+1}+\frac{1}{2(p-1-k)+1}\right)\equiv 1=g_0\pmod{p^2},$$

(1.12) in the case $x = 1$ yields (1.15). As

$$\begin{aligned} p\sum_{k=0}^{p-1}\frac{(-1)^k}{2k+1} =&(-1)^{(p-1)/2}+p\sum_{k=0}^{(p-3)/2}\left(\frac{(-1)^k}{2k+1}+\frac{(-1)^{p-1-k}}{2(p-1-k)+1}\right)\\ \equiv&\left(\frac{-1}{p}\right)\pmod{p^2}, \end{aligned}$$

(1.16) follows from (1.14) with $x = -1$. Combining (1.14) with (4.4) we obtain (1.17).

(ii) Note that

$$\begin{aligned} \sum_{l=1}^{p-1}\frac{g_l(x)}{l} =&\sum_{l=1}^{p-1}\frac{1}{l}\sum_{k=0}^{l}\binom{l}{k}f_k(x)=H_{p-1}+\sum_{l=1}^{p-1}\sum_{k=1}^{l}\frac{f_k}{l}\binom{l}{k}\\ \equiv&\sum_{k=1}^{p-1}\frac{f_k(x)}{k}\sum_{l=k}^{p-1}\binom{l-1}{k-1}=\sum_{k=1}^{p-1}\frac{f_k(x)}{k}\binom{p-1}{k}\\ \equiv&\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k(x)(1-pH_k)\pmod{p^2}. \end{aligned}$$

In view of (2.6), this implies (1.18).

Clearly,

$$\begin{aligned} \sum_{n=0}^{p-1}(-1)^n(2n+1)A_n =&\sum_{n=0}^{p-1}(2n+1)\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}(-1)^k g_k\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k g_k\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k g_k\frac{p(p-k)}{k+1}\binom{p+k}{2k}\\ =&p^2\sum_{k=0}^{p-1}\frac{(-1)^k g_k}{k+1}\binom{p-1}{k}\binom{p+k}{k}\\ =&p\,g_{p-1}\binom{2p-1}{p-1}+p^2\sum_{k=0}^{p-2}\frac{g_k}{k+1}\prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right)\\ \equiv&p\,g_{p-1}+p^2\sum_{k=0}^{p-2}\frac{g_k}{k+1}\pmod{p^4}. \end{aligned}$$

Combining this with (3.11) and (4.3), we obtain

$$p\left(\frac{p}{3}\right)\equiv p\left(\frac{p}{3}\right)(1+2p\,q_p(3))+p^2\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\pmod{p^3}$$

and hence (1.19) follows.

(1.20) follows from a combination of (1.15) and (1.22) in the case $n = p$. If we let u_n denote the left-hand side or the right-hand side of (1.22), then by applying the Zeilgerber algorithm (cf. [PWZ]) via `Mathematica` (version 7) we get the recurrence relation

$$\begin{aligned}&(n+2)(n+3)^2(2n+3)u_{n+3}\\=&(n+2)(22n^3+121n^2+211n+120)u_{n+2}\\&-(n+1)(38n^3+171n^2+229n+102)u_{n+1}+9n^2(n+1)(2n+5)u_n\end{aligned}$$

for $n = 1, 2, 3, \ldots$. Thus (1.22) can be proved by induction.

(iii) Finally we show (1.21). Observe that

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{g_l(x)}{l^2}=&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{l=k}^{p-1}\binom{l-1}{k-1}^2\\=&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}^2=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{-k}{j}^2\\\equiv&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2\pmod p.\end{aligned}$$

For any $k = 1, \ldots, p-1$, we have

$$\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2=\sum_{j=0}^{p-k}\binom{p-k}{j}\binom{p-k}{p-k-j}-1=\binom{2(p-k)}{p-k}-1$$

by the Chu-Vandermonde identity. Thus

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k^2}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\left(\binom{2(p-k)}{p-k}-1\right)\equiv-\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\pmod p$$

(Note that $\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv 0 \pmod p$ for $k = 1, \ldots, p-1$.) It is known that if $p > 5$ then

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv 0\pmod p$$

(cf. [T]). So (1.21) is valid.

In view of the above, we have completed the proof of Theorem 1.2. □

5. Open conjectural congruences

In this section we include various related conjectural congruences, some of which are refinements of our results in earlier sections.

Conjecture 5.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{f_k}{8^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},\qquad \sum_{k=1}^{p-1}\frac{f_k}{k8^k}\equiv-\frac{3}{2}H_{(p-1)/2}\pmod{p^2},$$

and

$$\sum_{n=0}^{p-1}(-1)^n\sum_{k=0}^{n}\binom{n}{k}^3(-8)^k\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

Also,

$$\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\equiv-\left(\frac{p}{3}\right)\frac{9^{p-1}-1}{p}\pmod{p^2}\ \textit{and}\ \sum_{k=0}^{p-1}\frac{g_k}{9^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

If $p > 5$, then

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k}\equiv 0\pmod{p^2}.$$

Conjecture 5.2. *For any positive integer n, we have*

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^kf_k\in\mathbb{Z},\quad \frac{1}{n^2}\sum_{k=0}^{n-1}(4k+1)g_k9^{n-1-k}\in\mathbb{Z},$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x)\in\mathbb{Z}[x].$$

If n is a power of two, then

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(3k+1)f_k(x)8^{n-1-k}\in\mathbb{Z}[x]\ \textit{and}\ \frac{1}{n}\sum_{k=0}^{n-1}(4k+1)g_k(x)9^{n-1-k}\in\mathbb{Z}[x].$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^kf_k\equiv 2p^2(2^p-1)^2\pmod{p^5},$$

$$\sum_{k=0}^{p-1}(4k+1)\frac{g_k}{9^k}\equiv\frac{p^2}{2}\left(3-\left(\frac{p}{3}\right)\right)-3p^3q_p(3)\pmod{p^4},$$

and

$$\sum_{k=0}^{p-1}(4k+3)g_k(x)\equiv p\pmod{p^2}\quad\textit{for any integer } x\not\equiv 1\pmod{p}.$$

Conjecture 5.3. (i) *For any integer $n > 1$, we have*

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{(n-1)n^2},$$
$$\sum_{k=0}^{n-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv 0 \pmod{4(n-1)n^3},$$
$$\sum_{k=0}^{n-1}(12k^3+34k^2+30k+9)g_k \equiv 0 \pmod{3n^3}.$$

(ii) *For each odd prime p we have*

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1)-16p^3q_p(2) \pmod{p^4},$$
$$\sum_{k=0}^{p-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv -4p^3 \pmod{p^4},$$
$$\sum_{k=0}^{p-1}(12k^3+34k^2+30k+9)g_k \equiv \frac{3p^3}{2}\left(1+3\left(\frac{p}{3}\right)\right) \pmod{p^4}.$$

For a 3-adic number x we let $\nu_3(x)$ denote the 3-adic valuation of x.

Conjecture 5.4. *Let n be any positive integer. Then*

$$\nu_3\left(\sum_{k=0}^{n-1}(-1)^k k f_k\right) \geqslant 2\nu_3(n),$$

and

$$\nu_3\left(\sum_{k=0}^{n-1}(-1)^k f_k^{(r)}\right) \geqslant 2\nu_3(n) \quad \text{if } r \equiv 2,3 \pmod 6.$$

We also have

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k\right) = 3\nu_3(n) \leqslant \nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right).$$

If n is a positive multiple of 3, then

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right) = 3\nu_3(n)+2.$$

Conjecture 5.5. (i) *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}(6k^3+9k^2+5k+1)(-1)^kA_k\equiv 0\pmod{n^3},$$

$$\sum_{k=0}^{n-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k\equiv 0\pmod{n^4}.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(6k^3+9k^2+5k+1)A_k\equiv p^3+2p^4H_{p-1}-\frac{2}{5}p^8B_{p-5}\pmod{p^9}.$$

If $p>5$, then

$$\sum_{k=0}^{p-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k$$
$$\equiv -2p^4+3p^5+(6p-8)p^5H_{p-1}-\frac{12}{5}p^9B_{p-5}\pmod{p^{10}},$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers.

References

[AAR] G. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$. Journees arithmétiques de Luminy*, Astérisque **61** (1979), 11–13.

[B] P. Barrucand, *A combinatorial identity, problem 75-4*, SIAM Review **17** (1975), 168.

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[C] D. Callan, *A combinatorial interpretation for an identity of Barrucand*, J. Integer Seq. **11** (2008), Article 08.3.4, 3pp.

[G] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972.

[GZ] V. J. W. Guo and J. Zeng, *Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials*, preprint, `http://arxiv.org/abs/1101.0983`.

[JV] F. Javis and H. A. Verrill, *Supercongruences for the Catalan-Larcombe-French numbers*, Ramanujan J. **22** (2010), 171–186.

[L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. **39** (1938), 350–360.

[O] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, Amer. Math. Soc., Providence, R.I., 2003.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, *A = B*, A K Peters, Wellesley, 1996.

[Po] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of* $\zeta(3)$, Math. Intelligencer **1** (1978/79), 195–203.

[Sl] N. J. A. Sloane, Sequence A000172 in OEIS (On-Line Encyclopedia of Integer Sequences), `http://oeis.org/A000172`.

[St92] V. Strehl, *Recurrences and Legendre transform*, Sém. Lothar. Combin. **29** (1992), 1-22.

[St94] V. Strehl, *Binomial identities–combinatorial and algorithmic aspects*, Discrete Math. **136** (1994), 309–346.

[S11a] Z. W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (1011), in press.

[S11b] Z. W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2012), 2509-2535.

[S11c] Z. W. Sun, *On sums of Apéry polynomials and related congruences*, submitted, arXiv:1101.1946. `http://arxiv.org/abs/1101.1946`.

[S12a] Z. W. Sun, *Arithmetic theory of harmonic numbers*, Proc. Amer. Math. Soc. **140** (2012), 415–428.

[S12b] Z. W. Sun, *On sums of binomial coefficients modulo* p^2, Colloq. Math., revised. `http://arxiv.org/abs/0910.5667`.

[ST1] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.

[ST2] Z. W. Sun and R. Tauraso, *On some new congruences for binomial coefficients*, Int. J. Number Theory **7** (2011), 645–662.

[T] R. Tauraso, *More congruences for central binomial congruences*, J. Number Theory **130** (2010), 2639–2649.

[W] J. Wolstenholme, *On certain properties of prime numbers*, Quart. J. Appl. Math. **5** (1862), 35–39.

[Z] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, in: Groups and Symmetries: from Neolithic Scots to John McKay, CRM Proc. Lecture Notes 47, Amer. Math. Soc., Providence, RI, 2009, pp. 349–366.