

Hankel transform of a sequence obtained by series reversion II - aerating transforms

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Abstract

This paper provides the connection between the Hankel transform and aerating transforms of a given integer sequence. Results obtained are used to establish a completely different Hankel transform evaluation of the series reversion of a certain rational function $Q(x)$ and shifted sequences, recently published in our paper [2]. For that purpose, we needed to evaluate the Hankel transforms of the sequences $(\alpha^2 C_n - \beta C_{n+1})_{n \in \mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n \in \mathbb{N}_0}$, where $C = (C_n)_{n \in \mathbb{N}_0}$ is the well-known sequence of Catalan numbers. This generalizes the results of Cvetković, Rajković and Ivković [4]. Also, we need the evaluation of Hankel-like determinants whose entries are Catalan numbers C_n and which is based on the recent results of Krattenthaler [8]. The results obtained are general and can be applied to many other Hankel transform evaluations.

Key words: Hankel transform, Catalan numbers, aerating transform, series reversion.

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1 Introduction

The Hankel transform of a given sequence $a = (a_n)_{n \in \mathbb{N}_0}$ is defined as the sequence $h = (h_n)_{n \in \mathbb{N}_0}$ of Hankel determinants, i.e.

$$h_n = \det ([a_{i+j}]_{0 \leq i, j \leq n}), \quad (n \in \mathbb{N}_0) \quad (1)$$

and denoted by $h = \mathcal{H}(a)$. The term ‘‘Hankel transform’’ was first introduced by Layman [9] in 2001. Despite that, many Hankel determinants evaluation were obtained much earlier, mostly due to their important combinatorial properties (see for example [3, 6, 13, 14]).

Papers [3, 4, 10] use a method based on orthogonal polynomials (or continued fractions) to provide a Hankel transform evaluation of different sequences. It is also used in our recently published paper [2] where we evaluated the Hankel transform of a series reversion of the function $\frac{x}{1+\alpha x+\beta x^2}$, as well as of the corresponding shifted sequences.

In this paper, we also show another evaluation for the same sequences, which is based on the application of the falling α -binomial transform [12] and aerating transforms. The method described in this paper provides us with more general results regarding the connection between Hankel transforms and aerating transforms.

For our purpose we need the Hankel transform evaluation of $(\alpha^2 C_n - \beta C_{n+1})_{n \in \mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n \in \mathbb{N}_0}$, where $C = (C_n)_{n \in \mathbb{N}_0}$ is the well-known sequence of Catalan numbers. This generalizes results of Cvetković, Rajković and Ivković [4]. We also need the evaluation of Hankel-like determinants whose entries are Catalan numbers C_n and which is based on the recent results of Krattenthaler [8].

For our further discussion we need to recall the definition of the series reversion of a (generating) function $f(x)$ which satisfies $f(0) = 0$ (see [1]).

Definition 1.1. For a given (generating) function $v = f(u)$ with the property $f(0) = 0$, the series reversion is the sequence $(s_n)_{n \in \mathbb{N}_0}$ such that

$$u = f^{-1}(v) = s_1 v + s_2 v^2 + \cdots + s_n v^n + \cdots,$$

where $u = f^{-1}(v)$ is the inverse function of $v = f(u)$. Note that since $f(0) = 0$, there must hold $s_0 = f^{-1}(0) = 0$.

2 The series reversion of $\frac{x}{1+\alpha x+\beta x^2}$

We recall a few basic properties and expressions of the sequence $(u_n)_{n \in \mathbb{N}_0}$ obtained by reverting

$$Q(x) = \frac{x}{1 + \alpha x + \beta x^2}.$$

This sequence is already investigated in [1, 2]. The generating function $U(x)$ satisfies $Q(U(x)) = x$ (Definition 1.1) and is given by

$$U(x) = \frac{1 - \alpha x - \sqrt{1 - 2\alpha x + (\alpha^2 - 4\beta)x^2}}{2\beta x}. \quad (2)$$

The general term of the sequence $(u_n)_{n \in \mathbb{N}_0}$ can be expressed in the following way (Proposition 9 in [1]):

$$u_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k \alpha^{n-2k-1} \beta^k. \quad (3)$$

Note that the sequence $(u_n)_{n \in \mathbb{N}_0}$ generalizes the sequence $(C_n + \delta_{n0})_{n \in \mathbb{N}_0}$ (for $\alpha = 2$ and $\beta = 1$) and more generally the sequence $((N_n(z) - \delta_{n0})/z)_{n \in \mathbb{N}_0}$ where $N_n(z)$ is the n -th Narayana polynomial (for $\alpha = z + 1$ and $\beta = z$).

Consider the shifted sequences $(u_n^*)_{n \in \mathbb{N}_0}$ and $(u_n^{**})_{n \in \mathbb{N}_0}$ defined by $u_n^* = u_{n+1}$ and $u_n^{**} = u_{n+2}$. Also denote by h_n , h_n^* and h_n^{**} , the Hankel transforms of the sequences u_n , u_n^* and u_n^{**} respectively. Our previous paper [2] provides the evaluation of h_n^* , h_n^{**} and h_n using the method based on orthogonal polynomials [4, 10]. The main results are the following theorems (Theorem 4.3, Theorem 4.4, and Corollary 5.4 in [2]):

Theorem 2.1. [2] *The Hankel transform of the sequence $(u_n^*)_{n \in \mathbb{N}_0}$ is given by*

$$h_n^* = \beta \binom{n+1}{2}. \quad (4)$$

Theorem 2.2. [2] *The Hankel transform of the sequence $(u_n^{**})_{n \in \mathbb{N}_0}$ is given by*

$$h_n^{**} = \frac{\beta \binom{n+1}{2}}{2^{n+1} \sqrt{\alpha^2 - 4\beta}} [(\alpha + \sqrt{\alpha^2 - 4\beta})^{n+2} - (\alpha - \sqrt{\alpha^2 - 4\beta})^{n+2}]. \quad (5)$$

Theorem 2.3. [2] *The Hankel transform of the sequence $(u_n)_{n \in \mathbb{N}_0}$ is given by*

$$h_n = \frac{\beta \binom{n}{2}}{2^n \sqrt{\alpha^2 - 4\beta}} \left[(\alpha - \sqrt{\alpha^2 - 4\beta})^n - (\alpha + \sqrt{\alpha^2 - 4\beta})^n \right]. \quad (6)$$

Note that the sequence $(u_n^*)_{n \in \mathbb{N}_0}$ reduces to the sequence $(C_n^a)_{n \in \mathbb{N}_0}$ of *aerated Catalan numbers* (A126120), by choosing $\alpha = 0$ and $\beta = 1$ (yields directly from (2)). The sequence $(C_n^a)_{n \in \mathbb{N}_0}$ is defined by

$$C_n^a = \begin{cases} C_{n/2}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}.$$

Also note that the Hankel transform of the aerated sequence $(C_n^a)_{n \in \mathbb{N}_0}$ is $(1)_{n \in \mathbb{N}_0}$ (which is proven in the section 4), the same as in the case of the Catalan sequence (see for example [8]). That result raises the more general question about the Hankel transform of *aerated sequences*, which we deal with in the rest of the paper.

3 The falling α -binomial transform

The following transform is a generalization of the well-known binomial transform and was introduced by Spivey and Steil [12]. We will use it in further considerations.

Definition 3.1. *For a given sequence $a = (a_n)_{n \in \mathbb{N}_0}$, its falling α -binomial transformation $b = \mathcal{B}(a; \alpha)$ is defined by*

$$b_n = \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} a_k.$$

Spivey and Steil [12] proved that the Hankel transform is invariant under the falling α -binomial transform for arbitrary α . In other words, the following lemma is valid.

Lemma 3.1. [12] *For an arbitrary sequence $a = (a_n)_{n \in \mathbb{N}_0}$ and number α , it holds that $\mathcal{H}(\mathcal{B}(a; \alpha)) = \mathcal{H}(a)$.*

The falling α -binomial transform can be written in the following matrix form

$$b = \mathbf{B}^\alpha a, \quad \mathbf{B}^\alpha = \left[\binom{n}{k} \alpha^{n-k} \right]_{n,k \in \mathbb{N}_0}$$

where we treat the sequences a and b as corresponding column vectors (we also use this notation in the rest of the paper). We call the matrix \mathbf{B}^α the α -binomial matrix. The following lemma shows the connection between the Hankel matrices

$$\mathbf{H}_a = [a_{i+j}]_{i,j \in \mathbb{N}_0}, \quad \mathbf{H}_b = [b_{i+j}]_{i,j \in \mathbb{N}_0}$$

and the matrix \mathbf{B}^α .

Lemma 3.2. *If $b = \mathcal{B}(a; \alpha)$ then there holds*

$$\mathbf{H}_b = \mathbf{B}^\alpha \mathbf{H}_a (\mathbf{B}^\alpha)^T. \quad (7)$$

Proof. Let us start from the general element b_{n+m} of the matrix \mathbf{H}_b :

$$b_{n+m} = \sum_t \binom{n+m}{t} \alpha^{n+m-t} a_t.$$

Using the well-known identity

$$\binom{n+m}{t} = \sum_{k=0}^n \binom{n}{k} \binom{m}{t-k}$$

we obtain

$$\begin{aligned} b_{n+m} &= \sum_{t=0}^{n+m} \sum_{k=0}^n \binom{n}{k} \binom{m}{t-k} \alpha^{n+m-t} a_t = \sum_{l=0}^m \sum_{k=0}^n \binom{n}{k} \binom{m}{l} \alpha^{n+m-k-l} a_{k+l} \\ &= \sum_{l=0}^m \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \cdot a_{k+l} \cdot \binom{m}{l} \alpha^{m-l} = \sum_{l=0}^m \sum_{k=0}^n (\mathbf{B}^\alpha)_{nk} \cdot a_{k+l} \cdot (\mathbf{B}^\alpha)_{ml}. \end{aligned}$$

This completes the proof of the lemma. \square

In the following sections, we give alternative proofs (to those given in [2]) of theorems regarding the Hankel transform of u_n^* , u_n^{**} and u_n (Theorem 2.1, Theorem 2.2 and Theorem 2.3).

4 The sequence u_n^*

We give an evaluation of the Hankel transform of u_n^* just using transformations and known results concerning the Hankel transform of the Catalan numbers [8]. For this purpose, we define the following *aerating transform*.

Definition 4.1. *For a given sequence $c = (c_n)_{n \in \mathbb{N}_0}$, we define its aerating transformation $p = \mathcal{A}(c)$ by*

$$p_n = \begin{cases} c_{n/2}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}.$$

In other words, if $p = \mathcal{A}(c)$ then $p = (c_0, 0, c_1, 0, c_2, 0, c_3, 0, \dots)$.

Hence $C^a = \mathcal{A}(C)$ where $C = (C_n)_{n \in \mathbb{N}_0}$ is the sequence of Catalan numbers. The following theorem shows the connection between the Hankel transform of a given sequence c and its aerated sequence $p = \mathcal{A}(c)$.

Theorem 4.1. Let $g = \mathcal{H}(p)$ and $h = \mathcal{H}(c)$ where $p = \mathcal{A}(c)$ is the aerated sequence of c . Then there holds

$$\det[p_{i+j}]_{0 \leq i, j \leq n} = \det[c_{i+j}]_{0 \leq i, j \leq \lfloor \frac{n}{2} \rfloor} \cdot \det[c_{i+j+1}]_{0 \leq i, j \leq \lfloor \frac{n-1}{2} \rfloor}.$$

In terms of Hankel transforms, this last equality can be written as

$$g_n = h_{\lfloor \frac{n}{2} \rfloor} h_{\lfloor \frac{n-1}{2} \rfloor}^*,$$

where h^* is the Hankel transform of the shifted sequence $c^* = (c_{n+1})_{n \in \mathbb{N}_0}$ and $h^* = \mathcal{H}(c^*)$.

Proof. By exchanging the rows and columns of the determinant $\det[p_{i+j}]_{0 \leq i, j \leq n-1}$ we obtain

$$\det[p_{i+j}]_{0 \leq i, j \leq n} = \begin{vmatrix} c_0 & 0 & c_1 & 0 & c_2 & \cdots \\ 0 & c_1 & 0 & c_2 & 0 & \\ c_1 & 0 & c_2 & 0 & c_3 & \\ 0 & c_2 & 0 & c_3 & 0 & \\ c_2 & 0 & c_3 & 0 & c_4 & \\ \vdots & & & & & \ddots \end{vmatrix} = \det \begin{bmatrix} \mathbf{A} & \\ & \mathbf{B} \end{bmatrix}$$

where $\mathbf{A} = [c_{i+j}]_{0 \leq i, j \leq \lfloor \frac{n}{2} \rfloor - 1}$ and $\mathbf{B} = [c_{i+j+1}]_{0 \leq i, j \leq \lfloor \frac{n-1}{2} \rfloor - 1}$. Now the statement of the theorem follows immediately. \square

It is known (see for example [7]) that $\mathcal{H}((C_n)_{n \in \mathbb{N}_0}) = \mathcal{H}((C_{n+1})_{n \in \mathbb{N}_0}) = (1)_{n \in \mathbb{N}_0}$. Using Theorem 4.1 (for $c_n = C_n$) we obtain the result $\mathcal{H}((C_n^a)_{n \in \mathbb{N}_0}) = (1)_{n \in \mathbb{N}_0}$.

We also need the following proposition:

Proposition 4.2. Let $c = (c_n)_{n \in \mathbb{N}_0}$ be an arbitrary sequence and $h = \mathcal{H}(c)$ its Hankel transform. Then $\mathcal{H}((r^n c_n)_{n \in \mathbb{N}_0}) = (r^{n(n+1)} h_n)_{n \in \mathbb{N}_0}$ where r is an arbitrary number.

Let $c_n = \beta^n C_n$ and let $p = \mathcal{A}(c)$. Recall that u_n^* can be expressed as follows (directly from (3)):

$$u_n^* = u_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \alpha^{n-2k} \beta^k C_k = \sum_{l=0}^n \binom{n}{l} \alpha^{n-l} p_l$$

which implies that $(u_n^*)_{n \in \mathbb{N}_0} = \mathcal{B}(p; \alpha)$.

Note that the sequence $p = \mathcal{A}(c)$ can be expressed as $p = (\beta^{n/2} C_n^a)_{n \in \mathbb{N}_0}$. Now using Lemma 3.1 and Proposition 4.2 we obtain the result of Theorem 2.1:

$$h^* = \mathcal{H}(u^*) = \mathcal{H}(p) = \mathcal{H}((\beta^{n/2} C_n^a)_{n \in \mathbb{N}_0}) = (\beta^{\binom{n+1}{2}})_{n \in \mathbb{N}_0}.$$

5 The Hankel transform of a linear combination of Catalan and shifted Catalan numbers

Cvetković, Rajković and Ivković [4] considered the Hankel transform of the sequence $(C_n + C_{n+1})_{n \in \mathbb{N}_0}$. In this section, we generalize their result, providing the Hankel transform evaluation of the sequences $(\alpha^2 C_n - \beta C_{n+1})_{n \in \mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n \in \mathbb{N}_0}$. We also need this result in the next section.

We use the method based on orthogonal polynomials, as used in [4, 10]. It assumes that a given sequence $(a_n)_{n \in \mathbb{N}_0}$ is a *moment sequence* with respect to some weight function (measure) $w(x)$, i.e. that there holds

$$a_n = \int_{\mathbb{R}} x^n w(x) dx.$$

If $h = \mathcal{H}(a)$ and $h_n \neq 0$ for all $n \in \mathbb{N}_0$, then there exists a sequence of monic orthogonal polynomials $(\pi_n(x))_{n \in \mathbb{N}_0}$, which satisfies a three-term recurrence relation

$$\pi_{n+1}(x) = (x - \alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x). \quad (8)$$

The Hankel transform h_n can be evaluated using the following Heilermann formula (see for example [7]):

$$h_n = a_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n. \quad (9)$$

In order to establish closed-form expression for coefficients α_n and β_n , the following transformation lemmas can be useful:

Lemma 5.1. [2] *Let $w(x)$ and $\tilde{w}(x)$ be weight functions and denote by $(\pi_n(x))_{n \in \mathbb{N}_0}$ and $(\tilde{\pi}_n(x))_{n \in \mathbb{N}_0}$ the corresponding orthogonal polynomials. Also denote by $(\alpha_n)_{n \in \mathbb{N}_0}$, $(\beta_n)_{n \in \mathbb{N}_0}$ and $(\tilde{\alpha}_n)_{n \in \mathbb{N}_0}$, $(\tilde{\beta}_n)_{n \in \mathbb{N}_0}$ the three-term relation coefficients corresponding to $w(x)$ and $\tilde{w}(x)$ respectively. The following transformation formulas are valid:*

- (1) *If $\tilde{w}(x) = Cw(x)$ where $C > 0$ then we have $\tilde{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = C\beta_0$, $\tilde{\beta}_n = \beta_n$ for $n \in \mathbb{N}$. Additionally there holds $\tilde{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.*
- (2) *If $\tilde{w}(x) = w(ax + b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\tilde{\alpha}_n = \frac{\alpha_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = \frac{\beta_0}{|a|}$ and $\tilde{\beta}_n = \frac{\beta_n}{a^2}$ for $n \in \mathbb{N}$. Additionally there holds $\tilde{\pi}_n(x) = \frac{1}{a^n} \pi_n(ax + b)$.*

Lemma 5.2. (Linear multiplier transformation) [5] *Consider the same notation as in Lemma 5.1. Let the sequence $(r_n)_{n \in \mathbb{N}_0}$ be defined by*

$$r_0 = c - \alpha_0, \quad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0). \quad (10)$$

If $\tilde{w}(x) = (x - c)w(x)$ where $c < \inf \text{supp}(w)$, then there holds

$$\begin{aligned} \tilde{\beta}_0 &= \int_{\mathbb{R}} \tilde{w}(x) dx, \quad \tilde{\beta}_n = \beta_n \frac{r_n}{r_{n-1}}, \quad (n \in \mathbb{N}), \\ \tilde{\alpha}_n &= \alpha_{n+1} + r_{n+1} - r_n, \quad (n \in \mathbb{N}_0). \end{aligned} \quad (11)$$

Now we prove the main theorem of this section. The proof is based on the sequential application of the previous two lemmas for a known weight function, i.e. a weight function whose coefficients α_n and β_n are known.

Theorem 5.3. *The Hankel transforms of the sequences $(\alpha^2 C_n - \beta C_{n+1})_{n \in \mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n \in \mathbb{N}_0}$ can be evaluated as follows*

$$\begin{aligned} \det[\alpha^2 C_{i+j} - \beta C_{i+j+1}]_{0 \leq i, j \leq n} &= \frac{1}{2^{2n+3} \sqrt{\alpha^2 - 4\beta}} \left[(\alpha + \sqrt{\alpha^2 - 4\beta})^{2n+3} - (\alpha - \sqrt{\alpha^2 - 4\beta})^{2n+3} \right] \\ \det[\alpha^2 C_{i+j+1} - \beta C_{i+j+2}]_{0 \leq i, j \leq n} &= \frac{1}{2^{2n+4} \alpha \sqrt{\alpha^2 - 4\beta}} \left[(\alpha + \sqrt{\alpha^2 - 4\beta})^{2n+4} - (\alpha - \sqrt{\alpha^2 - 4\beta})^{2n+4} \right] \end{aligned}$$

Proof. We use again the method based on orthogonal polynomials. It is well-known (see for example [4]) that the Catalan numbers have the following moment representation

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4}{x} - 1} dx.$$

This directly implies that the sequence $(\alpha^2 C_n - \beta C_{n+1})_{n \in \mathbb{N}_0}$ is the moment sequence of the weight function:

$$\hat{w}(x) = \frac{1}{2\pi} (\alpha^2 - \beta x) \sqrt{\frac{4}{x} - 1}.$$

In order to determine the three-term recurrence relation coefficients corresponding to $\hat{w}(x)$, we start from the weight function of the monic Chebyshev polynomials of the fourth kind:

$$w^{(0)}(x) = \sqrt{\frac{1-x}{1+x}}, \quad x \in [-1, 1].$$

The corresponding coefficients $\alpha_n^{(0)}$ and $\beta_n^{(0)}$ are

$$\alpha_0^{(0)} = -1/2, \quad \alpha_n^{(0)} = 0, \quad n \geq 1, \quad \beta_0^{(0)} = \pi, \quad \beta_n^{(0)} = 1/4, \quad n \geq 1.$$

Now we define a new weight function $w^{(1)}(x)$ by

$$w^{(1)}(x) = w^{(0)}\left(\frac{x}{2} - 1\right)$$

and use part (2) of Lemma 5.1 with $a = 1/2$ and $b = -1$. Hence we obtain

$$\alpha_0^{(1)} = 1, \quad \alpha_n^{(1)} = 2, \quad n \geq 1, \quad \beta_0^{(1)} = 2\pi, \quad \beta_n^{(1)} = 1, \quad n \geq 1.$$

The next transformation is

$$w^{(2)}(x) = -\frac{\beta}{2\pi} \cdot w^{(1)}(x).$$

From part (1) of Lemma 5.1 we see that

$$\alpha_0^{(2)} = 1, \quad \alpha_n^{(2)} = 2, \quad n \geq 1, \quad \beta_0^{(2)} = -\beta, \quad \beta_n^{(2)} = 1, \quad n \geq 1.$$

The final transformation is a linear multiplier transformation

$$\hat{w}(x) = \left(x - \frac{\alpha^2}{\beta}\right) \cdot w^{(2)}(x).$$

According to Lemma 5.1 ($c = \alpha^2/\beta$) we have to consider the following temporary sequence

$$r_0 = \frac{\alpha^2}{\beta} - 1, \quad r_n = \frac{\alpha^2}{\beta} - 2 - \frac{1}{r_{n-1}} \tag{12}$$

and the coefficients $\hat{\beta}_n$ are obtained by

$$\hat{\beta}_0 = \alpha^2 C_0 - \beta C_1 = \alpha^2 - \beta, \quad \hat{\beta}_n = \beta_n^{(2)} \frac{r_n}{r_{n-1}}, \quad n \in \mathbb{N}.$$

The Heilermann formula now yields

$$\frac{\hat{h}_{n+1}}{\hat{h}_n} = \hat{\beta}_0 \hat{\beta}_1 \cdots \hat{\beta}_{n+1} = \beta r_{n+1}.$$

Replacing the last expression into (12) we obtain the following linear difference equation

$$\hat{h}_n - (\alpha^2 - \beta)\hat{h}_{n-1} + \beta^2\hat{h}_{n-2} = 0, \quad (n \geq 2) \quad (13)$$

where the initial values are given by $\hat{h}_0 = \alpha^2 - \beta$ and $\hat{h}_1 = \alpha^4 - 3\alpha^2\beta + \beta^2$. By solving (13), we directly obtain the first statement of lemma.

To prove the second statement, let us observe that the weight function of the sequence $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n \in \mathbb{N}_0}$ is equal to

$$\check{w}(x) = \frac{1}{2\pi} x(\alpha^2 - \beta x) \sqrt{\frac{4}{x} - 1} = \frac{\beta}{\pi} \cdot \left(\frac{\alpha^2}{\beta} - x \right) \cdot \sqrt{1 - \left(\frac{x-2}{2} \right)^2}.$$

The initial weight function now is the weight function of Chebyshev polynomials of the second kind:

$$w^{(0)}(x) = \sqrt{1 - x^2}, \quad x \in [-1, 1].$$

As in the previous case, by applying the following sequence of transformations

$$w^{(1)}(x) = w^{(0)}\left(\frac{x-2}{2}\right), \quad w^{(2)}(x) = -\frac{\beta}{\pi} \cdot w^{(1)}(x), \quad \check{w}(x) = \left(x - \frac{\alpha^2}{\beta}\right) \cdot w^{(2)}(x)$$

we prove that the sequence $(\check{h}_n)_{n \in \mathbb{N}_0}$ satisfies the same linear difference equation

$$\check{h}_n - (\alpha^2 - \beta)\check{h}_{n-1} + \beta^2\check{h}_{n-2} = 0, \quad (n \geq 2) \quad (14)$$

but with different initial values $\check{h}_0 = \alpha^2 - 2\beta$ and $\check{h}_1 = \alpha^4 - 3\alpha^2\beta + 3\beta^2$. By solving (14) we obtain the second statement of lemma. \square

6 The sequence u_n^{**}

We can also express the sequence $u_n^{**} = u_{n+2}$ as the falling α -binomial transformation of a certain sequence, whose Hankel determinant will be evaluated. First we need to define the generalization of the aerating transform $\mathcal{A}(p)$.

Definition 6.1. For a given sequence $c = (c_n)_{n \in \mathbb{N}_0}$, we define its α -aerating transformation $b = \mathcal{A}(c; \alpha)$ by $a_n = \alpha p_n + p_{n+1}$, where $p = \mathcal{A}(c)$. In other words, if $a = \mathcal{A}(c; \alpha)$ then $a = (\alpha c_0, c_1, \alpha c_1, c_2, \alpha c_2, c_3, \alpha c_3, \dots)$.

Let $a = \mathcal{A}(c; \alpha)$, i.e. the α -aerating transform of the sequence $c_n = \beta^n C_n$. The sequence $a = (a_n)_{n \in \mathbb{N}_0}$ can be expressed as follows

$$a_n = \begin{cases} \alpha \beta^k C_k, & n = 2k \\ \beta^k C_k, & n = 2k - 1 \end{cases}. \quad (15)$$

According to (3) we have

$$\begin{aligned} u_n^{**} &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2k} \alpha^{n+1-2k} \beta^k C_k \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2k-1} \alpha^{n-(2k-1)} \beta^k C_k + \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2k} \alpha^{n-2k} (\alpha \beta^k C_k) \\ &= \sum_{l=0}^n \binom{n}{l} \alpha^{n-l} a_l. \end{aligned}$$

Hence $u^{**} = \mathcal{B}(a; \alpha)$ and from Lemma 3.1 we conclude that $\mathcal{H}(u^{**}) = \mathcal{H}(a)$. To evaluate $\mathcal{H}(a)$, we need the following theorem.

From now on, we denote by $[\mathbf{A}]_{m \times m}$ a matrix formed by first m rows and columns of the (infinite) matrix \mathbf{A} . Also we label rows and columns of matrices starting from 0 (i.e. 0, 1, 2, ...).

Theorem 6.1. *Let $g = \mathcal{H}(a)$ and $a = \mathcal{A}(c; \alpha)$. Then the following holds*

$$g_n = \det[a_{i+j}]_{0 \leq i, j \leq n} = \begin{cases} \det[\alpha^2 c_{i+j} - c_{i+j+1}]_{0 \leq i, j \leq k-1} \cdot \det[c_{i+j+1}]_{0 \leq i, j \leq k-1}, & n = 2k - 1 \\ \alpha \cdot \det[\alpha^2 c_{i+j+1} - c_{i+j+2}]_{0 \leq i, j \leq k-1} \cdot \det[c_{i+j+1}]_{0 \leq i, j \leq k}, & n = 2k \end{cases}. \quad (16)$$

Proof. The determinant $\det[a_{i+j}]_{0 \leq i, j \leq n}$ that we wish to evaluate, has the form

$$\det[a_{i+j}]_{0 \leq i, j \leq n} = \det \begin{bmatrix} \alpha c_0 & c_1 & \alpha c_1 & c_2 & \cdots \\ c_1 & \alpha c_1 & c_2 & \alpha c_2 & \\ \alpha c_1 & c_2 & \alpha c_2 & c_3 & \\ c_2 & \alpha c_2 & c_3 & \alpha c_3 & \\ \vdots & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}.$$

We distinguish two cases depending on the parity of n .

Case 1. $n = 2k - 1$ is odd. We multiply column $2j + 1$ by α^{-1} and subtract from the column $2j$, for every $j = 0, 1, \dots, k - 2$. That leads to the following determinant

$$\det[a_{i+j}]_{0 \leq i, j \leq n} = \det \begin{bmatrix} \alpha c_0 - \alpha^{-1} c_1 & c_1 & \alpha c_1 - \alpha^{-1} c_2 & c_2 & \cdots \\ 0 & \alpha c_1 & 0 & \alpha c_2 & \\ \alpha c_1 - \alpha^{-1} c_2 & c_2 & \alpha c_2 - \alpha^{-1} c_3 & c_3 & \\ 0 & \alpha c_2 & 0 & \alpha c_3 & \\ \vdots & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}$$

By exchanging rows and columns we get the block diagonal form:

$$\det[a_{i+j}]_{0 \leq i, j \leq n} = \det \begin{bmatrix} \mathbf{A} & * \\ & \mathbf{B} \end{bmatrix} = \det \mathbf{A} \cdot \det \mathbf{B},$$

where star (*) denotes the appropriate $k \times k$ matrix which does not have an influence in determinant computation. Matrices \mathbf{A} and \mathbf{B} are given by

$$\mathbf{A} = [\alpha c_{i+j} - \alpha^{-1} c_{i+j+1}]_{0 \leq i, j \leq k-1}, \quad \mathbf{B} = [\alpha c_{i+j+1}]_{0 \leq i, j \leq k-1}.$$

By taking α from each column of matrix \mathbf{B} and putting to the corresponding column of matrix \mathbf{A} , we get the first case of the expression (16).

Case 2. $n = 2k$ is even. Multiplying column $2j$ by α^{-1} and subtracting from the column $2j - 1$ (for every $j = 1, 2, \dots, k$) yields the determinant

$$\det[a_{i+j}]_{0 \leq i, j \leq n} = \det \begin{bmatrix} \alpha c_0 & 0 & \alpha c_1 & 0 & \alpha c_2 & \cdots \\ c_1 & \alpha c_1 - \alpha^{-1} c_2 & c_2 & \alpha c_2 - \alpha^{-1} c_3 & c_3 & \\ \alpha c_1 & 0 & \alpha c_2 & 0 & \alpha c_3 & \\ c_2 & \alpha c_2 - \alpha^{-1} c_3 & c_3 & \alpha c_3 - \alpha^{-1} c_4 & \alpha c_3 & \\ \alpha c_2 & 0 & \alpha c_3 & 0 & \alpha c_4 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}.$$

Again, by exchanging rows and columns we get the block diagonal form:

$$\det[a_{i+j}]_{0 \leq i, j \leq n} = \det \begin{bmatrix} \mathbf{A}' & * \\ & \mathbf{B}' \end{bmatrix} = \det \mathbf{A}' \cdot \det \mathbf{B}',$$

where

$$\mathbf{A}' = [\alpha c_{i+j+1}]_{0 \leq i, j \leq k}, \quad \mathbf{B}' = [\alpha c_{i+j+1} - \alpha^{-1} c_{i+j+2}]_{0 \leq i, j \leq k-1}.$$

By taking α from each column of the matrix \mathbf{B}' and putting the first k entries to the corresponding columns of the matrix \mathbf{A}' , we obtain the second part of (16). \square

According to the previous theorem, Proposition 4.2 and the fact that $\mathcal{H}((C_n)_{n \in \mathbb{N}_0}) = (1)_{n \in \mathbb{N}_0}$, for any odd $n = 2k - 1$, it holds that

$$\begin{aligned} \det[a_{i+j}]_{0 \leq i, j \leq n} &= \det[\alpha^2 \beta^{i+j} C_{i+j} - \beta^{i+j+1} C_{i+j+1}]_{0 \leq i, j \leq k-1} \cdot \det[\beta^{i+j+1} C_{i+j+1}]_{0 \leq i, j \leq k-1} \\ &= \beta^{\binom{n}{2}} \det[\alpha^2 C_{i+j} - \beta C_{i+j+1}]_{0 \leq i, j \leq k-1} \cdot \det[C_{i+j+1}]_{0 \leq i, j \leq k-1} \\ &= \beta^{\binom{n}{2}} \det[\alpha^2 C_{i+j} - \beta C_{i+j+1}]_{0 \leq i, j \leq k-1}. \end{aligned}$$

Similarly, for even $n = 2k$, we have

$$\begin{aligned} \det[a_{i+j}]_{0 \leq i, j \leq n} &= \alpha \det[\alpha^2 \beta^{i+j+1} C_{i+j+1} - \beta^{i+j+2} C_{i+j+2}]_{0 \leq i, j \leq k-1} \cdot \det[\beta^{i+j+1} C_{i+j+1}]_{0 \leq i, j \leq k} \\ &= \alpha \beta^{\frac{n^2+2n+2}{2}} \det[\alpha^2 C_{i+j+1} - \beta C_{i+j+2}]_{0 \leq i, j \leq k-1}. \end{aligned}$$

Now expression (5) (i.e. Theorem 2.2) is directly obtained using Theorem 5.3.

7 Hankel-like determinants based on Catalan numbers

Before we proceed to the reevaluation of the Hankel transform of the sequence $(u_n)_{n \in \mathbb{N}_0}$, we need to prove two lemmas concerning determinants which are generalizations of the Hankel determinants. Our main tool is the following theorem proven by Krattenthaler in [8] (Theorem 3):

Theorem 7.1. [8] *Let n be a positive integer and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ non-negative integers. Then*

$$\det[C_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = \prod_{0 \leq i < j \leq k-1} (\alpha_j - \alpha_i) \prod_{i=0}^{k-1} \frac{(i+k)!(2\alpha_i)!}{(2i)!\alpha_i!(\alpha_i+k)!}. \quad (17)$$

We use the notation $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise. Also, we assume that the sequence $c = (c_n)_{n \in \mathbb{N}_0}$ is defined by $c_n = \beta^n C_n$.

Lemma 7.2. *For every $l = 0, 1, \dots, k-1$ we have*

$$\det[c_{i+j+\chi(j \geq l)+1}]_{0 \leq i, j \leq k-1} = \beta^{k^2+k-l} \binom{l+k+1}{2l+1}.$$

Proof. Denote by $\alpha_i = i + \chi(i \geq l) + 1$. We need to evaluate the determinant

$$\det[c_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = \det[\beta^{\alpha_i+j} C_{\alpha_i+j}]_{0 \leq i, j \leq k-1}.$$

By taking β^{α_i} from row i ($i = 0, 1, \dots, k-1$) and then β^j from column j ($j = 0, 1, \dots, k-1$) we get the total power of β equal to

$$\sum_{i=0}^{k-1} \alpha_i + \sum_{j=0}^{k-1} j = k^2 + k - l.$$

Hence, our determinant reduces to

$$\det[c_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = \beta^{k^2+k-l} \det[C_{\alpha_i+j}]_{0 \leq i, j \leq k-1}. \quad (18)$$

According to the Theorem 7.1, we need to compute the following products

$$P_1 = \prod_{0 \leq i < j \leq k-1} (\alpha_j - \alpha_i), \quad P_2 = \prod_{i=0}^{k-1} \frac{(i+k)!(2\alpha_i)!}{(2i)!\alpha_i!(\alpha_i+k)!}.$$

By direct evaluation it can be shown that

$$P_1 = \binom{k}{l} \prod_{j=0}^{k-1} j!, \quad P_2 = \frac{k!(l+1)!}{\prod_{j=0}^{k+1} j!} \cdot \frac{(2k+2)(l+k+1)!}{(2l+2)!}. \quad (19)$$

The first product was evaluated considering three different cases ($i < j < l$, $l \leq i < j$, $i \leq l < j$), while for the second we only needed to distinguish between $i < l$ and $i \geq l$. Now using (19) and Theorem 7.1 we obtain

$$\det[C_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = P_1 \cdot P_2 = \binom{l+k+1}{2l+1}.$$

Now the statement of the lemma follows directly from the previous equation and (18). \square

Lemma 7.3. *For every $l = 0, 1, \dots, k-1$ we have*

$$\det[c_{i+j+\chi(j \geq l)}]_{0 \leq i, j \leq k-1} = \beta^{k^2-l} \binom{l+k}{2l}.$$

Proof. We use again the same procedure. Now we denote $\alpha_i = i + \chi(i \geq l)$ and obtain

$$\det[c_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = \det[\beta^{\alpha_i+j} C_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = \beta^{k^2-l} \det[C_{\alpha_i+j}]_{0 \leq i, j \leq k-1}.$$

The product P_1 has the same value as in the previous case, while P_2 is equal to

$$P_2 = \frac{(l+1)!}{\prod_{j=0}^k j!} \cdot \frac{(l+k)!}{(2l)!}$$

Again, by replacing $\det[C_{\alpha_i+j}]_{0 \leq i, j \leq k-1} = P_1 \cdot P_2$ (Theorem 7.1) we obtain the statement of the lemma. \square

8 The sequence u_n

In the section 6 we proved that $u^{**} = \mathcal{B}(a; \alpha)$, where a is the α -aerating transform of the sequence $c_n = \beta^n C_n$ ($a = \mathcal{A}(c; \alpha)$), i.e. (equation (15)):

$$a_n = \begin{cases} \alpha \beta^k C_k, & n = 2k \\ \beta^k C_k, & n = 2k - 1 \end{cases}.$$

According to Lemma 3.2, we have $\mathbf{H}_{u^{**}} = \mathbf{B}^\alpha \mathbf{H}_a (\mathbf{B}^\alpha)^T$. We have already proved that $u^* = \mathcal{B}(p; \alpha)$ (section 4) where $p = \mathcal{A}(c)$, i.e.

$$p_n = \begin{cases} \beta^k C_k, & n = 2k \\ 0, & n = 2k - 1 \end{cases}.$$

This can be written in matrix notation as $u^* = \mathbf{B}^\alpha p$. Now we have that the following matrix equality holds:

$$\begin{bmatrix} 1 & \\ & \mathbf{B}^\alpha \end{bmatrix} \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix} \begin{bmatrix} 1 & \\ & (\mathbf{B}^\alpha)^T \end{bmatrix} = \begin{bmatrix} 0 & p^T (\mathbf{B}^\alpha)^T \\ \mathbf{B}^\alpha p & \mathbf{B}^\alpha \mathbf{H}_a (\mathbf{B}^\alpha)^T \end{bmatrix} = \begin{bmatrix} 0 & (u^*)^T \\ u^* & \mathbf{H}_{u^{**}} \end{bmatrix} = \mathbf{H}_u \quad (20)$$

Hence, the determinant of the $(n+1) \times (n+1)$ principal minor of \mathbf{H}_u , formed by the rows and columns with indices $1, 2, \dots, n+1$, is equal to the same minor of the matrix

$$\mathbf{H}' = \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix}.$$

That minor is exactly h_n , i.e. n -th member of the Hankel transform $h = \mathcal{H}(u)$. In other words, we have to compute

$$h_n = \det[\mathbf{H}_u]_{(n+1) \times (n+1)} = \det \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix}_{(n+1) \times (n+1)} = \det \begin{bmatrix} 0 & c_0 & 0 & c_1 & 0 & \cdots \\ c_0 & \alpha c_0 & c_1 & \alpha c_1 & c_2 & \\ 0 & c_1 & \alpha c_1 & c_2 & \alpha c_2 & \\ c_1 & \alpha c_1 & c_2 & \alpha c_2 & c_3 & \\ 0 & c_2 & \alpha c_2 & c_3 & \alpha c_3 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}.$$

We distinguish two cases depending on the parity of n .

Case 1. $n = 2k - 1$ is odd. Multiplying the column $2j$ by α and subtracting from column $2j + 1$ ($j = 0, 1, \dots, k - 1$) yields

$$h_n = \det \begin{bmatrix} 0 & c_0 & 0 & c_1 & 0 & \cdots \\ c_0 & 0 & c_1 & 0 & c_2 & \\ 0 & c_1 & \alpha c_1 & c_2 - \alpha^2 c_1 & \alpha c_2 & \\ c_1 & 0 & c_2 & 0 & c_3 & \\ 0 & c_2 & \alpha c_2 & c_3 - \alpha^2 c_2 & \alpha c_3 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}$$

By exchanging rows and columns in the previous determinant we obtain

$$h_n = (-1)^k \det \begin{bmatrix} \mathbf{A} & * \\ & \mathbf{B} \end{bmatrix} = (-1)^k \det \mathbf{A} \cdot \det \mathbf{B} \quad (21)$$

where the matrices \mathbf{A} and \mathbf{B} are equal to

$$\mathbf{A} = \det \begin{bmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 - \alpha^2 c_1 & & c_{k-2} - \alpha^2 c_{k-1} \\ \vdots & & \ddots & \\ c_{k-1} & c_{k-2} - \alpha^2 c_{k-1} & & c_{2k-2} - \alpha^2 c_{2k-3} \end{bmatrix}, \quad \mathbf{B} = [c_{i+j}]_{0 \leq i, j \leq k-1}.$$

Since $c_n = \beta^n C_n$, using Proposition 4.2 we obtain

$$\det \mathbf{B} = \det[\beta^{i+j} C_{i+j}]_{0 \leq i, j \leq k-1} = \beta^{k(k-1)} \det[C_{i+j}]_{0 \leq i, j \leq k-1} = \beta^{k(k-1)}. \quad (22)$$

By adding column j to column $j+1$ of the matrix \mathbf{A} ($j = 0, 1, \dots, k-2$) we obtain the following determinant

$$\det \mathbf{A} = \det \begin{bmatrix} c_0 & \alpha^2 c_0 + c_1 & \alpha^4 c_0 + \alpha^2 c_1 + c_2 & \cdots \\ c_1 & c_2 & c_3 & \\ c_2 & c_3 & c_4 & \\ \vdots & & & \ddots \end{bmatrix}_{k \times k}.$$

Expanding over the first row yields

$$\det \mathbf{A} = \sum_{l=0}^{k-1} (-1)^l \left(\sum_{h=0}^l \alpha^{2h} c_{l-h} \right) \det[c_{i+j+\chi(j \geq l)+1}]_{0 \leq i, j \leq k-2}. \quad (23)$$

Using Lemma 7.2 together with the expressions (21), (22) and (23), we obtain

$$h_{2k-1} = \beta^{(k-1)(2k-1)} \sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^l \alpha^{2h} \beta^{k-1-h} C_{l-h} \right) \binom{l+k}{2l+1}. \quad (24)$$

Case 2. $n = 2k$ is even. Subtracting column $2j-1$ from column $2j$ ($j = 0, 1, \dots, k-1$) we obtain

$$h_n = \det \begin{bmatrix} 0 & c_0 & -\alpha c_0 & c_1 & 0 & \cdots \\ c_0 & \alpha c_0 & c_1 - \alpha^2 c_0 & \alpha c_1 & c_2 - \alpha^2 c_1 & \\ 0 & c_1 & 0 & c_2 & 0 & \\ c_1 & \alpha c_1 & c_2 - \alpha^2 c_1 & \alpha c_2 & c_3 - \alpha^2 c_2 & \\ 0 & c_2 & 0 & c_3 & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1) \times (n+1)}$$

By exchanging rows and columns in the previous determinant we obtain

$$h_n = (-1)^k \det \begin{bmatrix} \mathbf{A} & * \\ & \mathbf{B} \end{bmatrix} = (-1)^k \det \mathbf{A} \cdot \det \mathbf{B} \quad (25)$$

where the matrices \mathbf{A} and \mathbf{B} are equal to

$$\mathbf{A} = \det \begin{bmatrix} 0 & -\alpha c_0 & \cdots & -\alpha c_{k-1} \\ c_0 & c_1 - \alpha^2 c_0 & & c_k - \alpha^2 c_{k-1} \\ \vdots & & \ddots & \\ c_{k-1} & c_k - \alpha^2 c_{k-1} & & c_{2k-1} - \alpha^2 c_{2k-2} \end{bmatrix}, \quad \mathbf{B} = [c_{i+j+1}]_{0 \leq i, j \leq k-1}.$$

Since $c_n = \beta^n C_n$, using Proposition 4.2 we obtain

$$\det \mathbf{B} = \det[\beta^{i+j+1} C_{i+j+1}]_{0 \leq i, j \leq k-1} = \beta^{k^2} \det[C_{i+j}]_{0 \leq i, j \leq k-1} = \beta^{k^2}. \quad (26)$$

Again, by multiplying column j by α and adding to column $j + 1$ ($j = 0, 1, \dots, k - 1$), we obtain the following determinant

$$\det \mathbf{A} = \det \begin{bmatrix} 0 & -\alpha c_0 & \alpha^3 c_0 - \alpha c_1 & -\alpha^5 c_0 - \alpha^3 c_1 - \alpha c_2 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \\ c_1 & c_2 & c_3 & c_4 & \\ c_2 & c_3 & c_4 & c_5 & \\ \vdots & & & & \ddots \end{bmatrix}_{(k+1) \times (k+1)}.$$

which can be expanded by the first row in the following way

$$\det A = \sum_{l=1}^k (-1)^l \left(\sum_{h=0}^{l-1} \alpha^{2h+1} c_{l-1-h} \right) \det [c_{i+j+\chi(j \geq l)}]_{0 \leq i, j \leq k-1}. \quad (27)$$

Now using Lemma 7.3 and expressions (25), (26) and (27) we obtain

$$h_{2k} = \beta^{k(2k-1)} \sum_{l=1}^k (-1)^{k+l-1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} \beta^{k-1-h} C_{l-1-h} \right) \binom{l+k}{2l}. \quad (28)$$

Proof of Theorem 2.3. We can rewrite expressions (24) and (28) as follows (we exchanged the order of summation):

$$\begin{aligned} h_{2k-1} &= \beta^{(k-1)(2k-1)} \sum_{h=0}^{k-1} \alpha^{2h} \beta^{k-1-h} \sum_{l=h}^{k-1} (-1)^{k+l} C_{l-h} \binom{l+k}{2l+1}, \\ h_{2k} &= \beta^{k(2k-1)} \sum_{h=0}^{k-1} \alpha^{2h+1} \beta^{k-1-h} \sum_{l=h+1}^k (-1)^{k+l-1} C_{l-1-h} \binom{l+k}{2l}. \end{aligned} \quad (29)$$

Now let $z_n = \beta^{-\binom{n}{2}} h_n$ and in the second equation of (29) decrease the bounds for l by 1. Expressions for z_{2k} and z_{2k-1} are

$$\begin{aligned} z_{2k-1} &= \sum_{h=0}^{k-1} \alpha^{2h} \beta^{k-1-h} \sum_{l=h}^{k-1} (-1)^{k+l} C_{l-h} \binom{l+k}{2l+1}, \\ z_{2k} &= \sum_{h=0}^{k-1} \alpha^{2h+1} \beta^{k-1-h} \sum_{l=h}^{k-1} (-1)^{k+l} C_{l-h} \binom{l+k+1}{2l+2}. \end{aligned} \quad (30)$$

By direct verification we conclude that z_n satisfies the three-term linear difference equation

$$z_{n+2} - \alpha z_{n+1} + \beta z_n = 0$$

for all $n \in \mathbb{N}_0$, which directly implies the expression (6):

$$h_n = \beta^{\binom{n}{2}} z_n = \frac{\beta^{\binom{n}{2}}}{2^n \sqrt{\alpha^2 - 4\beta}} \left[\left(\alpha - \sqrt{\alpha^2 - 4\beta} \right)^n - \left(\alpha + \sqrt{\alpha^2 - 4\beta} \right)^n \right].$$

This completes the proof of Theorem 2.3.

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