Computer experiments
with Mersenne primes

Marek Wolf

Cardinal Stefan Wyszynski University
Department of Mathematics and Sciences
ul. Wóycickiego 1/3, Auditorium Maximum, (room 113)
PL-01-938 Warsaw, Poland
e-mail: m.wolf@uksw.edu.pl

Abstract

We have calculated on the computer the sum $\overline{\mathcal{B}}_M$ of reciprocals of all 47 known Mersenne primes with the accuracy of over 12000000 decimal digits. Next we developed $\overline{\mathcal{B}}_M$ into the continued fraction and calculated geometrical means of the partial denominators of the continued fraction expansion of $\overline{\mathcal{B}}_M$. We get values converging to the Khinchin's constant. Next we calculated the n-th square roots of the denominators of the n-th convergents of these continued fractions obtaining values approaching the Khinchin-Lèvy constant. These two results suggests that the sum of reciprocals of all Mersenne primes is irrational, supporting the common believe that there is an infinity of the Mersenne primes. For comparison we have done the same procedures with slightly modified set of 47 numbers obtaining quite different results. Next we investigated the continued fraction whose partial quotients are Mersenne primes and we argue that it should be transcendental.

1 Introduction.

The Mersenne primes $\mathcal{M}_n$ are primes of the form $2^p - 1$ where p must be a prime, see e.g. [18, Sect. 2.VII]. The set of Mersenne primes starts with $\mathcal{M}_1 = 2^2 - 1, \mathcal{M}_2 = 2^3 - 1, \mathcal{M}_3 = 2^5 - 1$ and only 47 primes of this form are currently known, see Great Internet Mersenne Prime Search (GIMPS) at www.mersenne.org. The largest known Mersenne prime has the value $\mathcal{M}_{47} = 2^{43112609} - 1 = 3.1647026933\ldots \times 10^{12978188}$. In general the largest known primes are the Mersenne primes, as the Lucas–Lehmer primality test applicable only to numbers of the form $2^p - 1$ needs a multiple of p steps, thus the complexity of checking primality of $\mathcal{M}_n$ is $\mathcal{O}(\log(\mathcal{M}_n))$. Let us remark that algorithm of Agrawal, Kayal and Saxena (AKS) for arbitrary prime p works in about $\mathcal{O}(\log^{7.5}(p))$ steps and modification by Lenstra and Pomerance has complexity $\mathcal{O}(\log^6(p))$.

There is no proof of the infinitude of $\mathcal{M}_n$, but a common belief is that as there are presumedly infinitely many even perfect numbers thus there is also an infinity of Mersenne primes. S. S. Wagstaff Jr. in [26] (see also [21, §3.5]) gave heuristic arguments, that $\mathcal{M}_n$ grow doubly exponentially:

$$\log_2 \log_2 \mathcal{M}_n \sim ne^{-\gamma}, \tag{1}$$

where $\gamma = 0.57721566\ldots$ is the Euler–Mascheroni constant. In the Fig. 1 we compare the Wagstaff conjecture with all 47 presently known Mersenne primes $\mathcal{M}_n$. Of these 47 known $\mathcal{M}_n = 2^p - 1$ there are 27 with $p \mod 4 = 1$ and 19 with $p \mod 4 = 3$. It is in opposite to the set of all primes where the phenomenon of Chebyshev bias is known: for initial primes there are more primes $p \equiv 3 \pmod 4$ than $p \equiv 1 \pmod 4$, [11], [19].

In this paper we are going to exploit two facts about the continued fractions to support the conjecture on the infinitude of Mersenne primes: the existence of the Khinchin constant and Khinchin–Lèvy constant. We calculate the sum of reciprocals of the Mersenne primes $\mathcal{B}_M = \sum_n 1/\mathcal{M}_n$; if there is infinity of Mersenne primes then this number $\mathcal{B}_M$ should be irrational (at least, because it is probably even transcendental, as it is difficult to imagine the polynomial with some mysterious integer coefficients whose one of roots should be $\mathcal{B}_M$).

There exists a method based on the continued fraction expansion which allows to detect whether a given number r can be irrational or not. Let

$$r = [a_0(r); a_1(r), a_2(r), a_3(r), \ldots] = a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{a_3(r) + \ddots}}} \tag{2}$$

be the continued fraction expansion of the real number r, where $a_0(r)$ is an integer and all $a_k(r)$ with $k \geq 1$ are positive integers. The quantities $a_k(r)$ are called partial quotients or the partial denominators. Khinchin has proved [13], see also [20], that

$$\lim_{n \to \infty} \left(a_1(r) \ldots a_n(r)\right)^{\frac{1}{n}} = \prod_{m=1}^{\infty} \left\{1 + \frac{1}{m(m+2)}\right\}^{\log_2 m} \equiv K \approx 2.685452001 \tag{3}$$

is a constant for almost all real r [8, §1.8] (the term a_0 is skipped in (3)). The exceptions are of the Lebesgue measure zero and include *rational numbers*, quadratic irrationals and some irrational numbers too, like for example the Euler constant $e = \lim_{n \to \infty}(1 + \frac{1}{n})^n = 2.7182818285\ldots$

for which the n-th geometrical mean tends to infinity like $\sqrt[3]{n}$, see [10, §14.3 (p.160)]. The constant K is called the Khinchin constant. If the sequence

$$K(r; n) = \big(a_1(r)a_2(r)\ldots a_n(r)\big)^{\frac{1}{n}} \tag{4}$$

for a given number r tend to K for $n \to \infty$ we can regard it as an indication that r is irrational — all rational numbers have finite number of partial quotients in the continued fraction expansion and hence starting with some n_0 for all $n > n_0$ will be $a_n = 0$. It seems to be possible to construct a sequence of rational numbers such that the geometrical means of partial quotients of their continued fraction will tend to the Khinchin constant.

The Khinchin—Lèvy's constant arises in the following way: Let the rational $P_n(r)/Q_n(r)$ be the n-th partial convergent of the continued fraction of r:

$$\frac{P_n(r)}{Q_n(r)} = [a_0(r); a_1(r), a_2(r), a_3(r), \ldots, a_n(r)]. \tag{5}$$

In 1935 Khinchin [12] has proved that for almost all real numbers r the denominators of the finite continued fraction approximations fulfill:

$$\lim_{n\to\infty} \big(Q_n(r)\big)^{1/n} \equiv \lim_{n\to\infty} \big(L(r; n) = L \tag{6}$$

and in 1936 Paul Levy [14] found an explicit expression for this constant L:

$$\lim_{n\to\infty} \sqrt[n]{Q_n(r)} = e^{\pi^2/12\log(2)} \equiv L = 3.27582291872\ldots \tag{7}$$

L is called the Khinchin—Lèvy's constant [8, §1.8]. Again the set of exceptions to the above limit is of the Lebesgue measure zero and it includes rational numbers, quadratic irrational etc.

2   First experiment

Let us define the sum of reciprocals of all Mersenne primes:

$$\mathcal{B}_M = \sum_{n=1}^{\infty} \frac{1}{\mathcal{M}_n}, \tag{8}$$

which can regarded as the analog of the Brun's constant, i.e. the sum of reciprocals of all twin primes:

$$\mathcal{B}_2 = \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \ldots. \tag{9}$$

In 1919 Brun [5] has shown that this constant $\mathcal{B}_2$ is finite, thus leaving the problem of infinity of twin primes not decided. Today's best numerical value is $\mathcal{B}_2 \approx 1.90216058$, see [16], [22]. Yet it is possible to prove that there is infinity of twins by showing that Brun's constant is irrational [27] (we believe it is even transcendental).

Using PARI [24] we have calculated the sum of reciprocals of all known 47 Mersenne primes $\overline{\mathcal{B}}_M$ with accuracy over 12 millions digits:

$$\overline{\mathcal{B}}_M = 0.5164541789407885653304873429715228588159685534154197\ldots. \tag{10}$$

This number is not recognized by the Symbolic Inverse Calculator (http://pi.lacim.uqam.ca) maintained by Simone Plouffe. The bar over $\mathcal{B}_M$ denotes the finite (at present consisting of 47 terms) approximation to the full sum defined in (8). It is not known, whether there are Mersenne prime numbers with exponent $20996011 < p < 43112609$ — currently confirmed by GIMPS is that $2^{20996011} - 1$ is the 40-th Mersenne prime $\mathcal{M}_{40}$— it is not known whether any undiscovered Mersenne primes exist between the 40th $\mathcal{M}_{40}$ and the 47th Mersenne prime $\mathcal{M}_{47}$. We have taken 12000035 digits of $\overline{\mathcal{B}}_M$ — it means that we assume that there are no unknown Mersenne primes with $p < 39863137$. Using the incredibly fast procedure $\texttt{ContinuedFraction}[\cdot]$ implemented in Mathematica$^{\copyright}$ we calculated the continued fraction expansion of $\overline{\mathcal{B}}_M$. The result was built from 11645012 partial denominators $a_1 = 1, a_2 = 1, a_3 = 14, \ldots, a_{11645012} = 4$. The n-th convergent $P_n(r)/Q_n(r)$, see (5), approximate the value of r with accuracy at least $1/Q_k Q_{k+1}$ [13, Theorem 9, p.9]:

$$\left| r - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}} < \frac{1}{Q_k^2 a_{k+1}} < \frac{1}{Q_k^2}. \tag{11}$$

From this it follows that if r is known with precision of d decimal digits we can continue with calculation of quotients a_n up to such n that the denominator of the n-th convergent $Q_n^2 < 10^d$. We have checked that $Q_{11645012} = 4.291385 \times 10^{6000016}$.

The largest denominator was $a_{9965536} = 716699617$. We have checked correctness of the continued fraction expansion of $\overline{\mathcal{B}}_M$ by calculating backwards from $[0; 1, 1, 14, \ldots, 4]$ the partial convergent. The Mathematica$^{\copyright}$ has build in the procedure $\texttt{FromContinuedFraction}[\cdot]$, but we have used our own procedure written in PARI and implementing the recurrence:

$$P_{n+1} = a_n P_n + P_{n-1}, \qquad Q_{n+1} = a_n Q_n + Q_{n-1}, \quad n \geq 1 \tag{12}$$

with initial values

$$P_0 = a_0, \quad Q_0 = 1, \quad P_1 = a_0 a_1 + 1, \quad Q_1 = a_1. \tag{13}$$

We have obtained the ratio of two mutually prime 6000018 decimal digits long integers (it means denominator was of the order $10^{6000018}$ and hence its square was smaller than $10^{12000035}$, see eq.(11)):

$$\overbrace{\frac{22163041091211233132511438 69 \ldots 2210}{42913857598492245346167160 35 \ldots 2813}}^{6000018 \ digits}$$

whose ratio had 12000033 digits the same as $\overline{\mathcal{B}}_M$. The decimal expansion of $\overline{\mathcal{B}}_M = P/Q$ is of course periodic (recurring), see [9, Th. 135], but the length of the period is much larger than 1.2×10^{12}. According to the Theorem 135 from [9] the period r of the decimal expansion of $\overline{\mathcal{B}}_M$ is equal to the order of 10 mod Q, i.e. it is the smallest positive r for which

$$10^r \equiv 1 (\mod Q). \tag{14}$$

Because Q being the product of all 47 Mersenne primes is of the order $3.509 \ldots \times 10^{86789810}$, we expect that the value of r is much larger than 10^{12}. Another argument is that we got over 11 500 000 partial quotients of the continued fraction of $\overline{\mathcal{B}}_M$ — the numbers with periodic decimal expansions have only finite number of partial quotients different from zero.

From the sequence of partial quotients $a_1 = 1, a_2 = 1, a_3 = 14, \ldots, a_{11645012} = 4$ we have calculated running geometrical means

$$K(n) = \left(\prod_{k=1}^{n} a_k \right)^{1/n} \tag{15}$$

for $n = 11, \ldots, 11645012$. The obtained numbers $K(n)$ quickly tend to the Khinchine constant thus in Fig. 2 we have plotted the differences $|K(n) - K|$. The power fit to the values for $n = 1000 \ldots 11645012$ gives the decrease of the form $|K(n) - K| \sim n^{-0.79}$ and it suggests that indeed $\lim_{n\to\infty} K(n) = K$ and thus $\mathcal{B}_M$ is irrational. Indices n for which the geometric means $K(n)$ produce progressively better approximations to Khinchin's constant are:

$$1, 3, 2, 16, 17, 21, 24, 26, 29, 412, 788, 1045, 369625, 369636, \ldots, 5137093, 10389989; \tag{16}$$

the smallest value of $|K(n) - K|$ was $4.455957\ldots \times 10^{-11}$. This sequence can be regarded as the counterpart to the A048613 at OEIS.org.

Next we calculated running (i.e. for $n = 11, \ldots, 11645012$) partial quotients P_n/Q_n and then the quantities $L(n) = \sqrt[n]{Q_n}$, which for almost all irrational numbers should tend to the Khinchine–Levy constant. The behaviour of $\sqrt[n]{Q_n}$ is shown in Fig.3. Again we see that these quantities tend to the limit L; the fitting of the power–like dependence for $n > 10$ gives that $|L(n) - L| \approx 175.39n^{-0.92}$. The shape of the plot in this figure is similar to the plot of $|K(n) - K|$ in Fig. 2.

Both differences $K(n) - K$ and $L(n) - L$ have a lot sign changes for $n < 11645012$. Figures 4 and 5 present the plots of these differences together with the number of sign changes.

The data presented in Figures 2 and 3 provide the hints that $\mathcal{B}_M$ is irrational and hence that there is infinity of Mersenne primes. But we are convinced $\mathcal{B}_M$ *is in fact transcendental.* In favor of this claim we recall here the result of A. J. van der Poorten and J. Shallit [25] that the following sum

$$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots + \frac{1}{2^{F_n}} + \ldots \tag{17}$$

where F_n are Fibonacci numbers, is transcendental. It is well known that the Liouville number

$$\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \frac{1}{2^{3!}} + \frac{1}{2^{4!}} + \cdots + \frac{1}{2^{n!}} + \ldots \tag{18}$$

is transcendental see [9, Theorem 192]. In $\mathcal{B}_M$, assuming the Wagstaff conjecture, unfortunately the terms decrease slower: $n! > 2^n > 2^{e^{-\gamma}n}$ for $n \geq 4$ but faster than $F_n = \left\lfloor \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$.

Let $\psi_n(m)$ denotes the number of partial quotients a_k with $k = 1, 2, \ldots, n$ which are equal to m:

$$\psi_n(m) = \sharp\{k : k \leq n \text{ and } a_k = m\}.$$

Then the Gauss–Kuzmin theorem (for excellent exposition see e.g. [10, §14.3]) asserts that

$$\lim_{n\to\infty} \frac{\psi_n(m)}{n} = \frac{\log\left(1 + \frac{1}{m(m+2)}\right)}{\log(2)} \tag{19}$$

for continued fractions of almost all real numbers. In other words, the probability to find the partial quotient $a_k = m$ is equal to $\log_2(1 + 1/m(m + 2))$. In Fig. 6 we present the plot of the $\frac{\psi_{11645013}(m)}{11645013}$ for the continued fraction of $\overline{\mathcal{B}}_M$ and $m = 1, 2, \ldots 1000$ together with prediction given by the Gauss–Kuzmin theorem finding excellent agreement.

Finally let us notice, that the number $\overline{\mathcal{B}}_M$ computed with 12000035 digits is normal in the base 10, see Table I.

TABLE I

Illustration of the normality of $\overline{\mathcal{B}}_M$: the numbers in second row gives the number of digits $0, 1, \ldots 9$ appearing in the decimal expansion of $\overline{\mathcal{B}}_M$ and the third row contains the ratio of numbers in second row divided by 12000035.

0	1	2	3	4
1200553	1199322	1199420	1200548	1199397
0.1000458	0.0999432	0.0999514	0.1000454	0.0999495

5	6	7	8	9
1198596	1200876	1200056	1201757	1199510
0.0998827	0.1000727	0.1000044	0.1001461	0.0999589

For comparison we have repeated the above procedure for artificial set of 47 numbers of the size corresponding to known Mersenne primes. We have simply skipped -1 in the Mersenne primes and using PARI we have computed with over 120000000 digits the sum:

$$\mathcal{S} = \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{42643801}} + \frac{1}{2^{43112609}}$$

This number $\mathcal{S}$ is the ratio of the form $A/2^{43112609}$, where $\gcd(A, 243112609) = 1$. From [9, §9.2] we know that $\mathcal{S}$ has *terminating* decimal expansion consisting of 43112609 decimal digits, thus calculating 12000000 digits of this sum makes sens as it does not contain recurring periodic patterns of digits. We have developed $\mathcal{S}$ into the continued fraction, what resulted in 10550114 partial quotients. The calculated quantities for this case we denote with the subscript 2: $Q_2(n), K_2(n), L_2(n)$ to distinguish them from earlier experiment for $\overline{\mathcal{B}}_M$. We have calculated the running geometrical averages of the partial quotients $K_2(n)$ and the results are presented in Figure 7. Next we calculated 10550114 partial convergents $P_2(n)/Q_2(n), n = 1, 2, \ldots, 10550114$ and from them the quantities $L_2(n) \equiv (Q_2(n))^{1/n}$, which should tend to the Levy constant L. In Figure 8 the differences $|L_2(n) - L|$ are plotted. Obtained plots are completely different from those seen in Figures 2 and 3 and they suggest $K_2(n)$ as well as $L_2(n)$ do not have the limit. In this artificial case we have encountered the phenomenon of extremely large partial denominators: there were a_n of the order $10^{70548}, 10^{97732}$ and 10^{279910}. These large partial denominators are responsible for the smaller number of a_k than for $\overline{\mathcal{B}}_M$, see (11).

3 Second experiment

Let us define the supposedly infinite and convergent continued fraction $u_{\mathcal{M}}$ by taking $a_n = \mathcal{M}_n$:

$$u_{\mathcal{M}} = [0; 3, \ 7, \ 31, \ 127, \ 8191, \ 131071, \ 524287, \ 2147483647, \ \ldots] \tag{20}$$

Using all 47 Mersenne primes $3, 7, 31, \ldots, 2^{43112609} - 1$ in a couple of minutes we have calculated $u_\mathcal{M}$ with the precision of 10000000 digits; first 50 digits of $u_\mathcal{M}$ are:

$$u_\mathcal{M} = 0.31824815840584486942596202748140694243806236564\ldots \qquad (21)$$

This number is not recognized by the Symbolic Inverse Calculator (http://pi.lacim.uqam.ca). Because $1/Q_{47}^2(u_\mathcal{M}) \approx 1.84313 \times 10^{-173579621}$ it follows from (11) that theoretically it is possible to obtain the value of $u_\mathcal{M}$ from presently known 47 Mersenne primes with over 170,000,000 decimal digits of accuracy. Of course $u_\mathcal{M}$ is the exception to the Khinchin and Levy Theorems in view of the very fast growth of $u_\mathcal{M}$ — see the Wagstaff [26] conjecture (1).

There is a vast literature concerning the transcendentality of continued fractions. For example the continued fraction

$$[0; 2^{1!}, 2^{2!}, 2^{3!}, \ldots, 2^{n!}, \ldots] \qquad (22)$$

is transcendental, see [9, Theorem 192], [23, Example 4].

The Theorem of H. Davenport and K.F. Roth [7] states, that if the denominators Q_n of convergents of the continued fraction $r = [a_0; a_1, a_2, \ldots]$ fulfill

$$\limsup_n \frac{\sqrt{\log(n)} \log(\log(Q_n(r)))}{n} = \infty \qquad (23)$$

then r is transcendental. This theorem requires for the transcendence of r very fast increase of denominators of the convergents: at least doubly exponential growth is required for (23). The set of continued fractions which can satisfy the Theorem of H. Davenport and K.F. Roth is of measure zero, as it follows from the Theorem 31 from the Khinchin's book [13], which asserts there exists an absolute constant B such that for *almost all* real numbers r and sufficiently large n the denominators of its continued fractions satisfy:

$$Q_n(r) < e^{Bn}. \qquad (24)$$

The paper of A. Baker [4] from 1962 contains a few theorems on the transcendentality of Maillet type continued fractions [15], i. e. continued fractions with bounded partial quotients which have transcencendental values. Besides Maillet continued fractions there are some specific families of other continued fractions of which it is known that they are transcendental. In the papers [17], [2] it was proved that the Thue–Morse continued fractions with bounded partial quotients are transcendental. Quite recently there appeared the preprint [6] where the transcendence of the Rosen continued fractions was established. For more examples see [3].

In the paper [1] B. Adamczewski and Y. Bugeaud, among others, have improved (23) to the form:

$$\text{If} \quad \limsup_n \frac{\log(\log(Q_n(r)))}{n^{2/3} \log(n)^{2/3} \log(\log(n))} = \infty \qquad (25)$$

then r is transcendental.

Assuming the Wagstaff conjecture $\mathcal{M}_n \sim 2^{2^{ne^{-\gamma}}}$ mentioned in the Introduction we obtain that for large n

$$Q_n > 2^{c2^{(n+1)e^{-\gamma}}}, \qquad c = \frac{1}{2^{e^{-\gamma}} - 1} = 2.101893933\ldots \qquad (26)$$

and thus the transcendence of $u_{\mathcal{M}}$ will follow from the Davenport–Roth Theorem (23):

$$\frac{\sqrt{\log(n)}\,\log(\log(Q_n(r)))}{n} \sim \sqrt{\log(n)} \to \infty. \tag{27}$$

We illustrate the inequality (26) in Figure 9 — the values of labels on the y–axis give an idea of the order of the fast grow of $Q_n(u_{\mathcal{M}})$: the largest for $n = 47$ is of the order $Q_{47} = e^{1.9984\ldots\times 10^8} = 2.32928\ldots \times 10^{86789810}$, see also Table II.

TABLE II

A sample of values of inverses of the squares of the n-th convergents of $u_{\mathcal{M}}$ giving an idea of the speed of convergence of $[0; \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n]$ to $u_{\mathcal{M}}$.

n	$1/Q_n^2$
3	$2.131173743 \times 10^{-6}$
4	$1.320662319 \times 10^{-10}$
5	$1.968416969 \times 10^{-18}$
6	$1.145786956 \times 10^{-28}$
7	$4.168364565 \times 10^{-40}$
8	$9.038699842 \times 10^{-59}$
9	$1.699990496 \times 10^{-95}$
$\vdots$	$\vdots$
17	$9.32543401 \times 10^{-4439}$
18	$1.38891910 \times 10^{-6375}$
19	$3.81534516 \times 10^{-8936}$
20	$4.67942175 \times 10^{-11599}$
$\vdots$	$\vdots$
40	$4.50116310 \times 10^{-31553835}$
41	$5.02100786 \times 10^{-46025300}$
42	$3.36434042 \times 10^{-61657758}$
43	$3.38166968 \times 10^{-79961861}$
44	$2.17906011 \times 10^{-99578575}$
45	$5.32688381 \times 10^{-121949118}$
46	$1.84595823 \times 10^{-147623244}$
47	$1.84313029 \times 10^{-173579621}$

One of the transcendence criterion is the Thue-Siegel-Roth Theorem, which we recall here in the following form:

Thue-Siegel-Roth Theorem: If there exist such $\epsilon > 0$ that for *infinitely* many fractions A_n/B_n the inequality

$$\left| r - \frac{A_n}{B_n} \right| < \frac{1}{B_n^{2+\epsilon}}, \qquad n = 1, 2, 3, \ldots, \tag{28}$$

holds, then r is transcendental.

Let us stress, that ϵ here does not depend on n — it has to be the same for all fractions A_n/B_n. We can test the criterion (28) for $u_{\mathcal{M}}$ using as the rational approximations A_n/B_n the convergents P_n/Q_n of the continued fraction (20).

In [23] J. Sondow has given the estimation for ϵ appearing in r.h.s.of (28); namely he proved that for irrational numbers with continued fraction expansion $[a_0; a_1, a_2, \ldots]$ and convergents P_n/Q_n:

$$\epsilon \leq \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log Q_n}. \tag{29}$$

Let us denote $\delta \equiv 2 + \epsilon$. From the Wagstaff conjecture we obtain that the exponent δ of B^δ appearing in on the r.h.s. of (28) should be of the order

$$\delta \approx 2 + 2^{e^{-\gamma}} - 1 = 2.47477\ldots \qquad (\text{i.e.} \quad \epsilon \approx 0.47477\ldots) \tag{30}$$

implying transcendence of $u_{\mathcal{M}}$. In Fig.10 we present actual values of $\delta(u_{\mathcal{M}}; n) = -\log|u_{\mathcal{M}} - P_n/Q_n|/\log(Q_n)$ for $n = 3, 4, \ldots, 45$ and indeed the values oscillate around $1 + 2^{e^{-\gamma}} = 2.47477\ldots$. First we have calculated $u_{\mathcal{M}}$ using all 47 Mersenne primes with accuracy 140000000 digits and for $n = 3, 4, \ldots, 45$ we have calculated convergents P_n/Q_n and next the differences $|u_{\mathcal{M}} - P_n/Q_n|$ with accuracy $1/Q_n^2$ (see Table II), from which we determined the $\delta(u_{\mathcal{M}}; n)$. The arithmetic average of 43 values $\delta(u_{\mathcal{M}}; n)$ is $2.5002\ldots$, quite close to the estimated value (30). It took a few months of CPU time to collect data presented in Fig. 10: It took 12 days of CPU time on the AMD Opteron 2700 MHz processor to collect data for $n \leq 40$; the point $n = 40$ needed precision of almost 40,000,000 digits, as $|u_{\mathcal{M}} - P_{40}/Q_{40}| = 1.5033 \times 10^{-38789567}$, while $1/Q_{40}^2 = 4.501\ldots \times 10^{-31553835}$. To calculate the difference $|u_{\mathcal{M}} - P_n/Q_n|$ for $n = 41, 42, 43$ the precision of 100000000 digits was needed. For $n = 44$ and $n = 45$ the difference $|u_{\mathcal{M}} - P_n/Q_n|$ was calculated with the precision 130000000 digits (see Table II for $n = 44$ and $n = 45$) and it took about one month of CPU time for each point.

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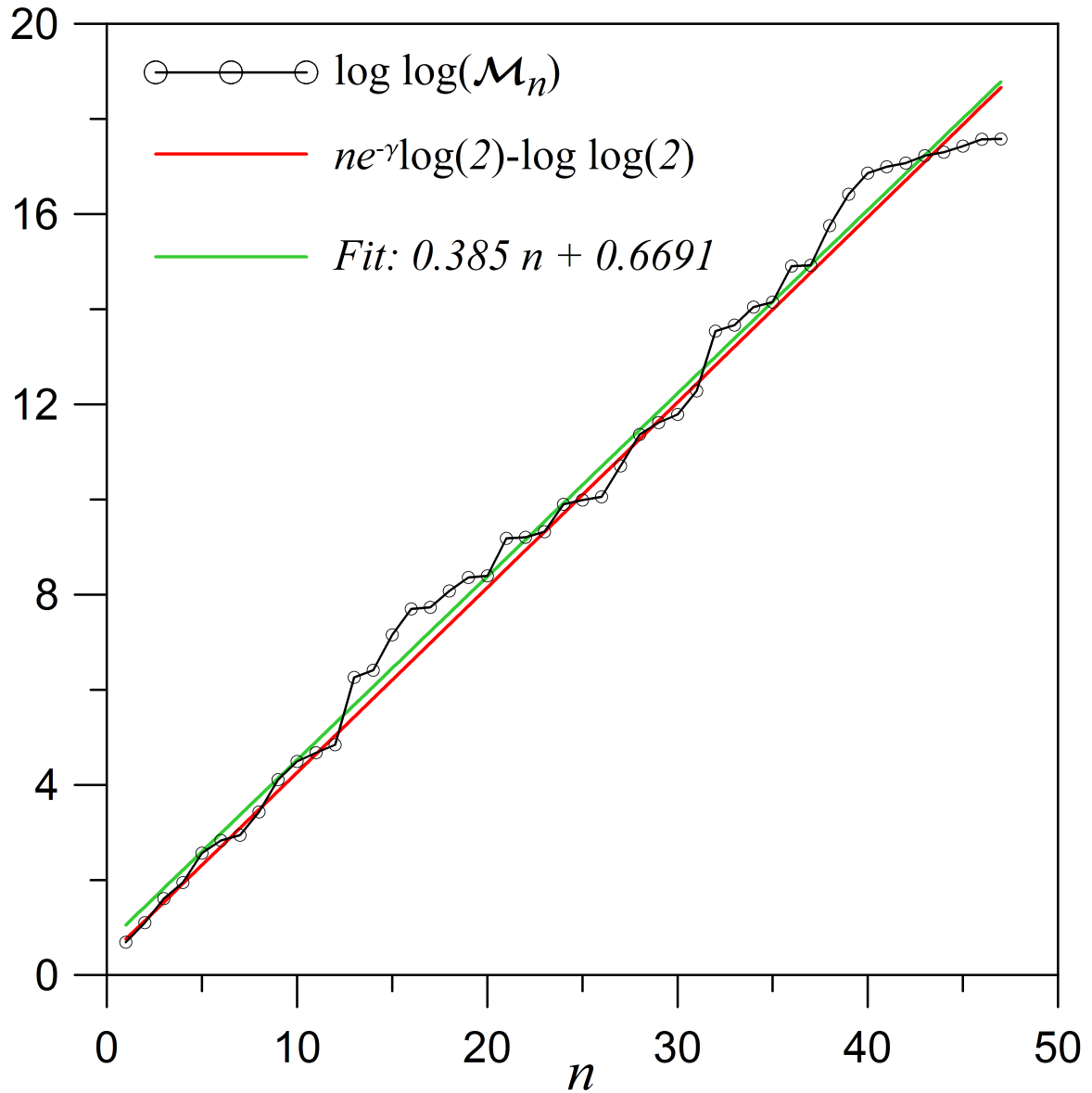[27] M. Wolf. Remark on the irrationality of the Brun's constant. *ArXiv: math.NT/1002.4174*, Feb. 2010.

Figure 1: The plot of $\log\log(\mathcal{M}_n)$ and the Wagstaff conjecture (1). The fit was made to all known $\mathcal{M}_n$ and it is $0.3854n + 0.6691$, while $ne^{-\gamma}\log(2) - \log\log(2) \approx 0.3892n + 0.3665$. The rather good coincidence of $\log\log(\mathcal{M}_n)$ and (1) is seeming, as to get original $\mathcal{M}_n$'s the errors are amplified to huge values by double exponentiation.
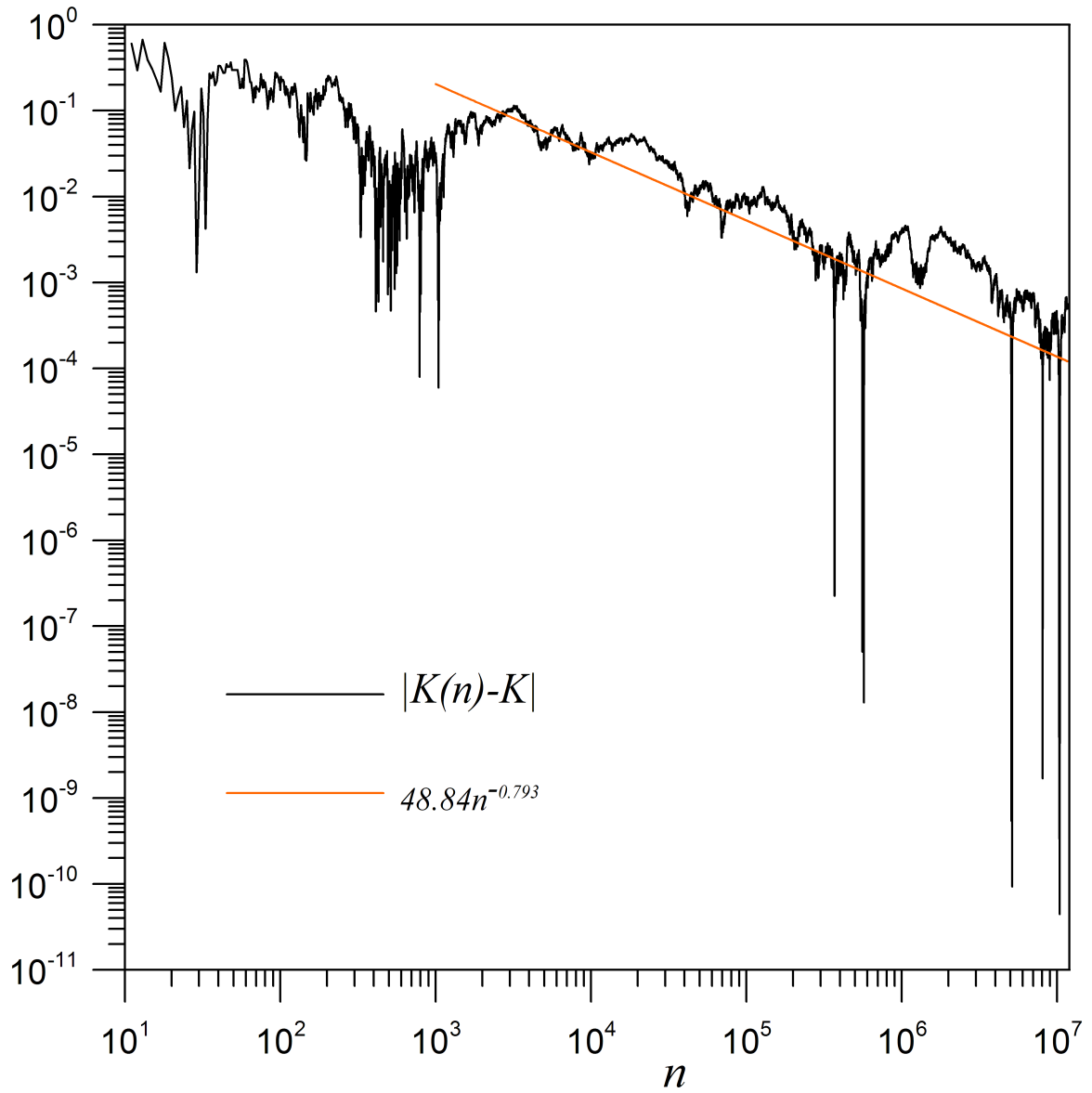
Figure 2: The plot showing the distance to K of the running geometrical averages $K(n) = (a_1 a_2 \cdots a_n)^{1/n}$ for the continued fraction of $\mathcal{B}_M$.
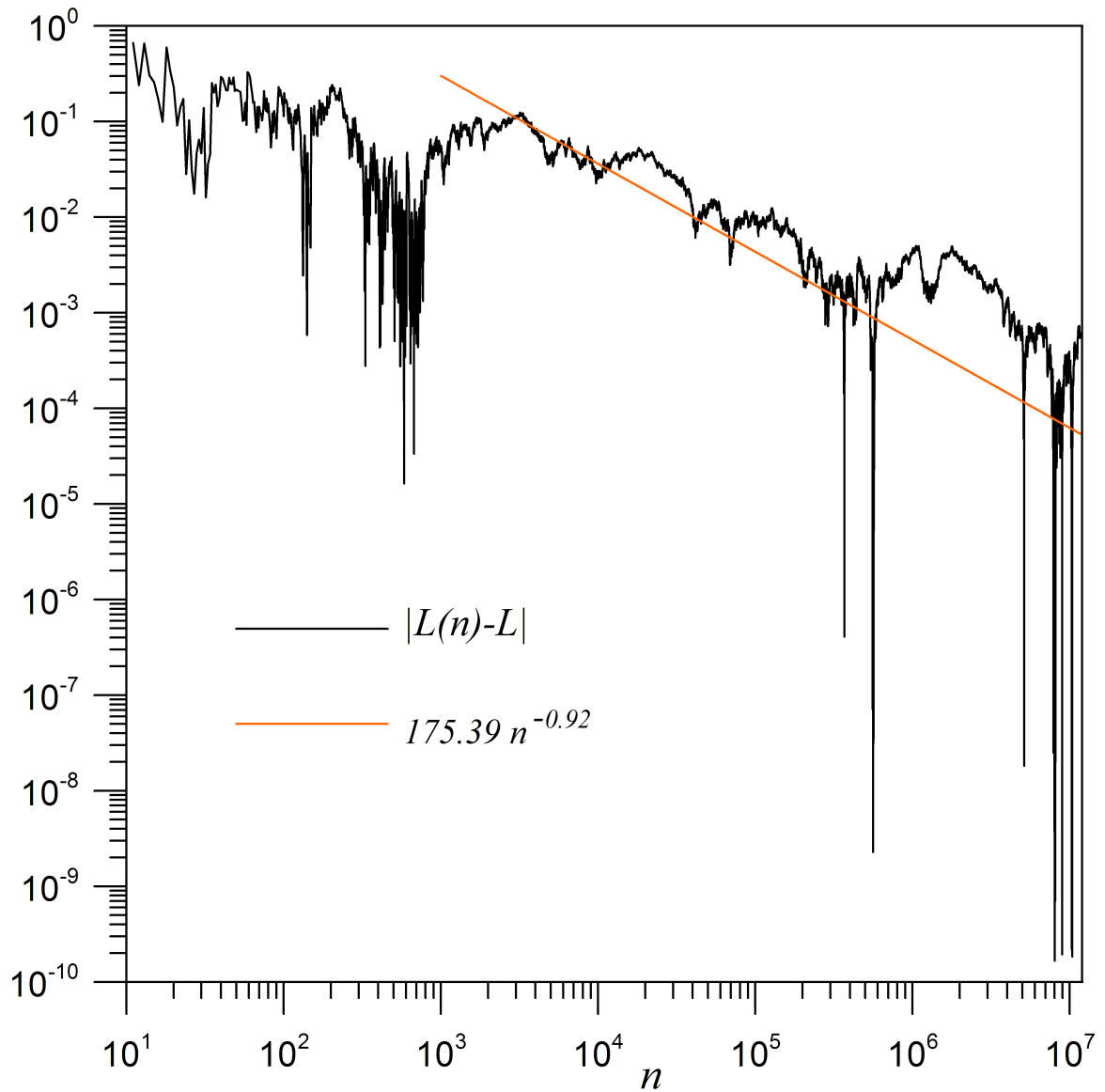
Figure 3: The plot showing the distance to L of the $(Q(n))^{1/n}$ obtained from the partial convergents of the continued fraction of $\mathcal{B}_M$ for $n = 11, \ldots, 11645012$.
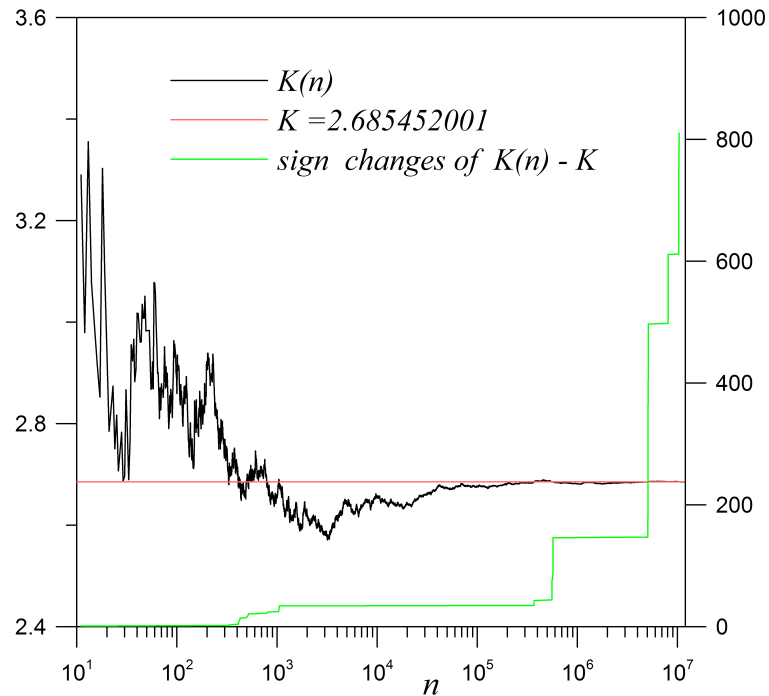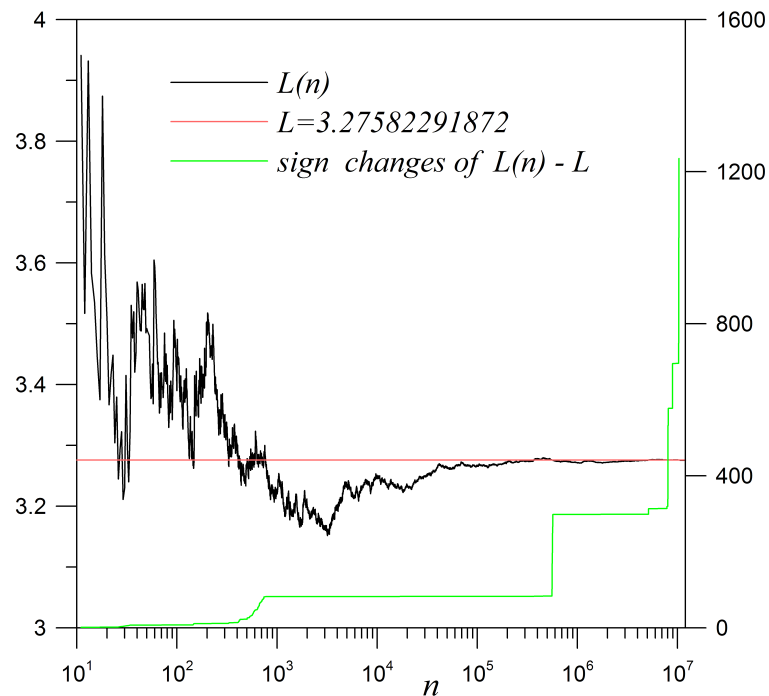
Figure 4: The plot of $K(n)$ in black approaching the Khinchine constant $K = 2.685452\ldots$ (in red) with values presented on left y-axis. In green are presented numbers of sign changes of $K(n) - K$ up to n — the right y-axis is for this plot.
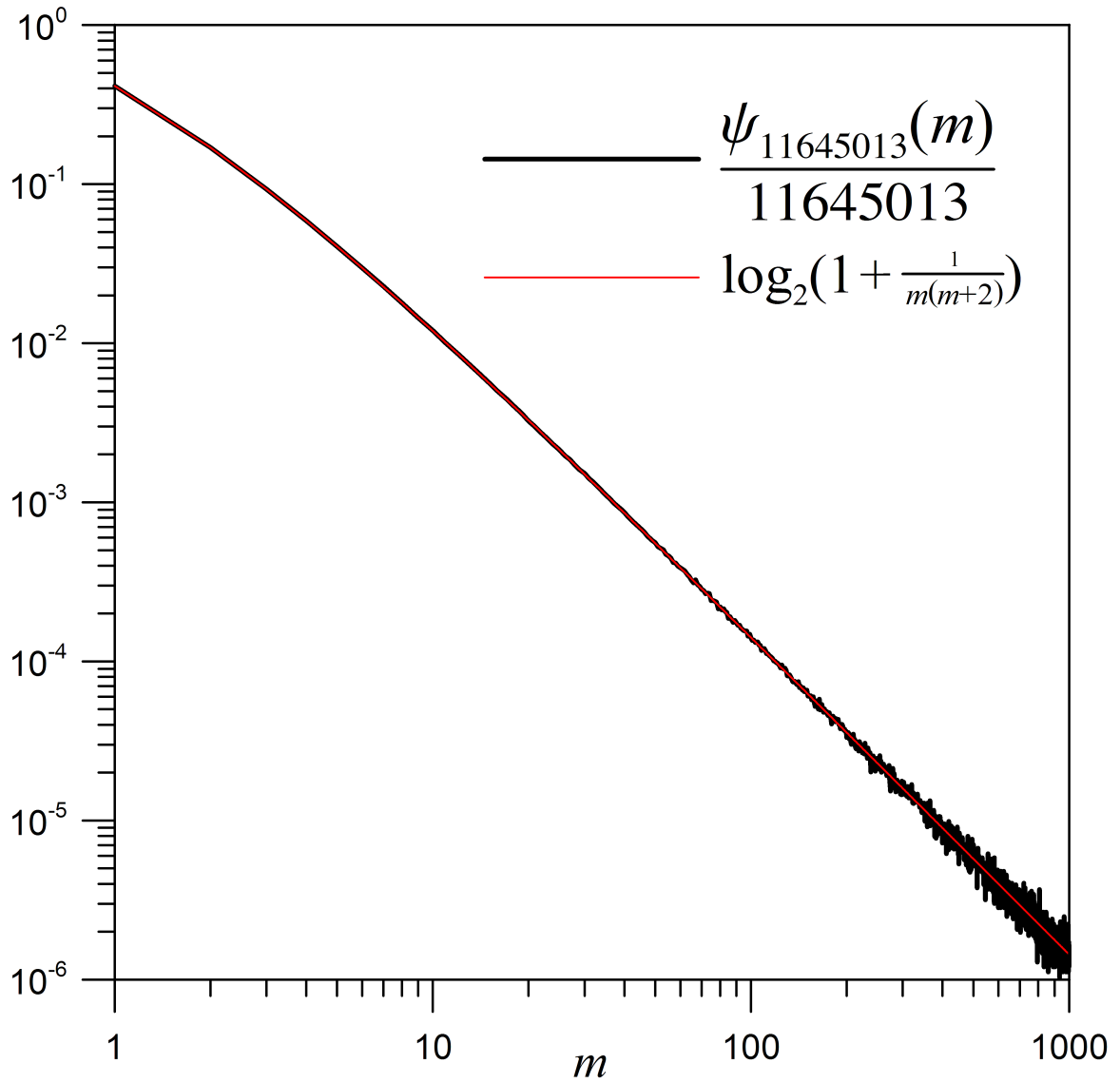

Figure 5: The plot of $L(n)$ in black approaching the Khinchine–Levy constant $L = 3.27582291872\ldots$ (in red) with values presented on left y-axis. In green are presented numbers of sign changes of $L(n) - L$ up to n — the right y-axis is for this plot.

Figure 6: The plot of the measured for the continued fraction of $\mathcal{B}_M$ probability to find the partial quotient $a_k = m$ for the continued fraction of $\mathcal{B}_M$.
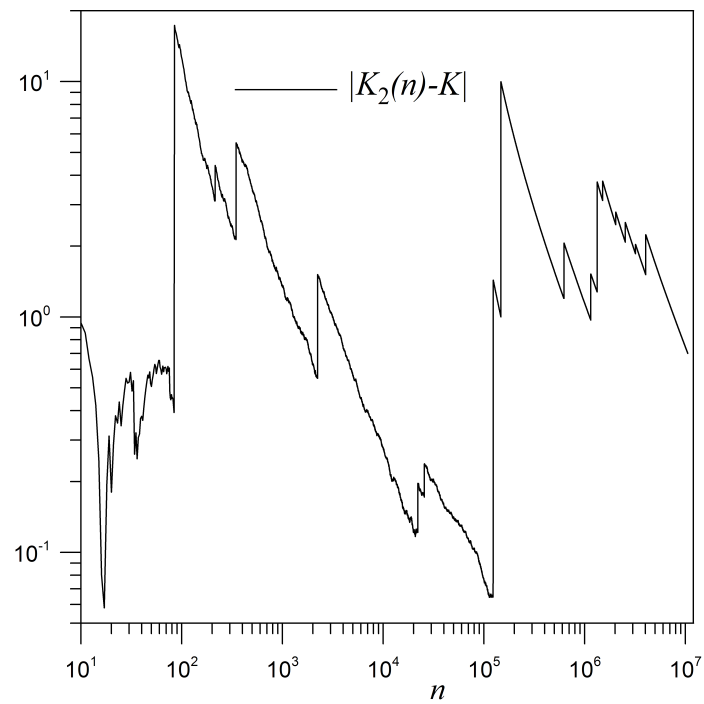
Figure 7: The plot showing the distance to K of the running geometrical averages $K_2(n)$ for the continued fraction of $\mathcal{S}$ for $n = 11, \ldots, 10550114$.
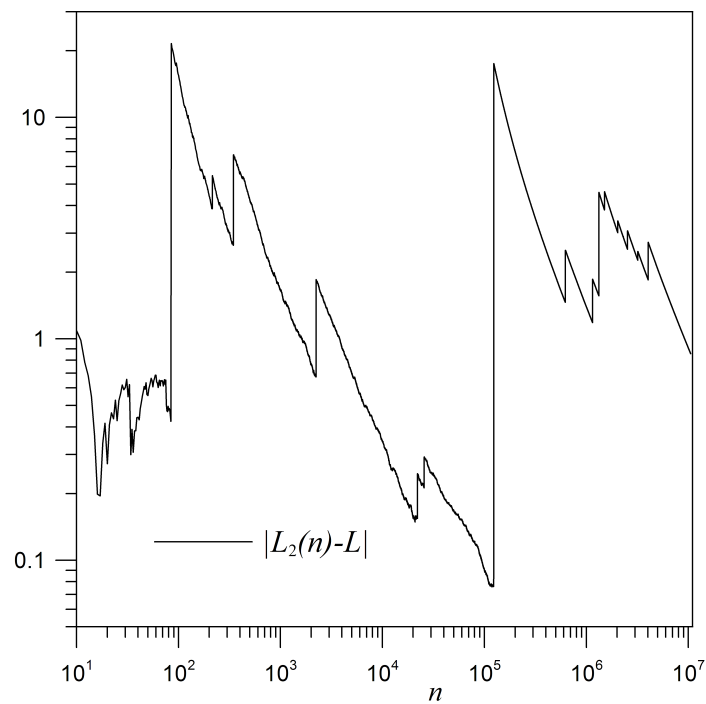

Figure 8: The plot showing the distance to L of the $(Q_2(n))^{1/n}$ obtained from the partial convergents of the continued fraction of $\mathcal{S}$ for $n = 11, \ldots, 10550114$.
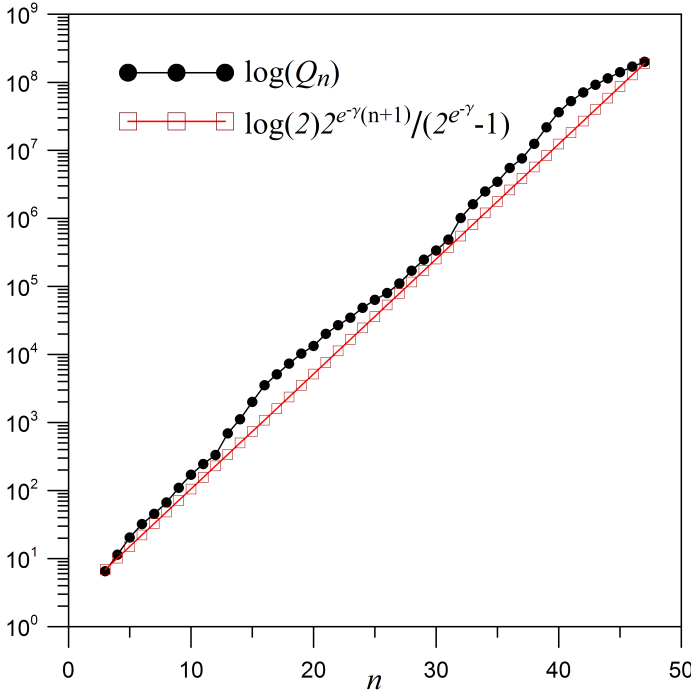
Figure 9: Illustration of the inequality (26) for $3 \leq n \leq 47$. Although the last points seem to coincide in fact $Q_{47} = 2.32928 \ldots \times 10^{86789810}$, while $2^{c2^{48e^{-\gamma}}} = 1.21513 \ldots \times 10^{82034318}$ — hundreds thousands orders of difference!
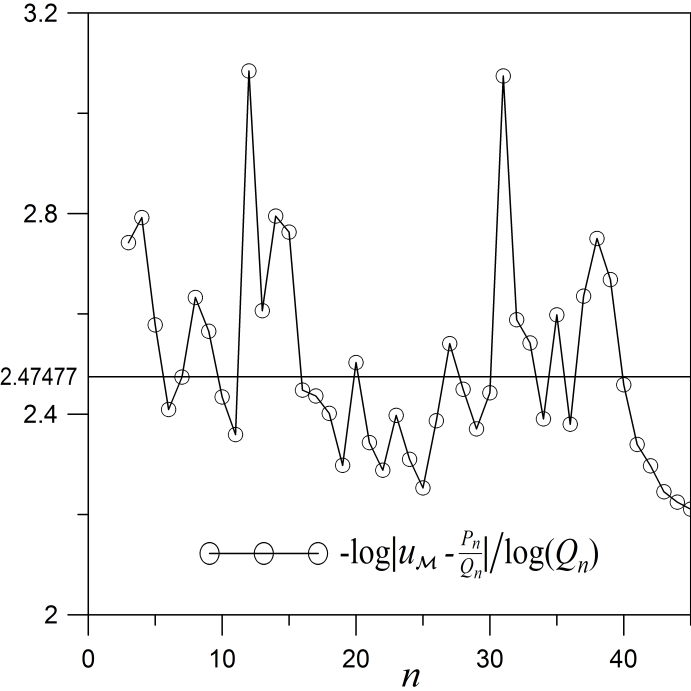

Figure 10: The plot of $-\log |u_{\mathcal{M}} - P_n/Q_n| / \log(Q_n)$ fluctuating around the estimation (30).