arXiv:1112.1278v3 [math.CO] 9 Jun 2012

DRESSIANS, TROPICAL GRASSMANNIANS, AND THEIR RAYS

SVEN HERRMANN, MICHAEL JOSWIG, AND DAVID SPEYER

Abstract. The Dressian $\mathrm{Dr}(k,n)$ parametrizes all tropical linear spaces, and it carries a natural fan structure as a subfan of the secondary fan of the hypersimplex $\Delta(k,n)$. We explore the combinatorics of the rays of $\mathrm{Dr}(k,n)$, that is, the most degenerate tropical planes, for arbitrary k and n. This is related to a new rigidity concept for configurations of $n-k$ points in the tropical $(k-1)$-torus. Additional conditions are given for $k=3$. On the way, we compute the entire fan $\mathrm{Dr}(3,8)$.

1. Introduction

The classical Grassmannian parametrizes all linear spaces over a fixed field. Its tropicalization, the *tropical Grassmannian* [22], parametrizes those tropical linear spaces which arise as tropicalizations of ordinary linear spaces over a field of Puiseux series. Studying the tropical Grassmannians and related concepts is motivated by questions in algebraic geometry, for example see [10], as well as by applications in algorithmic biology, for example see [18]. Here we are aiming at exploring the tropical Grassmannian $\mathrm{Gr}(k,n)$ from the combinatorial point of view. To this end, we study an outer approximation, the *Dressian* $\mathrm{Dr}(k,n)$, which is the polyhedral fan of those regular subdivisions of the hypersimplex $\Delta(k,n)$ which have the property that each cell is a matroid polytope. In this manner, $\mathrm{Dr}(k,n)$ is a subfan of the secondary fan of $\Delta(k,n)$. Both fans have a non-trivial n-dimensional lineality space and we will consider them (and all other fans in this paper) and their cones always modulo this lineality, meaning that the smallest non-trivial cones (of dimension $n+1$) are considered to be one-dimensional and called rays. Alternatively, as a set, the Dressian can also be described as the tropical pre-variety which arises as the intersection of the tropical hypersurfaces defined by the 3-term Plücker relations; for other characterizations see Proposition 3 below. Therefore, Dressians were called *tropical pre-Grassmannians* in [22]. From the description via the 3-term Plücker relations, it follows that $\mathrm{Dr}(k,n)$ contains $\mathrm{Gr}(k,n)$ as a set. Asymptotically, for fixed k and growing n the Dressians $\mathrm{Dr}(k,n)$ become much larger than the tropical Grassmannians $\mathrm{Gr}(k,n)$ [12, Thm. 3.6].

Products of simplices naturally arise in this context as the vertex figures of hypersimplices, and the relationship between the secondary fan structures of these two polytopes has been studied previously, for example, by Kapranov [16, Sec. 1.4]. Develin and Sturmfels [5, Thm. 1] showed that the regular subdivisions of products of simplices are dual to tropical convex hulls of finitely many points. Our first main result is Corollary 14:

Theorem. *For $n>k>1$ there is a piecewise-linear embedding τ of the secondary fan of the product of simplices $\Delta_{k-1}\times\Delta_{n-k-1}$ into the Dressian $\mathrm{Dr}(k,n)$. The image of τ is contained in the tropical Grassmannian $\mathrm{Gr}(k,n)$ as a set.*

The secondary fan of $\Delta_{k-1}\times\Delta_{n-k-1}$ has the same dimension $nk-n-k^2+1$ as $\mathrm{Gr}(k,n)$ (modulo lineality), and τ is a homeomorphism onto its image. For each of the $\binom{n}{k}$ vertices

Date: 12th June 2012.

The first author was supported by a fellowship within the Postdoc-Program of the German Academic Exchange Service (DAAD). The second author was supported by DFG Priority Program 1489 "Experimental Methods in Algebra, Geometry and Number Theory".

of $\Delta(k,n)$, we get an inner approximation of the tropical Grassmannians in this manner. Via the same approach we also relate non-regular subdivisions of products of simplices to non-regular matroid subdivisions of hypersimplices. After making a preprint version of our result available we learned that Rincón independently proved a similar statement [19].

The goal of the present paper is to combinatorially describe the Dressians as well as possible. Since, in a way, the Dressians encode all of matroid theory this is quite an endeavor. Our expectations must therefore be modest. So we focus on the rays of $\mathrm{Dr}(k,n)$, that is, on those tropical linear spaces corresponding to matroid decompositions of $\Delta(k,n)$ which can only be coarsened in a trivial way. We call a configuration of k points in the tropical torus $\mathbb{T}^{n-1}$ *tropically rigid* if it does correspond to a ray of the secondary fan of $\Delta_{k-1} \times \Delta_{n-1}$. Via the aforementioned Corollary 14, tropically rigid point configurations give rise to rays of the Dressians which are also rays of the tropical Grassmannians. The known coarsest matroid subdivisions of the hypersimplices are the *splits* (with precisely two maximal cells) [13, Prop. 5.2] and the 3-*splits* (with precisely three maximal cells sharing a common codimension-2 cell) [11, Cor. 6.4]. It turns out that all coarsest matroid subdivisions of $\Delta(3,8)$ but one (up to symmetry) are induced by coarsest subdivisions of a vertex figure; see Figures 3, 4 and 8 below.

Our second main result is the explicit computation of the entire fan $\mathrm{Dr}(3,8)$ via `polymake` [7]. This computation, in particular, leads to our Theorem 31:

Theorem. *The Dressian* $\mathrm{Dr}(3,8)$ *is a non-pure non-simplicial nine-dimensional polyhedral fan with* f*-vector*

$$(1; 15{,}470;\ 642{,}677;\ 8{,}892{,}898;\ 57{,}394{,}505;\ 194{,}258{,}750;$$
$$353{,}149{,}650;\ 324{,}404{,}880;\ 117{,}594{,}645;\ 113{,}400)\,.$$

Modulo the natural $\mathrm{Sym}(8)$*-symmetry, the* f*-vector reads*

$$(1;\ 12;\ 155;\ 1{,}149;\ 5{,}013;\ 12{,}737;\ 18{,}802;\ 14{,}727;\ 4{,}788;\ 14)\,.$$

There are 116,962,265 *maximal cones,* 113,400 *of dimension* 9 *and* 116,848,865 *of dimension* 8. *Up to symmetry, there are* 4,748 *maximal cones,* 14 *of dimension* 9 *and* 4,734 *of dimension* 8.

We are indebted to an anonymous referee for several very helpful suggestions.

2. Tropical Polytopes and Matroid Subdivisions

A map π from $\binom{[n]}{k}$, the set of k-subsets of $[n] = \{1, 2, \ldots, n\}$, to the set $\mathbb{R}$ is called a *(finite) tropical Plücker vector* if the minimum of the three numbers

$$\pi(\rho ij) + \pi(\rho\ell m)\,, \quad \pi(\rho i\ell) + \pi(\rho jm)\,, \quad \pi(\rho im) + \pi(\rho j\ell) \tag{1}$$

is attained at least twice for each choice ρ of a $(k-2)$-subset of $[n]$ and pairwise distinct $i, j, \ell, m \in [n] \setminus \rho$; here we use the common shorthand notation ρij for the set $\rho \cup \{i, j\}$. Condition (1) is equivalent to requiring that π is contained in the tropical pre-variety which arises as the intersection of the tropical hypersurfaces of all 3-term Plücker relations [22]; this tropical pre-variety is the *Dressian* $\mathrm{Dr}(k,n)$. Throughout this paper we assume that $n > k > 0$.

A particularly interesting class of finite tropical Plücker vectors comes about as follows. Consider a matrix $V \in \mathbb{R}^{k\times(n-k)}$. The *augmented matrix* of V is the $k\times n$-matrix $\bar{V} = (E_k|V)$, where E_k is the *tropical identity matrix* of rank k (the $k\times k$-matrix with 0 on the diagonal and coefficients equal to ∞ otherwise), and $(A|B)$ denotes the block column matrix formed from the columns of A and B. Each k-element subset $\sigma \subseteq [n]$ specifies a $k\times k$-submatrix $\bar{V}_\sigma$ by selecting the columns of $\bar{V}$ whose indices are in σ. Now the map

$$\tau_V \,:\, \binom{[n]}{k} \to \mathbb{R}\,, \quad \sigma \mapsto \mathrm{tdet}(\bar{V}_\sigma) \tag{2}$$

is a finite tropical Plücker vector. Here

$$\mathrm{tdet}(A) \;=\; \min_{\omega\in\mathrm{Sym}(k)} a_{1,\omega(1)} + a_{2,\omega(2)} + \cdots + a_{k,\omega(k)} \tag{3}$$

denotes the *tropical determinant* of the matrix $A = (a_{ij})_{i,j} \in \mathbb{R}^{k\times k}$, and $\mathrm{Sym}(k)$ is the symmetric group naturally acting on the set $[k]$. We obtain a map τ which sends the $k\times(n-k)$-matrix V to the vector τ_V of length $\binom{n}{k}$. Conversely, for $\pi \in \mathbb{R}^{\binom{[n]}{k}}$ we define a $k\times(n-k)$-matrix $\Phi(\pi) = (\phi_{ij})_{i,j}$ by letting

$$\phi_{ij} \;=\; \pi(([k]\setminus\{i\})\cup\{j+k\})\,.$$

That is, the coefficients of $\Phi(\pi)$ are the tropical determinants of those $k\times k$-submatrices of $\bar{V}$ which are formed by the first k columns, except for the i-th which is replaced by the $(j+k)$-th column of $\bar{V}$, and which is the same as the j-th column of V.

Lemma 1. *For an arbitrary matrix $V \in \mathbb{R}^{k\times(n-k)}$ we have $\Phi(\tau_V) = V$. In particular, the map τ is injective, and the map Φ is surjective onto $\mathbb{R}^{k\times(n-k)}$. Moreover, Φ is linear and τ is piecewise-linear.*

Proof. For $\sigma = ([k]\setminus\{i\})\cup\{j+k\}$ the matrix $\bar{V}_\sigma$ has precisely one column with finite entries only, namely the last one, corresponding to column j of the matrix V. Each of the first $k-1$ columns of $\bar{V}_\sigma$ has precisely one finite entry, which equals zero in all cases. Among the first $k-1$ columns the only row with only ∞ coefficients is the i-th one. Hence the tropical determinant of $\bar{V}_\sigma$ equals the coefficient v_{ij} of the matrix V. This is precisely the first claim.

The tropical determinant is a piecewise-linear map; see (3). Hence τ is piecewise-linear, too. The map Φ is a linear projection. □

In general, the map τ is not surjective; for details see Example 10 below.

Remark 2. The map τ is not linear: For instance, consider $k = 2$, $n = 4$,

$$V \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \text{and} \quad W \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

Then $\tau_V = (0,1,0,0,1,0)$, $\tau_W = (0,0,1,1,0,0)$, and

$$\tau_{V+W} \;=\; (0,1,1,1,1,2) \;\neq\; (0,1,1,1,1,0) \;=\; \tau_V + \tau_W\,,$$

where the coordinates of $\mathbb{R}^{\binom{4}{2}}$ are (lexicographically) labeled 12, 13, 14, 23, 24, 34.

The map τ_V can be read as a height function on the hypersimplex

$$\Delta(k,n) \;:=\; \mathrm{conv}\left\{ e_\sigma := \sum_{i\in\sigma} e_i \in \mathbb{R}^n \;\middle|\; \sigma \in \binom{[n]}{k} \right\}$$

via mapping the vertex e_σ to $\tau_V(\sigma)$. Similarly, the matrix V has a natural interpretation as a height function on the vertices of $\Delta_{k-1}\times\Delta_{n-k-1}$.

A *subpolytope* of a polytope with vertex set X is the convex hull of a subset of X. A subpolytope of $\Delta(k,n)$ whose edges are parallel to edges of $\Delta(k,n)$, that is, differences of standard basis vectors $e_i - e_j$, is a *(k,n)-matroid polytope*. By a result of Gel′fand, Goresky, MacPherson and Serganova [8, Thm. 4.1] the vertices of a (k,n)-matroid polytope correspond to the bases of a matroid of rank at most k on n elements; hence the name *matroid polytope*. A *(k,n)-matroid subdivision* is a polytopal subdivision (of $\Delta(k,n)$) such that each cell is a (k,n)-matroid polytope. A *regular* (matroid) subdivision is induced by a height function; see [3, Sec. 2.2.3] for details about regular subdivisions. The following characterization is crucial.

Proposition 3 (Kapranov [16] and [21, Prop. 2.2]). *Let Σ be a regular subdivision of $\Delta(k,n)$ induced by the lifting function π. Then the following are equivalent.*

(i) π *is a finite tropical Plücker vector,*
(ii) Σ *is a matroid subdivision,*
(iii) *the* 1*-cells of* Σ *are precisely the edges of* $\Delta(k,n)$.

For $\sigma, \sigma' \in \binom{[n]}{k}$ the vertices e_σ and $e_{\sigma'}$ are neighbors in the vertex-edge graph of $\Delta(k,n)$ if and only if the symmetric difference of σ and σ' consists of two elements. This means that the neighbors of e_σ are contained in the set

$$e_\sigma^\perp \ := \ \left\{ x \in \mathbb{R}^n \ \middle| \ \sum_{i=1}^n x_i = k \text{ and } \sum_{i \in \sigma} x_i = k-1 \right\} ,$$

which forms a hyperplane in the affine span of the hypersimplex. The vertex figure of each vertex in $\Delta(k,n)$ is isomorphic to the product of simplices $\Delta_{k-1} \times \Delta_{n-k-1}$. The map

$$\iota \, : \, \{e_i + e_j \mid i \in [k], j \in [n] \setminus [k]\} \to e_{[k]}^\perp \, , \ e_i + e_j \mapsto e_{[k] \setminus \{i\} \cup \{j\}}$$

describes a bijection from the vertices of $\Delta_{k-1} \times \Delta_{n-k-1}$ to the neighbors of the vertex $e_{[k]}$ of $\Delta(k,n)$. This naturally induces a map form the set of all subdivisions of $\Delta_{k-1} \times \Delta_{n-k-1}$ to the set of all subdivisions of the vertex figure $\Delta(k,n) \cap e_{[k]}^\perp$, which we also denote by ι. For any not necessarily regular subdivision Σ of $\Delta(k,n)$ we let $\Sigma \cap e_{[k]}^\perp$ denote the polytopal complex arising from intersecting each cell of Σ with the affine subspace $e_{[k]}^\perp$.

The following is the content of Corollary 1.4.14 in [16]; here we give an elementary proof.

Proposition 4. *Let* $V \in \mathbb{R}^{k \times (n-k)}$ *be a matrix,* Γ *the regular subdivision of* $\Delta_{k-1} \times \Delta_{n-k-1}$ *induced by* V*, and* Σ *the regular matroid subdivision of* $\Delta(k,n)$ *induced by* τ_V.

Then the polytopal complex $\Sigma \cap e_{[k]}^\perp$ *coincides with* $\iota(\Gamma)$.

Proof. The neighbors of the vertex $e_{[k]}$ lie in the common hyperplane $e_{[k]}^\perp$, and therefore the set $\Delta(k,n) \cap e_{[k]}^\perp$ serves as a model for the vertex figure of $e_{[k]}$. Since Σ is a matroid subdivision, Proposition 3 (iii) implies that each edge of Σ intersects the hyperplane $e_{[k]}^\perp$ in a vertex. This says that $\Sigma \cap e_{[k]}$ is a (regular) subdivision of the vertex figure $\Delta(k,n) \cap e_{[k]}^\perp$ (without any new vertices). This way the claim follows from Lemma 1. $\square$

The inclusion relation among the cells turns a pure polytopal complex Σ of dimension m into a partially ordered set. The dimension of a cell serves as a rank function on the poset: The poset elements of rank $\ell + 1$ are the ℓ-dimensional cells of Σ. The *tight-span* $\mathcal{T}(\Sigma)$ is the partially ordered set obtained by restricting the previously mentioned partial order to interior cells, and dualizing it. So the elements of the tight-span rank $\ell + 1$ are the interior cells of dimension $m - \ell$, and the partial ordering is given by reverse inclusion. If Σ is a polytopal subdivision of a polytope, the tight-span is isomorphic to the containment poset of a contractible cell complex.

If, additionally, this subdivision is regular, then this cell complex can be realized as a polytopal complex. Obviously, if Σ is a subdivision of a polytope P and Σ' a refinement of Σ, then $\mathcal{T}(\Sigma)$ is a subcomplex of $\mathcal{T}(\Sigma')$. Furthermore, if P' is a subpolytope of P, the tight-span of the subdivision $\Sigma|_{P'} := \{S \cap P' \mid S \in \Sigma\}$ is a subcomplex of $T(\Sigma)$. In general, $\Sigma|_{P'}$ may have vertices which do not occur in Σ; see Figure 1 for an example.

Proposition 5. *Let* P *be a polytope,* P' *a subpolytope of* P *and* Σ *a subdivision of* P *such that* $T(\Sigma)$ *and* $T(\Sigma|_{P'})$ *are isomorphic.*

(i) *If* Σ' *is a subdivision of* P *coarsening* Σ*, then* $T(\Sigma')$ *and* $T(\Sigma'|_{P'})$ *are isomorphic.*
(ii) *If* $\Sigma|_{P'}$ *is a coarsest subdivision of* P'*, then* Σ *is a coarsest subdivision of* P.

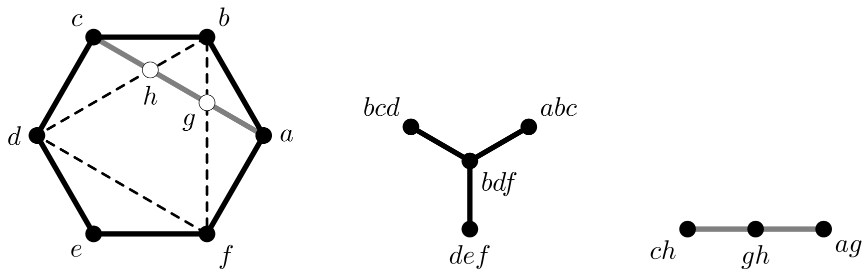

FIGURE 1. Subdivision Σ of the hexagon $P = abcbdef$ (left). The induced subdivision $\Sigma|_{P'}$ of the subpolytope $P' = ac$ has two new vertices, labeled g and h. Tight span of Σ (center) and tight span of $\Sigma|_{P'}$ (right)

Proof. We first prove (i). Suppose $T(\Sigma')$ and $T(\Sigma'|_{P'})$ are not isomorphic, that is, there exists a full-dimensional cell $F' \in \Sigma'$ such that $F' \cap P'$ is not full-dimensional. However, since $T(\Sigma)$ and $T(\Sigma|_{P'})$ are isomorphic, for a full-dimensional cell $F \subset F'$ of Σ, we have that $F \cap P' \subset F' \cap P'$ is full-dimensional, a contradiction.

Now suppose that there exists a non-trivial coarsening Σ' of Σ. By (i), $T(\Sigma')$ and $T(\Sigma'|_{P'})$ are isomorphic. This shows that $\Sigma'|_{P'}$ is a non-trivial coarsening of $\Sigma|_{P'}$, and this establishes (ii). $\square$

The following general uniqueness result applies to our main result below.

Proposition 6. *Let P be a polytope and v a vertex of P. Further, let N be the set of vertices neighboring v, and let $Q = \operatorname{conv} N$. Suppose that $\dim Q = \dim P - 1$. Then for each subdivision Γ of Q there is at most one subdivision Σ of P such that $\Sigma \cap Q = \Gamma$ and every maximal cell of Σ contains v.*

Proof. For each cell γ of Γ, let $C(\gamma)$ be the cone from v over γ. Let $D(\gamma) = C(\gamma) \cap P$. Then P is stratified into the $D(\gamma)$, but this may not be a polyhedral decomposition because $D(\gamma)$ may have vertices which are not vertices of P. We will show that, if there is any subdivision Σ with the required properties, then $\Sigma = \{D(\gamma) \mid \gamma \in \Gamma\}$.

First, let σ be any maximal cell of Σ. Then $\sigma \cap Q$, by dimensionality, is a maximal cell of Γ; call it $\gamma(\sigma)$. Since σ contains v, we have that σ contains $\operatorname{conv}(v, \gamma(\sigma))$. This means that there cannot be a second maximal cell $\sigma' \neq \sigma$ with $\gamma(\sigma') = \gamma(\sigma)$, as then they would overlap in the full dimensional set $\operatorname{conv}(v, \gamma(\sigma))$.

Also, σ has a vertex at v, and has no other vertices lying on the v-side of Q (since there are no such vertices in P). So σ is contained in $C(\gamma(\sigma))$ and is thus contained in $C(\gamma(\sigma)) \cap P = D(\gamma(\sigma))$.

Thus, we have shown that there is an injection from facets of Σ, to facets of Γ, such that $\sigma \subseteq D(\gamma(\sigma))$. But, since Σ is supposed to be a decomposition of all of P, we must have equality for every σ. As promised, we have shown that $\{D(\gamma) \mid \gamma \in \Gamma\}$ is the only decomposition with the properties required. $\square$

The subsequent result is of key relevance. Rincón independently proved a similar result for regular subdivisions [19].

Theorem 7. *Let Γ be a not necessarily regular subdivision of $\Delta_{k-1} \times \Delta_{n-k-1}$. Then there exists a subdivision Σ of $\Delta(k,n)$ such that:*

(i) $\Sigma \cap e_{[k]}^{\perp} = \iota(\Gamma)$.
(ii) *Each maximal cell of Σ contains the vertex $e_{[k]}$.*
(iii) *The tight-spans $\mathcal{T}(\Gamma)$ and $\mathcal{T}(\Sigma)$ are isomorphic.*

Furthermore, if Γ is induced by a lifting function $V \in \mathbb{R}^{k \times (n-k)}$ and thus regular, then Σ is also regular and induced by τ_V. If, however, Γ is not regular, then Σ is not regular either.

In particular, each tight-span of a (regular) subdivision of $\Delta_{k-1} \times \Delta_{n-k-1}$ also arises as the tight-span of a (regular) matroid subdivision of $\Delta(k,n)$. Notice that Theorem 7 also yields an independent proof of Proposition 4. It follows from Proposition 6 that Σ is uniquely determined by Γ.

Proof. For S a subset of a real vector space, write $\operatorname{pos} S$ for the positive real span of S.

Consider the polytope $T := \operatorname{conv}\{e_\sigma - e_{[k]} \mid \sigma \in \binom{[n]}{k}\}$ obtained by translating $\Delta(k,n)$. We equip the vertex figure U at the origin with the given subdivision Γ of $\Delta_{k-1} \times \Delta_{n-k-1}$ and define

$$\Sigma \;:=\; \{\operatorname{pos}\gamma \cap T \;|\; \gamma \in \Gamma\} \;. \tag{4}$$

Notice that Σ is constructed as in the proof of Proposition 6.

We will show that Σ is a valid subdivision of T. By construction and since Γ is a valid subdivision of U, it suffices to show that all zero-dimensional faces of Σ are actually vertices of T. So let $v \in \Sigma$ be zero-dimensional. Then v is the intersection of linear hyperplanes spanned by vertices of U and a face F of T. The vertices of U are all points of the form $e_i - e_j$ with $j \in [k]$ and $i \in [n] \setminus [k]$. Taking into account that U is contained in the hyperplane $\sum_{i=1}^n x_i = 0$, this implies that a hyperplane spanned by these vertices can be described by an equation of the form $\sum_{i \in A \cup B} x_i = 0$ for some non-empty $A \subset [k]$ and $B \subset [n] \setminus [k]$. On the other hand, F (as a face of T) is the intersection of hyperplanes of the form $x_i \in \{0,1,-1\}$. The only possibility for an intersection of these two types of hyperplanes to be zero-dimensional is to be a vertex of U. This shows (i).

It is immediate from the construction that each maximal cell of Σ contains the origin. This establishes (ii). To show (iii) first observe that for any two cells C and D of Γ we have $(\operatorname{pos} C \cap T) \cap (\operatorname{pos} D \cap T) = (\operatorname{pos} C \cap D) \cap T$. Second, $\dim(\operatorname{pos} C \cap T) = \dim C + 1$, and hence maximal cells in Γ correspond to maximal cells in Σ. This yields (iii).

We now turn to the situation where Γ is regular and induced by the lifting function V. By construction, this implies that Σ then is induced by the lifting function κ with

$$\kappa(e_\sigma - e_{[k]}) \;=\; \min_w \sum_{i,j} w(i,j) \bar{V}_{i,j} \;, \tag{5}$$

where w ranges over all functions $[k] \times [k+1,n] \to \mathbb{R}_{\geq 0}$ with $\sum_{i,j} w(i,j)(e_j - e_i) = e_B - e_A$. Here we set $A = [k] \setminus ([k] \cap \sigma)$ and $B = [k+1,n] \cap \sigma$ for all $\sigma \in \binom{[n]}{k}$.

On the other hand, we have

$$\tau_V(\sigma) \;=\; \operatorname{tdet}(\bar{V}_\sigma) \;=\; \min_\alpha \sum_{i \in A} \bar{V}_{i,\alpha(i)} \;, \tag{6}$$

where the minimum ranges over all bijections $\alpha : A \to B$. Note that $\#A = \#B$ by construction.

The minimum in (5) is obtained by the minimal cost flow in the complete bipartite graph with vertex set $A \cup B$ and edge set $\{\{i,j\} \mid i \in A, j \in B\}$ with one unit of fluid coming in at each of the sources in A and going out at each of the sinks in B; where $\bar{V}_{ij}$ is the cost of flowing one unit from i to j.

The minimum in (6) is the minimal cost flow in the same graph but restricting the flow values to 0 and 1. Solutions to minimal cost flow problems with integral constraints are always integral, showing that $\kappa(e_\sigma - e_{[k]}) = \tau_V(\sigma)$; see, e.g., [20, §10.2].

It remains to consider the situation when Γ is not regular. Suppose Σ were regular with lifting function λ. Then by restricting λ to the vertices of $\Delta(k,n)$ which are neighbors to the origin, we would obtain a regular subdivision of the vertex figure U. By construction this would agree with Γ, a contradiction. □

Remark 8. Let P be a face of Γ. Let G be the bipartite graph with vertex set $[n]$ and with an edge (i,j) if $e_{[k]} - e_i + e_j$ is a vertex of F. Let Q be the corresponding face of Σ. Then the above proof shows that the matroid corresponding to Q is the principal transversal matroid of the graph G; see [2].

The symmetric group $\mathrm{Sym}(n)$ acts linearly on the Euclidean space $\mathbb{R}^n$ by permuting the coordinate directions. This induces a transitive action on the set of vertices of the hypersimplex $\Delta(k,n)$. The stabilizer of a vertex acts transitively on the set of neighbors of this vertex. This induces a vertex-transitive action on $\Delta_{k-1} \times \Delta_{n-k-1}$. On the level of lifting functions written in matrix form, this action permutes the rows and the columns. Throughout we identify a $k{\times}(n-k)$-matrix with its ordered sequence of $n-k$ points in $\mathbb{T}^{k-1}$. Permuting the columns corresponds to translating the points in the configuration, and permuting the rows corresponds to the induced action on the k coordinate directions of the tropical torus $\mathbb{T}^{k-1}$. Now two point configurations (of $n-k$ points each) in $\mathbb{T}^{k-1}$ are *equivalent* if they are in the same orbit of the semi-direct product $\mathbb{R}^k \rtimes \mathrm{Sym}(k)$, where the additive group of $\mathbb{R}^k$ acts by translations. In this sense, two $k{\times}(n-k)$-matrices are equivalent if they can be transformed into one another by the following operations:

(i) permuting the rows,
(ii) permuting the columns,
(iii) adding an arbitrary vector in $\mathbb{R}^k$ to all the columns,
(iv) adding a constant multiple of $\mathbb{1}$ to any column.

Notice that the roles of the rows and the columns in the above is symmetric: adding a constant multiple of $\mathbb{1} = (1,1,\ldots,1)$ to the i-th row is the same as adding the vector e_i to all columns, and adding the vector $v \in \mathbb{R}^{n-k}$ to all the rows is the same as adding $v_j\mathbb{1}$ to the j-th column for all $1 \leq j \leq n-k$. The action of $\mathrm{Sym}(n)$ on $\mathbb{R}^n$ also induces natural actions on the Grassmannian $\mathrm{Gr}(k,n)$ and on the Dressian $\mathrm{Dr}(k,n)$.

In view of the transitivity of the $\mathrm{Sym}(n)$ action on the vertices of $\Delta(k,n)$, Proposition 4 and Theorem 7 generalize to arbitrary vertices e_σ of $\Delta(k,n)$ instead of $e_{[k]}$. In fact, the entire construction in the beginning of this section can be generalized for an arbitrary k-element subset σ of $[n]$ to define functions τ_V^σ and Φ^σ having the same properties as $\tau_V = \tau_V^{[k]}$ and $\Phi = \Phi^{[k]}$.

Since there is no restriction on the matrix V we obtain the following result.

Corollary 9. *For each regular subdivision Γ of $\Delta_{k-1} \times \Delta_{n-k-1}$ and for each vertex v of $\Delta(k,n)$ there exists a regular matroid subdivision Σ of $\Delta(k,n)$ such that the subdivision of the vertex figure of v induced by Σ coincides with Γ.*

It is now an interesting question to ask which regular matroid subdivisions of $\Delta(k,n)$ are induced by tropical Plücker vectors τ_V as defined in (2). This question has the following complete answer for $k=2$.

Example 10. Let $k=2$. Then $\Delta_1 \times \Delta_{n-3}$ is a prism over a simplex, and the tight-span of any of its regular subdivisions is a path. Equivalently, the tropical convex hull of two distinct points p and q in $\mathbb{T}^{n-3}$ is a one-dimensional polytopal complex. The tropical line segment $\mathrm{tconv}(p,q)$ is the union of at most $n-3$ ordinary line segments. If p and q are generic, equality is attained, and $V := (p|q) \in \mathbb{R}^{2\times(n-2)}$ induces a (regular) triangulation of $\Delta_1 \times \Delta_{n-3}$. In this case the tight-span of the matroid decomposition induced by τ_V corresponds to a *caterpillar tree* with precisely $n-3$ interior edges; see [12, Fig. 8 (left)]. In the non-generic case some of these interior edges shrink to points. The construction of τ_V from p and q is a special case of the construction of a tropical linear space from a set of tropically collinear points due to Develin [4]. For more details see also Example 20 below.

Notice that all trivalent trees with (at most) five leaves are caterpillar trees. For $n=6$ there is precisely one combinatorial type of tree which is not a caterpillar tree: the *snowflake tree*; see [12, Fig. 8 (right)]. For $n>6$ there is a greater variety of non-caterpillar trees. These trees correspond to elements of $\mathrm{Gr}(2,n)$ but are not be obtained from tropical line segments via the map τ.

For our investigations further below it is also instructive to look at one more special case.

Example 11. Let $k = 3$ and $n = 6$. The tropical Grassmannian $\mathrm{Gr}(3,6)$ has been analyzed in detail in [22] (see also [12, Fig. 1]). We will adopt their notation. Observe that $\mathrm{Gr}(3,6)$ and the Dressian $\mathrm{Dr}(3,6)$ have the same support, but the fan structures differ. Up to symmetry, there are seven distinct combinatorial types of finest matroid subdivisions of $\Delta(3,6)$, that is, there are seven types of generic 2-planes in 5-space. Five of these types arise from configurations of three points in $\mathbb{T}^2$ via the lifting $V \mapsto \tau_V$; we list five 3×3-matrices along with the types of the induced generic 2-planes; see Figure 2:

$$\underset{\text{EEEG}}{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\text{EEFG}}{\begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\text{EEFF(b)}}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}} \quad \underset{\text{EFFG}}{\begin{pmatrix} 0 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\text{FFFGG}}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

The two remaining types EEEE and EEFF(a) do not occur in this way. Notice that the tight-spans of EEFF(a) and EEFF(b) only differ in their labellings; both "look like" tropical polytopes, but one arises via the lifting $V \mapsto \tau_V$, while the other one does not. Figure 2 shows the tropical complexes, which coincide with the respective tight spans of the induced matroid subdivisions. The latter are explicitly described in Table 1, and this list shows that e_{123} is a vertex of each maximal cell in each of these matroid decompositions. All occurring matroids are graphical, defined by one of the two graphs in [12, Fig. 7] to the left.

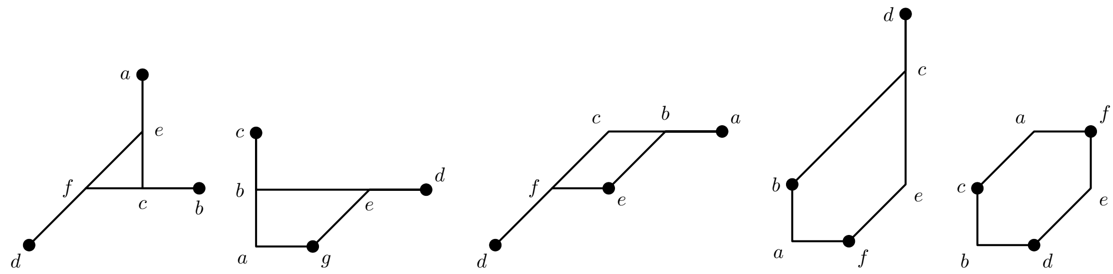

FIGURE 2. Tropical polytopes in $\mathbb{T}^2$ leading to generic tropical planes in $\mathbb{T}^5$. Pictures are drawn by taking the first two coordinates in the usual directions e_1, e_2 and the last coordinate in direction $-e_1 - e_2$. Point labels match the lists of matroids in Table 1

Proposition 12. *For every $V \in \mathbb{R}^{k \times (n-k)}$ the point τ_V is contained in the tropical Grassmannian $\mathrm{Gr}(k,n)$.*

Proof. Take any *lift* of $\bar{V}$ to a matrix $V^* \in K^{k \times n}$ with coefficients in a Puiseux series field K (see [21, Sec. 4] and [17] for details). Then the valuation map val of K takes V^*_{ij} to $\bar{V}_{ij}$ for all i and j. The first k columns of $\bar{V}$ form the $k \times k$-tropical identity matrix, and therefore the first k columns of V^* are linearly independent. We conclude that the column space $L(V^*)$ is a k-dimensional subspace of the vector space K^n. Without loss of generality we may assume that the lift to V^* is generic, so we have $\tau_V(\sigma) = \mathrm{val}(\mathfrak{p}(\sigma))$, where $\mathfrak{p} : \binom{[n]}{k} \to K$ are the classical Plücker coordinates of $L(V^*)$. Now [21, Prop. 4.2] implies that the tropicalization of $L(V^*)$ coincides with the tropical linear space defined by τ_V, that is, τ_V is a tropical Plücker vector. $\square$

Remark 13. This gives us a sufficient criterion to show that a given tropical Plücker vector $\pi \in \mathrm{Dr}(k,n)$ is actually contained in the tropical Grassmannian $\mathrm{Gr}(k,n)$: For each $\sigma \in \binom{[n]}{k}$ (that is, for each vertex of $\Delta(k,n)$) compute the matrix $\Phi^\sigma(\pi)$ and check if

EEEG:		
	a	123 125 134 135 136 145 156 235 345 356
	b	123 124 134 234 235 236 245 246 345 346
	c	123 124 134 136 146 235 236 245 246 345 346 356 456
	d	123 124 125 126 136 146 156 236 246 256
	e	123 124 125 134 136 145 146 156 235 245 345 356 456
	f	123 124 125 136 146 156 235 236 245 246 256 356 456
EEFG:		
	a	123 125 126 134 136 145 146 156 236 256 346 456
	b	123 125 134 136 145 156 235 236 256 345 346 356 456
	c	123 125 134 135 136 145 156 235 345 356
	d	123 124 134 234 235 236 245 246 345 346
	e	123 124 125 134 145 235 236 245 246 256 345 346 456
	f	123 124 125 126 134 145 146 236 246 256 346 456
EEFF(b):		
	a	123 124 134 234 235 236 245 246 345 346
	b	123 124 134 135 145 235 236 245 246 345 346 356 456
	c	123 124 134 135 136 145 146 236 246 346 356 456
	d	123 124 125 126 136 146 156 236 246 256
	e	123 124 125 135 145 235 236 245 246 256 356 456
	f	123 124 125 135 136 145 146 156 236 246 256 356 456
EFFG:		
	a	123 125 126 136 156 234 236 245 246 256 346 456
	b	123 125 136 156 234 235 236 245 256 346 356 456
	c	123 125 134 136 145 156 234 235 245 345 346 356 456
	d	123 125 134 135 136 145 156 235 345 356
	e	123 124 125 134 136 145 146 156 234 245 346 456
	f	123 124 125 126 136 146 156 234 245 246 346 456
FFFGG:		
	a	123 125 126 135 156 234 235 245 246 256 345 456
	b	123 126 135 136 156 234 236 246 345 346 356 456
	c	123 126 135 156 234 235 236 246 256 345 356 456
	d	123 126 134 135 136 146 156 234 246 345 346 456
	e	123 124 126 134 135 145 146 156 234 246 345 456
	f	123 124 125 126 135 145 156 234 245 246 345 456

TABLE 1. List of matroid bases per maximal cell for each type of generic tropical plane in $\mathbb{T}^5$ shown in Figure 2

$\tau^{\sigma}_{\Phi^{\sigma}(\pi)} = \pi$. If there is some vertex e_σ for which this is the case, Proposition 12 yields that $\pi \in \mathrm{Gr}(k,n)$. That this criterion is not necessary follows from the existence of the generic tropical planes of types EEEE and EEFF(a) from Example 11 which cannot be obtained via this construction. For EEEE this is easily seen, since the tight-span is a snowflake tree (that is, a tree with exactly one interior node) with six leaves. This does not correspond to a point configuration in $\mathbb{T}^2$.

Corollary 14. *The map τ induces a piecewise-linear embedding of the secondary fan of $\Delta_{k-1} \times \Delta_{n-k-1}$ into the Dressian $\mathrm{Dr}(k,n)$. The image of τ is a subset of the tropical Grassmannian $\mathrm{Gr}(k,n)$.*

Proof. It follows from Theorem 7 and, in particular, the construction (4) that τ induces a piecewise-linear embedding. The final claim is now a consequence of Proposition 12. $\square$

See also Example 20 below.

3. Tropically Rigid Point Configurations

Throughout the following let $V \in \mathbb{R}^{k\times(n-k)}$ be a $k \times (n-k)$-matrix with $n \geq 2k$. We will read (the columns of) V as a configuration of k labeled points in $\mathbb{T}^{n-k-1}$, possibly with repetitions. Associated with V is the regular subdivision Γ of $\Delta_{k-1} \times \Delta_{n-k-1}$ which is induced by lifting the vertex (e_i, e_j) to v_{ij}. The *secondary fan* $\mathfrak{S}$ of $\Delta_{k-1} \times \Delta_{n-k-1}$ is the polyhedral fan in $\mathbb{R}^{k\times(n-k)}$ which arises from grouping together those lifting functions which induce the same subdivision. The fan $\mathfrak{S}$ has a lineality space of dimension $n-1$. Therefore, by taking quotients we can view $\mathfrak{S}$ as a fan in $\mathbb{R}^{k(n-k)-n+1} = \mathbb{R}^{kn-n-k^2+1}$.

The (regular) subdivisions of $\Delta_{k-1} \times \Delta_{n-k-1}$ are partially ordered by refinement. The point configuration V is *generic* if V considered as a lifting function induces a triangulation, that is, a finest subdivision. At the other extreme we call V *tropically rigid* if it induces a coarsest (non-trivial) subdivision. The matrix V gives rise to a polyhedral subdivision of $\mathbb{T}^{k-1}$ according to type, and the bounded cells form the tropical polytope $\operatorname{tconv}(V)$; see Develin and Sturmfels [5, Sec. 3]. The bounded cells of the type decomposition are precisely the cells of the tight-span of Γ. The set $\operatorname{tconv}(V)$ endowed with its canonical cell decomposition by type is called the *tropical complex* of V. Its vertices are the *pseudo-vertices* of the point configuration V. They bijectively correspond to the maximal cells of Γ.

The purpose of the remainder of this section is to list many examples of tropically rigid point configurations since these will be used later to construct rays of the Dressians.

Example 15. Consider the point

$$p_\ell \;=\; (\underbrace{0,0,\dots,0}_{\ell},\underbrace{1,1,\dots,1}_{k-\ell})$$

in $\mathbb{T}^{k-1}$. The tropical line segment $\operatorname{tconv}(0, p_\ell)$ is the ordinary line segment from 0 to p_ℓ. The tropical complex has two vertices, 0 and p_ℓ, corresponding to the two maximal cells of the dual regular subdivision of $\Delta_{k-1} \times \Delta_1$. Subdivisions with precisely two maximal cells are called *splits*.

Example 16. The tropical complex Γ of the point configuration formed by the columns of the $k\times k$-matrix

$$V \;=\; \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$$

is a $(k-1)$-simplex. Its dual Γ is a coarsest subdivision of $\Delta_{k-1} \times \Delta_{n-k-1}$ (more precisely, a *k-split*) and hence the point configuration V is tropically rigid, see [11, Sec. 4]).

For a subdivision Γ of $\Delta_{k-1} \times \Delta_{n-k-1}$, one can give an explicit description of finitely many inequalities and equations describing the secondary cone of of all weight functions functions yielding this subdivision; see [3, Cor. 5.2.7]. So, given a matrix $V \in \mathbb{R}^{k\times(n-k)}$ we can decide if the subdivision Γ it describes is coarsest by solving a linear program to determine the dimension of the secondary cone of Γ.

Example 17. The columns of the matrix

$$V \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix},$$

form a configuration of five points in $\mathbb{T}^2$, see Figure 3 to the right. The origin $(0,0,0)$ is a pseudo-vertex, but it does not occur among the generators. The induced regular subdivision of $\Delta_2 \times \Delta_4$ is coarsest, and V is tropically rigid. The tropical Plücker vector τ_V induces a coarsest matroid subdivision of $\Delta(3, 8)$.

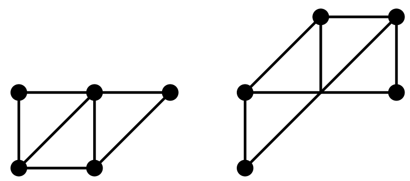

FIGURE 3. Two tropically rigid configurations of five points in $\mathbb{T}^2$ without multiple points

Duplicating points in a tropically rigid point configuration is a general way to create new tropically rigid point configurations from old ones, as the next result shows.

Proposition 18. *Let $V \in \mathbb{R}^{k\times(n-k)}$ be a tropically rigid point configuration, and w any column of V. Then the point configuration $V' := (V|w) \in \mathbb{R}^{k\times(n+1-k)}$ is also tropically rigid.*

Proof. Let Γ be the subdivision of $\Delta_{k-1} \times \Delta_{n-k-1}$ induced by V and Γ' the subdivision of $\Delta_{k-1} \times \Delta_{n-k}$ induced by V'. Suppose w is the i-th column of V. By looking at the lifted polytopes, it follows that the maximal cells of Γ' are exactly the polytopes

$$\operatorname{conv}\bigl(\{e_\ell + e_m \,|\, e_\ell + e_m \in C\} \cup \{e_\ell + e_{n+1} \,|\, e_\ell + e_i \in C\}\bigr)\,,$$

where C is a maximal cell of Γ. In particular, $\mathcal{T}(\Gamma)$ and $\mathcal{T}(\Gamma')$ are isomorphic. The equation $x_{n+1} = 0$ defines a facet F of $\Delta_{k-1} \times \Delta_{n-k}$ which is isomorphic to $\Delta_{k-1} \times \Delta_{n-k-1}$. The subdivision $\Gamma'|_F$ of F induced by Γ' is isomorphic to Γ, which is a coarsest subdivision by assumption. Hence the tight-spans of Γ and Γ' agree. Applying Proposition 5 (ii) to the subpolytope F of $\Delta_{k-1} \times \Delta_{n-k}$ shows that Γ' is a coarsest subdivision, too. This means that the point configuration V' is tropically rigid. $\square$

Figure 4 below shows tropically rigid configurations of five points in the plane which arise from configurations with fewer points by duplicating as described in Proposition 18.

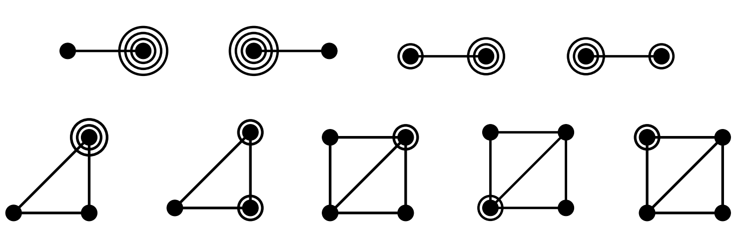

FIGURE 4. Tropically rigid configurations of five points in $\mathbb{T}^2$ with multiple points; multiple points circled

Remark 19. Up to equivalence, there are exactly eleven tropically rigid configurations of five points in $\mathbb{T}^2$. The only two configurations without multiple points are shown in Figure 3. In Figure 4 we depict the ones with at least one multiple point. Notice that all four splits in the upper row are pairwise not equivalent as point configurations in $\mathbb{T}^2$.

That this, indeed, is the complete list follows from the classification of the tight-spans of rays of $\mathrm{Dr}(3, n)$ in Example 30 below in connection with Theorem 7.

Example 20. Continuing our Example 10 here again we consider the case $k = 2$. All rays of $\mathrm{Dr}(2, n)$ correspond to splits of $\Delta(2, n)$, and these also give the only tropically rigid point configurations (without duplicates) in this case.

The $(n-3)$-dimensional permutahedron is a secondary polytope of $\Delta_1 \times \Delta_{n-3}$; see [3, §6.2.1]. That is, its normal fan is the secondary fan of that product of simplices. This secondary fan embeds into the Dressian $\mathrm{Dr}(2, n)$, which coincides with $\mathrm{Gr}(2, n)$, as described in Corollary 14. For $n = 5$ the Dressian $\mathrm{Dr}(2, 5)$ (as a spherical polytopal complex) is isomorphic to the Petersen graph (considered as a 1-dimensional spherical polytopal complex). The Dressian $\mathrm{Dr}(2, 5)$ is also the graph complement of the the vertex-edge graph of $\Delta(2, 5)$, and thus we can label its vertices with 2-element subsets of the set $\{1, 2, 3, 4, 5\}$. This comes about as each ray of the secondary fan of $\Delta(2, 5)$ is a vertex split; for instance, this follows from the classification of hypersimplex splits [13, §5]. The 2-dimensional permutahedron (as well as its dual) is a hexagon. The embedding with respect to the vertex e_{12} is shown in Figure 5.

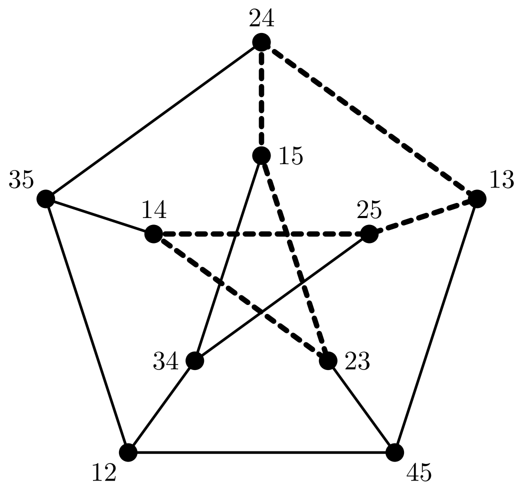

FIGURE 5. $\mathrm{Dr}(2, 5)$ with embedded hexagon

4. Combinatorial Properties of Coarsest Matroid Subdivisions

For the relevant definitions and basic properties of matroids we refer to White [23, 24]. Let $\mathcal{M}$ be a matroid on the set $[n]$. Two elements $i, j \in [n]$ are said to be *equivalent* if there exists a circuit C of $\mathcal{M}$ with $i, j \in C$. The equivalence classes of this relation are the *connected components* of $\mathcal{M}$. We denote by $c(\mathcal{M})$ the number of connected components of $\mathcal{M}$. A matroid is called *connected* if it has $c(\mathcal{M}) = 1$. In fact, there is a relation between the number of connected components of a matroid and the dimension of its matroid polytope.

Proposition 21 (Feichtner and Sturmfels [6, Prop. 2.4]). *Let $\mathcal{M}$ be a matroid with n elements. Then the dimension of the matroid polytope of $\mathcal{M}$ equals $n - c(\mathcal{M})$.*

If a connected component of $\mathcal{M}$ has cardinality 1, it is called *trivial* and its unique element is a *loop* of $\mathcal{M}$. Since bases are maximal with the property of not containing a circuit, it follows that a basis of $\mathcal{M}$ has to contain at least one element from each non-trivial connected component of $\mathcal{M}$. This says that the number of non-trivial connected components of a matroid is bounded by its rank.

The restriction of $\mathcal{M}$ to each of its connected components $C_1, C_2, \ldots, C_\nu$ is a matroid $\mathcal{M}(C_i)$ with ground set C_i and, obviously, one has

$$\sum_{i=1}^{\nu} \operatorname{rank} \mathcal{M}(C_\nu) = \operatorname{rank} \mathcal{M}.$$

This is also true for matroids with trivial components, as a trivial component has rank 0.

Let now Σ be a matroid subdivision of $\Delta(k,n)$ and $F \in \Sigma$ some interior cell. Then there exists a rank-k matroid $\mathcal{M}$ with n elements such that the matroid polytope of $\mathcal{M}$ is F. Furthermore (F being interior) $\mathcal{M}$ does not contain any loops so it has at most k connected components. Since the number of connected components of a loop-free matroid is at most the rank, translating Proposition 21 to the language of tight-spans gives us the following:

Lemma 22. *Tight-spans of matroid subdivisions of* $\Delta(k,n)$ *are at most* $(k-1)$*-dimensional.*

In particular, if F has codimension $k-1$, $\mathcal{M}$ has k connected components $C_1, C_2, \dots, C_k$ and the bases of $\mathcal{M}$ are the sets containing exactly one element from each C_i. Otherwise stated, $\mathcal{M}$ is the product of k rank-1 matroids. If F has codimension 1, its matroid $\mathcal{M}$ has two connected components C_1, C_2, F lies in the hyperplane $\sum_{i\in C_1} x_i = \operatorname{rank} \mathcal{M}(C_1) =: l$, and $\mathcal{M}$ is the product of a rank-l and a rank-$(k-l)$ matroid.

We will now examine the rays of $\mathrm{Dr}(k,n)$. By Lemma 22, these correspond to k-dimensional tropical linear spaces. However, in this least generic case we also have to deal with lower dimensional degenerations up to embeddings of tropical lines; these arise as degenerations of proper (tropical) planes.

Proposition 23 ([12, Prop. 3.4])**.** *Each split of the hypersimplex* $\Delta(k,n)$ *gives a ray of the Dressian* $\mathrm{Dr}(k,n)$*.*

This holds true since such a split is a regular matroid subdivision with precisely two maximal cells; so this must be a coarsest subdivision. Via tropical rigidity the same result can also be obtained as follows: Applying Proposition 18 to Example 15 gives us a procedure to generate splits of any hypersimplex. In fact, all hypersimplex splits arise via picking a vertex (figure), the parameter ℓ (to determine the point p_ℓ), and the duplication pattern. By [13, Thm. 5.3] the total number of splits of the hypersimplex $\Delta(k,n)$ with $n \geq 2k$ equals

$$(k-1)\left(2^{n-1}-(n+1)\right) - \sum_{i=2}^{k-1}(k-i)\binom{n}{i}. \tag{7}$$

Further known rays of $\mathrm{Dr}(k,n)$ are the 3-splits (subdivisions whose tight-span is a triangle) as shown in [11, Thm. 6.5]. Keeping the same notation as above, we now specialize to the case $k=3$. This means that we are looking at tropical point configurations in $\mathbb{T}^2$ and at the induced tropical planes in $\mathbb{T}^{n-1}$. For the number of splits of $\Delta(3,n)$ with $n \geq 6$ the formula (7) reads

$$(3-1)\left(2^{n-1}-(n+1)\right) - \sum_{i=2}^{3-1}(3-i)\binom{n}{i} \;=\; 2^n - \frac{1}{2}\left(n^2+3n+4\right).$$

The following is readily implied by Corollary 14.

Corollary 24. *Any tropically rigid configuration* V *of* n *points in* $\mathbb{T}^{k-1}$ *gives rise to a ray of the Dressian* $\mathrm{Dr}(k,n)$*, which is also a ray of the tropical Grassmannian* $\mathrm{Gr}(k,n)$*, such that the tight-span of the ray coincides with the tropical complex of* V*.*

Corollary 25. *For* $n \geq 2k$ *there are at least* $T(n-k,k)$ *combinatorially distinct rays of* $\mathrm{Dr}(k,n)$ *and* $\mathrm{Gr}(k,n)$*.*

Here $T(m,k)$ is the number of partitions of m into k positive parts, which is the same as the number of partitions of m in which the greatest part is k; see Integer Sequence: A008284 [1]. We have the recursion

$$T(m,k) \;=\; \sum_{i=1}^{k} T(m-k,i)$$

with $T(m,m)=1$.

Proof. Consider the tropical complex Γ from Example 16 which defines a $(k-1)$-dimensional simplex. We can now distribute $n-2k$ additional points arbitrarily among the k vertices of Γ, creating multiple points this way. Two such configurations are equivalent under the action of the group $\mathrm{Sym}(k)$ if and only if they correspond to the same partition of $n-k$ into k positive parts. By Proposition 18 the new configuration is tropically rigid and, by Corollary 24, corresponds to a ray of the $\mathrm{Dr}(k,n)$ and $\mathrm{Gr}(k,n)$. □

Remark 26. For $n \geq 6$ we have

$$T(n-3,3) \;=\; \lfloor 1/12(n-3)^2 + 1/2 \rfloor\,;$$

see Integer Sequence A001399 [1]. This establishes a quadratic lower bound for the number of combinatorially distinct configurations of $n-3$ points in $\mathbb{T}^2$ which are tropically rigid.

5. Most Degenerate Tropical Planes

We will now concentrate on the case $k=3$ and discuss the coarsest matroid subdivisions of the hypersimplices $\Delta(3,n)$. These are precisely the rays of the Dressian $\mathrm{Dr}(3,n)$. Generically, these correspond to tropical planes. In view of Lemma 22, we can give the following restriction on the tight-span of a coarsest matroid subdivision of $\mathrm{Dr}(3,n)$.

Proposition 27. *Let Σ be a coarsest matroid subdivision of* $\mathrm{Dr}(3,n)$*. Then $\mathcal{T}(\Sigma)$ is either a line segment or pure two-dimensional.*

Proof. By Lemma 22, $\mathcal{T}(\Sigma)$ is at most two-dimensional. If Σ is a split, $\mathcal{T}(\Sigma)$ is a line segment; otherwise, [11, Cor. 5.3] implies that all maximal faces of $\mathcal{T}(\Sigma)$ are at least two-dimensional. This shows the claim. □

Once again specializing our discussion in the previous section to the case $k=3$, it follows that a matroid $\mathcal{M}$ corresponding to a codimension-1 cell of a matroid subdivision of $\Delta(3,n)$ is the product of a rank-2 matroid and a rank-1 matroid. So, in particular, $\mathcal{M}$ is realizable over any infinite field $\mathbb{K}$.

The matroid $\mathcal{M}(C)$ of a codimension-2 cell C is even simpler. Such a matroid must be a rank 3 matroid with no loops and three connected components. Let the three components be $\{\alpha,\beta,\gamma\}$, with $[n]=\alpha\sqcup\beta\sqcup\gamma$. So the bases of $\mathcal{M}(c)$ are the triples (i,j,ℓ) with $i\in\alpha$, $j\in\beta$, $\ell\in\gamma$. Every codimension-1 cell F containing C lies in one of the three hyperplanes H_α, H_β or H_γ. So C is contained in at most six codimension-1 cells (each hyperplane is divided in half by C, and can contribute a cell from each side).

Let S be the two-dimensional cell of the tight-span dual to C. So S has at most 6 sides. If the subdivision of $\Delta(3,n)$ is regular, so that S comes with an embedding into $\mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$, then the edges of C are orthogonal to H_α, H_β and H_γ, meaning that they are parallel to the vectors $\sum_{i\in\alpha} e_i$, $\sum_{i\in\beta} e_i$ and $\sum_{i\in\gamma} e_i$. Note that, in $\mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$, these three vectors sum to 0. So C is either a triangle, a parallelogram, a trapezoid, a pentagon or a hexagon. (See Figure 6.) In fact, for any rank k, these are the only possible 2-faces in the tight-span of a tropical linear space. See [21, Prop. 2.6].

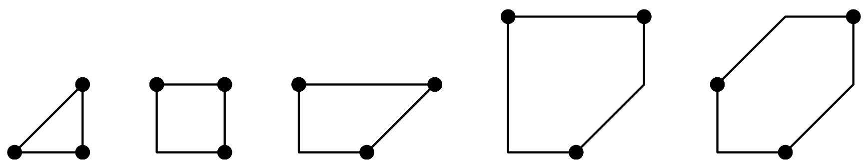

Figure 6. Shapes that might appear in the tight-span of a matroid subdivision; these precisely are the two-dimensional polytropes [15]

The vertices of the tight-span are in bijection with the maximal cells of the matroid subdivision. When we coarsen a matroid subdivision, adjacent cells merge into larger cells;

the dual effect is that edges of the tight-span contract down to length 0. We will say that such an edge *collapses*. We make the following observations:

Proposition 28. *If one edge of a triangle collapses, then all the edges of that triangle collapse.*

Proposition 29. *If one edge of a parallelogram collapses, so does the opposite edge.*

These conditions make it possible in many cases to recognize that a subdivision cannot be non-trivially coarsened. For example, if any edge in Figure 8 is collapsed, then all 7 edges must be. Collapsing all the edges gives the trivial subdivision. So the tight-span in Figure 8 cannot be non-trivially coarsened.

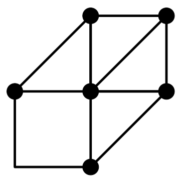

FIGURE 7. Configuration of six points corresponding to a coarsest tropical subdivision that contains a quadrangle

In a preprint version of this paper, it was claimed that coarsest subdivisions only contain triangles, not faces with higher numbers of edges. We can use the above propositions, together with the methods of Section 2, to refute this claim: Figure 7 shows six points in $\mathbb{R}^2$ whose tropical convex hull is a hexagon. The natural polyhedral subdivision of this hexagon is into four triangles and a parallelogram. The corresponding tropical 3-plane in 9-space thus also has a tight-span which consists of four triangles and a parallelogram arranged around a central point. Again, Propositions 28 and 29 show that, if we collapse any edge, we must collapse all of them. So this subdivision is coarsest, even though it has a quadrilateral face.

Example 30. There are 15,470 rays of $\mathrm{Dr}(3,8)$ coming in twelve symmetry classes. We begin with listing the ones that were known before:

- ▷ The simplest ones are the splits. According to (7) the total number of splits of $\Delta(3,8)$ equals 210. This gives four distinct splits up to symmetry, in the notation of [13] these correspond to the $(6,2;1)$-, $(3,5;1)$-, $(5,3;1)$-, and $(4,4;1)$-hyperplanes. We obtain 28, 56, 56 and 70 rays per orbit, respectively. The corresponding tropically rigid configuration are shown in the top part of Figure 4.
- ▷ Furthermore, in [11, Sec. 6] it is shown that certain 3-*splits*, that is, coarsest subdivisions with three maximal faces, (or, equivalently, subdivisions, whose tight-span is a triangle) correspond to rays of $\mathrm{Dr}(k,n)$. Specifically, [11, Cor. 6.4] tells us that there are 980 of these 3-splits coming in two equivalence classes. These are also obtained by the two tropically rigid configuration in Figure 4, where the left one gives 420 and the right one 560 rays.

Further rays are given by our tropically rigid point configurations of Section 3:

- ▷ The next tight-span that might occur are two triangles connected by an edge. The corresponding subdivisions come in three equivalence classes and may be obtained by the three tropical point configurations to the right of Figure 4. They give us 840, 1,260 and 1,680 rays, respectively, yielding a total of 3,780 with such a tight-span.
- ▷ The remaining rays of the Dressian coming from tropical point configurations are those depicted in Figure 3, hence corresponding to three or four connected triangles. Both of these give rise 5,040 rays.

However, not all tight-spans of rays of $\mathrm{Dr}(3,n)$ need to be planar, hence not all rays may be induced by tropically rigid point configurations. Indeed, our computations show that the simplest non-planar simplicial complex occurs as the tight-span of a ray of $\mathrm{Dr}(3,8)$. This tight-span is depicted in Figure 8 and it gives rise to the remaining 420 rays.

Altogether, Figures 3, 4 and 8 give the tight-spans of all rays of $\mathrm{Dr}(3,8)$. Explicit coordinates for the first eleven rays can be obtained by applying the map τ to the tropical point configurations in Figures 3 and 4. The last ray is a 0/1-vector of length $\binom{8}{3} = 56$ which maps the vertices of $\Delta(3,8)$ corresponding to the following 30 three-element subsets of $\{1,2,\ldots,8\}$ to 0, and the 26 remaining ones to 1:

$$123\ 124\ 126\ 127\ 128\ 134\ 136\ 137\ 138\ 234\ 235\ 236\ 237\ 238\ 245$$
$$247\ 248\ 256\ 257\ 258\ 267\ 268\ 345\ 347\ 348\ 356\ 357\ 358\ 367\ 368\,.$$

These sets form the bases of a matroid.

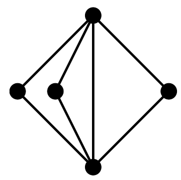

FIGURE 8. Non-planar tight-span of a ray of $\mathrm{Dr}(3,8)$. This is a 2-dimensional pure simplicial complex of three triangles sharing a common edge.

6. Computational Results

Here we explain how we computed the entire fan $\mathrm{Dr}(3,8)$ with `polymake` [7]. Fundamentally, we used a similar approach as for computing $\mathrm{Dr}(3,7)$ in [12]. However, we re-implemented that algorithm with a number of modifications. The key new idea is to have a short canonical description for each matroid subdivision such that it is fast to recognize duplicates. For $\mathrm{Dr}(3,7)$ this gives a speed up factor of about four. At the same time, our new approach is more efficient with respect to memory consumption. We verified the computational results for $\mathrm{Dr}(3,6)$ from [22] and $\mathrm{Dr}(3,7)$ from [12] with our new implementation.

A cone of the polyhedral fan $\mathrm{Dr}(k,n)$ is defined by deciding for each 3-term Plücker relation (1) which two of the three terms attain the maximum. This is to say, such a cone is defined by equations and inequalities of the form

$$\pi(\rho ij) + \pi(\rho \ell m) \ = \ \pi(\rho i\ell) + \pi(\rho jm) \ \geq \ \pi(\rho im) + \pi(\rho j\ell) \tag{8}$$

where ρ is a subset of $[n]$ of cardinality $k-2$ and $i,j,\ell,m \in [n] \setminus \rho$ are pairwise distinct. Enumerating all maximal cones in $\mathrm{Dr}(k,n)$ is now equivalent to going through all possible 3^ν combinations, where $\nu = \binom{n}{4} \cdot \binom{n-4}{k-2} = \binom{n}{k-2} \cdot \binom{n-k+2}{4}$ is the number of 3-term Plücker relations. This direct but naïve approach is, of course, infeasible in terms of complexity.

For $\rho \in \binom{[n]}{k-2}$ and $i,j,\ell,m \in [n] \setminus \rho$ the six points

$$e_{\rho ij}\,, \quad e_{\rho i\ell}\,, \quad e_{\rho im}\,, \quad e_{\rho j\ell}\,, \quad e_{\rho jm}\,, \quad e_{\rho \ell m}$$

are the vertices of a regular octahedron O, which forms a 3-face of the hypersimplex $\Delta(k,n)$. The three choices to pick the maximum in (8) correspond to the three ways to split O into two square pyramids. These are called *octahedral splits*. Each octahedral split gives one equation and one inequality.

Geometrically, the cones of $\mathrm{Dr}(k,n)$ correspond to matroid decompositions of $\Delta(k,n)$. The three-dimensional faces of $\Delta(k,n)$ are either simplices or octahedra such as O. Each matroid decomposition of $\Delta(k,n)$ induces a matroid decomposition on each of its faces.

Simplices do not admit any non-trivial subdivision (without new vertices). So only the three splits of the regular octahedron give rise to a non-trivial matroid subdivisions of any three-dimensional matroid polytope. This way, each cone of $\mathrm{Dr}(k,n)$ or, equivalently, each matroid subdivision of $\Delta(k,n)$ yields splits on a subset of its octahedral 3-faces. We encode this information as a *sequence of octahedral splits*: for a fixed linear ordering of all octahedral faces of $\Delta(k,n)$ we list if that face is subdivided or not, and if so, by which of the three possible splits. The amount of storage required is two bits per octahedral 3-face, and this totals to 2ν bits.

A slight variation of the naïve algorithm, conceptually, now leads to a first backtracking scheme to compute $\mathrm{Dr}(k,n)$ with the fan structure imposed by the Plücker relations.

(i) Start with the entire space $C = \mathbb{R}^{\binom{n}{k}}$ and the empty sequence L of octahedral splits.
(ii) Iterate through all octahedral faces.
(iii) For each new octahedron, add one of the three possible splits to L and intersect C by the corresponding hyperplane and halfspace.
(iv) If $\dim C$ is too small for a maximal cone, backtrack and try another possibility for the same octahedron.
(v) If all three splits for an octahedron have been tried, backtrack and try another octahedron.
(vi) If we are at the last octahedron, output C and L continue backtracking.

The output will be a list of all maximal cones of $\mathrm{Dr}(k,n)$ along with their description as sequences of octahedral splits. However, due to the vast amount of maximal cones in $\mathrm{Dr}(3,8)$, the computation is still infeasible. Therefore, we here give the following modified algorithm, using the same idea, but computing $\mathrm{Dr}(k,n)$ iteratively from $\mathrm{Dr}(k,n-1)$ and taking the known boundary into account. This works since each facet of $\Delta(k,n)$ is again a hypersimplex, either of type $\Delta(k-1,n)$ or of type $\Delta(k,n-1)$. Moreover, each sequence of octahedral splits of a hypersimplex induces a sequence of octahedral splits on each facet (and thus, inductively, on each lower dimensional face).

(i) After arriving at a new octahedral face and adding a split, consider the octahedral splits induced in each $\Delta(k,n-1)$ boundary face of $\Delta(k,n)$.
(ii) Test if there is some matroid subdivision of $\Delta(k,n-1)$ defining these octahedral splits.
(iii) If this is not the case go back and try another of the three possibilities to split the octahedron.

With this algorithm we were able to compute all maximal cones of $\mathrm{Dr}(3,7)$ in around 15 minutes (from $\mathrm{Dr}(3,6)$ which is computed with either algorithm in a few seconds). All timings are taken single-threaded on a machine with AMD Athlon(tm) 64 X2 Dual Core Processor 4200+ (4433.05 bogomips per core) 4 GB main memory, running Ubuntu 10.04 (Lucid Lynx). To compute $\mathrm{Dr}(3,8)$, we made the further modification to assume that for the first Plücker relations one of the three possibilities is fixed. In this way, the algorithm does not compute all maximal cones any more, but it is still guaranteed that we get at least one maximal cone in each symmetry class. With this reduction, the computation for $\mathrm{Dr}(3,8)$ took about 200 hours. The output of this algorithm, however, has to be processed further to yield anything useful.

To get a complete description of $\mathrm{Dr}(3,8)$ from this result, we first produced a list containing one representative of each symmetry class of maximal cones together with the corresponding orbit size. This is obtained by the following algorithm.

(i) Initialize L as the empty list.
(ii) For each cone compute the lexicographic first cone C in the same orbit.
(iii) If C is in L proceed to the next cone, otherwise add C to L and compute the orbit of C and store the size.

This computation took around 230 hours. Of course, it would be faster to store all cones from all orbits during the computation, however this not feasible in terms of space.

With this list of maximal cones the following further steps were necessary to compute the f-vector and the f-vector up to symmetry:

(i) For one maximal cone from each orbit, compute the rays.
(ii) Compute one ray form each orbit and then all rays and store them in a list R.
(iii) Each maximal cone is no translated in a description by the indices of its rays in R.
(iv) For one maximal cone in each orbit compute all faces of a fixed dimension d, and represent it by the indices of its rays in R.
(v) For each possible dimension d, go through all faces so computed and compute one from each orbit together with the orbit size similar as above.

For dimension 6, the computations took about 14.5 hours, this was the maximum. The final result can be summed up as follows.

Theorem 31. *The Dressian* $\mathrm{Dr}(3,8)$ *is a non-pure non-simplicial nine-dimensional polyhedral fan with f-vector*

$$(1; 15{,}470;\ 642{,}677;\ 8{,}892{,}898;\ 57{,}394{,}505;\ 194{,}258{,}750;$$
$$353{,}149{,}650;\ 324{,}404{,}880;\ 117{,}594{,}645;\ 113{,}400)\,.$$

Modulo the natural $\mathrm{Sym}(8)$*-symmetry, the f-vector reads*

$$(1;\ 12;\ 155;\ 1{,}149;\ 5{,}013;\ 12{,}737;\ 18{,}802;\ 14{,}727;\ 4{,}788;\ 14)\,.$$

There are 116,962,265 *maximal cones,* 113,400 *of dimension* 9 *and* 116,848,865 *of dimension* 8*. Up to symmetry, there are* 4,748 *maximal cones,* 14 *of dimension* 9 *and* 4,734 *of dimension* 8*.*

Corollary 32. *There are* 4,748 *distinct combinatorial types of generic tropical planes in* $\mathbb{T}^6$*.*

Proposition 33. *All rays of* $\mathrm{Dr}(3,8)$ *are rays of* $\mathrm{Gr}(3,8)$*.*

Proof. By Proposition 12 all rays coming from tropically rigid point configuration are elements of the Grassmannian. So it remains to show that the last ray is contained in $\mathrm{Gr}(3,8)$. This is done by explicit computation with the computer algebra software `Macaulay2` [9]. We computed the initial ideal I_r of the Plücker ideal defined by the weight vector corresponding to this ray and verified that I_r does not contain any monomials. Note, however, that the tropical Grassmannian depends on the characteristic of the field considered. Therefore, we computed over the polynomial ring $\mathbb{Z}[p_S]$ with integer coefficients. It turns out that all polynomials in the integral Gröbner basis of I_r only have non-vanishing coefficients ± 1, yielding the result for arbitrary characteristic. ☐

The data computed as available at `http://svenherrmann.net/DR38/dr38.html`.

7. Questions

We would like to close this paper with some open questions.

Question 34. *Can all coarsest matroid subdivisions of* $\Delta(3,n)$ *with planar tight-spans be constructed via tropical point configurations?*

Our computations show that this question has a positive answer for all $n \leq 8$.

There is a very crude parameter to estimate how complicated a subdivision is from the combinatorial point of view. The *spread* of an element of $\mathrm{Dr}(k,n)$ is defined in as the number of maximal cells of the corresponding subdivision. It was shown in [12, Prop. 3.7] that the spread is not bounded if n increases (with k fixed).

Question 35. *What is the maximal* spread *of a ray of* $\mathrm{Dr}(k,n)$ *(at least for $k = 3$)?*

For $k = 3$ and $n = 5, 6, 7, 8$, the values are $2, 3, 4, 6$, respectively.

It is known that the natural fan structures of the tropical Grassmannians $\mathrm{Gr}(k, n)$ and the Dressians $\mathrm{Dr}(k, n)$ differ for $k \geq 3$. However, the following is unclear to us.

Question 36. *Are the rays of* $\mathrm{Dr}(k, n)$ *always rays of* $\mathrm{Gr}(k, n)$*?*

This was known to be true previously for $k = 3$ and $n \leq 7$. It follows from our results in Section 6 that it also holds for $k = 3$ and $n = 8$. The converse is not true: there are rays of $\mathrm{Gr}(3, 7)$ that are not rays of $\mathrm{Dr}(3, 7)$.

There is also a number of obvious computational challenges.

Question 37. *Is it feasible to compute* $\mathrm{Gr}(3, 8)$*?*

With currently available implementations of the relevant algorithms, such as `Gfan` [14], the answer seems to be *no*, even for some fixed characteristic. It should be feasible, however, to sample (few) points from each maximal face of the Dressian $\mathrm{Dr}(3, 8)$ and check if they are contained in the tropical Grassmannian of a fixed characteristic. Doing the necessary Gröbner bases computations over the integers, to obtain results valid for all characteristics, is more demanding.

Even if we cannot tell exactly how $\mathrm{Dr}(3, 9)$, $\mathrm{Dr}(3, 10)$, etc. look like, by now we have gathered substantial information about the Dressians $\mathrm{Dr}(3, n)$ for arbitrary n. However, it is expected that entirely new features arise if one replaces the first parameter by 4. So another very serious challenge is to answer the following.

Question 38. *Is it feasible to compute* $\mathrm{Dr}(4, 8)$*?*

References

1. *The on-line encyclopedia of integer sequences*, 2011, published electronically at `http://oeis.org`.
2. Richard A. Brualdi, *Transversal matroids*, Combinatorial Geometries, Encyclopedia of Mathematics and its Applications, vol. 29, Cambridge University Press, Cambridge, 1987, pp. 72–97. MR 0921064 (88g:05048)
3. Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations: Structures for algorithms and applications*, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010. MR 2743368 (2011j:52037)
4. Mike Develin, *The moduli space of n tropically collinear points in* $\mathbb{R}^d$, Collect. Math. **56** (2005), no. 1, 1–19. MR 2131129 (2005m:55025)
5. Mike Develin and Bernd Sturmfels, *Tropical convexity*, Doc. Math. **9** (2004), 1–27 (electronic), correction: ibid., pp. 205–206. MR 2054977 (2005i:52010)
6. Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), no. 4, 437–468. MR 2191630 (2006j:05036)
7. Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR 1785292 (2001f:52033)
8. Israil M. Gel′fand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63** (1987), no. 3, 301–316. MR 877789 (88f:14045)
9. Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.
10. Paul Hacking, Sean Keel, and Jenia Tevelev, *Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces*, Invent. Math. **178** (2009), no. 1, 173–227. MR 2534095 (2010i:14062)
11. Sven Herrmann, *On the facets of the secondary polytope*, J. Combin. Theory Ser. A **118** (2011), no. 2, 425–447. MR 2739494
12. Sven Herrmann, Anders Jensen, Michael Joswig, and Bernd Sturmfels, *How to draw tropical planes*, Electron. J. Combin. **16** (2009), no. 2, Special volume in honor of Anders Björner, Research Paper 6, 26. MR 2515769
13. Sven Herrmann and Michael Joswig, *Splitting polytopes*, Münster J. Math. **1** (2008), 109–141. MR 2502496 (2010h:52015)

14. Anders Jensen, *`Gfan` – a software package for Gröbner fans and tropical varieties*, `home.imf.au.dk/jensen/software/gfan/gfan.html`.
15. Michael Joswig and Katja Kulas, *Tropical and ordinary convexity combined*, Adv. Geom. **10** (2010), no. 2, 333–352. MR 2629819 (2011c:14162)
16. Mikhail M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110. MR 1237834 (95g:14053)
17. Thomas Markwig, *A field of generalised Puiseux series for tropical geometry*, Rend. Semin. Mat. Torino **68** (2010), no. 1, 79–92.
18. Lior Pachter and Bernd Sturmfels (eds.), *Algebraic statistics for computational biology*, Cambridge University Press, New York, 2005. MR 2205865 (2006i:92002)
19. Felipe Rincón, *Local tropical linear spaces*, in preparation.
20. Alexander Schrijver, *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003, Paths, flows, matchings, Chapters 1–38. MR 1956924 (2004b:90004a)
21. David E Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558. MR 2448909
22. David E Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411. MR 2071813 (2005d:14089)
23. Neil White (ed.), *Theory of matroids*, Encyclopedia of Mathematics and its Applications, vol. 26, Cambridge University Press, Cambridge, 1986. MR 849389 (87k:05054)
24. Neil White (ed.), *Matroid applications*, Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press, Cambridge, 1992. MR 1165537 (92m:05004)

Sven Herrmann, School of Computing Sciences, University of East Anglia, Norwich, NR4 7TJ, UK
E-mail address: `sherrmann@mathematik.tu-darmstadt.de`

Michael Joswig, Fachbereich Mathematik, TU Darmstadt, 64289 Darmstadt, Germany
E-mail address: `joswig@mathematik.tu-darmstadt.de`

David Speyer, Department of Mathematics, 530 Church Street, Ann Arbor, MI 48109-1043 USA
E-mail address: `speyer@umich.edu`